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Rough approximations in non-commutative residuated lattices

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Abstract Based on Pawlak's rough set theory, we study and investigate the roughness in non-commutative residuated lattices, which are generalizations of non-commutative fuzzy structures such as MV-algebras and BL-algebras. We give many theorems and examples to describe the rough approximations. Also, to investigate the properties of roughness of subsets (and of course filters) more closely, we consider some different kinds of filters such as Boolean filters and prime filters. Especially, we prove that with respect to some certain filters, the obtained approximations form a Boolean algebra or a pseudo MTL-algebra.

Keywords Algebras of fuzzy logics · Residuated lattice · Boolean algebra · Rough approximation

Mathematics Subject Classification 03G25 · 08A72

1 Introduction

The theory of rough sets, introduced by Pawlak [15], is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key concept in Pawlak's rough set model is that of equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The rough sets theory has often proved to be an excellent mathematical tool for the analysis of a vague description of objects called actions in decision problems. Many different problems can be addressed by rough sets theory. During the last few years this formalism has been approached as a tool used in connection with many different areas of research. There have been investigations of the relations between rough sets theory and fuzzy sets. Rough sets theory has also provided the necessary formalism and ideas for the

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development of some propositional machine learning systems. It has also been used for, among many others, knowledge representation, data mining, dealing with imperfect data, reducing knowledge representation and for analyzing attribute dependencies. The notions of rough relations and rough functions are based on rough sets theory and can be applied as a theoretical basis for rough controllers, among others.

An algebraic approach to rough sets has been given by Iwinski [9]. He also suggested the lattice theoretical approach to rough sets. Estaji et al. [8] also studied rough set theory in lattice theory. In 1994, Biswas and Nanda [2] introduced and discussed the concept of rough groups and rough subgroups (see also [12,13]). Roughness in the other algebraic structures such as semigroups and rings was studied by many authors (see [5,6,11,12,21]). Pomykala and Pomykala [16] showed that the set of rough sets forms a Stone algebra. Comer [4] presented an interesting discussion of rough sets and various algebras related to the study of algebraic logic, such as Stone algebras and relation algebras. Jun applied rough set theory to BCK-algebras [10]. Recently, Rasouli et al. [18] introduced and studied the notion of roughness in MV-algebras. Also, Torkzadeh et al. studied the roughness of BL-algebras [19]. Roughness in (commutative) residuated lattices was also investigated by Rachůneck et al. [17].

In this paper, we consider non-commutative residuated lattices [7,20], which are common structures among algebras associated with fuzzy logics, as the universal set. We give a complete description of algebraic properties of roughness by giving many theorems and examples, and so we extend the results of [17–19]. We characterize the approximations with respect to some types of filters and give some interesting results: we consider different kinds of filters (prime filters and Boolean filters) to study the approximations of sets (or filters) and characterize the rough approximations. Particularly, we show that based on the kinds of filters, the obtained approximations can be a Boolean algebra or a pseudo MTL-algebra, as mentioned in the abstract.

2 Preliminaries

This section is devoted to give some definitions and results from the literature. For more details, we refer to the references [1,3,8,15,20].

2.1 Residuated lattices

Definition 2.1 A structure $(\mathcal{L}, \vee, \wedge, *, \to, \rightsquigarrow, 0, 1)$ of type (2, 2, 2, 2, 2, 0, 0) is called a (non-commutative) *residuated lattice* if

(RL1) $(\mathcal{L}, \vee, \wedge, 0, 1)$ is a bounded lattice,
(RL2) $(\mathcal{L}, *, 1)$ is a monoid,
(RL3) $x * y \leq z$ if and only if $x \leq y \to z$ if and only if $y \leq x \rightsquigarrow z$, for all $x, y, z \in \mathcal{L}$.

- The elements $x \to 0$ and $x \rightsquigarrow 0$ are denoted by $\neg x$ and $\sim x$, respectively.
- A residuated lattice $\mathcal{L}$ is called *involutive* if $x^{-\sim} = x^{\sim -} = x$, for all $x \in \mathcal{L}$.
- A residuated lattice $\mathcal{L}$ is called a *pseudo* MTL-*algebra* if it satisfies (PL) $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$, for all $x, y \in \mathcal{L}$.

Proposition 2.2 *In any residuated lattice $\mathcal{L}$, the following properties hold:*

(1) $x \leq y$ *if and only if* $x \to y = 1$ *if and only if* $x \rightsquigarrow y = 1$,
(2) $x \to x = x \rightsquigarrow x = 1$, $1 \to x = 1 \rightsquigarrow x = x$,

(3) $x \le y$ *implies* $x * z \le y * z$ *and* $z * x \le z * y$,
(4) $x \le y$ *implies* $z \to x \le z \to y$ *and* $z \rightsquigarrow x \le z \rightsquigarrow y$,
(5) $x \le y$ *implies* $y \to z \le x \to z$ *and* $y \rightsquigarrow z \le x \rightsquigarrow z$, *particularly,* $x \le y$ *implies that* $y^- \le x^-$ *and* $y^\sim \le x^\sim$,
(6) $y \le x \to y$ *and* $y \le x \rightsquigarrow y$, $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
(7) $x * y \le x * (x \rightsquigarrow y) \le x \wedge y$, $y * x \le (x \to y) * x \le x \wedge y$,
(8) $x \to y \le (y \to z) \rightsquigarrow (x \to z)$, *particularly,* $x \to y \le y^- \rightsquigarrow x^-$,
(9) $x \rightsquigarrow y \le (y \rightsquigarrow z) \to (x \rightsquigarrow z)$, *particularly,* $x \rightsquigarrow y \le y^\sim \to x^\sim$,
(10) $x \to y \le (z \to x) \to (z \to y)$, $x \rightsquigarrow y \le (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,
(11) $(x * y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$, $(x * y) \to z = x \to (y \to z)$,
(12) $x * (y \to z) \le y \to (x * z)$, $(y \rightsquigarrow z) * x \le y \rightsquigarrow (z * x)$,

Definition 2.3 A nonempty subset F of residuated lattice $\mathcal{L}$ is called a *filter* if

(i) $x, y \in F$ imply $x * y \in F$,
(ii) $x \le y$ and $x \in F$ imply $y \in F$

or equivalently, $1 \in F$, and $x \to y \in F$ (or $x \rightsquigarrow y \in F$) and $x \in F$ imply $y \in F$.

We mention that for nonempty subset X of residuated lattice $\mathcal{L}$, the intersection of any nonempty family of filters of $\mathcal{L}$ which contain X is again a filter, denoted $[x)$, which is called *the filter generated by* X. Also, the set of all filters of $\mathcal{L}$ together with set inclusion, as the partial ordering, forms a complete lattice in which for a family $\{F_i\}_{i\in I}$ of filters, $\wedge F_i = \cap F_i$ and $\vee F_i = [\cup_{i\in I} F_i)$ (see [3]).

The next proposition gives a characterization of these filters.

Proposition 2.4 *If* $X \subseteq \mathcal{L}$, *then*

$$[X) = \{y \in \mathcal{L} : y \ge g_1 * g_2 * \cdots * g_k, g_i \in X, 1 \le i \le k\}.$$

Definition 2.5 A filter F of $\mathcal{L}$ is called a *Boolean filter*

(i) of type 1 if $x \in F$ or $\neg x, \sim x \in F$, for all $x \in \mathcal{L}$,
(ii) of type 2 if $x \vee \sim x \in F$ and $x \vee \neg x \in F$, for all $x \in \mathcal{L}$.

Filter F is said to be *normal* if it satisfies

$$x \to y \in F \Leftrightarrow x \rightsquigarrow y \in F.$$

Every normal filter F of residuated lattice $\mathcal{L}$ induces a congruence θ_F on $\mathcal{L}$ as

$$x\theta_F y \Leftrightarrow x \to y, y \to x \in F.$$

$\mathcal{L}/F$, the set of congruence classes of θ_F together with those operations induced from $\mathcal{L}$ forms a residuated lattice. The induced partial ordering on $\mathcal{L}/F$ is defined as

$$x/\theta_F \le y/\theta_F \Leftrightarrow x \to y \in F.$$

Theorem 2.6 *Let F be a normal filter of* $\mathcal{L}$.

