Afr. Mat. (2017) 28:643–660
DOI 10.1007/s13370-016-0472-6

Square-mean μ-pseudo almost periodic and automophic solutions for a class of semilinear integro-differential stochastic evolution equations

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Received: 23 April 2015 / Accepted: 29 November 2016 / Published online: 18 December 2016

Abstract In this work, we use the concept of μ-square-mean pseudo almost periodic and automorphic processes introduced by Diop et al. (Afr Mat 26(5):779–812, 2015) to discuss the existence and uniqueness of solutions for some semilinear integro-differential stochastic evolution equations. We provide an example to illustrate ours results.

Keywords μ-Pseudo almost periodic solution · μ-Pseudo almost automorphic solution · Completeness · Composition theorem · Stochastic processes · Stochastic integro-differential equation

Mathematics Subject Classification 34C27 · 34K14 · 34K30 · 34K50 · 35B15 · 35K55 · 43A60 · 60G20

1 Introduction

Integro-differential equations arose naturally in mechanics, electromagnetic theory, heat flow, nuclear reactor dynamics and population dynamics. Ding et al. [14] investigated the existence of pseudo almost periodic solutions for an equation arising in the study of heat conduction in materials with memory, which could be transformed into the following abstract integro-differential equation

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + h(t, x(t)) \quad \text{for all} \quad t \geq 0. \tag{1.1}$$

Furthermore, noise or stochastic perturbation is unavoidable and omnipresent in nature as well as in man-made systems. This paper is mainly focused on the existence and uniqueness of μ-pseudo almost periodic and automorphic solutions to the semilinear integro-differential

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stochastic evolution equations in a Hilbert space $\mathbb{H}$

$$x'(t) = Ax(t) + g(t, x(t)) + \int_{-\infty}^{t} B_1(t-s) f(s, x(s))ds + \int_{-\infty}^{t} B_2(t-s) h(s, x(s))dW(s) \quad \text{for all} \quad t \in \mathbb{R}, \tag{1.2}$$

where $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is densely defined closed operator (possibly unbounded). B_1 and B_2 are convolution-type kernels in $L^1(0,\infty)$ and $L^2(0,\infty)$ respectively. $f, g, h : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ are two stochastic processes and $W(t)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(u) - W(v) | u, v \leq t\}$.

We assume $(\mathbb{H}, \|\cdot\|)$ is real separable Hilbert space and $L^2(\Omega, \mathbb{H})$ represents the space of all $\mathbb{H}$-valued random variables x such that

$$\mathbb{E}\|x\|^2 = \int_\Omega \|x\|^2 dP < +\infty.$$

The concept of almost automorphic is a natural generalization of the almost periodicity that was introduced by Bochner [8]. For more details about the almost automorphic functions we refer the reader to the book [27] where the author gave an important overview about the theory of almost automorphic functions and their applications to differential equations. In the last decade, many authors have produced extensive literature on the theory of almost automorphy and its applications to differential equations, more details can be found in [9,17–19,22–24,26,31,32] and the references therein. Then a generalization of almost automorphic functions gives pseudo almost automorphic functions. Also weighted pseudo almost automorphic functions which are more general than weighted pseudo almost periodic functions in [2,12,13] and the references therein.

In recent years, the study of almost periodic and almost automorphic solutions to some stochastic differential equations have been considerably investigated in lots of publications [4,6,7,10,20,21,28,30] because of its significance and applications in physics, mechanics and mathematical biology. The concept of square-mean almost pseudo automorphic stochastic processes was introduced by Chen and Lin in [11]. One say that a continuous stochastic process (see Definition 2.1) x is μ-pseudo almost periodic (resp. automorphic) in the square-mean sense if

$$x = x_1 + x_2,$$

where x_1 is almost periodic (resp. automorphic) and x_2 is μ-ergodic in the sense that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|x_2(t)\|^2 d\mu(t) = 0,$$

where μ is a positive measure on $\mathbb{R}$, $\mu([-r,r])$ is the measure of the set $[-r,r]$. One can observe that a square-mean pseudo almost periodic (resp. automorphic) process is a square-mean μ-pseudo almost periodic (resp. automorphic) process in the particular case where the measure μ is the Lebesgue measure on $\mathbb{R}$. For more details about the μ-pseudo almost periodic (resp. automorphic) processes, one can refer to [15,16]. Note that in [3,25], a new concept of almost periodic (resp. automorphic) process in a distribution sense was introduced in the literature. The authors use this concept to study some stochastic differential equations.

Recently, Bezandry in [5] investigate the existence and uniqueness of the square-mean almost periodic solution of the Eq. (1.2). However, to the best of the author knowledge, the existence of μ-square-mean pseudo almost periodic and automorphic solutions of the Eq. (1.2) remains an untreated question, which is the mean motivation of this paper.

The organization of this work is as follows. In Sect. 2, we make a recalling on μ-pseudo almost periodic and automorphic processes. In Sect. 3, using the results obtain in Sect. 2 and some suitable conditions, we prove the existence and the uniqueness of the square-mean μ-pseudo almost periodic and automorphic mild solution of the Eq. (1.2). In Sect. 4, we provided an example to illustrate our results.

