Afr. Mat. (2017) 28:831–839
DOI 10.1007/s13370-017-0490-z

Almost continuity and its applications on weak structures

A. M. Zahran[1] · **A. Ghareeb**[2] · **A. K. Mousa**[1,3]

Received: 4 October 2015 / Accepted: 30 January 2017 / Published online: 9 February 2017

Abstract In this paper, we introduce and study the concept of almost continuity in weak structures (Császár in Acta Math Hung 131(1–2):193–195, 2011) and discuss some of its characteristic properties. Finally, we give some applications of this new kind of continuity.

Keywords Weak structures · Almost $\mathcal{W}$-continuous function · Almost $\mathcal{W}$-regular · Almost $\mathcal{W}$-normal · $\mathcal{W}$-compact · Nearly $\mathcal{W}$-compact

1 Introduction and preliminaries

Császár [1] defined generalized topology and studied some of its concepts like generalized open sets and continuity. Later on, Maki et al. [6] introduced minimal structures and investigated some of its concepts. Lately, Császár [2] presented the weak structures (A family $\mathcal{W} \subset P(X)$ is called a weak structure on X (briefly, WS) iff $\emptyset \in \mathcal{W}$). A non-empty set X with a weak structure $\mathcal{W}$ is called simply a space $(X, \mathcal{W})$. The members of $\mathcal{W}$ are $\mathcal{W}$-open subsets and their complements are $\mathcal{W}$-closed subsets. Moreover, Császár [2] presented the operations $c_{\mathcal{W}}(A)$ and $i_{\mathcal{W}}(A)$ in WS as the intersection of all $\mathcal{W}$-closed set containing A and the union of all $\mathcal{W}$-open subsets of A. Also, the properties of $c_{\mathcal{W}}(A)$ and $i_{\mathcal{W}}$ are introduced and discussed. For more details about weak structures, the readers should refer [4,5,7–9].

Theorem 1.1 [2] *For any space $(X, \mathcal{W})$ and $A, B \subseteq X$, we have:*

(1) $A \subseteq c_{\mathcal{W}}(A)$ *and* $A \supseteq i_{\mathcal{W}}(A)$;
(2) *If* $A \subseteq B$, *then* $c_{\mathcal{W}}(A) \subset c_{\mathcal{W}}(B)$ *and* $i_{\mathcal{W}}(A) \subset i_{\mathcal{W}}(B)$;
(3) *If* A *is* $\mathcal{W}$*-closed, then* $A = c_{\mathcal{W}}(A)$, *and if* A *is* $\mathcal{W}$*-open, then* $A = i_{\mathcal{W}}(A)$;

✉ A. Ghareeb
nasserfuzt@hotmail.com, a.ghareeb@sci.svu.edu.eg

1 Department of Mathematics, Faculty of Science, Al-Azhar University, Assuit, Egypt

2 Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt

3 Department of Mathematics, College of Al-Wajh, University of Tabuk, Tabuk, Saudi Arabia

(4) $c_{\mathcal{W}}(c_{\mathcal{W}}(A)) = c_{\mathcal{W}}(A)$ *and* $i_{\mathcal{W}}(i_{\mathcal{W}}(A)) = i_{\mathcal{W}}(A)$*;*
(5) $c_{\mathcal{W}}(X - A) = X - i_{\mathcal{W}}(A)$ *and* $i_{\mathcal{W}}(X - A) = X - c_{\mathcal{W}}(A)$*;*
(6) $i_{\mathcal{W}}(c_{\mathcal{W}}(i_{\mathcal{W}}(c_{\mathcal{W}}(A)))) = i_{\mathcal{W}}(c_{\mathcal{W}}(A))$ *and* $c_{\mathcal{W}}(i_{\mathcal{W}}(c_{\mathcal{W}}(i_{\mathcal{W}}(A)))) = c_{\mathcal{W}}(i_{\mathcal{W}}(A))$*;*
(7) $x \in c_{\mathcal{W}}(A)$ *iff* $V \cap A \neq \emptyset$ *for every* $\mathcal{W}$*-open subset* V *containing* x*;*
(8) $x \in i_{\mathcal{W}}(A)$ *iff there exists a* $\mathcal{W}$*-open subset* V *such that* $x \in V \subset A$*.*

Theorem 1.2 [3] *For a space* $(X, \mathcal{W})$ *and* $U, V \subseteq X$*, we have:*

(1) $i_{\mathcal{W}}(U \bigcap V) \subseteq i_{\mathcal{W}}(U) \bigcap i_{\mathcal{W}}(V)$*;*
(2) $c_{\mathcal{W}}(U) \cup c_{\mathcal{W}}(V) \subseteq c_{\mathcal{W}}(U \cup V)$*.*

Definition 1.1 Let $(X, \mathcal{W})$ be a space and $A \subseteq X$. Then

(1) $A \in \alpha(\mathcal{W})$ [2] if $A \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(i_{\mathcal{W}}(A)))$;
(2) $A \in \pi(\mathcal{W})$ [2] if $A \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}((A)))$;
(3) $A \in \sigma(\mathcal{W})$ [2] if $A \subseteq c_{\mathcal{W}}((i_{\mathcal{W}}(A))$;
(4) $A \in \beta(\mathcal{W})$ [2] if $A \subseteq c_{\mathcal{W}}(i_{\mathcal{W}}(c_{\mathcal{W}}(A)))$;
(5) $A \in \rho(\mathcal{W})$ [2] if $A \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(A)) \bigcup c_{\mathcal{W}}(i_{\mathcal{W}}(A))$;
(6) $A \in r(\mathcal{W})$ [3] if $A = i_{\mathcal{W}}(c_{\mathcal{W}}(A))$;
(7) $A \in rc(\mathcal{W})$ [3] if $A = c_{\mathcal{W}}(i_{\mathcal{W}}((A))$.