(i) *F is a Boolean filter of type 1 if and only if* $|\mathcal{L}/F| \le 2$.
(ii) *F is a Boolean filter of type 2 if and only if* $\mathcal{L}/F$ *is a Boolean algebra.*

Definition 2.7 Let F be a filter of $\mathcal{L}$. F is called a *prime filter*

(i) *of type 1*, if $x \to y \in F, x \rightsquigarrow y \in F$ or $y \to x \in F, y \rightsquigarrow x \in F$,

(ii) *of type 2*, if $x \vee y \in F$ implies $x \in F$ or $y \in F$,
(iii) *of type 3*, if $(x \rightarrow y) \vee (y \rightarrow x) \in F$ and $(x \rightsquigarrow y) \vee (y \rightsquigarrow x) \in F$.

Theorem 2.8 *Let F be a normal filter of $\mathcal{L}$.*

(i) *F is a prime filter of type 1 if and only if $\mathcal{L}/F$ is a linearly ordered residuated lattice.*
(ii) *F is a prime filter of type 2 if and only if $1_{\mathcal{L}/F}$ is $\vee$-irreducible.*
(iii) *F is a prime filter of type 3 if and only if $\mathcal{L}/F$ is a pseudo MTL-algebra.*

2.2 Rough sets

Definition 2.9 Let U be a universal set and θ an equivalence relation on U. The pair (U, θ) is called the Pawlak's approximation space (or briefly, an *approximation space*).

Definition 2.10 Let (U, θ) be an approximation space. A mapping $Apr : 2^U \longrightarrow 2^U \times 2^U$ with $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where $\underline{Apr}(X) = \{u \in U : [u]_\theta \subseteq X\}$ and $\overline{Apr}(X) = \{u \in U : [u]_\theta \cap X \neq \emptyset\}$ is called a *rough approximation*. In this case, $\underline{Apr}(X)$ and $\overline{Apr}(X)$ are called, respectively, the *lower approximation* and the *upper approximation* of X in (U, θ).

Also, $Bnd_\theta(X) = \overline{Apr}(X) - \underline{Apr}(X)$ is called the *boundary area* of X and $Apr^c(X) = (U - \underline{Apr}(X), U - \overline{Apr}(X))$ is called the *complement* of X in (U, θ).

Definition 2.11 For approximation space (U, θ), $(A, B) \in 2^U \times 2^U$ is called a *rough set* if there exists $X \subseteq U$ such that $(A, B) = Apr(X)$.

Definition 2.12 Let (U, θ) be an approximation space and $X \subseteq U$. Then, with respect to θ, X is called

(i) *definable* if $\underline{Apr}(X) = \overline{Apr}(X)$,
(ii) *empty-interior* if $\underline{Apr}(X) = \emptyset$,
(iii) *empty-exterior* if $\overline{Apr}(X) = \emptyset$.

Proposition 2.13 *In any approximation space (U, θ), the following hold:*

(1) $\underline{Apr}(X) \subseteq X \subseteq \overline{Apr}(X)$,
(2) *$\emptyset$ and U are definable with respect to every equivalence relation on U,*
(3) *if $X \subseteq Y$, then $\underline{Apr}(X) \subseteq \underline{Apr}(Y)$ and $\overline{Apr}(X) \subseteq \overline{Apr}(Y)$,*
(4) $\underline{Apr}(X \cap Y) = \underline{Apr}(X) \cap \underline{Apr}(Y)$,
(5) $\overline{Apr}(X \cap Y) \subseteq \overline{Apr}(X) \cap \overline{Apr}(Y)$,
(6) $\overline{Apr}(X \cup Y) = \overline{Apr}(X) \cup \overline{Apr}(Y)$,
(7) $\underline{Apr}(X \cup Y) \supseteq \underline{Apr}(X) \cup \underline{Apr}(Y)$,
(8) $\underline{Apr}(\underline{Apr}(X)) = \underline{Apr}(X)$,
(9) $\overline{Apr}(\overline{Apr}(X)) = \overline{Apr}(X)$,
(10) $\overline{Apr}(\underline{Apr}(X)) = \underline{Apr}(X)$,
(11) $\underline{Apr}(\overline{Apr}(X)) = \overline{Apr}(X)$.

3 Properties of approximations in residuated lattices

First of all, we mention that throughout the paper, $\mathcal{L}$ will denote a residuated lattice, F a normal filter of $\mathcal{L}$ and θ_F the congruence induced by F. Hence, the approximation space is denoted by $(\mathcal{L}, F)$. The upper approximation and lower approximation with respect to θ_F (or briefly, with respect to F) are denoted by $\underline{Apr}_F$ and $\overline{Apr}_F$, respectively. Hence, for a nonempty set X, if $\overline{Apr}_F(X) = \underline{Apr}_F(X)$, we say that X is definable with respect to F.

Remark 3.1 (1) We observe that for the trivial filter $F = \{1\}$, every equivalence class $[x]_F$ is singleton, more precisely, $[x]_F = \{x\}$. Hence, for every $X \subseteq \mathcal{L}$,

$$Apr_F(X) = (X, X)$$

(2) For $F = \mathcal{L}$ we have $[u]_{\mathcal{L}} = \mathcal{L}$, for all $u \in \mathcal{L}$, and so for every proper subset X of $\mathcal{L}$ we get $\underline{Apr}_{\mathcal{L}}(X) = \emptyset$ and $\overline{Apr}_{\mathcal{L}}(X) = \mathcal{L}$. Hence,

$$Apr_{\mathcal{L}}(X) = (\emptyset, \mathcal{L})$$

Thus, in a residuated lattice $\mathcal{L}$

(1) any nonempty subset of $\mathcal{L}$ is definable with respect to the filter $\{1\}$.
(2) any proper subset of $\mathcal{L}$ is empty-interior with respect to $\mathcal{L}$.

From the defn we know that for a set to be definable with respect to a filter, it must $\underline{Apr}_F(X) = X = \overline{Apr}_F(X)$. We show that one of the equalities can be omitted. Indeed, for a set X to be definable, with respect to a filter F, it suffices $\underline{Apr}_F(X) = X$ or $\overline{Apr}_F(X) = X$.

Theorem 3.2 *For nonempty subset X of $\mathcal{L}$,*

$$\underline{Apr}_F(X) = X \quad \Leftrightarrow \quad \overline{Apr}_F(X) = X.$$

Proof Assume that $\underline{Apr}_F(X) = X$ and $x \in \overline{Apr}_F(X)$. Then, $[x]_F \cap X \neq \emptyset$, whence $x \in [x]_F = [u]_F \subseteq X$, for $u \in X$. This implies that $\overline{Apr}_F(X) \subseteq X$. Hence, $\overline{Apr}_F(X) = X$; i.e., X is definable with respect to F.

Now, we assume that $\overline{Apr}_F(X) = X$ and $x \in X$. Then, $[x]_F \cap X \neq \emptyset$. Let $a \in [x]_F$. Then, $[a]_F = [x]_F$, whence $[a]_F \cap X \neq \emptyset$. This implies that $a \in \overline{Apr}_F(X) = X$, whence $[x]_F \subseteq X$. Then, $x \in \underline{Apr}_F(X)$, i.e., $X \subseteq \underline{Apr}_F(X)$, proving $\underline{Apr}_F(X) = X$. □

Corollary 3.3 *Nonempty subset X of $\mathcal{L}$ is definable, with respect to normal filter F, if and only if $\underline{Apr}_F(X) = X$ or $\overline{Apr}_F(X) = X$* (or both).

From now on, in this paper, the set of all definable subsets of $\mathcal{L}$ is denoted by $DF(\mathcal{L})$.
Notation. For nonempty subset X of $\mathcal{L}$, we let

$$X^- = \{x^- : x \in X\} \text{ and } X^\sim = \{x^\sim : x \in X\}.$$

Obviously, in an involutive residuated lattice $\mathcal{L}$, $X^{-\sim} = X = X^{\sim -}$, for any nonempty subset X of $\mathcal{L}$.

Proposition 3.4 *Assume that $\mathcal{L}$ is an involutive residuated lattice and X a nonempty subset of $\mathcal{L}$. Then*

(i) $\overline{Apr}_F(X^-) = \overline{Apr}_F(X)^-$,
(ii) $\underline{Apr}_F(X^-) = \underline{Apr}_F(X)^-$,
(iii) $\overline{Apr}_F(X^\sim) = \overline{Apr}_F(X)^\sim$,
(iv) $\underline{Apr}_F(X^\sim) = \underline{Apr}_F(X)^\sim$.