2 Preliminaries

2.1 Square-mean μ-ergodic process

Denote by $\mathcal{B}$ the Lebesgue σ-field $\mathbb{R}$ of and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$ for all $a, b \in \mathbb{R}$ $(a \leq b)$.

$L^2(\Omega, \mathbb{H})$ is Hilbert space equipped with the following norm

$$\|x\|_{L^2} = \left(\int_\Omega \|x\|^2 dP\right)^{\frac{1}{2}}.$$

Definition 2.1 Let $x : \mathbb{R} \to L^2(\Omega, \mathbb{H})$ be a stochastic process.

(1) x is said to be stochastically bounded if there exists $M > 0$ such that

$$\mathbb{E}\|x(t)\|^2 \leq M \quad \text{for all} \quad t \in \mathbb{R}.$$

(2) x is said to be stochastically continuous if

$$\lim_{t\to s} \mathbb{E}\|x(t) - x(s)\|^2 = 0 \quad \text{for all} \quad s \in \mathbb{R}.$$

We denote by $SBC(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ the space of all the stochastically bounded and continuous process. Clearly, the space $SBC(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ is a Banach space equipped with the following norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} (\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}.$$

Definition 2.2 [15] Let $\mu \in \mathcal{M}$. A stochastic process x is said to be square-mean μ-ergodic if $x \in SBC(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ and satisfied

$$\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \mathbb{E}\|x(t)\|^2 d\mu(t) = 0.$$

We denote the space of all such process by $\mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$.

For $\mu \in \mathcal{M}$, we denote

$$\mathcal{E}(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}), \mu) = \{\varphi(\cdot, x) \in \mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu) \quad \text{for any} \quad x \in L^2(\Omega, \mathbb{H})\}.$$

Proposition 2.1 [15] *Let $\mu \in \mathcal{M}$. Then $\mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ is a Banach space with the norm $\|\cdot\|_\infty$.*

Example 2.1 Let ρ be a nonnegative $\mathcal{B}$-measure function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t)dt \quad \text{for} \quad A \in \mathcal{B}, \tag{2.1}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. the function ρ which occurs in (2.1) is called the Radon–Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$ [29, p. 130]. In this case, $\mu \in \mathcal{M}$ if and only if its Radon–Nikodym derivative ρ is locally Lebesgue-integrable on $\mathbb{R}$ and it satisfies

$$\int_{-\infty}^{+\infty} \rho(t)dt = +\infty.$$

Definition 2.3 [1] Let μ_1 and $\mu_2 \in \mathcal{M}$. μ_1 is said to be equivalent to μ_2 ($\mu_1 \sim \mu_2$) if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that

$$\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A)$$

for $A \in \mathcal{B}$ satisfying $A \cap I = \varnothing$.

Theorem 2.1 [15] *Let μ_1 and $\mu_2 \in \mathcal{M}$. If μ_1 and μ_2 are equivalent, then $\mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu_1) = \mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu_2)$.*

For $\mu \in \mathcal{M}$ and $\tau \in \mathbb{R}$, we denote by μ_τ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\tau(A) = \mu(a + \tau : a \in A) \text{for} A \in \mathcal{B}.$$

From $\mu \in \mathcal{M}$, we formulate the following hypothesis.

(H) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu_\tau(A) \leq \beta\mu(A) \quad \text{when } A \in \mathcal{B} \text{ satisfies } A \cap I = \varnothing.$$

Lemma 2.1 [1] *Let $\mu \in \mathcal{M}$. Then μ satisfies **(H)** if and only if μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 2.2 [1] *Hypothesis **(H)** implies that for all $\sigma > 0$,*

$$\limsup_{r \to +\infty} \frac{\mu([-r - \sigma, r + \sigma])}{\mu([-r, r])} < +\infty.$$

Remark 2.1 [1] For Example 2.1, Hypothesis **(H)** holds if and only if for all $\tau \in \mathbb{R}$, there exist a constant $\beta > 0$ and a bounded interval I such that

$$\rho(t + \tau) \leq \beta\rho(t) \quad \text{a.e} \quad \mathbb{R}\backslash I.$$

When μ is given by a density as follows

$$\mu(t) = \rho(t)dt,$$

where ρ satisfies the condition of Example 2.1, then Hypothesis **(H)** is equivalent to say

$$\text{for all} \quad \tau \in \mathbb{R}, \quad \limsup_{|t| \to +\infty} \frac{\rho(t + \tau)}{\rho(t)} < +\infty.$$

Example 2.2

$$\rho(t) = \begin{cases} \exp(t) & if \quad t \leq 0 \\ 1 \quad \text{if} & t > 0. \end{cases}$$

In fact

$$\lim_{t \to +\infty} \frac{\rho(t + \tau)}{\rho(t)} = 1 \quad \text{and} \quad \lim_{t \to -\infty} \frac{\rho(t + \tau)}{\rho(t)} = \exp(\tau) \quad \text{for} \quad \tau \in \mathbb{R}.$$

Let $f \in SBC(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ and $\tau \in \mathbb{R}$. We denote by f_τ the function defined by $f_\tau(t) = f(t+\tau)$ for $t \in \mathbb{R}$.