2 Almost $\mathcal{W}$-continuity

Lemma 2.1 *For any space* $(X, \mathcal{W})$ *and* $\mathcal{W}$*-open set* V*, if* $U \bigcap V = \emptyset$*, then* $c_{\mathcal{W}}(U) \bigcap V = \emptyset$ *for each subset* U *of* X*.*

Proof Let V be an $\mathcal{W}$-open set and $U \subseteq X$. Suppose $U \bigcap V = \emptyset$ and $c_{\mathcal{W}}(U) \bigcap V \neq \emptyset$, then there exists $x \in X$ such that $x \in c_{\mathcal{W}}(U)$ and $x \in V$. Since $x \in c_{\mathcal{W}}(U)$ and V is an $\mathcal{W}$-open set containing x, then $U \bigcap V \neq \emptyset$. This is a contradiction. Therefore if $U \bigcap V = \emptyset$, then $c_{\mathcal{W}}(U) \bigcap V = \emptyset$ for each subset U of X. □

Definition 2.1 For any space $(X, \mathcal{W})$ and $S \subseteq X$, a point $x \in X$ is said to be:

(1) $\mathcal{W}_\theta$-cluster point of S if $c_{\mathcal{W}}(V) \bigcap S \neq \emptyset$ for every $\mathcal{W}$-open set V containing x.
(2) $\mathcal{W}_\delta$-cluster point of S if $i_{\mathcal{W}}(c_{\mathcal{W}}(V)) \bigcap S \neq \emptyset$ for every $\mathcal{W}$-open set V containing x.

The set of all $\mathcal{W}_\theta$ (resp. $\mathcal{W}_\delta$)-cluster points of S is called the $\mathcal{W}_\theta$ (resp. $\mathcal{W}_\delta$)-closure of S and is denoted by $c^\theta_{\mathcal{W}}(S)$ (resp. $c^\delta_{\mathcal{W}}(S)$). If $S = c^\theta_{\mathcal{W}}(S)$ (resp. $S = c^\delta_{\mathcal{W}}(S)$), then A is called $\mathcal{W}_\theta$ (resp. $\mathcal{W}_\delta$)-closed. The complement of an $\mathcal{W}_\theta$ (resp. $\mathcal{W}_\delta$)-closed set is called $\mathcal{W}_\theta$ (resp. $\mathcal{W}_\delta$)-open.

The union of all $\mathcal{W}_\theta$ (resp. $\mathcal{W}_\delta$)-open sets contained in S is called the $\mathcal{W}_\theta$ (resp. $\mathcal{W}_\delta$)-interior of S and is denoted by $i^\theta_{\mathcal{W}}(S)$ (resp. $i^\delta_{\mathcal{W}}(S)$). The class of $\mathcal{W}_\theta$ (resp. $\mathcal{W}_\delta$)-open sets in $\mathcal{W}$ is denoted by $\theta(\mathcal{W})$ (resp. $\delta(\mathcal{W})$) and the class of $\mathcal{W}_\theta$ (resp. $\mathcal{W}_\delta$)-closed sets in $\mathcal{W}$ is denoted by $\theta c(\mathcal{W})$ (resp. $\delta c(\mathcal{W})$).

Remark 2.1 One may notice that if $(X, \mathcal{W})$ a space and $V \subseteq X$, then

$$c_{\mathcal{W}}(V) \subseteq c^\delta_{\mathcal{W}}(V) \subseteq c^\theta_{\mathcal{W}}(V).$$

Lemma 2.2 *Let* $(X, \mathcal{W})$ *be a space. Then*

$$c_{\mathcal{W}}(V) = c^\delta_{\mathcal{W}}(V) = c^\theta_{\mathcal{W}}(V)$$

for each $\mathcal{W}$*-open set* V *in* X*.*

Proof We aim to prove that $c^{\theta}_{\mathcal{W}}(N) \subseteq c_{\mathcal{W}}(N)$. Let N be an $\mathcal{W}$-open set in X and let $x \notin c_{\mathcal{W}}(N)$. Then there exists $M \in \mathcal{W}$ such that $x \in M$ and $M \bigcap N = \emptyset$. By Lemma 2.1, we have $c_{\mathcal{W}}(M) \bigcap N = \emptyset$ and hence $x \notin c^{\theta}_{\mathcal{W}}(N)$. Thus $c^{\theta}_{\mathcal{W}}(N) \subseteq c_{\mathcal{W}}(N)$. Since $c_{\mathcal{W}}(N) \subseteq c^{\theta}_{\mathcal{W}}(N)$ for each subset N in X, then $c_{\mathcal{W}}(N) = c^{\theta}_{\mathcal{W}}(N)$. □

Definition 2.2 A function $f : (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ from a space $(X, \mathcal{W}_X)$ to a space $(Y, \mathcal{W}_Y)$ is called almost $\mathcal{W}$-continuous at $x \in X$ if for every $\mathcal{W}_Y$-open set N containing $f(x)$, there is a $\mathcal{W}_X$-open set M including x such that $f(M) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(N))$. A map f is called almost $\mathcal{W}$-continuous if it is almost $\mathcal{W}$-continuous at each $x \in X$.