Proof (i) Let $x \in \overline{Apr}_F(X^-)$. Then, there exists $z \in X^-$ such that $x \rightsquigarrow z \in F$ and $z \rightsquigarrow x \in F$. From $x \rightsquigarrow z \leq z^\sim \to x^\sim$ it follows that $z^\sim \to x^\sim \in F$. Similarly, from $z \rightsquigarrow x \leq x^\sim \to z^\sim$ it follows that $x^\sim \to z^\sim \in F$. Since $z^\sim \in X^{-\sim} = X$ we get $x^\sim \in \overline{Apr}_F(X)$, whence $x = x^{\sim -} \in \overline{Apr}_F(X)^-$, proving $\overline{Apr}_F(X^-) \subseteq \overline{Apr}_F(X)^-$.

Conversely, let $x \in \overline{Apr}_F(X)^-$. Then, $x = y^-$, where $[y]_F \cap X \neq \emptyset$. This implies that $y \to z, z \to y \in F$, for $z \in X$. From $z \to y \leq y^- \rightsquigarrow z^-$ it follows that $x \rightsquigarrow z^- = y^- \rightsquigarrow z^- \in F$. Similarly, we can deduce that $z^- \rightsquigarrow x \in F$. This implies that $z^- \in [x]_F \cap X^-$, showing that $x \in \overline{Apr}_F(X^-)$. Hence, $\overline{Apr}_F(X)^- \subseteq \overline{Apr}_F(X^-)$. Therefore, (i) holds.

(ii) Let $x \in \underline{Apr}_F(X^-)$, i.e., $[x]_F \subseteq X^-$. Let $a \in [x^\sim]_F$. Then, $a \to x^\sim \in F$ and $x^\sim \to a \in F$. Now, from $a \to x^\sim \leq x^{\sim -} \rightsquigarrow a^- = x \rightsquigarrow a^-$ and that $a \to x^\sim \in F$ it follows that $x \rightsquigarrow a^- \in F$. Similarly, $a^- \rightsquigarrow x \in F$. Thus, $a^- \in [x]_F \subseteq X^-$, whence $a \in X$. Hence, $[x^\sim]_F \subseteq X$ and so $x^\sim \in \underline{Apr}_F(X)$, whence $x = x^{\sim -} \in \underline{Apr}_F(X)^-$, proving $\underline{Apr}_F(X^-) \subseteq \underline{Apr}_F(X)^-$.
Now, let $x \in \underline{Apr}_F(X)^-$. Then, $x = y^-$, where $[y]_F \subseteq X$. Let $a \in [x]_F$. From $a \rightsquigarrow x \leq x^\sim \to a^\sim$ and that $a \rightsquigarrow x \in F$ it follows that $y \to a^\sim = x^\sim \to a^\sim \in F$. By a similar way, it is shown that $a^\sim \to y \in F$. Hence, $a^\sim \in [y]_F \subseteq X$, whence $a \in X^-$. This implies that $[x]_F \subseteq X^-$, i.e., $x \in \underline{Apr}_F(X^-)$, proving $\underline{Apr}_F(X)^- \subseteq \underline{Apr}_F(X^-)$. Therefore, $\underline{Apr}_F(X)^- = \underline{Apr}_F(X^-)$, proving (ii).
The proofs of (iii) and (iv) are similar to the proofs of (i) and (ii), respectively. □

Proposition 3.5 *Let X be a nonempty subset and F be a normal filter of $\mathcal{L}$. Then,*

(i) *if $F \cap X \neq \emptyset$, then $F \subseteq \overline{Apr}_F(X)$,*
(ii) *$F \subseteq X$ if and only if $F \subseteq \underline{Apr}_F(X)$,*
(iii) *$X \subseteq F$ if and only if $\overline{Apr}_F(X) = F$,*
(iv) *if X is a filter, then X is definable with respect to F if and only if $F \subseteq X$.*
(v) *if X is a subalgebra of $\mathcal{L}$, then so is $\overline{Apr}_F(X)$,*
(vi) *if $\mathcal{L}$ is linearly ordered and X is a filter of $\mathcal{L}$, then so is $\overline{Apr}_F(X)$.*

Proof (i), (ii) and (v) are proved easily. We prove (iii) and (iv).

(iii) Assume that $X \subseteq F$ and $a \in \overline{Apr}_F(X)$. Then, $[a]_F \cap X \neq \emptyset$ and so $[a]_F \cap F \neq \emptyset$, whence $y \to a \in F$, for some $y \in F$. This implies that $a \in F$, proving that $\overline{Apr}_F(X) \subseteq F$. Now, for $z \in F$ we have $[z]_F = F$, whence $[z]_F \cap X = F \cap X = X \neq \emptyset$. i.e, $z \in \overline{Apr}_F(X)$. Thus, $F \subseteq \overline{Apr}_F(X)$, proving $\overline{Apr}_F(X) = F$. The converse is obvious.

(iv) Assume that $F \subseteq X$ and $x \in \overline{Apr}_F(X)$. Then, there exists $y \in X$ such that $y \to x \in F \subseteq X$, whence $x \in X$. This implies that $\overline{Apr}_F(X) \subseteq X$, proving X is definable with respect to F, by Corollary 3.3. The converse follows from (ii).

(vi) It is similar to the proof of [19, Theorem 3.14(3)]. □

Proposition 3.6 *Let G be a normal filter of $\mathcal{L}$ containing F. Then for every $X \subseteq \mathcal{L}$ we have*

(i) $\overline{Apr}_F(X) \subseteq \overline{Apr}_G(X)$,
(ii) $\underline{Apr}_G(X) \subseteq \underline{Apr}_F(X)$.

Proof (i) Let $a \in \overline{Apr}_F(X)$. Then, there exists $x \in X$ such that $a \to x \in F$ and $x \to a \in F \subseteq G$. This implies that $x \in [a]_G$ and so $a \in \overline{Apr}_G(X)$.

The proof of (ii) is similar. □

The next corollary follows immediately from Proposition 3.6 and so we omit the proof.

Corollary 3.7 *Let $\{G_i\}_{i\in I}$ be a nonempty family of normal filters of $\mathcal{L}$ and X be a nonempty subset of $\mathcal{L}$. Then,*

(i) $\overline{Apr}_{\cap G_i}(X) \subseteq \cap_{i\in I}\overline{Apr}_{G_i}(X) \subseteq \cup_{i\in I}\overline{Apr}_{G_i}(X) \subseteq \overline{Apr}_{\vee G_i}(X)$,
(ii) $\underline{Apr}_{\vee G_i}(X) \subseteq \cap_{i\in I}\underline{Apr}_{G_i}(X) \subseteq \cup_{i\in I}\underline{Apr}_{G_i}(X) \subseteq \underline{Apr}_{\cap G_i}(X)$.

Theorem 3.9 gives some conditions under which the equalities in Corollary 3.7 maybe hold. Before we state and prove it we give a lemma (Lemma 3.8) which states, in any linearly ordered residuated lattice, every two filters are comparable with respect to set inclusion as the partial ordering.

Lemma 3.8 *Assume that $\mathcal{L}$ is linearly ordered. Then, for any two filters F and G of $\mathcal{L}$ we have $F \subseteq G$ or $G \subseteq F$.*

Proof Assume that $F \nsubseteq G$. Then, there exists $f \in F$ such that $f \notin G$. Let $g \in G$. Since $\mathcal{L}$ is linear, so $f \leq g$ or $g \leq f$. If $g \leq f$, since G is a filter we must have $f \in G$, which is a contradiction. Thus, $f \leq g$, which implies that $g \in F$, proving $G \subseteq F$. □

Theorem 3.9 *Let $\{G_i\}_{i\in I}$ be a nonempty family of normal filters of $\mathcal{L}$ and $X \subseteq \mathcal{L}$ be nonempty.*

(i) *If X is definable with respect to G_j, for some $j \in I$, or $\mathcal{L}$ is linearly ordered then*

$$\overline{Apr}_{\cap G_i}(X) = \cap\overline{Apr}_{G_i}(X).$$

(ii) *If X is a filter of $\mathcal{L}$ containing G_i* (for all $i \in I$)*, then*

$$\overline{Apr}_{\cap G_i}(X) = \cap_{i\in I}\overline{Apr}_{G_i}(X) = \cup_{i\in I}\overline{Apr}_{G_i}(X) = \overline{Apr}_{\vee G_i}(X) = X$$

and

$$\underline{Apr}_{\cap G_i}(X) = \cap_{i\in I}\underline{Apr}_{G_i}(X) = \cup_{i\in I}\underline{Apr}_{G_i}(X) = \underline{Apr}_{\vee G_i}(X) = X.$$

