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Fuzzy independent topological spaces generated by fuzzy γ^*-open sets and their applications

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Abstract To the best of our knowledge, till date there are only two directions along which fuzzy topology (general topology) can be compared, one is the generalization of fuzzy topology (such as fuzzy supra topology, generalized fuzzy topology etc.) and the other is the stronger form of fuzzy topology which is known as Alexandroff fuzzy topology. It means that a given topology is always linear. This paper aims to propose the third direction that is a new parallel form of fuzzy topology (or a non-linear topology) called fuzzy independent topology which is neither a generalization nor a stronger form of the given fuzzy topology though it is a unique natural offshoot of the given fuzzy topology but very rare in existence and that has been shown by defining fuzzy γ^*-open set in the sense of Dimitrije Andrijevic, by proving that fuzzy γ^*-open set and fuzzy open set are completely independent of each other though the collection of fuzzy γ^*-open sets are themselves a fuzzy topology therein. At the same time we claim that it is beyond the scope of general topology with the existing generalized open sets in literature and consequently we move one more step towards learning the difference between topology and fuzzy topology. To this end, we study the fundamental properties of the new structure. Also we study some of the basic properties and characterizations of fuzzy γ^*-open sets and few of their applications. The investigation enables us to present a new covering property of the given fuzzy topological space and the preservation theory is obtained. Finally to illustrate the advantage of the proposed concept, we compare the obtained results with some already existing ones.

Keywords Fuzzy topology · Fuzzy γ^*-open set · Fuzzy independent topology (or a non-linear topology) · Fuzzy I-open set · Fuzzy I-open cover

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1 Introduction and preliminaries

Since fuzzy set theory (as a generalization of ordinary set) was proposed by Zadeh [52], it has been developed in the theory and application of science over the last 52 years. Fuzzy topology is a generalization of topology in classical mathematics, but it also has its own special characteristics. Furthermore, it offers new methods and results and it also has applications in the important area of science and technology.

Various generalization of fuzzy open sets have been reported in the literature since 1968, fuzzy topological spaces were first introduced by Chang [16] and many authors have established different kinds of fuzzy compactness [7–9,15,27,30,31,46,49,50,54].

In 1983, Mashhour et al. [32,33] introduced the idea of supra topological spaces and Ghanim et al. [23] explained the gradation of supra-openness induced by a gradation of supra-proximity. The concepts of semi-open and semi-closed sets were first introduced in a fuzzy topological space by Azad [6]. The same concept has been introduced in bitopological spaces [14] and fuzzy bitopological spaces [48] respectively. Also in 1987, Abd El-Monsef et al. [1] introduced the notion of fuzzy supra topological spaces and studied fuzzy supra continuous function and obtained some properties and characterizations of those. Bhaumik and Mukherjee [11] started in on induced fuzzy supra-topological spaces. Mukherjee et al. [38] introduced the notion of pre-induced L-fuzzy supra topological spaces. The concepts of Alexandroff space [2], Neighbourhood space [28], Bornological space [25] and infra space [44] are in literature. Generalization of fuzzy continuous functions are studied by Balasubramanian et al. [10] and Park et al. [42]. The study of generalised open sets and generalised topology are carried out by Á. Császár, [17–19].

The stratiform structure by Xingfang Zhang et al. [53] is the most outstanding characteristic of L-fuzzy topological space as for every fixed stratification, it is proved that all L-fuzzy stratopen sets just form a new L-fuzzy topology, which is called stratiform L-fuzzy topology where new direction contains more information than the old one.

It is observed in the last few years that a large number of papers are devoted to the study of generalised open-like sets of a topological space, containing the class of open sets and possessing properties more or less similar to those of open sets. Extensive research on generalized open sets is still being continued by different mathematicians for the new results and applications.

Till date the study of general topology is regarded as a special case of fuzzy topology where all fuzzy sets in question take the values 0_X and 1_X only and many mathematicians [26,33,51] continued their investigation in fuzzy topological spaces and in most of the cases it is observed that topology is the generalization of the new spaces. These works of the above researchers on fuzzy topology motivated us to investigate a new topology itself in a given fuzzy topological space. In the present treatise the author aims to show that there exists a parallel topology in a fuzzy topological space which is neither a generalization nor stronger form of the original given fuzzy topology. That is to say that the goal of this paper is to study the spaces those have topologies which satisfy corresponding conditions but with unsurpassed nature. We also show that such an idea exists in fuzzy topological space but beyond the scope of a general topological space. This may not come as a shock to many people, because fuzzy topology is the generalization of topology. So the results illustrate an essential difference between topology and fuzzy topology in the many valued setting.

The objective of this present treatise is to study spaces those having topologies but with an incomparable nature. In this paper we propose a definition of a new parallel topology or independent topology or a non-linear topology for a fuzzy topological space mainly and call it the fuzzy independent topology and study fuzzy I-open sets and fuzzy I-closed sets and then we discuss about the notion of fuzzy I-continuity which is not a generalization of fuzzy continuity. Existence of such notions is discussed in section 3 by defining fuzzy γ^*-open sets in the sense of Dimitrije Andrijevic and we claim that it is a stronger form of fuzzy γ-open set due to Hanafy. The main emphasis is laid on the fact of intersection property of a fuzzy open set and a fuzzy pre-open set. The beauty of this revision is to present a non-linear form of fuzzy topology and we contemplate that this study may be applicable in parallel topology related to parallel circuit of electric networks and consequently in fuzzy graph theory also. In order to make the contents of the paper as self contained as possible we momentarily depict certain definition and for those not described, we refer to [16,52].

In what follows, by a space X we shall always mean a fuzzy topological space (in short, fts) (X, τ) due to Chang [16]. A fuzzy point x_p in X is a fuzzy set having support $x \in X$ and value $p[\in 0, 1]$ [43]. The complement of a fuzzy set λ is denoted by λ^c.

For two fuzzy sets λ, μ, we shall write $\lambda q \mu$ (resp. $\lambda \overline{q} \mu$) to mean that λ is quasi coincident (resp. not quasi coincident) with μ, that is there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$ (resp. $\lambda(x) + \mu(x) \leq 1$) [43].

The union $\vee \lambda_i$ (resp. the intersection $\wedge \lambda_i$) of a family $\{\lambda_i\}$ of fuzzy sets of X is defined to be the mapping $sup\lambda_i$ (resp. $inf\lambda_i$). For any two members λ and μ of I^X; $\lambda \geq \mu$ iff $\lambda(x) \geq \mu(x)$ for each $x \in X$, and in this case λ is said to contain μ or μ is said to be contained in λ.

Chang [16]. Let $f : X \to Y$ be a mapping. If λ is a fuzzy set of X, we defined $f(\lambda)$ as

$$f(\lambda)(y) = \begin{cases} \underset{x \in f^{-1}(y)}{sup} \lambda(x) & ; if\ f^{-1}(y) \neq \phi \\ 0 & ; otherwise. \end{cases}$$

For each $y \in Y$; and if μ is fuzzy set of Y, we define $f^{-1}(\mu)$ as $f^{-1}(\mu)(x) = \mu(f(x)), \forall x \in X$.

Let us recall the listed definitions which we shall require later for better understanding of the work in the following sections.

Definition 1.1 Chang [16]. A fuzzy topology τ on X is a collection of subsets of I^X, such that

(i) $0_X, 1_X \in \tau$
(ii) If $\lambda, \mu \in \tau$ then $\lambda \wedge \mu \in \tau$,
(iii) if $\mu_i \in \tau$, for each $i \in \Gamma$, then $\bigvee_{i \Gamma \in} \mu_i \in \tau$.

The pair (X, τ) is called a fuzzy topological space or a fuzzy space, in short.

Definition 1.2 Chang [16]. The closure of a fuzzy set μ is denoted by $cl(\mu)$ and is given by

$$cl(\mu) = \wedge\{\lambda : \lambda \text{ is a fuzzy closed set and } \mu \leq \lambda\}.$$

The interior of μ is denoted by $int(\mu)$ and is given by
$int(\mu) = \vee\{\beta : \beta \text{ is a fuzzy open set and } \mu \geq \beta\}$.

Let τ_1 and τ_2 be two fuzzy topologies for X. If the inclusion $\tau_1 \subset \tau_2$ holds, we say that τ_2 is finer than τ_1 an τ_1 is coarser than τ_2.

Definition 1.3 [6]. Let μ be a fuzzy set of a fuzzy topological space (X, τ). Then μ is called a fuzzy semiopen (resp. semi closed) set of X if $\mu \leq (cl(int(\mu)))$ $(resp.\ (int(cl(\mu)) \leq \mu)$.