Theorem 2.1 *For a function $f : (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$. The following statements are equivalent:*

(1) *f is almost $\mathcal{W}$-continuous;*
(2) $f^{-1}(V) \subseteq i_{\mathcal{W}}(f^{-1}(V))$ *for each* $V \in r(\mathcal{W}_Y)$;
(3) $c_{\mathcal{W}}(f^{-1}(F)) \subseteq f^{-1}(F)$ *for each* $F \in rc(\mathcal{W}_Y)$;
(4) $f^{-1}(V) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$ *for each $\mathcal{W}$-open set V in Y;*
(5) $c_{\mathcal{W}}(f^{-1}(c_{\mathcal{W}}(i_{\mathcal{W}}(F)))) \subseteq f^{-1}(F)$ *for each $\mathcal{W}$-closed set F in Y;*
(6) $f^{-1}(V) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$ *for each* $V \in \pi(\mathcal{W}_Y)$;
(7) $c_{\mathcal{W}}(f^{-1}(c_{\mathcal{W}}(i_{\mathcal{W}}(F)))) \subseteq f^{-1}(F)$ *for each* $F \in \pi c(\mathcal{W}_Y)$;
(8) $f^{-1}(V) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$ *for each* $V \in \alpha(\mathcal{W}_Y)$;
(9) $c_{\mathcal{W}}(f^{-1}(c_{\mathcal{W}}(i_{\mathcal{W}}(F)))) \subseteq f^{-1}(F)$ *for each* $F \in \alpha c(\mathcal{W}_Y)$;
(10) *For each a point $x \in X$ and a $V \in \pi(\mathcal{W}_Y)$ containing $f(x)$, there exists an $\mathcal{W}_X$-open set U containing x such that $f(U) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V))$;*
(11) *For each a point $x \in X$ and a $V \in \alpha(\mathcal{W}_Y)$ containing $f(x)$, there exists an $\mathcal{W}_X$-open set U containing x such that $f(U) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V))$;*
(12) $c_{\mathcal{W}}(f^{-1}(V)) \subseteq f^{-1}(c_{\mathcal{W}}(V))$ *for each* $V \in \beta(\mathcal{W}_Y)$;
(13) $f^{-1}(i_{\mathcal{W}}(F)) \subseteq i_{\mathcal{W}}(f^{-1}(F))$ *for each* $F \in \beta c(\mathcal{W}_Y)$;
(14) $c_{\mathcal{W}}(f^{-1}(V)) \subseteq f^{-1}(c_{\mathcal{W}}(V))$ *for each* $V \in \sigma(\mathcal{W}_Y)$;
(15) $f^{-1}(i_{\mathcal{W}}(F)) \subseteq i_{\mathcal{W}}(f^{-1}(F))$ *for each* $F \in \sigma c(\mathcal{W}_Y)$;
(16) $c_{\mathcal{W}}(f^{-1}(V)) \subseteq f^{-1}(c_{\mathcal{W}}(V))$ *for each* $V \in \pi(\mathcal{W}_Y)$;
(17) $f^{-1}(i_{\mathcal{W}}(V)) \subseteq i_{\mathcal{W}}(f^{-1}(V))$ *for each* $V \in \pi c(\mathcal{W}_Y)$;
(18) $c_{\mathcal{W}}(f^{-1}(V)) \subseteq f^{-1}(c_{\mathcal{W}}(V))$ *for each* $V \in \alpha(\mathcal{W}_Y)$;
(19) $f^{-1}(i_{\mathcal{W}}(V)) \subseteq i_{\mathcal{W}}(f^{-1}(V))$ *for each* $V \in \alpha c(\mathcal{W}_Y)$.

Proof (1) ⇒ (2): Suppose that $V \in r(\mathcal{W}_Y)$ and $x \in f^{-1}(V)$. Then $V = i_{\mathcal{W}}(c_{\mathcal{W}}(V))$ and $f(x) \in V = i_{\mathcal{W}}(c_{\mathcal{W}}(V))$, then there exists $U \in \mathcal{W}_Y$ containing $f(x)$ such that $f(x) \in U \subseteq c_{\mathcal{W}}(V)$. By (1), there exists an $\mathcal{W}_X$-open set W in X containing x such that $f(W) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(U))$. Thus

$$f(x) \in f(W) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(U)) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V))$$

and hence $x \in W \subseteq f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))$. Therefore

$$x \in W \subseteq i_{\mathcal{W}}(W) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))) = i_{\mathcal{W}}(f^{-1}(V)).$$

(2) ⇒ (1): Suppose that V be an $\mathcal{W}_Y$-open set such that $f(x) \in V$. Then $x \in f^{-1}(V) \subseteq f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))$. Since $i_{\mathcal{W}}(c_{\mathcal{W}}(V)) \in r(\mathcal{W}_Y)$. By (2),

$$x \in i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))),$$

then there is $x \in U \in \mathcal{W}_X$ provided that $x \in U \subseteq f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))$. Thus there exists an $\mathcal{W}_X$-open set U containing x and $f(U) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V))$. Therefore f is almost $\mathcal{W}$-continuous.

(2) $\Leftrightarrow$ (3): Obvious.
(1) $\Rightarrow$ (4): Let V be an $\mathcal{W}_Y$-open set and $x \in f^{-1}(V)$. Then V is an $\mathcal{W}_Y$-open set containing $f(x)$. From (1), there is a $\mathcal{W}_X$-open set U containing x such that $f(U) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V))$. Thus $x \in U \subseteq f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))$. Since U is $\mathcal{W}_X$-open, then $x \in i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$. Therefore

$$f^{-1}(V) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))).$$

(4) $\Rightarrow$ (1): Let V be an $\mathcal{W}_Y$-open set containing $f(x)$. Then