Proof We assume that $\mathcal{L}$ is linearly ordered. By Lemma 3.8, for each $i, j \in I$, $G_i \subseteq G_j$ or $G_j \subseteq G_i$. Anyway, G_i's form a chain

$$G_{i_1} \subseteq G_{i_2} \subseteq \cdots,$$

where $i_1, i_2, \ldots \in I$. In this case,

$$\overline{Apr}_{\cap G_i}(X) = \overline{Apr}_{G_{i_1}}(X) \text{ and } \cap\,\overline{Apr}_{G_i}(X) = \overline{Apr}_{G_{i_1}}(X).$$

The other cases follows from Theorem 3.2 and Proposition 3.5(iv). □

Example 3.10 Let $\mathcal{L} = \{0, a, b, c, d, e, f, 1\}$ be a (non-linear) lattice whose Hasse diagram is below (see Fig. 1). We define the operations $*$, $\to=\rightsquigarrow$ on $\mathcal{L}$ as shown in Tables 1 and 2. Routine calculations show that $(\mathcal{L}; \vee, \wedge, *, \to, 0, 1)$ is a (commutative) residuated lattice in which

$$F_1 = \{1, f\},\ F_2 = \{1, e\},\ F_3 = \{1, d, e, f\} \text{ and } F_4 = \{1, a, c, e\}$$

are (normal) filters of $\mathcal{L}$ and the equivalence classes are

$$[0]_{F_1} = \{0, a\},\ [b]_{F_1} = \{b, c\},\ [d]_{F_1} = \{d, e\},\ [1]_{F_1} = \{1, f\} = F_1,$$
$$[0]_{F_2} = \{0, b\},\ [a]_{F_2} = \{a, c\},\ [d]_{F_2} = \{d, f\},\ [1]_{F_2} = \{1, e\} = F_2,$$
$$[0]_{F_3} = \{0, a, b, c\},\ [1]_{F_3} = F_3,$$
$$[0]_{F_4} = \{0, b, d, f\},\ [1]_{F_4} = F_4.$$

Fig. 1 The Hasse diagram of $\mathcal{L}$

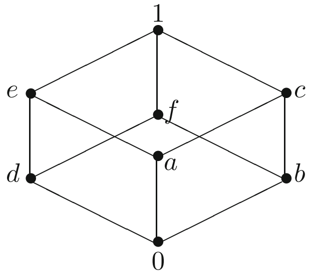

Table 1 The action of '$*$' on $\mathcal{L}$

$*$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	0	a	0	a
b	0	0	b	b	0	0	b	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	0	a	d	e	f	e
f	0	0	b	b	d	d	f	f
1	0	a	b	c	d	e	f	1

Table 2 The action of '$\rightarrow$' on $\mathcal{L}$

$\rightarrow=\rightsquigarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	f	1	f	1	f	1	f	1
b	e	e	1	1	e	e	1	1
c	d	e	f	1	d	d	d	1
d	c	a	b	c	1	1	1	c
e	b	c	b	c	f	1	f	1
f	a	a	c	c	e	e	1	1
1	0	a	b	c	d	e	f	1

(i) Let $X = \{b, e\}$. We can see that X does not contain F_3 and F_4. It is easily checked that $\overline{Apr}_{F_3}(X) = \mathcal{L} = \overline{Apr}_{F_4}(X)$, whereas $\overline{Apr}_{F_3 \cap F_4}(X) = \overline{Apr}_{F_2}(X) = \{0, b, e, 1\}$. Thus,

$$\overline{Apr}_{F_3 \cap F_4}(X) \neq \overline{Apr}_{F_3}(X) \cap \overline{Apr}_{F_4}(X).$$

(ii) Let $X = \{a\}$, which is not a filter of $\mathcal{L}$ and also does not contain F_3 and F_4. Then, $\overline{Apr}_{F_3}(X) = \{0, a, b, c\}$ and $\overline{Apr}_{F_4}(X) = \{1, a, c, e\}$ and so $\overline{Apr}_{F_3}(X) \cup \overline{Apr}_{F_4}(X) = \{0, a, b, c, e, 1\}$, whereas $\overline{Apr}_{F_3 \cap F_4} = \overline{Apr}_{F_2}(X) = \{a, c\}$. Furthermore, $\overline{Apr}_{F_3 \vee F_4}(X) = \overline{Apr}_{\mathcal{L}}(X) = \mathcal{L}$. Thus,

$$\overline{Apr}_{F_3}(X) \cap \overline{Apr}_{F_4}(X) \neq \overline{Apr}_{F_3}(X) \cup \overline{Apr}_{F_4}(X) \neq \overline{Apr}_{F_3 \vee F_4}(X).$$

(iii) Let $X = \{a, b, c\}$. Then, $\underline{Apr}_{F_1}(X) = \{b, c\}$ and $\underline{Apr}_{F_2}(X) = \{a, c\}$, whence $\underline{Apr}_{F_1}(X) \cap \underline{Apr}_{F_2}(X) = \{c\}$, whereas $\underline{Apr}_{F_1 \cap F_2}(X) = \underline{Apr}_{\{1\}}(X) = X$. Hence,

$$\underline{Apr}_{F_1}(X) \cap \underline{Apr}_{F_2}(X) \neq \underline{Apr}_{F_1 \cap F_2}(X).$$

Moreover, $\underline{Apr}_{F_4}(X) = \emptyset$ and so $\underline{Apr}_{F_4}(X) \cup \underline{Apr}_{F_2}(X) = \{a, c\}$ and $\underline{Apr}_{F_2 \vee F_4}(X) = \underline{Apr}_{F_4}(X) = \emptyset$, showing that

$$\underline{Apr}_{F_2}(X) \cap \underline{Apr}_{F_4}(X) \neq \underline{Apr}_{F_4}(X) \cup \underline{Apr}_{F_2}(X) \neq \underline{Apr}_{F_2 \vee F_4}(X).$$

For nonempty subsets $X_1, \ldots, X_n$ of $\mathcal{L}$, let

$$X_1 * \cdots * X_n = \{a \in \mathcal{L} : a \geq x_1 * \cdots * x_n, \text{ for } x_i \in X_i, i = 1, 2, \ldots, n\}.$$

Remark 3.11 We mention that when X_i $(1 \leq i \leq n)$ is a normal filter of $\mathcal{L}$, $X_1 * \cdots * X_n$ is also a normal filter of $\mathcal{L}$ (see [3]).

Theorem 3.14 gives a characterization of $\overline{Apr}_F(X_1 * X_2 * \cdots * X_n)$. For this, we first give two lemmas.

Lemma 3.12 *Let F be a normal filter of $\mathcal{L}$. Then, for each $f \in F$ and each $x \in \mathcal{L}$ we have $[f * x]_F = [x]_F$.*

Proof It follows from Proposition 2.2(11) and (12). □

Lemma 3.13 *Assume that $\mathcal{L}$ is linearly ordered and F a filter of $\mathcal{L}$. If $x, y \in \mathcal{L}$ be such that $[x]_F \neq [y]_F$ and $x \leq y$, then for all $t \in [x]_F$ and $s \in [y]_F$ we have $t \leq s$.*

Proof The proof is the same as the proof of [19, Lemma 3.13]. □

Theorem 3.14 *Let X_i $(1 \leq i \leq n)$ be a nonempty subset of $\mathcal{L}$. Then,*

(i) $\underline{Apr}_F(X_1) * \cdots * \underline{Apr}_F(X_n) \subseteq \underline{Apr}_F(X_1 * \cdots * X_n)$,
(ii) *if X_i $(1 \leq i \leq n)$ is definable, with respect to F,*

$$\underline{Apr}_F(X_1) * \cdots * \underline{Apr}_F(X_n) = \underline{Apr}_F(X_1 * \cdots * X_n),$$

(iii) $\overline{Apr}_F(X_1 * X_2 * \cdots * X_n) \subseteq \overline{Apr}_F(X_1) * \overline{Apr}_F(X_2) * \cdots * \overline{Apr}_F(X_n)$,
(iv) *under each of the conditions*