Definition 1.4 [47]. Let μ be a fuzzy set of a fuzzy topological space (X, τ). Then μ is called a fuzzy pre-open (resp. pre-closed) set of X if $\mu \leq int(cl\,(\mu))$ (resp. $cl\,(int\,(\mu)) \leq \mu$). The collection of all fuzzy pre-open (resp. fuzzy pre-closed) sets is denoted by $FPO\,(X)$(resp. $FPC\,(X)$).

Definition 1.5 [3] A subset A of a topological space (X, T) is said to be γ-open if $A \cap S \in PO(X)$ for each $S \in PO(X)$. The class of all γ-open sets in (X, T) is denoted by T_γ.

Theorem 1.6 *[3] A subset A of a topological space (X, T) is closed in (X, T) if and only if $A \cup B$ is pre-closed for every pre-closed set B.*

Here $cl_\gamma(A)$ and $int_\gamma\,(A)$ stand for the closure of A and the interior of A with respect to (X, T_γ).

2 Fuzzy independent topology

Generalizations of topology and the stronger form of it are always linear for any ordinary topological space. But here we study that in an fts this direction is of different kind and gives us some important result with new flavour. We divide this section into the following subsections as we are to study the basic and fundamental tools of the mentioned concept and to emphasis the difference with the existing ones.

2.1 Fuzzy I-open sets and some of their properties

Topology (resp. fuzzy topology) is the weaker form of Alexandroff topology (resp. Alexandroff fuzzy topology) and at the same time supra topology (resp. supra topology), infra topology (resp. infra fuzzy topology) and generalized topology (resp. fuzzy generalized topology) are some of the generalizations of it and respectively. Means that it is the study of generalized open-like sets of a topological space (resp. fts (X, τ)), containing the class of open sets those possessing properties more or less similar to that of open (resp. fuzzy open) sets. It is well known that corresponding to every fuzzy topological space there is associated a fuzzy semi regular space [37], coarser than the space itself. Naturally question arises as follows:

Is there any other topology (or fuzzy topology) which is a parallel topology associated to the given topology violating the above statement in a given topological space or fts ? That is seeking for a new topology which is neither a generalization nor a stronger form of the given topology. In order to answer this question, we first need a definition and further we place an additional structure on a fts which is proposed for a given space as follows:

Definition 2.1 Let (X, τ) be a fts and τ_I (depending on τ) be a family of sub sets of I^X such that τ and τ_I are incomparable (that is τ is not finer nor coarser than τ_1) and the following axioms are satisfied:

(I_1) 0_X and $1_X \in \tau_I$,
(I_2)τ_I is closed under arbitrary union and
(I_3)τ_I is closed under finite intersection.

Then τ_I is called the fuzzy independent topology (for short, fit) associated with the fuzzy topology τ and τ is called the reference topology. The space (X, τ_I) is called the fuzzy independent topological space (in short, fits) with respect to the fts (X, τ) and this fts is known as the reference space. The members of τ_I are called fuzzy independent open sets (in short, f.I-open sets) and the complement of the same are called as fuzzy independent closed sets (in short, f.I-closed sets). Unless otherwise stated, τ_I always represents the fit with respect to the fuzzy topology τ.

Recall that an fts (X, τ) is called fuzzy semi regular [6] if and only if the collection of all fuzzy regularly open sets of X forms a basis for the fuzzy topology τ on X.

Obviously both fits and fuzzy semi regular space are also independent of each other.

According to the above definition, fits is the natural offshoot of fts but very rare in existence. The following remark is a direct implication of the above definition.

Remark 2.2 Unlike fts the notion of discrete and indiscrete fuzzy topology are beyond the scope of fits.

Theorem 2.3 *The collection τ_{I_c} of all f.I-closed sets in a fits (X, τ_I) satisfies:*

(i) 0_X *and* $1_X \in \tau_{I_c}$,
(ii) *and if $\mu_i \in \tau_{I_c}$, then μ_i is closed under finite union and arbitrary intersection.*

Proof It is straight forward from the definition and hence omitted. □

Fuzzy I-open sets (resp. fuzzy I-closed sets) play a special role in the study of fit. Thus we are logically interested in the properties of those sets. It is not a mimic but the requirement of the study. The following theorem holds:

Theorem 2.4 *Let (X, τ_I) be a fits. Then the collection τ_{I_c} of all fuzzy I-closed sets in the given fits is a fuzzy co-topology, and is called fuzzy independent co-topology.*

Proof The proof, being straightforward, is omitted. □

Definition 2.5 A fuzzy set λ in a fits (X, τ) is called an independent neighbourhood (I-nbh, as an acronym) of a fuzzy point x_p iff $\exists$ a $\mu \in \tau_1$ such that $x_p \epsilon \mu \leq \lambda$; a I-nbh λ is said to be I-open iff λ is I-open. The family consisting of all the I-nbh of x_p is called the system of I-nbhs of x_p and is denoted by $N^I_{x_p}$.

Theorem 2.6 *In a fits (X, τ_I), a fuzzy subset λ is I-open iff for each fuzzy point $x_p \in \lambda$, $\lambda \in N^I_{x_p}$.*

Proof Let λ be a fuzzy I-open set, then by the above definition we have, $\lambda \in N^I_{x_p}$ for each fuzzy point $x_p \in \lambda$. On the other hand, let $\lambda \in N^I_{x_p}$ for each fuzzy point, then there is a $\mu_p \in \tau_I$ with $x_p \in \mu_p \leq \lambda$. But $\lambda = \vee x_p, \lambda_p = \bigvee_{x_p \in \lambda} \mu_p \leq \lambda$, hence $\lambda \leq \bigvee_{x_p \in \lambda} \mu_p \leq \lambda$. And therefore

$$\lambda = \vee \mu_p \in \tau_I.$$

□

Theorem 2.7 *If τ_I be a fuzzy independent topology on a set X, then the family $\{N^I_{x_p}$: x_p is a fuzzy point in $X\}$ have the following properties:*

(i) $1_X \in N^I_{x_p}$;
(ii) $\lambda \in N^I_{x_p} \Rightarrow x_p \leq \lambda$;
(iii) $\lambda \geq \mu \epsilon N^I_{x_p} \Rightarrow \lambda \in N^I_{x_p}$;
(iv) *For the support p of x_p, $N^I_{x_p} = \cap\{N^I_{x_q} : 0 \leq x_p \leq x_q$;*
(v) *If $\lambda \in N^I_{x_p}$, then there exists $\lambda_1 \in N^I_{x_p}$ with $\lambda_1 \leq \lambda$ such that $\lambda \in N^I_{x_p}$*

Proof It is straight forward and hence omitted. □

Definition 2.8 For a fuzzy set λ in a fits X, the independent interior (for short, I-interior) of λ is denoted by $I_{int}(\lambda)$, defined to be the union of all fuzzy I-open sets contained in λ, and the independent closure (for short, I-closure) of λ is denoted by $I_{cl}(\lambda)$, defined to be the intersection of all fuzzy I-closed sets containing λ.

Theorem 2.9 *If λ, μ are two fuzzy sets in a fits (X, τ_1), then*

(i) λ *is a fuzzy I-open (resp. fuzzy I-closed) set iff* $\lambda = I_{int}(\lambda)(\text{resp.}\ \lambda = I_{cl}(\lambda))$.
(ii) *If* $\lambda \le \mu$, *then* $I_{int}(\lambda) \le I_{int}(\mu)$ *and* $I_{cl}(\lambda) \le I_{cl}(\mu)$. *Also* $I_{int}(I_{int}(\lambda)) = I_{int}(\lambda)$ *and* $I_{cl}(I_{cl}(\lambda)) = I_{cl}(\lambda)$.
(iii) $I_{cl}(\lambda) \vee I_{cl}(\mu) = I_{cl}(\lambda \vee \mu)$ *and* $I_{cl}(\lambda) \wedge I_{cl}(\mu) \ge I_{cl}(\lambda \wedge \mu)$.
(iv) $I_{int}(\lambda) \wedge I_{int}(\mu) = I_{int}(\lambda \wedge \mu)$ *and* $I_{int}(\lambda) \vee I_{int}(\mu) \le I_{int}(\lambda \vee \mu)$
(v) $I_{int}(1_X - \mu) = 1_X - I_{cl}(\mu)$.
(vi) $I_{cl}(1_X - \mu) = 1_X - I_{int}(\mu)$.

