Afr. Mat. (2018) 29:73–79
https://doi.org/10.1007/s13370-017-0527-3

Extensions of Vandermonde determinant by computing divided differences

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Received: 22 February 2016 / Accepted: 2 September 2017 / Published online: 8 September 2017

Abstract As extensions of Vandermonde determinant, we establish a general determinant evaluation formula by means of the Laplace expansion formula. Several interesting determinant identities are consequently derived by computing divided differences.

Keywords Vandermonde determinant · Divided differences · Symmetric functions

Mathematics Subject Classification Primary 15A15; Secondary 05E05

1 Introduction and motivation

Divided differences (cf. [7, Chapter 1]) are useful tools in mathematics and physics. For example, their applications to symmetric functions and approximation theory can be found respectively in [6, § 1.2] and [8, § 2.4]. For a complex function $f(y)$ and uneven spaced grid points $\{x_k\}_{k=0}^n$, the divided differences with respect to y are defined in succession as follows:

$$\Delta[x_0, x_1]f(y) = \frac{f(x_0) - f(x_1)}{x_0 - x_1},$$
$$\Delta[x_0, x_1, x_2]f(y) = \frac{\Delta[x_0, x_1]f(y) - \Delta[x_1, x_2]f(y)}{x_0 - x_2},$$

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$$\cdots \vdots \cdots$$
$$\Delta[x_0, x_1, \ldots, x_n]f(y) = \frac{\Delta[x_0, x_1, \ldots, x_{n-1}]f(y) - \Delta[x_1, x_2, \ldots, x_n]f(y)}{x_0 - x_n}.$$

In general, they can be expressed through the Newton formula:

$$\Delta[x_0, x_1, \ldots, x_n]f(y) = \sum_{k=0}^{n} \frac{f(x_k)}{\prod_{i\neq k}(x_k - x_i)}. \tag{1}$$

Between the divided differences of monomials and complete symmetric functions, there holds the following well–known formula due to Sylvester (1839), whose proofs can be found in [1–3]:

$$\Delta[x_0, x_1, \ldots, x_n]y^m = \sum_{k=0}^{n} \frac{x_k^m}{\prod_{i\neq k}(x_k - x_i)}$$
$$= \begin{cases} 0, & m = 0, 1, \ldots, n-1; \\ h_{m-n}(x_0, x_1, \ldots, x_n), & m = n,\ n+1,\ \ldots; \\ \frac{(-1)^n}{x_0 x_1 \cdots x_n} h_{-m-1}\left(\frac{1}{x_0}, \frac{1}{x_1}, \ldots, \frac{1}{x_n}\right), & m = -1, -2, -3, \ldots. \end{cases} \tag{2}$$

Here and below, we shall denote the elementary and complete symmetric functions (cf. Macdonald [6, § 1.2]) of variables $X = \{x_0, x_1, \ldots, x_n\}$, respectively, by

$$e_0(X) = 1 \quad \text{and} \quad e_m(X) = \sum_{0\leq k_1<k_2<\cdots<k_m\leq n} x_{k_1}x_{k_2}\cdots x_{k_m} \quad \text{for } m \in \mathbb{N};$$
$$h_0(X) = 1 \quad \text{and} \quad h_m(X) = \sum_{0\leq k_1\leq k_2\leq\cdots\leq k_m\leq n} x_{k_1}x_{k_2}\cdots x_{k_m} \quad \text{for } m \in \mathbb{N}.$$

They admit the following generating functions:

$$\sum_{k=0}^{n+1} e_k(X)\, y^k = \prod_{k=0}^{n}(1 + x_k y), \tag{3}$$
$$\sum_{k=0}^{\infty} h_k(X)\, y^k = \prod_{k=0}^{n} \frac{1}{1 - x_k y}. \tag{4}$$

Recently, divided differences have been employed by the first author [4,5] and Tang–Xu [9] to investigate determinant evaluations. In this paper, we shall utilize the Laplace expansion formula to establish an extension of Vandermonde determinant. By computing divided differences, it will further be specialized to several interesting Vandermonde—like determinant identities.