(a) *$\mathcal{L}$ is linearly ordered,*
(b) *$X_i \subseteq F$, for all $i = 1, 2, \ldots, n$,*
(c) *X_i is definable, for all $i = 1, 2, \ldots, n$,*
(d) *$0 \in X_j$, for some $j \in \{1, 2, \ldots, n\}$, we have*

$$\overline{Apr}_F(X_1 * X_2 * \cdots * X_n) = \overline{Apr}_F(X_1) * \overline{Apr}_F(X_2) * \cdots * \overline{Apr}_F(X_n).$$

Proof (i) Let $x \in \underline{Apr}_F(X_1) * \cdots * \underline{Apr}_F(X_n)$. Then,

$$x \geq x_1 * \cdots * x_n, \tag{3.1}$$

where $[x_i]_F \subseteq X_i$, for $i = 1, 2, \ldots, n$. Now, let $a \in [x]_F$. From (3.1) and that $a \geq (a \to x) * x$ we get

$$a \geq (a \to x) * x_1 * \cdots * x_n = [(a \to x) * x_1] * x_2 * \cdots * x_n. \tag{3.2}$$

From Lemma 3.12 and that $a \to x \in F$ we deduce that

$$(a \to x) * x_1 \in [(a \to x) * x_1]_F = [x_1]_F \subseteq X_1,$$

whence, combining (3.2), we get $a \in X_1 * \cdots * X_n$. Hence, $x \in \underline{Apr}_F(X_1 * \cdots * X_n)$, proving $\underline{Apr}_F(X_1) * \cdots * \underline{Apr}_F(X_n) \subseteq \underline{Apr}_F(X_1 * \cdots * X_n)$.

(ii) It is easy.

(iii) Let $a \in \overline{Apr}_F(X_1 * X_2 * \cdots * X_n)$. Then, $[a]_F \cap (X_1 * X_2 * \cdots * X_n) \neq \emptyset$, whence for $x_i \in X_i$ $(1 \leq i \leq n)$ and $b \in [a]_F$ we have

$$b \geq x_1 * x_2 * \cdots * x_n. \tag{3.3}$$

From Proposition 2.2 we know that $a \geq (b \to a) * b$, combining (3.3) we get

$$a \geq (b \to a) * x_1 * x_2 * \cdots * x_n = [(b \to a) * x_1] * x_2 * \cdots * x_n. \tag{3.4}$$

Lemma 3.12 and that $b \to a \in F$ imply that $[(b \to a) * x_1]_F = [x_1]_F$, showing that $(b \to a) * x_1 \in \overline{Apr}_F(X_1)$. Considering (3.4) and that $x_i \in \overline{Apr}_F(X_i)$, for $i = 2, 3, \ldots, n$, we get $a \in \overline{Apr}_F(X_1) * \overline{Apr}_F(X_2) * \cdots * \overline{Apr}_F(X_n)$. Hence,

$$\overline{Apr}_F(X_1 * X_2 * \cdots * X_n) \subseteq \overline{Apr}_F(X_1) * \overline{Apr}_F(X_2) * \cdots * \overline{Apr}_F(X_n).$$

(iv) (a) We assume that $\mathcal{L}$ is linearly ordered. Let $a \in \overline{Apr}_F(X_1) * \cdots * \overline{Apr}_F(X_n)$. Then, $a \geq x_1 * \cdots * x_n$, where $x_i \in \overline{Apr}_F(X_i)$, for $i = 1, 2, \ldots, n$, and so there exists $b_i \in X_i$ such that $b_i \in [x_i]_F$. If $[a]_F = [x_1 * \cdots * x_n]_F$, since

$$x_1 * \cdots * x_n \in X_1 * \cdots * X_n \subseteq \overline{Apr}_F(X_1 * \cdots * X_n),$$

we must have $a \in \overline{Apr}_F(X_1 * \cdots * X_n)$. Otherwise, from Lemma 3.13 it follows that $a \geq b_1 * \cdots * b_n \in X_1 * \cdots * X_n$. Hence,

$$\overline{Apr}_F(X_1) * \cdots * \overline{Apr}_F(X_n) \subseteq \overline{Apr}_F(X_1 * \cdots * X_n).$$

(b) If $X_i \subseteq F$, for $i = 1, 2, \ldots, n$, on one hand $\overline{Apr}_F(X_i) = F$, by Proposition 3.5, and on the other hand $X_1 * X_2 * \cdots * X_n \subseteq F$. Hence,

$$\overline{Apr}_F(X_1 * X_2 * \cdots * X_n) = F = \overline{Apr}_F(X_1) * \overline{Apr}_F(X_2) * \cdots * \overline{Apr}_F(X_n).$$

(c) If X_i is definable, for $1 \leq i \leq n$, so

$$\overline{Apr}_F(X_1) * \overline{Apr}_F(X_2) * \cdots * \overline{Apr}_F(X_n) = X_1 * \cdots * X_n$$

and hence

$$\overline{Apr}_F(X_1 * X_2 * \cdots * X_n) = X_1 * \cdots * X_n.$$

(d) We observe that if $0 \in X_j$, for some $j \in \{1, 2, \ldots, n\}$, then $X_1 * \cdots * X_n = \mathcal{L}$, and so $\overline{Apr}_F(X_1 * \cdots * X_n) = \mathcal{L}$. By (iii), we get $\overline{Apr}_F(X_1) * \cdots * \overline{Apr}_F(X_n) = \mathcal{L}$. □

Examples 3.15 and 3.16 show that the equalities in Theorem 3.14 may not hold, in general.

Example 3.15 Let $\mathcal{L} = \{0, a, b, c, d, 1\}$ be a lattice whose Hasse diagram is below (see Fig. 2). We define the operations $*$, $\to$, $\rightsquigarrow$ on $\mathcal{L}$ as shown in Tables 3, 4 and 5. Routine calculations show that $(\mathcal{L}; \vee, \wedge, *, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice in which $F = \{1, d\}$ is a normal filter of $\mathcal{L}$ (see [14]). It is easy to see that

$$[0]_F = \{0\},\ [a]_F = \{a, b\},\ [c]_F = [d]_F = [1]_F = F.$$

Now, let $X = \{d\}$ and $Y = \{c\}$. Then, $X * Y = \{c, d, 1\}$ and $\underline{Apr}_F(X * Y) = X * Y$, whereas $\underline{Apr}_F(X) * \underline{Apr}_F(Y) = \emptyset$, because $\underline{Apr}_F(X) = \emptyset = \underline{Apr}_F(Y)$.

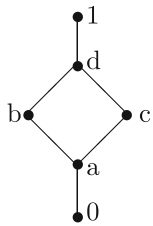

Fig. 2 The Hasse diagram of $\mathcal{L}$

Table 3 The action of '$*$' on $\mathcal{L}$

$*$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a
b	0	a	a	a	a	a	b
c	0	a	a	c	c	c	c
d	0	a	a	c	c	c	d
e	0	a	b	c	d	e	e
1	0	a	b	c	d	e	1

Table 4 The action of '$\rightarrow$' on $\mathcal{L}$

$\rightarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	d	1	1	1
b	0	d	1	d	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	d	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

Table 5 The action of '$\rightsquigarrow$'on $\mathcal{L}$

$\rightsquigarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	e	1	e	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	e	1	1	1
e	0	a	b	c	d	1	1
1	0	a	b	c	d	e	1

Example 3.16 Consider the residuated lattice $\mathcal{L}$ given in Example 3.10. It is easy to check that $F_5 = \{1, c\}$ is a filter of $\mathcal{L}$ and

$$[0]_{F_5} = \{0\},\ [a]_{F_5} = \{a, e\},\ [b]_{F_5} = \{b, f\},\ [d]_{F_5} = \{d\},\ [1]_{F_5} = F_5.$$

Now, let $X = \{e\}$ and $Y = \{f\}$. Then, $X * Y = \{1, f\}$ and $\overline{Apr}_{F_5}(X * Y) = \{1, b, c, f\}$, whereas $\overline{Apr}_{F_5}(X) * \overline{Apr}_{F_5}(Y) = \{a, e\} * \{b, f\} = \mathcal{L}$, showing that

$$\overline{Apr}_{F_5}(X) * \overline{Apr}_{F_5}(Y) \neq \overline{Apr}_{F_5}(X * Y).$$