Proof We show the proof of (v) and the rest are easy.

$$\begin{aligned}
1_X - I_{cl}(\mu) &= 1_X - \wedge\{\eta : 1_X - \eta \in \tau, \eta > \mu\}. \\
&= \vee\{1_X - \eta : 1_X - \eta \in \tau, \eta > \mu\} \\
&= \vee\{\theta : \theta \in \tau, 1_X - \theta > \mu\} \\
&= \vee\{\theta : \theta \in \tau, 1_X - \mu > \theta\} \\
&= I_{int}(1_X - \mu).
\end{aligned}$$

□

Definition 2.10 A fuzzy set λ in a fits (X, τ_I) is called an fuzzy independent locally closed set if $\lambda = \mu \wedge \beta$, where μ is an fuzzy I-open set and β is an fuzzy I-closed set. The complement of and fuzzy independent locally closed set is called fuzzy independent locally open set.

Remark 2.11 Every fuzzy I-open (resp. fuzzy I-closed) set is a fuzzy independent locally closed set.

Definition 2.12 A fuzzy set λ in a fits (X, τ_I) is called an fuzzy independent regular open set if $\lambda = I_{int}(I_{cl}(\lambda))$. The complement of a fuzzy independent regular open set is said to be fuzzy independent regular closed set.

Remark 2.13 Every fuzzy independent regular open (resp. fuzzy independent regular closed) set is I-open (resp. I-closed) set. But it can be easily verified that the converse is not true in general.

2.2 Fuzzy I-continuity

As we proceed with our study of fit, it is observed that much of the terminology has a fuzzy topological connotation. In this section we define six distinct notions of fuzzy continuity and also study some of their immediate consequences. Before concluding the section on fuzzy I-continuity, we consider the relation of reference space to fit to an extent which will at least indicate the origin of the term fit.

Definition 2.14 A function $f : (X, \tau_I) \rightarrow (Y, \sigma_I)$ from a fits X to a fits Y is fuzzy I-continuous (resp. fuzzy I-irresolute, fuzzy I-continuous of second kind, fuzzy contra I-continuous, fuzzy contra I*-continuous and fuzzy contra I-continuous of second kind) iff the inverse of each fuzzy open (resp. fuzzy I-open, fuzzy I-open, fuzzy open, fuzzy I-open and fuzzy I-open) set in Y is fuzzy I-open (resp. fuzzy I-open, fuzzy open, fuzzy I-closed, fuzzy I-closed and fuzzy closed) set in X.

Obviously, the above defined notions are independent among themselves and the composition of any function with itself is not the same function except fuzzy I-irresolute function. However the following compositions hold true in particular.

Theorem 2.15 *The composition of fuzzy I-continuous function and fuzzy I-irresolute function (resp. fuzzy I-continuous and fuzzy I-continuous of second kind, fuzzy I-continuous and fuzzy contra I*-continuous, fuzzy I-continuous and fuzzy contra I-continuous of second kind, fuzzy I-irresolute and fuzzy I-continuous of second kind, fuzzy I-irresolute and, fuzzy contra I*-continuous, fuzzy I-continuous of second kind and fuzzy I-continuous, fuzzy I-continuous of second kind and fuzzy contra I-continuous) is a fuzzy I-continuous (resp. fuzzy continuous, fuzzy contra I-continuous, fuzzy contra continuous, fuzzy I-continuous of second kind, fuzzy contra I*-continuous, fuzzy I-irresolute, fuzzy I-irresolute) function.*

Theorem 2.16 *The composition of fuzzy contra I-continuous function and fuzzy contra I-continuous of second kind (resp., fuzzy contra I*- continuous and fuzzy contra I-continuous of second kind, fuzzy contra I-continuous of second kind and fuzzy contra I-continuous) is a fuzzy continuous function (resp. fuzzy I-continuous of second kind, fuzzy I-irresolute) function.*

For the hypothetical definition and related results, existence is proved in section 3 with suitable example. Next we state some results without proof as those are trivial from a natural argument.

Theorem 2.17 *Let $f : (X, \tau_I) \rightarrow (Y, \sigma_I)$ be a function from a fits X to another fits Y, then the followings are equivalent:*

(i) *For each fuzzy set λ in X, the inverse of every neighbourhood of $f(\lambda)$ is an I-nbd of λ.*
(ii) *For each fuzzy set λ in X and each neighbourhood η of $f(\lambda)$, there is an I-nbh. μ of λ such that $f(\mu) < \eta$.*

Theorem 2.18 *Let $f : (X, \tau_I) \rightarrow (Y, \sigma_I)$ be a function from a fits X to another fits Y, then the followings are equivalent:*

(i) *The function f is fuzzy I-irresolute.*
(ii) *The inverse of every fuzzy I-closed set is again a fuzzy I-closed set.*

A fuzzy I-homomorphism is a fuzzy I-continuous one-to-one map of a fits X onto a fits Y such that the inverse of the map is also fuzzy I-continuous. If there exists a fuzzy I-homomorphism of one fits onto another fits, then these two fits are said to be fuzzy I-homomorphic to the other. Two fits are topologically fuzzy I-equivalent iff they are fuzzy I-homomorphic.

Definition 2.19 For two fits's (X, τ_I) and (Y, σ_I) the product fuzzy independent topology on $X = X_1 \times X_2$ is the one having the family $\{\mu_1 \times \mu_2 : \mu_1 \in \tau_1, \mu_2 \in \sigma_1\}$ as a basis.

Theorem 2.20 *Let $\left(X_1, \tau_I^1\right), \left(X_2, \tau_I^2\right), \left(Y_1, \sigma_I^1\right), \left(Y_2, \sigma_I^2\right)$ be fits's. If $f_i : X_i \rightarrow Y_i, i = 1, 2$ are fuzzy I-continuous, then $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is fuzzy I-continuous.*

Definition 2.21 A function $f : (X, \tau_I) \rightarrow (Y, \sigma_I)$ is called a fuzzy I-open (resp. fuzzy I-closed) if $f(\lambda)$ is fuzzy I-open (resp. fuzzy I-closed) for each fuzzy I-open (resp. fuzzy I-closed) set $\lambda \in \tau_I$.

It is to be noted that a fuzzy open mapping and a fuzzy I-open mapping are necessarily independent to each other.

Theorem 2.22 *Let $f : (X, \tau_I) \rightarrow (Y, \sigma_I)$ be a mapping from a fits X into another fits Y fits. Then f is fuzzy I-open iff for each fuzzy point x_p of X and each fuzzy open set λ in X containing x_p, there exists a fuzzy I-open set $\mu_{f(x)}$ containing $f(x_p)$ such that $\mu_{f(x)} \leq f(\lambda)$.*

Proof The proof is obvious and hence omitted. □

2.3 Fuzzy I-compact

Definition 2.23 A collection $\{\mu_i : i \in \Gamma\}$ of fuzzy I-open sets in a fts (X, τ) is called fuzzy I -open cover of a set λ of X if $\lambda \leq \{\mu_i : i \in \Gamma\}$ holds.

Definition 2.24 An fts (X, τ) is said to be fuzzy I-compact if every fuzzy I-open cover of X has finite sub cover.

Definition 2.25 A subset λ of an fts (X, τ) is said to be fuzzy I-compact relative to X, if for every collection $\{\mu_i : i \in \Gamma\}$ of fuzzy I-open sets of X such that $\lambda \leq \vee\{\mu_i : i \in \Gamma\}$, there exists a finite subset Γ_0 of Γ such that $\lambda \leq \vee\{\mu_i : i \in \Gamma_o\}$.

Remark 2.26 It is clear that the fuzzy compactness and fuzzy I-compactness are completely independent of each other and hence fuzzy I-compactness and fuzzy θ-compactness [35,36] are also completely independent of each other.

Theorem 2.27 *A fts (X, τ) is fuzzy I-compact iff every family of I-closed subsets of X with the finite intersection property has a non-empty intersection.*

Proof Let X be a fuzzy I-compact space and $\{\beta_i : i \in \Gamma\}$ be a family of fuzzy I-closed subsets of X with the finite intersection property and suppose that $\wedge\{\beta_i : i \in \Gamma\} = 0_X$. Then $\{\beta_i^c : i \in \Gamma\}$ is a fuzzy I-open cover of X. But since X is fuzzy I -compact, so it contains a finite sub cover $\{\beta_{i_j}^c : j = 1, 2, \ldots, n\}$ for X. This shows that $\wedge \left\{\beta_{i_j} : j = 1, 2, \ldots, n\right\} = 0_X$. This contradicts that $\{\beta_i : i \in \Gamma\}$ has the finite intersection property.