$$x \in f^{-1}(V) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))$$

and then there is a $\mathcal{W}_X$-open set U containing x such that

$$x \in U \subseteq f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))).$$

Thus $x \in U$ and $f(U) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V))$. Therefore f is almost $\mathcal{W}$-continuous.
(4) $\Leftrightarrow$ (5): It is clear.
(2) $\Rightarrow$ (6): Let $V \in \pi(\mathcal{W}_Y)$. Then $V \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V))$ and hence $f^{-1}(V) \subseteq f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))$. Since $i_{\mathcal{W}}(c_{\mathcal{W}}(V)) \in r(\mathcal{W}_Y)$, then by (2), we get $f^{-1}(V) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$.
(6) $\Leftrightarrow$ (7): It is clear.
(6) $\Rightarrow$ (4): It follows from $\mathcal{W} \subseteq \pi(\mathcal{W})$.
(7) $\Rightarrow$ (9) $\Leftrightarrow$ (8): It follows from $\alpha(\mathcal{W}) \subseteq \pi(\mathcal{W})$.
(8) $\Rightarrow$ (4): It follows from $\mathcal{W} \subseteq \alpha(\mathcal{W})$.
(1) $\Rightarrow$ (10): Let $x \in X$ and $V \in \pi(\mathcal{W}_Y)$ containing $f(x)$. Then $f(x) \in i_{\mathcal{W}}(c_{\mathcal{W}}(V))$ and hence there exists $U \in \mathcal{W}_Y$ such that $f(x) \in U \subseteq c_{\mathcal{W}}(V)$. By (1), there exists a $\mathcal{W}_X$-open set W containing x such that $f(W) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(U))$. Hence $f(W) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(U)) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V))$.
(10) $\Rightarrow$ (11): It follows from $\alpha(\mathcal{W}) \subseteq \pi(\mathcal{W})$.
(11) $\Rightarrow$ (1): It follows from $\mathcal{W} \subseteq \alpha(\mathcal{W})$.
(3) $\Rightarrow$ (12): Let $V \in \beta(\mathcal{W}_Y)$. Then $c_{\mathcal{W}}(V) = c_{\mathcal{W}}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))$ and hence $c_{\mathcal{W}}(V) \in rc(\mathcal{W}_Y)$. By (3), we get $c_{\mathcal{W}}(f^{-1}(V)) \subseteq c_{\mathcal{W}}(f^{-1}(c_{\mathcal{W}}(V))) \subseteq f^{-1}(c_{\mathcal{W}}(V))$.
(12) $\Rightarrow$ (3): Let $H \in rc(\mathcal{W}_Y)$. Then $H \in \beta(\mathcal{W}_Y)$ and hence by (12), we get $c_{\mathcal{W}}(f^{-1}(H) \subseteq f^{-1}(c_{\mathcal{W}}(H)) = f^{-1}(c_{\mathcal{W}}(i_{\mathcal{W}}(H))) = f^{-1}(H)$.
(12) $\Leftrightarrow$ (13): Obvious.
(12) $\Rightarrow$ (14) $\Leftrightarrow$ (15): It follows from $\beta(\mathcal{W}) \subseteq \sigma(\mathcal{W})$.
(14) $\Rightarrow$ (3): It is similar to that of (12) $\Rightarrow$ (3).
(12) $\Rightarrow$ (16) $\Leftrightarrow$ (17) $\Rightarrow$ (18) $\Leftrightarrow$ (19): It follows from $\beta(\mathcal{W}) \subseteq \pi(\mathcal{W})$ and $\pi(\mathcal{W}) \subseteq \alpha(\mathcal{W})$.
(19) $\Rightarrow$ (3): It is similar to that of (12) $\Rightarrow$ (3). □

Theorem 2.2 *For a function $f : (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$. Consider the following statements:*

(1) *f is almost $\mathcal{W}$-continuous;*
(2) *$f(c_{\mathcal{W}}(V)) \subseteq c^{\delta}_{\mathcal{W}}(f(V))$ for each subset V of X;*
(3) *$c_{\mathcal{W}}(f^{-1}(U)) \subseteq f^{-1}(c^{\delta}_{\mathcal{W}}(U))$ for each subset U of Y;*
(4) *$f^{-1}(i_{\mathcal{W}_\delta}(U)) \subseteq i_{\mathcal{W}}(f^{-1}((U))$ for each subset U of Y;*
(5) *$c_{\mathcal{W}}(f^{-1}(F)) \subseteq f^{-1}(F)$ for each $F \in \delta c(\mathcal{W}_Y)$;*
(6) *$f^{-1}(V) \subseteq i_{\mathcal{W}}(f^{-1}(V))$ for each $V \in \delta(\mathcal{W}_Y)$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

Proof (1) $\Rightarrow$ (2): Let $x \in c_{\mathcal{W}}(V)$ and U be an $\mathcal{W}$-open set of Y containing $f(x)$. By (1), there exists an $\mathcal{W}$-open set W containing x such that $f(W) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(U))$ and hence $W \bigcap V \neq \emptyset$. Thus $f(W) \bigcap f(V) \neq \emptyset$ which implies $i_{\mathcal{W}}(c_{\mathcal{W}}(U)) \bigcap f(V) \neq \emptyset$. Then $f(x) \in c_{\mathcal{W}_\delta}(f(V))$ and hence $x \in f^{-1}(c_{\mathcal{W}_\delta} f(V))$ which implies $c_{\mathcal{W}}(V) \subseteq f^{-1}(c^{\delta}_{\mathcal{W}} f(V))$. Therefore $f(c_{\mathcal{W}}(V)) \subseteq c^{\delta}_{\mathcal{W}} f(V))$.
(2) $\Rightarrow$ (3) $\Leftrightarrow$ (4): It is clear.
(3) $\Rightarrow$ (5): Let $H \in \delta c(\mathcal{W}_Y)$. Then $H = c^{\delta}_{\mathcal{W}}(H)$. By (3), we get $c_{\mathcal{W}}(f^{-1}(H) \subseteq f^{-1}(c^{\delta}_{\mathcal{W}}(H)) = f^{-1}(H)$.
(5) $\Leftrightarrow$ (6): It is clear.