Lemma 3.17 *Let $F_1, F_2, \ldots, F_n$ be filters of $\mathcal{L}$. Then, $F_i \subseteq F_1 * \cdots * F_n$, for all $i = 1, 2, \ldots, n$. Particularly, when F_i's are normal, then $[x]_{F_i} \subseteq [x]_{F_1 * \cdots * F_n}$, for all $x \in \mathcal{L}$.*

Proof Straightforward. □

Theorem 3.18 *Let F_i $(1 \leq i \leq n)$ be a normal filter and X a nonempty subset of $\mathcal{L}$. Then,*

(i) $\overline{Apr}_{F_1 * \cdots * F_n}(X) \subseteq \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X)$,
(ii) $\underline{Apr}_{F_1 * \cdots * F_n}(X) \subseteq \underline{Apr}_{F_1}(X) * \cdots * \underline{Apr}_{F_n}(X)$,
(iii) $\overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X) \subseteq \overline{Apr}_{F_1 * \cdots * F_n}(X) * \cdots * \overline{Apr}_{F_1 * \cdots * F_n}(X)$ *(n times)*,

Proof (i) Let $z \in \overline{Apr}_{F_1 * \cdots * F_n}(X)$. Then $x \to z \geq f_1 * \cdots * f_n$, where $x \in X$ and $f_i \in F_i$, for $i = 1, 2, \ldots, n$. Now, from (RL3) and Proposition 2.2(6) it follows that

$$z \geq f_1 * \cdots * f_n * x \geq (f_1 * x) * \cdots * (f_n * x). \tag{3.5}$$

On the other hand, from Lemma 3.12, we know that $[f_i * x]_{F_i} = [x]_{F_i}$, for all $i = 1, 2, \ldots, n$. Since $x \in X$, so $[f_i * x]_{F_i} \cap X \neq \emptyset$, whence $f_i * x \in \overline{Apr}_{F_i}(X)$. Combining (3.5) it follows that $z \in \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X)$, proving

$$\overline{Apr}_{F_1 * \cdots * F_n}(X) \subseteq \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X).$$

(ii) and (iii) follow from Lemma 3.17 and Propositions 2.2(6) and 3.6.
(iii) It follows from Lemma 3.17 and Proposition 3.6.

□

Theorem 3.19 *Let F_i $(1 \leq i \leq n)$ be a normal filter and X a nonempty subset of $\mathcal{L}$.*

(i) *If $\mathcal{L}$ is linearly ordered and X is a filter of $\mathcal{L}$, then*

$$\overline{Apr}_{F_1 * \cdots * F_n}(X) = \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X).$$

(ii) *If X is a filter of $\mathcal{L}$ containing F_i, for all $i = 1, 2, \ldots, n$, or $\mathcal{L}$ is commutative and X a filter of $\mathcal{L}$, then*

$$\underline{Apr}_{F_1 * \cdots * F_n}(X) = \underline{Apr}_{F_1}(X) * \cdots * \underline{Apr}_{F_n}(X).$$

(iii) *If X is a filter of $\mathcal{L}$ which is definable with respect to F_i, for $i = 1, 2, \ldots, n$, or $\mathcal{L}$ is commutative and X a subalgebra of $\mathcal{L}$, then*

$$\overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X) = \overline{Apr}_{F_1 * \cdots * F_n}(X) * \cdots * \overline{Apr}_{F_1 * \cdots * F_n}(X) \text{ (n times)}.$$

Proof (i) Let $x \in \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X)$. Then, $x \geq x_1 * \cdots * x_n$, where $[x_i]_{F_i} \cap X \neq \emptyset$. From Lemma 3.17, it follows that $[x_i]_{F_1 * \cdots * F_n} \cap X \neq \emptyset$, means that $x_i \in \overline{Apr}_{F_1 * \cdots * F_n}(X)$, whence $x_1 * \cdots * x_n \in \overline{Apr}_{F_1 * \cdots * F_n}(X)$, by Proposition 3.5(vi). This implies that $x \in \overline{Apr}_{F_1 * \cdots * F_n}(X)$, proving

$$\overline{Apr}_{F_1 * \cdots * F_n}(X) = \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X).$$

(ii) We assume that X is a filter of $\mathcal{L}$ containing F_i, for $i = 1, 2, \ldots, n$. Then, X is definable with respect to F_i and also with respect to $F_1 * \cdots * F_n$. Hence,

$$\underline{Apr}_{F_1*\cdots*F_n}(X) = X = \underline{Apr}_{F_1}(X) * \cdots * \underline{Apr}_{F_n}(X).$$

Now, we assume that $\mathcal{L}$ is commutative and X is a filter of $\mathcal{L}$. Let X be an element of $\underline{Apr}_{F_1}(X) * \cdots * \underline{Apr}_{F_n}(X)$. Then,

$$x \geq x_1 * \cdots * x_n, \tag{3.6}$$

where $[x_i]_{F_i} \subseteq X$. Let $a \in [x]_{F_1*\cdots*F_n}$. Then, there is $f_i \in F_i$, for all $i = 1, 2, \ldots, n$, such that $x \to a \geq f_1 * \cdots * f_n$, whence $a \geq f_1 * \cdots * f_n * x$. Combining (3.6) and observing the commutativity of $\mathcal{L}$, we get

$$a \geq (f_1 * \cdots * f_n) * (x_1 * \cdots * x_n) = (f_1 * x_1) * \cdots * (f_n * x_n).$$

On the other hand, from Lemma 3.12 we get $f_i * x_i \in [f_i * x_i]_{F_i} = [x_i]_{F_i} \subseteq X$, whence $(f_1 * x_1) * \cdots (f_n * x_n) \in X$. Hence, $a \in X$, proving $[x]_{F_1*\cdots*F_n} \subseteq X$. Thus,

$$\underline{Apr}_{F_1}(X) * \cdots * \underline{Apr}_{F_n}(X) \subseteq \underline{Apr}_{F_1*\cdots*F_n}(X).$$

(iii) We observe that when X is a filter which is definable with respect to F_i, for all $i = 1, 2, \ldots, n$, so $F_i \subseteq X$, by Proposition 3.5(iv). Hence, $F_1 * \cdots * F_n \subseteq X$ and so X is definable with respect to $F_1 * \cdots * F_n$. Thus,

$$\overline{Apr}_{F_1*\cdots*F_n}(X) * \cdots * \overline{Apr}_{F_1*\cdots*F_n}(X) = X = \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X).$$

Now, assume that $\mathcal{L}$ is commutative and X is a subalgebra of $\mathcal{L}$. Let

$$x \in \overline{Apr}_{F_1*\cdots*F_n}(X) * \cdots * \overline{Apr}_{F_1*\cdots*F_n}(X).$$

Then,

$$x \geq x_1 * \cdots * x_n, \tag{3.7}$$

where $x_i \in \overline{Apr}_{F_1*\cdots*F_n}(X)$. Hence, for $a_i \in X$ we have $a_i \to x_i \geq f_1 * \cdots * f_n$ and so

$$x_i \geq (f_1 * \cdots * f_n) * a_i \geq (f_1 * a_i) * \cdots * (f_n * a_i),$$

where $f_j \in F_j$, for $j = 1, 2, \ldots, n$. Since $[f_j * a_i]_{F_j} = [a_i]_{F_j}$ and $[a_i]_{F_j} \cap X \neq \emptyset$, so $f_j * a_i \in \overline{Apr}_{F_j}(X)$, whence $x_i \in \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X)$. Now, since X is a subalgebra of $\mathcal{L}$, then so is $\overline{Apr}_{F_j}(X)$, by Proposition 3.5(v), whence combining the commutativity of $\mathcal{L}$ we deduce that $x_1 * \cdots * x_n \in \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X)$. Combining (3.7) we get $x \in \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X)$, proving

$$\overline{Apr}_{F_1*\cdots*F_n}(X) * \cdots * \overline{Apr}_{F_1*\cdots*F_n}(X) \subseteq \overline{Apr}_{F_1}(X) * \cdots * \overline{Apr}_{F_n}(X).$$

□

Example 3.20 shows that the conditions given in Theorem 3.19 are necessary.

Example 3.20 Consider the residuated lattice $\mathcal{L}$ given in Example 3.10.