Conversely, we suppose that $\{\mu_i : i \in \Gamma\}$ be an I-open cover of X. Then we consider the family $\{\mu_i^c : i \in \Gamma\}$ of fuzzy I-closed sets. Since $\{\mu_i : i \in \Gamma\}$ is an I-open of X, then intersection of all members of $\{\mu_i^c : i \in \Gamma\}$ is null. Hence $\{\mu_i^c : i \in \Gamma\}$ does not have the finite intersection property. Equivalently, there are finite number of fuzzy I-open sets $\mu_{i_1}, \mu_{i_2}, \ldots\ldots\ldots\ldots.\mu_{i_n}$, such that $\mu_{i_1}^c \wedge \mu_{i_2}^c\wedge, \ldots \wedge \mu_{i_n}^c = 0_X$. This implies that $\{\mu_{i_1}, \mu_{i_2}, \ldots \mu_{i_n}\}$ is a finite sub cover of X. Consequently X is fuzzy I-compact. □

Theorem 2.28 *Let τ_I be the fit in a fts (X, τ). Then the fts (X, τ) is fuzzy I-compact iff (X, τ_I) is fuzzy compact.*

Proof It is straight forward and hence omitted. □

2.4 Fuzzy I-connectedness

In this part we study the concept of connectedness in the context of fits following the direction of [45] and prove that it is preserved under fuzzy I-irresolute mapping though it is proper for fuzzy continuous image in an fts.

Definition 2.29 Let (X, τ_I) be a fits. Let A be an ordinary subset of X. Then the relative independent topology of A is defined as follows:

The subset A of X (in the ordinary sense) has a characteristic function say μ_A such that

$$\mu_A(X) = \begin{cases} 1 & if\, x \in A \\ 0 & if\, x \notin A. \end{cases}$$

Suppose $\tau_{I_A} = \{B \wedge A : B \in \tau_I\}$.

Then τ_{I_A} is called a fuzzy independent subspace topology on A.

Because if we choose $\phi \in \tau_{I_A}$ with membership function μ_ϕ, then as $\min\left(\mu_\phi(x), \mu_A(x)\right) = \mu_\phi(x)$, So we have $\phi \in \tau_{I_A}$.

To show that $A \in \tau_{I_A}$, we choose μ_X, the characteristic function for X, then $\min(\mu_X(x), \mu_A(x)) = \mu_A(x)$.

Also $\beta_A, \lambda_A \in \tau_{I_A}$ then $\beta_A \wedge \lambda_A = (\beta \wedge \lambda) \wedge A$, and

For $\beta_A^i \in \tau_{I_A}$, then $\bigvee_i \beta_A^i = \bigvee_i (\beta_i \wedge A) = (\bigvee \beta_i) \wedge A$.

Accordingly, τ_{I_A} defines the fuzzy independent subspace topology for A.

Definition 2.30 A fuzzy independent topological space (X, τ_I) is said to be I-disconnected if $X = \lambda \wedge \mu$, where λ and μ are non-empty I-open sets such that $0_X = \lambda \wedge \mu$.

Therefore, an I-connected space can be defined as follows:

Definition 2.31 A fuzzy independent topological space (X, τ_I) is said to be I-connected if X cannot be represented as the union of two non-empty disjoint I-open fuzzy sets on X.

Remark 2.32 If $X = \lambda \wedge \mu$, where $\lambda \wedge \mu = 0_X$, μ and λ are non-empty fuzzy I-open sets of X, then they are complement to each other, and hence, both are I-open and I-closed sets.

Theorem 2.33 *Let $f : X \to Y$ be a fuzzy I-irresolute map from X to Y where (X, τ_I) and $\left(Y, \tau_{I_{f(x)}}\right)$ are fuzzy independent topological spaces and X is I-connected. Then $f(X)$ is I-connected.*

Proof Suppose $f(X)$ is not I-connected. Then there exists non-empty I-open fuzzy sets λ and $\beta \in \tau_{I_{f(x)}}$ such that $f(X) = \lambda \vee \beta$. It means that λ and β are obtained from non-empty I-open fuzzy sets say λ_Y, β_Y in Y such that

$$\lambda = \lambda_Y \wedge f(X); \beta = \beta_Y \wedge f(X).$$

Now we are to show that $f^{-1}(\lambda)$ and $f^{-1}(\beta)$ give a I-disconnectedness for X. That is,

$$X = f^{-1}(\lambda) \vee f^{-1}(\beta).$$

But

$$\begin{aligned} &\max(\mu_{f^{-1}(\lambda)}(x), \mu_{f^{-1}(\beta)}(x)) \\ &= \max(\mu_\lambda(f(x)), \mu_\beta(f(x))) \\ &= 1_X(\text{since}\, \lambda \vee \beta) = f(X) = \mu_X(X). \end{aligned}$$

□

This is a contradiction to the fact that X is I-connected. As a consequence, $f(X)$ is I-connected.

The results in this section show that the condition of being an fuzzy independent topological space severely limits the type of spaces we are considering. Therefore, we must look to more exotic examples of fuzzy topologies to useful spaces that are independent. The tools in this section also allow us to construct new fuzzy topologies from the given ones. We assume that for any fts (X, τ), there always exists an fuzzy independent topology which is incomparable with the space itself. In the next section we show that though fuzzy open set and fuzzy I-open sets are completely independent of each other but both generate the same generalized open set.

3 Fuzzy γ^*-open sets and their fundamental properties

The idea of γ-open set [3] was defined as the intersection of pre-open sets giving a pre-open set as a result. In this section we are to define the notion of fuzzy γ^*-open set by using fuzzy pre-open [47] sets to show that the set of all fuzzy γ^*-open sets is also a fuzzy topology on (X, τ), but that fuzzy topology is neither a weaker nor a stronger form of the given topology τ. No doubt should be there regarding fuzzy γ-open sets due to Hanafy [24] because it is based on γ-open [21] or b-open [4] or sp-open [20] and as already proposed in the introduction that our main concern is to introduce fuzzy γ^*-open set in the sense of Dimitrije Andrijevic [3] for the purpose mentioned above and this section is presented as follows:

3.1 Fuzzy γ^*-topology

Definition 3.1 A subset λ of a fts (X, τ) is said to be fuzzy γ^*-open if $\lambda \wedge \mu \in FPO(X)$ for each $\mu \in FPO(X)$. The complement of a fuzzy γ^*-open set is called a fuzzy γ^*-closed.

The family of all fuzzy γ^*-open sets in X is denoted by $F\gamma^*O(X)$ and that of fuzzy γ^*-closed sets is denoted by $F\gamma^*C(X)$.

Proposition 3.2 *A subset β of a fuzzy topological space (X, τ) is fuzzy γ^*-closed if $\beta \vee \mu^c \in FPC(X)$ for each $\mu \in FPO(X)$.*

Proposition 3.3 *Every fuzzy γ^*-open set is fuzzy pre-open set. But the converse is not necessarily true, in general.*

Proof The proof of the first part is obvious from the definition. The converse is not necessarily true, which is verified through the following example. □

Example 3.4 The example taken from [47], and it shows that a fuzzy pre-open set need not necessarily be a fuzzy γ^*-open set.

Let $\lambda_1, \lambda_2, \lambda_3$ be fuzzy sets of I^X defined as:

$$\lambda_1 = \begin{cases} 1 - 2x; & if\ 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}; & if\ \frac{1}{2} \leq x \leq 1; \end{cases}$$
$$\lambda_2 = \begin{cases} 0; & if\ 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}; & if\ \frac{1}{2} \leq x \leq 1; \end{cases}$$
$$\lambda_3 = x, \quad \forall x \in X.$$

Clearly $\tau = \{0_X, \lambda_1, \lambda_2, 1_X\}$ is a fuzzy topology on X. Here λ_3 and λ_1 are both fuzzy pre-open sets, but their intersection $\lambda_3 \wedge \lambda_1$ is not a pre-open set in the fts (X, τ).

Remark 3.5 Similarly it is also easy to show that every fuzzy γ^*-closed set is a pre-closed set. But the converse may not be true in general.

It is well established that every open set is a γ-open set in an ordinary topological space but the converse is not necessarily true which is shown in the example below.

Example 3.6 Let (X, T) be a topological space, where $X = \{a, b, c\}$ and $T = \{\phi, \{a\}, X\}$. Here the collection of all γ-open sets is $T_\gamma = \{\phi, \{a\}, X, \{a, b\}, \{a, c\}\}$.