Lemma 2.3 *Let $(X, \mathcal{W})$ be a space and $i_{\mathcal{W}}(V)$ be $\mathcal{W}$-open for each $V \in rc(\mathcal{W})$. Then $rc(\mathcal{W}) \subseteq \delta c(\mathcal{W})$.*

Proof Let $V \in rc(\mathcal{W})$ and let $x \notin V$. Then $x \notin c_{\mathcal{W}}(i_{\mathcal{W}}(V))$ and hence there exists an $\mathcal{W}$-open set U containing x such that $U \bigcap i_{\mathcal{W}}(V) = \emptyset$. Since $V \in rc(\mathcal{W})$, then $i_{\mathcal{W}}(V)$ is $\mathcal{W}$-open. Then by Lemma 2.1, we have $c_{\mathcal{W}}(U) \bigcap i_{\mathcal{W}}(V) = \emptyset$ and hence $i_{\mathcal{W}}(c_{\mathcal{W}}(U)) \bigcap i_{\mathcal{W}}(V) = \emptyset$. Since U is $\mathcal{W}$-open, then $c_{\mathcal{W}}(U) \in rc(\mathcal{W})$ and hence $i_{\mathcal{W}}(c_{\mathcal{W}}(U))$ is $\mathcal{W}$-open. By Lemma 2.1, we have $i_{\mathcal{W}}(c_{\mathcal{W}}((U)) \bigcap c_{\mathcal{W}}(i_{\mathcal{W}}(V)) = \emptyset$. Thus $i_{\mathcal{W}}(c_{\mathcal{W}}(U)) \bigcap V = \emptyset$ and hence $x \notin c^{\delta}_{\mathcal{W}}(V)$. Therefore $c^{\delta}_{\mathcal{W}}(V) \subseteq V$ and hence $V \in \delta c(\mathcal{W})$. □

Theorem 2.3 *Let $(X, \mathcal{W})$ be a space and $V \in rc(\mathcal{W})$. If $i_{\mathcal{W}}(V)$ is $\mathcal{W}$-open set, it leads to the equality of the statements in Theorem 2.2.*

Proof It is clear from Lemma 2.3 and Theorem 2.1(3). □

Theorem 2.4 *Let $f : (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ be an almost $\mathcal{W}$-continuous function and let V be an $\mathcal{W}$-open set of Y. If $x \in c_{\mathcal{W}}(f^{-1}(V)) - (f^{-1}(V))$, then $f(x) \in c_{\mathcal{W}}(V)$.*

Proof Let $x \in X$ and V be an $\mathcal{W}$-open set of Y such that $x \in c_{\mathcal{W}}(f^{-1}(V)) - f^{-1}(V)$ and $f(x) \notin c_{\mathcal{W}}(V)$. Then there exists an $\mathcal{W}$-open set U containing $f(x)$ such that $U \bigcap V = \emptyset$. Since f is almost $\mathcal{W}$-continuous, then there exists an $\mathcal{W}$-open set W containing x such that $f(W) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(U))$. Since $U \bigcap V = \emptyset$, then by Lemma 2.1, we have $c_{\mathcal{W}}(U) \bigcap V = \emptyset$ and hence $i_{\mathcal{W}}(c_{\mathcal{W}}(U)) \bigcap V = \emptyset$. Thus $f(W) \bigcap V = \emptyset$. Since $x \in c_{\mathcal{W}}(f^{-1}(V))$ and W is an $\mathcal{W}$-open set containing x, then $W \bigcap f^{-1}((V) \neq \emptyset$ and hence $f(W) \bigcap V \neq \emptyset$. This is a contradiction. Therefore $f(x) \in c_{\mathcal{W}}(V)$. □

Definition 2.3 For any WS $\mathcal{W}$ on X and $A \subset X$. A point $x \in X$ is called $\mathcal{W}$-boundary point of A iff $x \in c_{\mathcal{W}}(A) \bigcap c_{\mathcal{W}}(X - A)$. The family of all $\mathcal{W}$-boundary points of A is denoted by $Bd_{\mathcal{W}}(A)$.

Theorem 2.5 *For any space $(X, \mathcal{W})$ and $A \subseteq X$, we have:*

(1) $Bd_{\mathcal{W}}(A) = Bd_{\mathcal{W}}(X - A)$;
(2) $Bd_{\mathcal{W}}(A) = c_{\mathcal{W}}(A) - i_{\mathcal{W}}(A)$;
(3) $A \cap Bd_{\mathcal{W}}(A) = \emptyset$ *if* $A \in \mathcal{W}$;
(4) $Bd_{\mathcal{W}}(A) \subset A$ *if* $X - A \in \mathcal{W}$.

Proof It follows from Theorem 1.1 and Definition 2.3. □

Remark 2.2 It is clear that the converse of (3) and (4) in the above theorem are not correct in general as illustrated by the next example.

Example 2.1 Let $X = \{x, y, z\}$ and $\mathcal{W} = \{\emptyset, \{x\}, \{y\}, \{z\}\}$. It is clear that:

(1) $A = \{x, z\}$ achieves $A \cap Bd_{\mathcal{W}}(A) = \emptyset$, but $A \notin \mathcal{W}$;
(2) $A = \{z\}$ achieves $Bd_{\mathcal{W}}(A) \subset A$, but A is not $\mathcal{W}$-closed.