2 Extensions of Vandermonde determinant

For the indeterminates $X := \{x_k\}_{k=0}^n$, denote the divided differences by

$$\Delta[X]f(y) := \Delta[x_0, x_1, \ldots, x_n]f(y)$$

and the Vandermonde determinant by

$$V(X) := \det_{0\leq i,j\leq n}\left[x_i^j\right] = \prod_{0\leq i<j\leq n}(x_j - x_i).$$

Theorem 1 (Extension of Vandermonde determinant)

$$\det_{0\le i,j\le n}\left[x_i^j+u_iv_j\right]=V(X)\left\{1+\sum_{\imath,\jmath=0}^{n}(-1)^{n+\jmath}\frac{u_\imath v_\jmath e_{n-\jmath}\big(X\backslash\{x_\imath\}\big)}{\prod_{k\neq\imath}(x_\imath-x_k)}\right\}.$$

Proof Consider the extended matrix of order $(n+2)\times(2+n)$ given explicitly by

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline u_i & x_i^j+u_iv_j\ {\scriptstyle (0\le i,j\le n)} \end{array}\right]$$

whose determinant is obviously equal to the determinant stated in the theorem.
Now subtracting v_j times the first column from each other column, we transform the matrix into the following one:

$$\left[\begin{array}{c|c} 1 & -v_j \\ \hline u_i & x_i^j\ {\scriptstyle (0\le i,j\le n)} \end{array}\right].$$

Applying the Laplace expansion formula to the last matrix with respect to the first column and then to the first row, we have

$$\begin{aligned}\det_{0\le i,j\le n}\left[x_i^j+u_iv_j\right]&=\det_{0\le i,j\le n}\left[x_i^j\right]+\sum_{\imath=0}^{n}(-1)^\imath u_\imath\det_{0\le i,j\le n}\begin{bmatrix}v_j\\ \cdots\\ x_i^j\end{bmatrix}_{i\neq\imath}\\ &=V(X)+\sum_{\imath,\jmath=0}^{n}(-1)^{\imath+\jmath}u_\imath v_\jmath\det_{0\le i,j\le n}\left[x_i^j\right]_{\substack{i\neq\imath\\ j\neq\jmath}}.\end{aligned}\tag{5}$$

Let $[x^k]f(x)$ stand for the coefficient of x^k in the formal power series $f(x)$. Then the determinant displayed in (5) can be evaluated as

$$\begin{aligned}\det_{0\le i,j\le n}\left[x_i^j\right]_{\substack{i\neq\imath\\ j\neq\jmath}}&=(-1)^{\imath+\jmath}\,[x_\imath^\jmath]\,V(X)=(-1)^\jmath[x_\imath^\jmath]\prod_{\substack{0\le i<j\le n\\ i,j\neq\imath}}(x_j-x_i)\prod_{\substack{0\le k\le n\\ k\neq\imath}}(x_k-x_\imath)\\ &=e_{n-\jmath}\big(X\backslash\{x_\imath\}\big)\prod_{\substack{0\le i<j\le n\\ i,j\neq\imath}}(x_j-x_i)=(-1)^{n-\imath}\frac{e_{n-\jmath}\big(X\backslash\{x_\imath\}\big)}{\prod_{k\neq\imath}(x_\imath-x_k)}V(X).\end{aligned}$$

Substituting this expression into (5) and then simplifying the result, we obtain the determinant identity stated in the theorem. □

When $u_k=1$, Theorem 1 leads immediately to the following determinant identity.

Corollary 2

$$\det_{0\le i,j\le n}\left[x_i^j+v_j\right]=V(X)(1+v_0).$$

Proof According to (3), there holds the expression

$$e_{n-\jmath}(X\backslash\{x_\imath\})=[y^{n-\jmath}]\frac{\prod_{k=0}^n(1+yx_k)}{1+yx_\imath}=\sum_{k=0}^{n-\jmath}(-1)^kx_\imath^ke_{n-k-\jmath}(X).$$

Then we have, from (2), the relation

$$\Delta[X]y^k = \Delta[x_0, x_1, \ldots, x_n]y^k = \chi(k = n) \quad \text{for } k = 0, 1, \ldots, n$$

where χ is the logical function defined by $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Therefore, the double sum displayed in Theorem 1 with $u_k = 1$ can be rewritten as follows

$$\sum_{\imath=0}^{n}\sum_{\jmath=0}^{n}(-1)^{n-\jmath}\frac{v_\jmath e_{n-\jmath}(X\backslash\{x_\imath\})}{\prod_{k\neq\imath}(x_\imath - x_k)} = \sum_{\jmath=0}^{n}(-1)^{n-\jmath}v_\jmath\sum_{k=0}^{n-\jmath}(-1)^k e_{n-k-\jmath}(X)\Delta[X]y^k$$

which reduces to the single term v_0, corresponding to $k = n$ and $\jmath = 0$. □

For $v_k = v^k$ with $0 \le k \le n$, observing that

$$\sum_{\jmath=0}^{n}(-1)^{n-\jmath}v^\jmath e_{n-\jmath}(X\backslash\{x_\imath\}) = \prod_{\substack{k=0\\k\neq\imath}}^{n}(v - x_k) \quad \text{and} \quad \Delta[X]\frac{1}{v-y} = \frac{1}{\prod_{k=0}^{n}(v-x_k)},$$

we derive from Theorem 1 another determinant identity.