Definition 3.7 A topological space (X, T_1) is said to be an expansion of an topological space (X, T_2) if T_1 is finer than T_2 (that is $T_2 \subset T_1$).

Dimitrije Andrijevic [3] also proved that $T \subset T_\gamma \subset PO(X)$ in an ordinary topological space (X, T).

Proposition 3.8 *For every topological space (X, T), there exists an associated γ-topology which is an expansion of the given topology.*

Theorem 3.9 *[34] If U is an open set and A is a pre-open set, then $A \cup U$ is pre-open set.*

But a fuzzy set does not play the same role of an ordinary subset A like any open set A is a γ-open set. Amazingly, it is obtained that in a fts a fuzzy open set may not be a fuzzy γ^*-open set and that has already been verified in the above example (Example 3.4).

Now we generalize these results as an important theorem which is the main result of the whole work of this paper. It is the immediate consequence of the above examples 3.4 and 3.6.

Theorem 3.10 *In a fts (X, τ), the notions of fuzzy open set and fuzzy γ^*-open set are completely independent of each other.*

Remark 3.11 Recall that a fuzzy subset λ of a fts (X, τ), is called a fuzzy regular open [6] set if $int\, cl\, (\lambda) = \lambda$. It is clear that every fuzzy regular open (closed) set is a fuzzy open (closed) set. As a consequence of the above result we can say that fuzzy regular open set and fuzzy γ^*-open set are also completely independent of each other and that is verified in the following example.

Example 3.12 Let (X, τ) be a fts, where $X = \{a, b\}$ and $\tau = \{0_X, 1_X, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. The fuzzy sets are defined as follows: $\lambda_1 = \{(a, 0.2), (b, 0.5)\}$, $\lambda_2 = \{(a, 0.7), (b, 0.4)\}$, $\lambda_3 = \{(a, 0.7), (b, 0.5)\}$, $\lambda_4 = \{(a, 0.2), (b, 0.4)\}$ and $\lambda_5 = \{(a, 0.3), (b, 0.7)\}$. Here λ_3 is a fuzzy regular open set and λ_5 is a fuzzy pre-open set, but their intersection $\lambda_3 \wedge \lambda_5 = \{(a, 0.3), (b, 0.5)\}$, which is not a fuzzy pre-open set.

The next statements from [3] relate the closure and interior operator in (X, T_γ) to the operators in a general topological space (X, T).

Theorem 3.13 [3] *Let A be a subset of X. Then:*

(a) $int_\gamma\, (clA) = int\, (clA)$,
(b) $cl_\gamma (intA) = cl\, (intA)$,
(c) $RO\, (X, T) \in RO\left(X, T_\gamma\right)$.

But these are beyond the scope of the work in this paper, as the equality does not hold because of the above Theorem 3.10.

Remark 3.14 This important characterization of fuzzy γ^*-open set is quite different from all the existing various type generalization of fuzzy open sets in a fts. One can verify that fuzzy θ-open [36] (resp. fuzzy δ-open) set and fuzzy γ^*-open are also independent of each other.

Lemma 3.15 *Let (X, τ) be a fts and λ be a subset of it. Then λ is a fuzzy γ^*-open set iff for each fuzzy point $x_p \in \lambda$, there exists a fuzzy τ^*-open set μ such that $x_p \in \mu \leq \lambda$.*

Proof Easy, so omitted. □

Definition 3.16 Let (X, τ) be a fts and λ be a subset of it. Then λ is a fuzzy γ^*-neighbourhood of a fuzzy point x_p of X if there exist a fuzzy γ^*-open set μ such that $x_p \in \mu \leq \lambda$.

Definition 3.17 Let (X, τ) be a fts and λ be a subset of X. Then λ is a fuzzy γ^*-quasi neighbourhood of a fuzzy point x_p of X if there exist a fuzzy γ^*-open set μ such that $x_p q \mu \leq \lambda$.

Theorem 3.18 *Let (X, τ) be a fts. A fuzzy set λ is fuzzy γ^*-open set is iff for each fuzzy point $x_p \in \lambda$, λ is a fuzzy γ^*-neighborhood of x_p.*

Theorem 3.19 *Arbitrary (resp. finite) union of fuzzy γ^*-open (resp. fuzzy γ^*-closed) sets is also fuzzy γ^*-open (resp. fuzzy γ^*-closed) set.*

Proof Let $\lambda_i, i \in \Gamma$, be an arbitrary collection of fuzzy γ^*-open sets, then for each $i \in \Gamma$, $\lambda_i \bigwedge \beta \in FPO(X)$, for every $\beta \in FPO(X)$. So, $(\bigvee_{i\in\Gamma} \lambda_i) \bigwedge \beta = \bigvee_{i\in\Gamma}(\lambda_i \wedge \beta)$ is a fuzzy pre-open set that belongs to $FPO(X)$. □

Theorem 3.20 *The finite (resp. arbitrary) intersection fuzzy γ^*-open (resp. fuzzy γ^*-closed) sets is also fuzzy γ^*-open (fuzzy γ^*-closed) set.*

Proof If λ_1 and λ_2 are fuzzy γ^*-open sets, then $(\lambda_1 \wedge \lambda_2) \wedge \beta = \lambda_1 \wedge (\lambda_2 \wedge \beta)$ for all $\beta \in FPO(X)$. And consequently $(\lambda_1 \wedge \lambda_2) \in F\gamma^*O(X)$. □

We formulate an important result in the following theorem:

Theorem 3.21 *The family of all fuzzy γ^*-open sets in X forms a fuzzy topology.*

Proof It is always true that 0_X and 1_X are in $F\gamma^*O(X)$. The remaining part is proved by Theorems 3.19 and 3.20. □

Definition 3.22 The set of all fuzzy γ^*-open sets of a fts (X, τ) forms a fuzzy topology on X which is denoted by τ_{γ^*} and is called a fuzzy γ^*-topology on X. An ordered pair (X, τ_{γ^*}) is called the fuzzy γ^*-topological space. The elements of τ_{γ^*} are called the fuzzy τ_{γ^*}-open set and their complements are known as fuzzy τ_{γ^*}-closed sets.

It is to be noted that τ_{γ^*} is not the expansion of τ of a fts (X, τ) and this indicates a difference between a topological space and a fts.

Remark 3.23 For any fuzzy topology on X (except the discrete topology and indiscrete topology) there always exists a unique fuzzy γ^*-topology and this completes the existence of fuzzy independent topology and the required space.

Ganster [22] proved that for a general topological space (X, T), the collection of all pre-open sets that is $PO(X, T) = T$ is a topology iff the intersection of any two dense sets is pre-open set. But the topology obtained there is not an independent topology. We emphasize that no such scheme exists in the context of topology as open sets are always included in the collection of generalized open sets.

Question. Is there any other collection of sets or any other form of topology which structures an independent topology for a given topological space (X, T)?

Definition 3.24 For a fuzzy subset λ of X,

(i) the γ^*-closure of λ is the intersection of all fuzzy γ^*-closed sets containing λ and is denoted by $cl_{\gamma^*}(\lambda) = \bigwedge\{\mu \in I^X | \lambda \leq \mu, \mu \in F_{\gamma^*}C(X)\}$.
(ii) the γ^*-interior of λ is the union of all fuzzy γ^*-open sets contained in λ and written as

$$int_{\gamma^*}(\lambda) = \bigvee\{\beta \in I^X | \beta \leq \lambda, \beta \in F_{\gamma^*}O(X)\}.$$

The following theorem is the general consequence of the above definitions.

Theorem 3.25 *If λ and β are two fuzzy sets of X and x_p be a fuzzy point in X, then*

(i) λ is fuzzy γ^*-open iff $\lambda = int_{\gamma^*}(\lambda)$,
(ii) β is fuzzy γ^*-closed iff $\beta = cl_{\gamma^*}\{\beta\}$ and
(iii) $x_p \in cl_{\gamma^*}(\beta)$ iff $\beta \wedge \mu \neq 0_X$, for every fuzzy γ^*-open set μ containing x_p.