Theorem 2.6 *Let $f\colon (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ be a function and let $A = \{x \in X : f$ is not almost $\mathcal{W}$-continuous at $x\}$. Then $A = Bd_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$ for each $\mathcal{W}$-open set V containing $f(x)$.*

Proof Let $x \in A$. Then f is not almost $\mathcal{W}$-continuous at x and hence there exists an $\mathcal{W}$-open set V of Y containing $f(x)$ such that $U \bigcap (X - f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))) \neq \emptyset$ for each $\mathcal{W}$-open set U of X containing x and hence $x \in c_{\mathcal{W}}(X - f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$. Since $f(x) \in V$, then $x \in f^{-1}(V) \subseteq f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))$ and thus $x \in c_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$. Thus we get $x \in Bd_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$. Therefore

$$A \subseteq Bd_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))) \tag{1}$$

Let $x \notin A$ and V be an $\mathcal{W}$-open set containing $f(x)$. Then f is almost $\mathcal{W}$-continuous at x and V is an $\mathcal{W}$-open set containing $f(x)$ and hence there exists an $\mathcal{W}$-open set U containing x such that $U \subseteq f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))$. Thus $x \in i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$ and hence $x \notin X - i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))) = c_{\mathcal{W}}(X - f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$ which implies $x \notin Bd_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))$. Therefore

$$A \supseteq Bd_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V)))) \tag{2}$$

From (1) and (2) we have $A = Bd_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(V))))$. □

Definition 2.4 A map $f\colon (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ is said to be $r(\mathcal{W})$-continuous iff $f^{-1}(H)$ is an $\mathcal{W}$-open set in X for every $H \in r(\mathcal{W}_Y)$.

Remark 2.3 One may notice that each $r(\mathcal{W})$-continuous function is almost $\mathcal{W}$-continuous, but the converse need not be true in general as shown by the following example.

Example 2.2 Let $X = \{a, b, c\}$, $\mathcal{W}_X = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, $Y = \{x, y\}$, $\mathcal{W}_Y = \{\emptyset, \{x\}, \{y\}\}$ and $f\colon (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ be a map defined by $f(a) = f(b) = x$, $f(c) = y$. One may notice that:

(1) $A = \{x\} \in r(\mathcal{W}_Y)$ and $f^{-1}(A) = \{a, b\}$ which is not an $\mathcal{W}$-open set in X.
(2) For $\mathcal{W}_Y$-open set $\emptyset$, we have $c_{\mathcal{W}}(\emptyset) = \emptyset$ and hence $i_{\mathcal{W}}(c_{\mathcal{W}}(\emptyset)) = \emptyset$ which implies $f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(\emptyset))) = f^{-1}(\emptyset) = \emptyset$. Thus $f^{-1}(\emptyset) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(\emptyset))))$. For $\mathcal{W}_Y$-open set $\{x\}$, we have $c_{\mathcal{W}}(\{x\}) = \{x\}$ and hence $i_{\mathcal{W}}(c_{\mathcal{W}}(\{x\})) = \{x\}$ which implies $f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(\{x\}))) = f^{-1}(\{x\}) = \{a, b\}$. Thus $i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(\{x\})))) = i_{\mathcal{W}}(\{a, b\}) = \{a, b\}$ and hence $f^{-1}(\{x\}) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(\{x\}))))$. For $\mathcal{W}_Y$-open set $\{y\}$, we have $c_{\mathcal{W}}(\{y\}) = \{y\}$ and hence $i_{\mathcal{W}}(c_{\mathcal{W}}(\{y\})) = \{y\}$ which implies $f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(\{y\}))) = f^{-1}(\{y\}) = \{c\}$. Thus $i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(\{y\})))) = i_{\mathcal{W}}(\{c\}) = \{c\}$ and hence $f^{-1}(\{y\}) \subseteq i_{\mathcal{W}}(f^{-1}(i_{\mathcal{W}}(c_{\mathcal{W}}(\{y\}))))$. Then f satisfy (4) in Theorem 2.1 and hence f is almost $\mathcal{W}$-continuous. Hence f is almost $\mathcal{W}$-continuous but it is not $r(\mathcal{W})$-continuous.

Theorem 2.7 *If a function $f\colon (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ is almost $\mathcal{W}$-continuous and $c_{\mathcal{W}}(V)$ is $\mathcal{W}$-closed for each $V \subseteq X$, then f is $r(\mathcal{W})$-continuous.*

Proof Let $H \in rc(\mathcal{W}_Y)$. Then by Theorem 2.1(3) we have $c_{\mathcal{W}}(f^{-1}(H) \subseteq f^{-1}(H)$ and hence $c_{\mathcal{W}}(f^{-1}(H) = f^{-1}(H)$. Since $c_{\mathcal{W}}(V)$ is $\mathcal{W}$-closed for each $V \subseteq X$, then $c_{\mathcal{W}}(f^{-1}(H)$ is $\mathcal{W}$-closed in X and hence $(f^{-1}(H)$ is $\mathcal{W}$-closed in X. Therefore f is $r(\mathcal{W})$-continuous. □

Theorem 2.8 *If a function $f: (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ is $r(\mathcal{W})$-continuous, then for each $V \in r(\mathcal{W}_Y)$ such that $f(x) \in V$, there is a $\mathcal{W}_X$-open set U such that $x \in U$ and $f(U) \subseteq V$.*

Proof Let $V \in r(\mathcal{W}_Y)$ containing $f(x)$. Then $f^{-1}(V)$ is an $\mathcal{W}_X$-open set containing x and hence there exists an $\mathcal{W}_X$-open set U such that $x \in U \subseteq f^{-1}(V)$. Thus there exists an $\mathcal{W}_X$-open set U containing x such that $f(U) \subseteq V$. □

Remark 2.4 By the following example, we show that the converse of the above theorem need not be true in general.