Proposition 3

$$\det_{0\le i,j\le n}\left[x_i^j + u_i v^j\right] = V(X)\sum_{i=0}^{n}(1+u_i)\prod_{\substack{k=0\\k\neq i}}^{n}\frac{v-x_k}{x_i-x_k}.$$

Suppose further that $u_k := u(x_k)$ is a function of x_k for $0 \le k \le n$, we may state Proposition 3 equivalently in terms of divided differences.

Corollary 4

$$\det_{0\le i,j\le n}\left[x_i^j + u(x_i)v^j\right] = V(X)\Delta[X]\left\{\frac{1+u(y)}{v-y}\right\}\prod_{k=0}^{n}(v-x_k).$$

For $m \le n$, letting $u(x_i) = \lambda\prod_{k=1}^{m}(x_i - \gamma_k)$ and appealing to the formula

$$\Delta[X]\frac{\prod_{k=1}^{m}(y-\gamma_k)}{v-y} = \frac{\prod_{k=1}^{m}(v-\gamma_k)}{\prod_{k=0}^{n}(v-x_k)},$$

we derive from Corollary 4 the following determinant identity.

Example 5

$$\det_{0\le i,j\le n}\left[x_i^j + \lambda v^j\prod_{k=1}^{m}(x_i-\gamma_k)\right] = V(X)\left\{1+\lambda\prod_{k=1}^{m}(v-\gamma_k)\right\}.$$

Instead, letting $u(x_i) = \lambda\prod_{k=0}^{n}(x_i-\gamma_k)$ and then applying the formula

$$\Delta[X]\frac{\prod_{k=0}^{n}(y-\gamma_k)}{v-y} = \prod_{k=0}^{n}\frac{v-\gamma_k}{v-x_k} - 1,$$

we establish from Corollary 4 another determinant identity.

Example 6

$$\det_{0\le i,j\le n}\left[x_i^j + \lambda v^j\prod_{k=0}^{n}(x_i-\gamma_k)\right] = V(X)\left\{1-\lambda\prod_{k=0}^{n}(v-x_k)+\lambda\prod_{k=0}^{n}(v-\gamma_k)\right\}.$$

Specifying $\lambda = -1/\prod_{k=0}^{n}(v-\gamma_k)$ further in the last example, we obtain a more symmetric determinant formula:

$$\det_{0\le i,j\le n}\left[x_i^j - v^j\prod_{k=0}^{n}\frac{x_i-\gamma_k}{v-\gamma_k}\right] = V(X)\prod_{k=0}^{n}\frac{v-x_k}{v-\gamma_k}. \tag{6}$$

Finally, if we let in Corollary 4

$$u(y) = -\prod_{k=0}^{n+1}\frac{y-\gamma_k}{v-\gamma_k}$$

and applying the formula

$$\Delta[X]\frac{\prod_{k=0}^{n+1}(y-\gamma_k)}{v-y} = \gamma_{n+1} - v + \sum_{k=0}^{n}(\gamma_k - x_k) + \frac{\prod_{k=0}^{n+1}(v-\gamma_k)}{\prod_{k=0}^{n}(v-x_k)},$$

we would derive further the following determinant identity.

Example 7

$$\det_{0\le i,j\le n}\left[x_i^j - v^j\prod_{k=0}^{n+1}\frac{x_i-\gamma_k}{v-\gamma_k}\right] = V(X)\left\{1+\sum_{i=0}^{n}\frac{x_i-\gamma_i}{v-\gamma_{n+1}}\right\}\prod_{k=0}^{n}\frac{v-x_k}{v-\gamma_k}.$$

3 Further variants of Vandermonde determinant

For the sake of brevity, this section will use the indeterminates $\mathbb{X} := \{x_k\}_{k=1}^{n}$ and the following notation for the corresponding Vandermonde determinant

$$V(\mathbb{X}) := \det_{1\le i,j\le n}\left[x_i^{j-1}\right] = \prod_{1\le i<j\le n}(x_j - x_i).$$

Then Theorem 1 can be reformulated equivalently as follows:

$$\det_{1\le i,j\le n}\left[x_i^j + u_i v_j\right] = V(\mathbb{X})e_n(\mathbb{X})\left\{1+\sum_{\imath,\jmath=1}^{n}(-1)^{n+\jmath}\frac{u_\imath v_\jmath e_{n-\jmath}(\mathbb{X}\backslash\{x_\imath\})}{x_\imath\prod_{k\neq\imath}(x_\imath - x_k)}\right\}. \tag{7}$$

For $u_k = 1$, this equality becomes the following determinant identity.