(i) Let $X = \{a, c\}$. We can see that X is not a filter of $\mathcal{L}$, because $c \leq 1$ and $c \in X$, whereas $1 \notin X$. Now, $\overline{Apr}_{F_1}(X) = \{0, a, b, c\}$ and $\overline{Apr}_{F_2}(X) = X$ and so $\overline{Apr}_{F_1}(X) * \overline{Apr}_{F_2}(X) = \mathcal{L}$, whereas

$$\overline{Apr}_{F_1 * F_2}(X) = \overline{Apr}_{F_3}(X) = \{0, a, b, c\}.$$

Thus,

$$\overline{Apr}_{F_1}(X) * \overline{Apr}_{F_2}(X) \neq \overline{Apr}_{F_1 * F_2}(X).$$

(ii) Let $X = \{0, a, b, c\}$, which is not a filter of $\mathcal{L}$. Then,

$$\underline{Apr}_{F_1}(X) = \underline{Apr}_{F_2}(X) = \underline{Apr}_{F_1 * F_2}(X) = X,$$

whereas $\underline{Apr}_{F_1}(X) * \underline{Apr}_{F_2}(X) = \mathcal{L}$. Hence,

$$\underline{Apr}_{F_1}(X) * \underline{Apr}_{F_2}(X) \neq \underline{Apr}_{F_1 * F_2}(X).$$

(iii) Let $X = \{d\}$ and $F_5 = \{c, 1\}$ be the filter of $\mathcal{L}$ as mentioned in Example 3.16. Then, $\overline{Apr}_{F_1}(X) = \{d, e\}$ and $\overline{Apr}_{F_5}(X) = \{d\}$ and so $\overline{Apr}_{F_1}(X) * \overline{Apr}_{F_5}(X) = \{1, f, e, d\}$, whereas

$$\overline{Apr}_{F_1 * F_5}(X) = \overline{Apr}_{\{1,c,f,b\}}(X) = \{0, a, c, e\}.$$

Hence, $\overline{Apr}_{F_1 * F_5}(X) * \overline{Apr}_{F_1 * F_5}(X) = \mathcal{L}$, proving

$$\overline{Apr}_{F_1}(X) * \overline{Apr}_{F_5}(X) \neq \overline{Apr}_{F_1 * F_5}(X) * \overline{Apr}_{F_1 * F_5}(X).$$

Theorem 3.21 *Let $\{X_i\}_{i \in I}$ be a nonempty family of nonempty subsets of $\mathcal{L}$ such that $\cap X_i \neq \emptyset$. Then,*

(i) $\overline{Apr}_F(\cap X_i) \subseteq \cap \overline{Apr}_F(X_i) \subseteq \cup \overline{Apr}_F(X_i) \subseteq \overline{Apr}_F(\vee X_i)$,
(ii) $\underline{Apr}_F(\cap X_i) \subseteq \cap \underline{Apr}_F(X_i) \subseteq \cup \underline{Apr}_F(X_i) \subseteq \underline{Apr}_F(\vee X_i)$.

Proof It follows from Proposition 2.13. □

Theorem 3.22 *Let $\{X_i\}_{i \in I}$ be a nonempty family of nonempty subsets of $\mathcal{L}$ such that $\cap X_i \neq \emptyset$.*

(i) *If X_i (for all $i \in I$) is definable with respect to F, then*

$$\overline{Apr}_F(\cap X_i) = \cap \overline{Apr}_F(X_i) = \underline{Apr}_F(\cap X_i) = \cap \underline{Apr}_F(X_i) = \cap X_i.$$

(ii) *If $X_i \subseteq F$, for all $i \in I$, then*

$$\overline{Apr}_F(\cap X_i) = \cap \overline{Apr}_F(X_i) = \cup \overline{Apr}_F(X_i) = \overline{Apr}_F(\vee X_i) = F.$$

(iii) *If $F \subseteq X_j$, for some $j \in I$, then $\overline{Apr}_F(\vee X_i) = \vee X_i = \underline{Apr}_F(\vee X_i)$.*

Proof (i) It follows from the defn.
(ii) It follows from Proposition 3.5(iii) and Theorem 3.21.
(iii) Let $x \in \overline{Apr}_F(\vee X_i)$. Then, there exists $a \in \vee X_i$ such that $a \to x \in F$. Hence, $a \geq x_1 * x_2 * \cdots * x_n$, where $n \in \mathbb{N}$ and $x_i \in \cup X_i$, and so

$$(x_1 * x_2 * \cdots * x_n) \to x \geq a \to x \in F.$$

This implies that $(x_1 * x_2 * \cdots * x_n) \to x \in F \subseteq X_j \subseteq \vee X_i$. Since $\vee X_i$ is a filter and $x_i \in \cup X_i \subseteq \vee X_i$, for all $i = 1, 2, \ldots, n$, so $x_1 * \cdots * x_n$ and hence $x \in \vee X_i$. Thus, $\overline{Apr}_F(\vee X_i) = \vee X_i$. The other case follows from Theorem 3.2.

□

Fig. 3 The lattice of Example 3.24

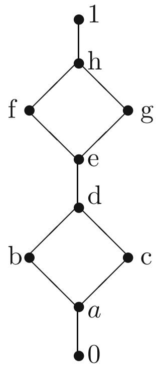

Table 6 The action of '$*$' on $\mathcal{L}$

$*$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	0	0	0	0	c	0	c	c
d	0	0	0	0	0	0	c	b	d	d
e	0	0	0	0	0	e	e	e	e	e
f	0	0	b	0	b	e	f	e	f	f
g	0	0	0	c	c	e	e	g	g	g
h	0	0	b	c	d	e	f	g	h	h
1	0	a	b	c	d	e	f	g	h	1

From Theorem 3.22 it follows that

Corollary 3.23 *If $\{X_i\}_{i\in I}$ is a family of nonempty subsets of $\mathcal{L}$ such that each X_i is definable with respect to F and $F \subseteq X_j$, for some $j \in I$, then $(DF(\mathcal{L}), \subseteq)$ forms a complete lattice.*

Example 3.24 shows that the equalities in Theorem 3.22 may not hold, in general.

Example 3.24 Let $\mathcal{L} = \{0, a, b, c, d, e, f, g, h, 1\}$ be a lattice whose Hasse diagram is below (see Fig. 3). Define $*$, $\rightarrow$, $\rightsquigarrow$ on $\mathcal{L}$ as shown in Tables 6, 7 and 8. Then, $\mathcal{L}$ is a residuated lattice in which $F = \{1, h, f, g, e\}$ is a normal filter (see [14]). It is easily seen that $[x]_F = \begin{cases} \{0, a, b, c, d\}, & x \in \mathcal{L}\backslash F, \\ F, & x \in F. \end{cases}$

(i) Let $X = \{a, f\}$ and $Y = \{a, c\}$. Then, $X \cap Y = \{a\}$ and so $\overline{Apr}_F(X \cap Y) = \{0, a, b, c, d\}$, whereas $\overline{Apr}_F(X) \cup \overline{Apr}_F(Y) = \mathcal{L}$. We observe that $X \nsubseteq F$ and $Y \nsubseteq F$.

(ii) Let $X = \{a, f\}$ and $Y = \{b, c\}$. Then, $X \cap Y = \emptyset$, and $X \nsubseteq F$ and $Y \nsubseteq F$. Also, $\overline{Apr}_F(X) = \mathcal{L}$ and $\overline{Apr}_F(Y) = \{0, a, b, c, d\}$, means that X and Y are not definable with respect to F. We can see that $\overline{Apr}_F(X \cap Y) = \emptyset$, $\overline{Apr}_F(X) \cap \overline{Apr}_F(Y) = \{0, a, b, c, d\}$ and $\overline{Apr}_F(X) \cup \overline{Apr}_F(Y) = \mathcal{L}$. Thus,

$$\overline{Apr}_F(X \cap Y) \neq \overline{Apr}_F(X) \cap \overline{Apr}_F(Y) \neq \overline{Apr}_F(X) \cup \overline{Apr}_F(Y).$$

Table 7 The action of '→' on $\mathcal{L}$

→	0	a	b	c	d	e	f	g	h	1
0	1	1	1	1	1	1	1	1	1	1
a	h	1	1	1	1	1	1	1	1	1
b	g	g	1	g	1	1	1	1	1	1
c	f	f	f	1	1	1	1	1	1	1
d	e	e	f	g	1	1	1	1	1	1
e	d	d	d	d	d	1	1	1	1	1
f	b	b	b	d	d	g	1	g	1	1
g	c	c	d	c	d	f	f	1	1	1
h	a	a	b	c	d	e	f	g	1	1
1	0	a	b	c	d	e	f	g	h	1