Example 2.3 Let $X = \{a, b, c\}$, $\mathcal{W}_X = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, $Y = \{x, y\}$, $\mathcal{W}_Y = \{\emptyset, \{x\}, \{y\}\}$ and $f: (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ be a function defined by $f(a) = f(b) = x$, $f(z) = c$. One may notice that:

(1) $A = \{x\} \in r(\mathcal{W}_Y)$ and $f^{-1}(A) = \{a, b\}$ which is not an $\mathcal{W}$-open set in X.
(2) For $a \in X$, $f(a) = x \in \{x\} = V$ which is an $\mathcal{W}_Y$-open set, there exists an $\mathcal{W}_X$-open set $U = \{a\}$ containing a such that $f(U) = f(\{a\}) = \{x\} \subseteq \{x\} = i_{\mathcal{W}}(c_{\mathcal{W}}\{x\})) = i_{\mathcal{W}}(c_{\mathcal{W}}(V))$. For $b \in X$, $f(b) = x \in \{x\} = V$ which is an $\mathcal{W}_Y$-open set, there exists an $\mathcal{W}_X$-open set $U = \{b\}$ containing b such that $f(U) = f(\{b\}) = \{x\} \subseteq \{x\} = i_{\mathcal{W}}(c_{\mathcal{W}}\{x\})) = i_{\mathcal{W}}(c_{\mathcal{W}}(V))$. For $c \in X$, $f(c) = y \in \{y\} = V$ which is an $\mathcal{W}_Y$-open set, there exists an $\mathcal{W}_X$-open set $U = \{c\}$ containing c such that $f(U) = f(\{c\}) = \{y\} \subseteq \{y\} = i_{\mathcal{W}}(c_{\mathcal{W}}\{y\})) = i_{\mathcal{W}}(c_{\mathcal{W}}(V))$.

Theorem 2.9 *Let $f: (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ be a function. If for each $x \in X$ and each $V \in r(\mathcal{W}_Y)$ containing $f(x)$, there exists an $\mathcal{W}_X$-open set U containing x such that $f(U) \subseteq V$, then f is almost $\mathcal{W}$-continuous.*

Proof Let $x \in X$ and V be an $\mathcal{W}_Y$-open set such that $f(x) \in V$. Then $i_{\mathcal{W}}(c_{\mathcal{W}}(V)) \in r(\mathcal{W})$ containing $f(x)$ and hence there is $U \in \mathcal{W}_X$ such that $x \in U$ and $f(U) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V))$. Therefore f is almost $\mathcal{W}$-continuous. □

Theorem 2.10 *Let $f: (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ be a map. If $i_{\mathcal{W}}(V)$ is an $\mathcal{W}$-open set for each $V \in r(\mathcal{W})$, then the converse of Theorem 2.9 is true.*

Proof Let $x \in X$ and $V \in r(\mathcal{W}_Y)$ containing $f(x)$. Then $i_{\mathcal{W}}(c_{\mathcal{W}}(V)) = i_{\mathcal{W}}(V)$ and hence $i_{\mathcal{W}}(c_{\mathcal{W}}(V)$ is an $\mathcal{W}_Y$ -open set containing $f(x)$ and hence there exists an $\mathcal{W}_X$-open set U containing x such that $f(U) \subseteq i_{\mathcal{W}}(c_{\mathcal{W}}(V)) = V$. □

Remark 2.5 If we replaced a space $(Y, \mathcal{W}_Y)$ by a topological space (Y, τ) in Theorems 2.7, 2.8, and in $r(\mathcal{W})$-continuity definition, then the statements in these theorem are equivalents.

Definition 2.5 A family of sets $\xi = \{\lambda_\alpha : \alpha \in \Delta\}$ in a space $(X, \mathcal{W})$ is said to be a cover of X if $\bigcup_{\lambda_\alpha \in \Delta} \lambda_\alpha = X$ and a subfamily of ξ having a similar property is called a subcover of ξ.

Definition 2.6 A space $(X, \mathcal{W})$ is called:

(1) $\mathcal{W}$-regular if for every $x \in X$ and $\mathcal{W}$-closed set U such that $x \notin U$, there exist $M, N \in \mathcal{W}$ such that $x \in M$, $U \subseteq N$ and $M \bigcap N = \emptyset$.

(2) Almost $\mathcal{W}$-regular if for every $x \in X$ and $F \in rc(\mathcal{W})$ with $x \notin F$, there exist $M, N \in \mathcal{W}$ such that $x \in M$, $F \subseteq N$ and $M \bigcap N = \emptyset$.
(3) $\mathcal{W}$-normal if for every two $\mathcal{W}$-closed sets U and V with $U \bigcap V = \emptyset$, there exist $M, V \in \mathcal{W}$ such that $U \subseteq M$, $V \subseteq N$ and $M \bigcap N = \emptyset$.
(4) Almost $\mathcal{W}$-normal if for every $U, V \in rc(\mathcal{W}_Y)$ with $U \bigcap V = \emptyset$, there exist $M, N \in \mathcal{W}$ such that $U \subseteq M$, $V \subseteq N$ and $M \bigcap N = \emptyset$.
(5) $\mathcal{W}$-compact if every $\mathcal{W}$-open cover of X has a finite subcover.
(6) Nearly $\mathcal{W}$-compact if every cover $\xi = \{\lambda_\alpha : \alpha \in \Delta, \lambda_\alpha \in r(\mathcal{W})\}$ of X has a finite subcover.