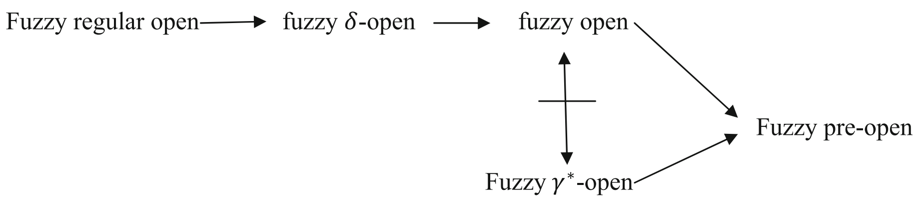
Corollary 3.26 *If $\lambda \leq \beta$ in X, then $int_{\gamma^*}(\lambda) \leq int_{\gamma^*}(\beta)$ and $cl_{\gamma^*}\{\lambda\} \leq cl_{\gamma^*}\{\beta\}$.*

Remark 3.27 The fuzzy γ^*-interior is equivalently defined as follows:

$$int_{\gamma^*}(\lambda) = 1_X - cl_{\gamma^*}(1_X - \lambda).$$

From [47], it is obvious that every fuzzy open (resp. closed) set is a fuzzy pre-open (resp. pre-closed) set. But the converse is false.

From this section, we can conclude that though fuzzy open set and fuzzy γ^*-open set are completely independent of each other but both of them imply the same generalized open set, namely fuzzy pre-open set and first time such type of relation is presented by the following diagram. The beauty of this relation emphasizes that the generalization of fuzzy open sets constitutes a new non-linear fuzzy topology in a given fts.



Question Is there any stronger form of open set that implies open set as well as its associated independent open set simultaneously in a fts ?

3.2 Fuzzy γ^*-continuous mappings

In this section we study some application of fuzzy γ^*-open sets and some related results which show the existence of composition of irresolute mapping and homeomorphism in this context.

Definition 3.28 A function $f : X \to Y$ is said to be fuzzy γ^*-continuous if and only if for each point x_p in X and for any fuzzy open q-neighbourhood η of $f(x_p)$ in Y, there exists a fuzzy γ^*-open q-neighbourhood μ of x_p such that $f(\mu) \leq \eta$.

Remark 3.29 Obviously the fuzzy continuity and fuzzy γ^*-continuity are independent notions.

The concept of fuzzy γ^*- continuity is presented by using fuzzy γ^* neighbourhood, and therefore also by using fuzzy γ^*-open sets as below:

Theorem 3.30 *A function $f : X \rightarrow Y$ is fuzzy γ^*-continuous if and only if for each point x_p in X and each fuzzy open neighbourhood η of $f(x_p)$, $f^{-1}(\eta)$ is a fuzzy γ^*-neighbourhood of x_p.*

Proof Let $x_p \in X$ and η be a fuzzy open neighbourhood of $f(x_p)$. So that there exists a q-neighbourhood β of $f(x_p)$ such that $\beta \leq \eta$. Since f is fuzzy γ^*-continuous, then there exists open q-neighbourhood μ of x_p such that $f(\mu) \leq \beta$ and hence $\mu \leq f^{-1}(\beta)$ But, since $f^{-1}(\beta) \leq f^{-1}(\eta)$, so $f^{-1}(\eta)$ is a fuzzy γ^*-neighbourhood of x_p.

Conversely, suppose $x_p \in X$ and η be a open neighbourhood of $f(x_p)$. Then η is a fuzzy neighbourhood of $f(x_p)$. But by the hypothesis, $f^{-1}(\eta)$ is a fuzzy γ^*-neighbourhood of x_p such that $\mu \leq f^{-1}(\beta)$ and so that $f(\mu) \leq f(f^{-1}(\eta)) \leq \eta$ Therefore, f is a fuzzy γ^*-continuous function. □

Corollary 3.31 *A function $f : X \rightarrow Y$ is fuzzy γ^*-mapping if and only if for each fuzzy open set λ of Y, $f^{-1}(\lambda)$ is a fuzzy γ^*-open set in X.*

The idea of fuzzy pre-continuity [13] was initiated in the following way:

A function $f : X \rightarrow Y$ is said to be fuzzy pre-continuous if $f^{-1}(V)$ is fuzzy pre-open set in X for each fuzzy open set V in Y.

Fuzzy continuity always implies fuzzy pre-continuity but the converse is not true in general.

Remark 3.32 Obviously every fuzzy γ^*-continuous function is a fuzzy pre-continuous function but the converse does not necessarily hold.

Example 3.33 Let $X = Y = \{x, y\}$, $\tau = \{0_X, 1_X, \{(x, 0.8), (y, 0.1)\}, \{(x, 0.9), (y, 0.7)\}\}$, and $\sigma = \{0_Y, 1_Y, \{(x, 0.7), (y, 0.9)\}\}$. We consider a function $f : (X, \tau) \rightarrow (Y, \sigma)$ such that $f(x) = y$ and $f(y) = x$. . Here f is a fuzzy pre-continuous function but not fuzzy γ^*-continuous function as the intersection of the fuzzy open set $\{(x, 0.9), (y, 0.7)\}$ and the fuzzy pre-open set $\{(x, 0.1), (y, 0.9)\}$ is not a fuzzy pre-open set in the fts (X, τ).

We have the following diagram:

$$\begin{tikzcd}
\text{fuzzy continuity} \arrow[dr] & \not\Leftrightarrow & \text{fuzzy } \gamma^*\text{-continuous} \arrow[dl] \\
& \text{fuzzy pre-continuity} &
\end{tikzcd}$$

Theorem 3.34 *A function $f : X \rightarrow Y$ is fuzzy γ^*-continuous iff the inverse image of each fuzzy closed set of Y is fuzzy γ^*-closed set in X.*

Proof This can be proved by usual technique. □

Definition 3.35 Let $f : X \rightarrow Y$ be a fuzzy mapping. Then

(i) f is said to be fuzzy γ^*-open if for each fuzzy open set λ in X, $f(\lambda)$ is a fuzzy γ^*-open set in Y.
(ii) f is said to be fuzzy γ^*-closed if for each fuzzy closed set λ in X, $f(\lambda)$ is fuzzy γ^* -closed set in Y.

Irresoluteness and its stronger form [55] are some of the main topics in topology. Fuzzy pre-irresolute function was defined in [38] as below:

A function $f : X \to Y$ is said to be fuzzy pre- irresolute if $f^{-1}(\mu)$ is a fuzzy pre-open set in X for each fuzzy pre-open set μ in Y.

We extend the concept and study a new class of fuzzy irresolute function, termed as fuzzy γ^*- irresolute function and show that fuzzy pre-irresoluteness is the generalization of it.

Definition 3.36 A function $f : X \to Y$ is said to be fuzzy γ^*-irresolute if $f^{-1}(\mu)$ is fuzzy γ^*- open set in X for each fuzzy γ^*-open set μ in Y.

Remark 3.37 Every fuzzy γ^*-irresolute function is a fuzzy pre- irresolute function but evidently the converse is not true in general.

Theorem 3.38 *If $f : X \to Y$ is a fuzzy mapping then the followings are equivalent:*

(i) *f is fuzzy γ^*- irresolute.*
(ii) *For each fuzzy set λ in X, $f(cl_{\gamma^*}(\lambda)) \le cl_{\gamma^*}(f(\lambda))$.*
(iii) *For each fuzzy set β in Y, $cl_{\gamma^*}(f^{-1}(\beta)) \le f^{-1}\left(cl_{\gamma^*}(\beta)\right)$.*
(iv) *For each fuzzy γ^*-closed set β in Y, $f^{-1}(\beta)$ is fuzzy γ^* -closed set in X.*
(v) *For each fuzzy γ^*-open set β in Y, $f^{-1}(\beta)$ is fuzzy γ^*-open set in X.*

Proof $(i) \Rightarrow (ii)$ Let $x_p \in cl_{\gamma^*}(\lambda)$ and η be a γ^*-open q-neighborhood of $f(x_p)$. Then there exists a γ^*-open neighbourhood μ of x_p such that $f(\mu) \le \eta$. Since $x_p \in cl_{\gamma^*}(\lambda)$, we have $\eta q \lambda$. Then $f(\mu) q f(\lambda)$. Thus $\eta q f(\lambda)$ and hence $f(x_p) \in cl_{\gamma^*}(f(\lambda))$. So $f(cl_{\gamma^*}(\lambda)) \le cl_{\gamma^*}(f(\lambda))$.

$(ii) \Rightarrow (iii)$ By (ii), $f(cl_{\gamma^*}(f^{-1}(\beta))) \le cl_{\gamma^*}\left(f\left(f^{-1}(\beta)\right)\right) \le cl_{\gamma^*}(\beta)$.

Therefore, $cl_{\gamma^*}(f^{-1}(\beta)) \le f^{-1}(cl_{\gamma^*}(\beta))$.