Table 8 The action of '⇝' on $\mathcal{L}$

⇝	0	a	b	c	d	e	f	g	h	1
0	1	1	1	1	1	1	1	1	1	1
a	h	1	1	1	1	1	1	1	1	1
b	f	f	1	f	1	1	1	1	1	1
c	g	g	g	1	1	1	1	1	1	1
d	e	e	g	f	1	1	1	1	1	1
e	d	d	d	d	d	1	1	1	1	1
f	c	c	d	c	d	g	1	g	1	1
g	b	b	b	d	d	f	f	1	1	1
h	a	a	b	c	d	e	f	g	1	1
1	0	a	b	c	d	e	f	g	h	1

(iii) Let $X = \{a\}$ and $Y = \{b\}$. Then, $\overline{Apr}_F(X) = \overline{Apr}_F(Y) = \{0, a, b, c, d\}$, whereas $\overline{Apr}_F(X \vee Y) = \overline{Apr}_F(\mathcal{L}) = \mathcal{L}$, proving

$$\overline{Apr}_F(X) \cup \overline{Apr}_F(Y) \neq \overline{Apr}_F(X \vee Y).$$

(iv) Let $X = \{1\}$ and $Y = \{h\}$. It is seen that $F \not\subseteq X$ and $F \not\subseteq Y$. Now, $X \vee Y = \{1, h\}$ and so $\overline{Apr}_F(X \vee Y) = F$ while $\underline{Apr}_F(X \vee Y) = \emptyset$. Thus,

$$\overline{Apr}_F(X \vee Y) \neq \underline{Apr}_F(X \vee Y).$$

We terminate our results by investigating the properties of rough approximations based on different types of filters.

Theorem 3.25 *Let X be a nonempty subset of $\mathcal{L}$. If $\mathcal{L}$ is linearly ordered and X is a Boolean filter of type 1 or of type 2, then so is $\overline{Apr}_F(X)$, respectively.*

Proof It follows from Proposition 3.5 and that $X \subseteq \overline{Apr}_F(X)$. □

We mention that when X is a subalgebra of $\mathcal{L}$, $\overline{Apr}_F(X)$ is also a subalgebra of $\mathcal{L}$ containing F (see Proposition 3.5). This means that $\overline{Apr}_F(X)/F$, together with those operations induced from $\mathcal{L}$, forms a residuated lattice. Here, we give some characterizations of this residuated lattice based on different types of filters.

For normal filter F and nonempty subset X of $\mathcal{L}$, let

$$\overline{\overline{Apr}}_F(X) = \{[x]_F \in \mathcal{L}/F : [x]_F \cap X \neq \emptyset\}$$

Obviously,

$$\overline{\overline{Apr}}_F(X) = \{[x]_F : x \in \overline{Apr}_F(X)\} = \overline{Apr}_F(X)/F$$

Theorem 3.26 *Let X be a subalgebra and F a normal filter of $\mathcal{L}$.*

(i) *If F is a Boolean filter of type 1, $|\overline{\overline{Apr}}_F(X)| \leq 2$, whence $\overline{Apr}_F(X) = F$ or $\overline{Apr}_F(X) = \mathcal{L} \backslash F$.*
(ii) *If F is a Boolean filter of type 2, $\overline{\overline{Apr}}_F(X)$ is a Boolean algebra.*
(iii) *If F is a prime filter of type 1, $\overline{\overline{Apr}}_F(X)$ is linearly ordered.*
(iv) *If F is a prime filter of type 2, $1_{\overline{\overline{Apr}}_F(X)}$ is $\vee$-irreducible.*
(v) *If F is a prime filter of type 3, $\overline{\overline{Apr}}_F(X)$ is a pseudo MTL-algebra.*

Proof It follows from Theorems 2.6 and 2.8. □

4 Discussion

We investigated the properties of roughness of a set X with respect to some types of filters such as Boolean filters (equivalently, implicative filters, see [14]) and also prime filters. There is a question: What we can say about the approximation of a set with respect to other types of filters such as positive implicative filters and fantastic filters (see [14,22]) or about the approximation of different kinds of filters with respect to a filter. At present, we know that if G is a positive implicative filter (or fantastic filter), $\overline{Apr}_F(G)$ is also a positive implicative filter (fantastic filter) if and only if G is definable with respect to F.

Open Problem 4.1 What under conditions, otherwise definability, the rough approximation of a set, or a filter, with respect to a positive implicative filter (fantastic filter) is again a positive implicative filter (fantastic filter, respectively)?

5 Conclusions

We investigated rough approximations in residuated lattices. Many theorems and propositions were given that stated the properties of them. We characterized rough approximations based on different kinds of filters and gave some structural theorems. Furthermore, we showed that the set of all definable sets of a residuated lattice forms a complete lattice. This approach is different from and more general than the usual one on algebras of fuzzy logics.

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References

1. Bakhshi, M.: Boolean filters and prime filters of non-commutative residuated lattices. In: Extended Abstract of the 42nd Annual Iraninan Mathematics Conference, Sep. 5–8, 2011, Vali-e-Asr University of Rafsanjan, Iran

2. Biswas, R., Nanda, S.: Rough groups and rough subgroups. Bull. Pol. Acad. Sci. Math. **42**(3), 251–254 (1994)
3. Ciungu, L.C.: Classes of residuated lattices. Ann. Univ. Craiova Math. Comput. Sci. Ser. **33**, 189–207 (2006)
4. Comer, S.D.: On connections between information systems, rough sets and algebraic logic. Algebraic Methods Logic Comput. Sci. **28**, 117–124 (1993)
5. Davvaz, B.: Roughness in rings. Inf. Sci. **164**, 147–163 (2004)
6. Davvaz, B.: Roughness based on fuzzy ideals. Inf. Sci. **176**, 2417–2437 (2006)
7. Dilworth, R.P.: Non-commutative residuated lattices. Trans. Am. Math. Soc. **46**(3), 426–444 (1939)
8. Estaji, A.A., Hooshmandasl, M.R., Davvaz, B.: Rough set theory applied to lattice theory. Inf. Sci. **200**, 108–122 (2012)
9. Iwinski, T.B.: Algebraic approach to rough sets. Bull. Pol. Acad. Sci. **35**, 673–683 (1987)
10. Jun, Y.B.: Roughness of ideals in BCK-algebras. Sci. Math. Jpn. **7**, 115–119 (2002)
11. Kazanci, O., Davvaz, B.: On the structure of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings. Inf. Sci. **178**, 1343–1354 (2008)
12. Kuroki, N.: Rough ideals in semigroups. Inf. Sci. **100**, 139–163 (1997)
13. Kuroki, N., Mordeson, J.N.: Structure of rough sets and rough groups. J. Fuzzy Math. **5**, 183–191 (1997)
14. Lianzhen, L., Kaitai, K.: Boolean filters and positive implicative filters of residuated lattices. Inf. Sci. **177**, 5726–5738 (2007)
15. Pawlak, Z.: Rough sets. Int. J. Comput. Inf. Sci. **11**(5), 341–357 (1982)
16. Pomykala, J., Pomykala, J.A.: The Stone algebra of rough sets. Bull. Pol. Acad. Sci. Math. **36**, 495–508 (1988)
17. Rachůek, J., Šalounová, D.: Roughness in residuated lattices. Advances in computational intelligence. In: Greco, S., Bouchon-Meunier, B., Coletti, G., Fedrizzi, M., Matarazzo, B., Yager, R.R. (eds.) Proc. 14th Int. Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, pp. 596–603. IPMU 2012, Catania, Italy (2012)
18. Rasouli, S., Davvaz, B.: Roughness in MV-algebras. Inf. Sci. **180**, 737–747 (2010)
19. Torkzadeh, L., Ghorbani, S.: Rough filters in BL-algebras. Int. J. Math. Math. Sci. doi:10.1155/2011/474375 (2011)
20. Ward, M., Dilworth, R.P.: Residuated lattices. Trans. Am. Math. Soc. **45**, 335–354 (1939)
21. Xiao, Q.M., Zhang, Z.L.: Rough prime ideals and rough fuzzy prime ideals in semigroups. Inf. Sci. **176**, 725–733 (2006)
22. Zhu, Y., Xu, Y.: On filter theory of residuated lattices. Inf. Sci. **180**, 3614–3632 (2010)