Theorem 2.11 *If $f: (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ is an $r(\mathcal{W})$-continuous, $\mathcal{W}$-open function and $(X, \mathcal{W}_X)$ is $\mathcal{W}$-regular, then $(Y, \mathcal{W}_Y)$ is almost $\mathcal{W}$-regular.*

Proof Let f be $r(\mathcal{W})$-continuous and $\mathcal{W}$-open function and $F \in rc(\mathcal{W}_Y)$ with $x \notin F$. Then $f^{-1}(H)$ are $\mathcal{W}$-closed set in X with $f^{-1}(x) \notin f^{-1}(F)$. Since $(X, \mathcal{W}_X)$ is $\mathcal{W}$-regular, then there exist $M, N \in \mathcal{W}_X$ such that $f^{-1}(F) \subseteq M$, $f^{-1}(x) \in N$ and $M \bigcap N = \emptyset$ and hence $F \subseteq f(M)$, $x \in f(N)$ and $f(M) \bigcap f(N) = \emptyset$. Since f is $\mathcal{W}$-open, then $f(M)$ and $f(N)$ are $\mathcal{W}_Y$-open sets. Therefore $(Y, \mathcal{W}_Y)$ is almost $\mathcal{W}$-regular. □

Theorem 2.12 *If $f: (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ is an $r(\mathcal{W})$-continuous, $\mathcal{W}$-open function and $(X, \mathcal{W}_X)$ is $\mathcal{W}$-normal, then $(Y, \mathcal{W}_Y)$ is almost $\mathcal{W}$-normal.*

Proof Let f be $r(\mathcal{W})$-continuous and $F_1, F_2 \in rc(\mathcal{W}_Y)$ such that $F_1 \bigcap F_2 = \emptyset$, then $f^{-1}(F_1)$, $f^{-1}(F_2)$ are $\mathcal{W}$-closed sets in X with $f^{-1}(F_1) \bigcap f^{-1}(F_2) = \emptyset$. Since $(X, \mathcal{W}_X)$ is $\mathcal{W}$-normal, then there exist two $\mathcal{W}_X$-open sets M and N such that $f^{-1}(F_1) \subseteq M$, $f^{-1}(F_2) \subseteq N$ and $M \bigcap N = \emptyset$ and then $F_1 \subseteq f(M)$, $F_2 \subseteq f(N)$ and $f(M) \bigcap f(N) = \emptyset$. Since f is $\mathcal{W}$-open, then $f(M)$ and $f(N)$ are $\mathcal{W}_Y$-open sets. Therefore $(Y, \mathcal{W}_Y)$ is almost $\mathcal{W}$-normal. □

Theorem 2.13 *If $f: (X, \mathcal{W}_X) \to (Y, \mathcal{W}_Y)$ is an $r(\mathcal{W})$-continuous surjective function and $(X, \mathcal{W}_X)$ is $\mathcal{W}$-compact, then $(Y, \mathcal{W}_Y)$ is nearly $\mathcal{W}$-compact.*

Proof Let f be $r(\mathcal{W})$-continuous,$\mathcal{W}$-open and $\xi = \{\lambda_\alpha : \alpha \in \Delta, \lambda_\alpha \in r(\mathcal{W})\}$ be a cover of Y. Then $f^{-1}(\xi) = \{f^{-1}(\lambda_\alpha) : \alpha \in \Delta, \lambda_\alpha \in r(\mathcal{W})\}$ is an $\mathcal{W}$-open cover of X. Since $(X, \mathcal{W}_X)$ is $\mathcal{W}$-compact, then $\{f^{-1}(\lambda_\alpha) : \alpha = 1, 2, 3, \ldots, n\}$ is a finite subcover of $f^{-1}(\xi)$ and hence $X = \bigcup_{\alpha=1}^{n} f^{-1}(\lambda_\alpha)$. Thus $Y = \bigcup_{\alpha=1}^{n} f(f^{-1}(\lambda_\alpha)) = \bigcup_{\alpha=1}^{n} \lambda_\alpha$. Therefore $(Y, \mathcal{W}_Y)$ is nearly $\mathcal{W}$-compact. □

References

1. Császár, A.: Generalized topology, generalized continuity. Acta Math. Hung. **96**, 351–357 (2002)
2. Császár, A.: Weak structure. Acta Math. Hung. **131**(1–2), 193–195 (2011)
3. Ekici, E.: On weak struture due to csaszar. Acta Math. Hung. **134**(4), 565–570 (2012)
4. Ghareeb, A., Güldürdek, Aslı: On modification of weak structures via hereditary classes. Boletim da Sociedade Paranaense de Matemática **35**(1), 229–235 (2017)
5. Ghareeb, A., Khalaf, M.M.: On weak structures. Analele Universităţii Oradea Fasc. Matematica **TOM XXII**(1), 129–134 (2015)
6. Umehara, J., Maki, H., Noiri, T.: Every topological space is pre $T_{\frac{1}{2}}$. Mem. Fac. Sci. Kochi. Univ. Ser. A Math. **17**, 53–66 (1996)
7. Zahran, A.M., El-Saady, K., Ghareeb, A.: Modification of weak structures via hereditary classes. Appl. Math. Lett. **25**(5), 869–872 (2012)

8. Zahran, A.M., Mousa, A.K., Ghareeb, A.: Generalized closed sets and some separation axioms on weak structure. Hacet. J. Math. Stat. **44**(3), 669–677 (2015)
9. Zakari, A.H., Ghareeb, A., Omran, S.: On soft weak structures. Soft Comput. 1–7 (2016). doi:10.1007/s00500-016-2136-8