$(iii) \Rightarrow (iv)$ We take $\beta = cl_{\gamma^*}(\beta)$. Now by (iii) $cl_{\gamma^*}\left(f^{-1}(\beta)\right) \le f^{-1}\left(cl_{\gamma^*}(\beta)\right) = f^{-1}(\beta)$.

So $cl_{\gamma^*}\left(f^{-1}(\beta)\right) = f^{-1}(\beta)$. Hence $f^{-1}(\beta)$ is a fuzzy γ^*-closed set.

$(iv \Rightarrow) (v)$ Let μ be a fuzzy γ^*-open set in Y. Then μ^c is fuzzy γ^*-closed set in Y. By (iv) $f^{-1}(\mu^c)$ is fuzzy γ^*-closed set in X. Since $f^{-1}(\mu^c) = 1_X - f^{-1}(\mu)$, $f^{-1}(\mu)$, is fuzzy γ^*-open set in X.

$(v) \Rightarrow (i)$ The proof is clear from the definition. □

Theorem 3.39 *Let (X, τ) and (Y, σ) be two fuzzy topological spaces. Let τ_{γ^*} and σ_{γ^*} be the fuzzy γ^*-topologies on X and Y respectively. Then if $f : (X, \tau) \to (Y, \sigma)$ is fuzzy γ^*-continuous then $f' : (X, \tau_{\gamma^*}) \to (Y, \sigma_{\gamma^*})$ is fuzzy continuous.*

Theorem 3.40 *If $f : (X, \tau_1) \to (Y, \tau_2)$ is a fuzzy γ^*-irresolute and $g : (Y, \tau_2) \to (Z, \tau_3)$ is also fuzzy γ^*-irresolute then gof is fuzzy γ^*-irresolute.*

Example 3.41 Let $X = Y = Z = \{x, y\}$.

Also let, $\tau_1 = \{\{(x, 0.9), (y, 0.7)\}, 0_X, 1_X\}$, $\tau_2 = \{\{(x, 0.8), (y, 0.7)\}, 0_Y, 1_Y\}$ and $\tau_3 = \{\{(x, 0.7), (y, 0.5)\}, 0_Z, 1_Z\}$.

$F_{\gamma^*}O(X) = \{\{(x, \alpha), (y, \beta)\}, \alpha > 0.1, \beta > 0.3\}$, $F_{\gamma^*}O(Y) = \{\{(x, \alpha), (y, \beta)\}, \alpha > 0.2, \beta > 0.3\}$ and $F_{\gamma^*}O(Z) = \{\{(x, \alpha), (y, \beta)\}, \alpha > 0.3, \beta > 0.5\}$.

Now, we consider the function $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ defined by $f(x) = x$, $f(y) = y$, f is a fuzzy γ^*-irresolute function and the inverse image of fuzzy γ^*-open set in Y is fuzzy γ^*-open set in X.

Again the function $g : (Y, \tau_2) \rightarrow (Z, \tau_3)$ is defined as $g(y) = y$, $g(z) = z$, then g is fuzzy γ^*-irresolute function as the inverse image of γ^*-open set in Z is γ^*-open set in Y.

Now let $\eta = \{\{(x, \alpha), (y, \beta)\}, \alpha > 0.3, \beta > 0.5\}$ be any fuzzy γ^*-open set in Z. Then For any fuzzy γ^*-open set η in Z, $(gof)^{-1}(\eta)$ is a fuzzy γ^*-open set in X. Thus composition of two fuzzy γ^*-irresolute function is a γ^*-irresolute function.

A fuzzy γ^*-homeomorphism is a fuzzy γ^*-continuous one-to-one map of a fits X onto another fits Y such that the inverse of the map is also fuzzy γ^*-continuous.

Example 3.42 Let $X = Y = Z = \{x, y\}$.

Also let, $\tau = \{\{(x, 0.9), (y, 0.7)\}, 0_X, 1_X\}$, $\sigma = \{\{(x, 0.8), (y, 0.7)\}, 0_Y, 1_Y\}$.

Here, $F_{\gamma^*}O(X) = \{\{(x, \alpha), (y, \beta)\}, \alpha > 0.1, \beta > 0.3\}$, $F_{\gamma^*}O(Y) = \{\{(x, \alpha), (y, \beta)\}, \alpha > 0.2, \beta > 0.3\}$.

Now if we consider the function $f : (X, \tau) \rightarrow (Y, \sigma)$ defined as $f(x) = x$, $f(y) = y$, then f is a fuzzy γ^*-continuous function as the inverse image of every fuzzy open set in Y is fuzzy γ^*-open set in X.

Again, since inverse image of any fuzzy open set in X under the function f^{-1} is fuzzy γ^*-open set in Y, it implies that f^{-1} is a fuzzy γ^*-continuous fucntion.

The weak form and strong form of fuzzy pre-irresoluteness and the fuzzy strong pre-continuity are shown in [40] and [29] respectively.

Definition 3.43 A fuzzy topological space X is said to be γ^*-closed if and only if for every family Γ of fuzzy γ^*-open sets λ such that $\bigvee_{\lambda \in \Gamma} \lambda = 1_X$, there exists a finite sub family $\Gamma_0 \subset \Gamma$ such that $\bigvee_{\lambda \in \Gamma_0} (1_Y(\lambda))(x) = 1_X$, for every $x \in X$.

Theorem 3.44 *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a fuzzy γ^*-irresolute surjection function. If X is a γ^*-closed space then Y is also a γ^*-closed space.*

Proof Let $\{\mu_i : i \in \Gamma\}$ be a fuzzy γ^*-open cover of Y. Then $\{f^{-1}(\mu_i) : i \in \Gamma\}$ is a fuzzy γ^*-open cover of X. Then there exists a finite sub set $\Gamma_0 \subset \Gamma$ such that $\bigvee\limits_{i \in \Gamma_0} cl_{\gamma^*}(f^{-1}(\mu_i)) = 1_x$ as X is fuzzy γ^*-closed. By the surjectivity of f and the above theorem 3.38, we have

$$1_X = f(1_X) = f(\bigvee\nolimits_{i \in \Gamma_0} cl_{\gamma^*}(f^{-1}(\mu_i))) \leq \bigvee\nolimits_{i \in \Gamma_0} cl_{\gamma^*}\left(f\left(f^{-1}(\mu_i)\right)\right) = \bigvee\nolimits_{i \in \Gamma_0} cl_{\gamma^*}(\mu_i)$$

Hence Y is γ^*-closed. □

3.3 Fuzzy γ^*-compactness

This section is devoted to study fuzzy γ^*-compactness in a given fts to show that it is a stronger shade of the space than fuzzy pre-compactness.

Definition 3.45 A collection $\{\mu_i : i \in \Gamma\}$ of fuzzy γ^*-open sets in a fts (X, τ) is called fuzzy γ^*-open cover of a set λ of X if $\lambda \leq \vee\{\mu_i : i \in \Gamma\}$ holds.

Definition 3.46 A fts (X, τ) is said to be fuzzy γ^*-compact if every fuzzy γ^*-open cover of X has finite subcover. A fuzzy subset λ of a fts (X, τ) is said to be fuzzy γ^*-compact in X, if for every collection $\{\mu_i : i \in \Gamma\}$ of fuzzy γ^*-open sets of X such that $\lambda \leq \vee\{\mu_i : i \in \Gamma\}$, there exists a finite subset Γ_0 of Γ such that $\lambda \leq \vee\{\mu_i : i \in \Gamma_0\}$.

In [12], fuzzy pre-compactness was defined as follows:

A fts (X, τ) is said to be fuzzy pre-compact if every fuzzy pre-open cover of X has finite sub cover.

Theorem 3.47 *Every fuzzy pre-compact space is fuzzy γ^*-compact.*

Proof Let $\{\mu_i : i \in \Gamma\}$ be a fuzzy -open cover of a fuzzy topological space (X, τ). Since any fuzzy γ^*-open is fuzzy pre-open set then $\{\mu_i : i \in \Gamma\}$ is fuzzy pre-open cover of (X, τ). But X is fuzzy pre-compact, so there exists a finite sub set Γ_0 of Γ such that $\{\mu_i : i \in \Gamma_0\}$. Hence X is a fuzzy γ^*-compact space. □

Question. Is there an example which shows that fuzzy γ^*-compactness does not imply fuzzy pre-compactness?

We state the following theorem without proof.