Proposition 8

$$\det_{1\le i,j\le n}\left[x_i^j + v_j\right] = V(\mathbb{X})\left\{e_n(\mathbb{X}) - \sum_{k=1}^{n}(-1)^k v_k e_{n-k}(\mathbb{X})\right\}.$$

Proof Recalling (3), we have the expression

$$e_{n-\jmath}(\mathbb{X}\backslash\{x_\imath\}) = [y^{n-\jmath}]\frac{\prod_{k=1}^{n}(1+yx_k)}{1+yx_\imath} = \sum_{k=0}^{n-\jmath}(-1)^k x_\imath^k e_{n-k-\jmath}(\mathbb{X}).$$

According to (2), it is trivial to check the relations

$$\Delta[\mathbb{X}]\frac{1}{y} = \frac{(-1)^{n-1}}{e_n(\mathbb{X})} \quad\text{and}\quad \Delta[\mathbb{X}]y^k = \chi(k=n-1) \quad\text{for}\quad k=0,1,\ldots,n-1.$$

Then for $u_k = 1$, we may reformulate the double sum displayed in (7) as

$$\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n}(-1)^{n-\jmath}\frac{v_\jmath e_{n-\jmath}(\mathbb{X}\backslash\{x_\imath\})}{x_\imath\prod_{k\neq\imath}(x_\imath-x_k)} = \sum_{\jmath=1}^{n}(-1)^{n-\jmath}v_\jmath e_{n-\jmath}(\mathbb{X})\Delta[\mathbb{X}]\frac{1}{y}$$
$$+\sum_{\jmath=1}^{n}(-1)^{n-\jmath}v_\jmath\sum_{k=1}^{n-\jmath}(-1)^k e_{n-k-\jmath}(\mathbb{X})\Delta[\mathbb{X}]y^{k-1}.$$

Observing that the last double sum vanishes and then evaluating the divided differences in the penultimate line, we confirm the identity stated in the Proposition. □

Letting $v_j = v^j$ in Proposition 8, we get the simplified determinant identity.

Example 9

$$\det_{1\leq i,j\leq n}\left[x_i^j + v^j\right] = V(\mathbb{X})\left\{2e_n(\mathbb{X}) - \prod_{k=1}^{n}(x_k - v)\right\}.$$

Notice that the determinant in Corollary 4 can be expressed equivalently as

$$\det_{1\leq i,j\leq n}\left[x_i^j + u(x_i)v^j\right] = V(\mathbb{X})e_n(\mathbb{X})\left\{1 + \Delta[\mathbb{X}]\frac{vu(y)}{y(v-y)}\prod_{k=1}^{n}(v - x_k)\right\}. \tag{8}$$

For $m \leq n$, letting $u(x_i) = \lambda\prod_{k=1}^{m}(x_i - \gamma_k)$ and appealing to the formula

$$\Delta[\mathbb{X}]\frac{v\prod_{k=1}^{m}(y-\gamma_k)}{y(v-y)} = (-1)^{m+n+1}\frac{\prod_{k=1}^{m}\gamma_k}{\prod_{k=1}^{n}x_k} + \frac{\prod_{k=1}^{m}(v-\gamma_k)}{\prod_{k=1}^{n}(v-x_k)},$$

we obtain from (8) another determinant identity.

Corollary 10

$$\det_{1\leq i,j\leq n}\left[x_i^j + \lambda v^j\prod_{k=1}^{m}(x_i-\gamma_k)\right] = V(\mathbb{X})\left\{e_n(\mathbb{X}) + \lambda\prod_{k=1}^{n}x_k\prod_{k=1}^{m}(v-\gamma_k)\right.$$
$$\left.-(-1)^{m+n}\lambda\prod_{k=1}^{m}\gamma_k\prod_{k=1}^{n}(v-x_k)\right\}.$$

When $m = n$ and $\lambda = -1/\prod_{k=1}^{n}(v-\gamma_k)$, the last corollary yields further to a more symmetric determinant identity:

$$\det_{1\leq i,j\leq n}\left[x_i^j - v^j\prod_{k=1}^{n}\frac{x_i-\gamma_k}{v-\gamma_k}\right] = V(\mathbb{X})\prod_{k=1}^{n}\frac{\gamma_k(v-x_k)}{v-\gamma_k}. \tag{9}$$

Finally, for $v_k = h_k(\mathbb{X})$, Proposition 8 reduces to the strange looking identity:

$$\det_{1\leq i,j\leq n}\left[x_i^j + h_j(\mathbb{X})\right] = 2V(\mathbb{X})e_n(\mathbb{X}). \tag{10}$$

Acknowledgements The authors are sincerely grateful to the anonymous referees for their careful reading and valuable comments that have significantly contributed to improve the manuscript during the revision.

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