Theorem 3.48 *An fts (X, τ) is fuzzy γ^*-compact iff the corresponding fuzzy γ^*-topological space (X, τ_{γ^*}) is fuzzy compact.*

Theorem 3.49 *An fts (X, τ) is fuzzy γ^*-compact iff every family of fuzzy γ^*-closed subset of X with the finite intersection property has a non empty intersection.*

Proof The proof is similar to that of theorem 2.27. □

Corollary 3.50 *An fuzzy γ^*-topological space (X, τ_{γ^*}) is fuzzy compact iff every family of fuzzy τ_{γ^*}-closed subsets of X with the finite intersection property has a non-empty intersection.*

Hence one can notice that fuzzy γ^*-compactness of a fuzzy topological space is equivalent to fuzzy compactness of fuzzy γ^*-topological space. But two covers of X are still different with a common shade.

Theorem 3.51 *Let β be a fuzzy γ^*-closed subset of a fuzzy γ^*-compact space (X, τ), then β is also fuzzy γ^*-compact set in X.*

Proof Let β be any fuzzy γ^*-closed subset of X and $\{\mu_i : i \in \Gamma\}$ be a fuzzy γ^*-open cover of X. Since β^c is fuzzy γ^*-open set then $\{\mu_i : i \in \Gamma\} \vee \beta^c$ is a fuzzy γ^*-open cover of X. But since X is fuzzy γ^*-compact, then there exists a finite subset Γ_0 of Γ such that $X \leq \{\mu_i : i \in \Gamma_0\} \vee \beta^c$. However, β and β^c are disjoint, and hence $\beta \leq \{\mu_i : i \in \Gamma_0\}$. Therefore, β is a fuzzy γ^*-compact set in X. □

Theorem 3.52 *Let λ and β are two fuzzy subsets of a fts (X, τ), such that λ is γ^*-compact set in X and β is fuzzy γ^*-closed set in.X. Then $\lambda \wedge \beta$ is a fuzzy γ^*-compact set in X.*

Proof Suppose $\{\mu_i : i \in \Gamma\}$ be a fuzzy γ^*-open cover of $\lambda \wedge \beta$. As β^c is a fuzzy γ^*-open set then $\{\mu_i : i \in \Gamma\} \vee \beta^c$ is a γ^*-open cover of λ. Since λ is a fuzzy γ^*-compact set in X, then there exists a finite subset Γ_0 of Γ such that $\lambda \leq \{\mu_i : i \in \Gamma_0\} \vee \beta^c$. Therefore, $\lambda \wedge \beta \leq \{\mu_i : i \in \Gamma_0\}$. Hence $\lambda \wedge \beta$ is a fuzzy γ^*-compact set in X. □

Theorem 3.53 *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a fuzzy γ^*-irresolute and surjection mapping. If X is fuzzy γ^*-compact space, then Y is also a fuzzy γ^*-compact space.*

Proof Let $\{\mu_i : i \in \Gamma\}$ be a fuzzy γ^*-open cover of Y . Then $\{f^{-1}(\mu_i) : i \in \Gamma\}$ is a fuzzy cover of X. Since f is fuzzy γ^*-irresolute then $f^{-1}(\mu_i)$ is a fuzzy γ^*-open set and hence $\{f^{-1}(\mu_i) : i \in \Gamma\}$ is γ^*-open cover of X. As X is fuzzy γ^*-compact space, then there exists a finite subset Γ_0 of Γ such that $X \leq \vee\{f^{-1}(\mu_i) : i \in \Gamma_0\}$. Thus

$$f(X) \leq f\left\{\vee\left\{f^{-1}(\mu_i) : i \in \Gamma_0\right\}\right\} = \vee\left\{f\left\{f^{-1}(\mu_i) : i \in \Gamma_0\right\}\right\} = \vee\{(\mu_i) : i \in \Gamma_0\}.$$

Since f is surjective, then $Y = f(x) \leq \bigvee\{(\mu_i) : i \in \Gamma_0\}$ and consequently Y is fuzzy γ^*-compact . □

Theorem 3.54 *Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy γ^*-irresolute function. If a fuzzy subset λ is fuzzy γ^*-compact set in X, then the image $f(\lambda)$ is a fuzzy γ^*-compact set in Y.*

Proof Let $\{\mu_i : i \in \Gamma\}$ be a γ^*-open cover of $f(\lambda)$. Since f is fuzzy γ^*-continuous, then $f^{-1}(\mu_i)$ is a fuzzy γ^*-open set in X for all $i \in \Gamma$. Thus $\{f^{-1}(\mu_i) : i \in \Gamma\}$ is a cover of λ by fuzzy γ^*-open set in X. Since λ is fuzzy γ^*-compact in X, so there is a finite subset Γ_0 of Γ such that $\lambda \leq \vee\{f^{-1}(\mu_i) : i \in \Gamma\}$. Therefore, $f(\lambda) \leq f\{\vee f^{-1}(\mu_i) : i \in \Gamma\}$, and hence, $f(\lambda) \leq \vee\{\mu_i : i \in \Gamma_0\}$. Therefore, $f(\lambda)$ is fuzzy γ^*-compact set in Y. □

Theorem 3.55 *Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy γ^*-irresolute, fuzzy γ^*-open and injective mapping. If a fuzzy subset β of Y is fuzzy γ^*-compact set in Y, then $f^{-1}(\beta)$ is fuzzy γ^*-compact set in X.*

Proof Let $\{\mu_i : i \in \Gamma\}$ be a fuzzy γ^*-open cover of $f^{-1}(\beta)$ in X. Then $f^{-1}(\beta) \leq \vee\{\mu_i : i \in \Gamma\}$. and hence $\beta \leq f(f^{-1}(\beta)) \leq f(\vee\{(\mu_i) : i \in \Gamma\}) = \vee\{f(\mu_i) : i \in \Gamma\}$. As β is fuzzy γ^*-compact in Y there is a finite subset Γ_0 of Γ such that $\beta \leq \vee\{f(\mu_i) : i \in \Gamma_0\}$. Hence $f^{-1}(\beta) \leq f^{-1}(\vee\{f(\mu_i) : i \in \Gamma_0\}) = \vee\left\{f^{-1}(f(\mu_i)) : i \in \Gamma_0\right\} = \vee\{\mu_i : i \in \Gamma_0\}$. □

Theorem 3.56 *Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy γ^*-open bijective function and Y is a fuzzy γ^*-compact space then X is fuzzy γ^*-compact space.*

Proof Let $\{\mu_i : i \in \Gamma\}$ be a collection of fuzzy γ^*-open sets that covers X, then $\{f(\mu_i) : i \in \Gamma\}$ is a fuzzy γ^*-open covering of Y. Then we have $1_X - f^{-1}(1_Y) = f^{-1}(f(\bigvee_{i\in\Gamma} \mu_i)) = \bigvee_{i\in\Gamma} \mu_i$ and hence X is fuzzy γ^*-compact. □

4 Concluding remarks

Before three decades, Olav Njastad [39] shown that for a topological space (X, T), the topology T_α consists of exactly those sets A for which $A \cap^B \in SO(X)$ for all $B \in SO(X)$, the collection of all semi-open sets of the given topological space (X, T). Putting $PO(X)$, the collection of all pre-open sets of the given topological space (X, T), instead of $SO(X)$. Dimitrije Andrijevic emphasised that $T \subset T_\alpha \subset T_\gamma \subset PO(X)$ (refer, 2.2, 2.3, [3]). That is both of these collections are generalizations of topology and it is always linear. Certainly it is very interesting to prove that topology does not imply γ^*-topology in the context of fuzzy topology and in the history of topology, first time it is proved that a topology can generate a new topology which is incomparable with the given space itself that is a non-linear topology is appeared. Fuzzy γ^*-topology is not the only fit but one can show that the collection of all fuzzy semi-open [6] sets where the intersection is closed is also a fit and a unification theory may be studied for the same. The concept may be generalized in an L-fuzzy topological space as the space attracts more and more attention of the scientific community and many interesting properties of L-fts have been investigated [5]. We emphasize that no such idea exists in the context of topology as open sets are always included in the collection of generalized open sets. A good number of exercises can be done in the area of fit related to extremally disconnectedness and generalized I-open sets may be studied in various forms with a unified theory. The notion of separation axioms may be more important based on fuzzy γ^*-open sets. The idea of fit can be investigated in a fuzzy supra topological space, generalised fuzzy topological space and also in a fuzzy Alexanforff space. Mukherjee [37] showed that for any fts (X, τ), its semiregularization (X, τ_s) is fuzzy semi regular. Similar type of results may be obtained in fts by using fuzzy pre-open sets. Positively this notion of fit will give a new dimension in the application of fuzzy topology as well as fuzzy mathematics as a promise of having extensive claim in its embryonic stage.

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