A subset $\mathfrak{F}$ of $SBC(\mathbb{R}, L^2(\Omega, \mathbb{H})$ is said to translation invariant if for all $f \in \mathfrak{F}$ we have $f_\tau \in \mathfrak{F}$ for all $\tau \in \mathbb{R}$.

Theorem 2.2 [15] *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Then $\mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ is translation invariant.*

2.2 μ-Pseudo almost periodic process

Definition 2.4 [6] Let $x : \mathbb{R} \to L^2(\Omega, \mathbb{H})$ be a continuous stochastic process. x is said be square-mean almost periodic process if for each $\epsilon > 0$ there exists $l > 0$ such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + l]$ satisfying

$$\sup_{t \in \mathbb{R}} \mathbb{E}\|x(t+\tau) - x(t)\|^2 < \epsilon.$$

We denote the space of all such stochastic processes by $SAP(\mathbb{R}, L^2(\Omega, \mathbb{H}))$.

Theorem 2.3 [6] *$SAP(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space.*

Definition 2.5 [15] Let $\mu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(\Omega, \mathbb{H})$ be a continuous stochastic process. f is said be μ-square-mean pseudo almost periodic process if it can be decomposed as follows

$$f = g + \varphi,$$

where $g \in SAP(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ and $\varphi \in \mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$.

We denote the space of all such stochastic processes by $SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$. Then we have

$$SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu) \subset SBC(\mathbb{R}, L^2(\Omega, \mathbb{H})).$$

Theorem 2.4 [15] *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Then $SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ is translation invariant.*

Theorem 2.5 [15] *Let $\mu \in \mathcal{M}$ and $f \in SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ be such that*

$$f = g + \varphi,$$

where $g \in SAP(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ and $\varphi \in \mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$. If $SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ is translation invariant, then

$$\overline{\{f(t) | t \in \mathbb{R}\}} \supset \{g(t) | t \in \mathbb{R}\}.$$

Theorem 2.6 [15] *Let $\mu \in \mathcal{M}$. Assume that $SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ is translation invariant. Then $SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ is a Banach space with the norm $\|\cdot\|_\infty$.*

Definition 2.6 [6] Let $f : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$, $(t, x) \mapsto f(t, x)$ be continuous. f is said be square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $x \in K$ where $K \subset L^2(\Omega, \mathbb{H})$ is a compact if for any $\epsilon > 0$, there exists $l(\epsilon, K) > 0$ such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + l(\epsilon, K)]$ satisfying

$$\sup_{t \in \mathbb{R}} \mathbb{E}\|f(t+\tau, x) - f(t, x)\|^2 < \epsilon$$

for each stochastic process $x : \mathbb{R} \to K$.

We denote the space of such stochastic processes by

$$SAP(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H})) = \{g(\cdot, x) \in SAP(\mathbb{R}, L^2(\Omega, \mathbb{H})) \text{ for any } x \in L^2(\Omega, \mathbb{H})\}.$$

Definition 2.7 [15] Let $\mu \in \mathcal{M}$. A continuous function $f(t, x) : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ is said to be square-mean $\mu-$pseudo almost periodic in $t \in \mathbb{R}$ for any $x \in L^2(\Omega, \mathbb{H})$ if it can be decomposed as $f = g + \varphi$, where $g \in SAP(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}))$ and $\varphi \in \mathcal{E}(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}), \mu)$. Denote the set of all such stochastically continuous processes by $SPAP(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}), \mu)$.

Theorem 2.7 [15] *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Suppose that $f \in SPAP(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}), \mu)$ and that there exists a positive number L such that, for any $x, y \in L^2(\Omega, \mathbb{H})$,*

$$\mathbb{E}\|f(t, x) - f(t, y)\|^2 \leq L \cdot \mathbb{E}\|x - y\|^2$$

for $t \in \mathbb{R}$. Then $t \mapsto f(t, x(t)) \in SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ for any $x \in SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$.

2.3 μ-Pseudo almost automorphic process

Definition 2.8 [21] Let $x : \mathbb{R} \to L^2(\Omega, \mathbb{H})$ be a continuous stochastic process. x is said be square-mean almost automorphic process if for every sequence of real numbers $(t_n^{'})_n$ we can extract a subsequence $(t_n)_n$ such that, for some stochastic process $y : \mathbb{R} \to L^2(\Omega, \mathbb{H})$, we have

$$\lim_{n \to +\infty} \mathbb{E}\|x(t + t_n) - y(t)\|^2 = 0 \quad \text{for all} \quad t \in \mathbb{R}$$

and

$$\lim_{n \to +\infty} \mathbb{E}\|y(t - t_n) - x(t)\|^2 = 0 \quad \text{for all} \quad t \in \mathbb{R}.$$

We denote the space of all such stochastic processes by $SAA(\mathbb{R}, L^2(\Omega, \mathbb{H}))$.

Theorem 2.8 [21] *$SAA(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space.*

Definition 2.9 Let $\mu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(\Omega, \mathbb{H})$ be a continuous stochastic process. f is said be μ-square-mean pseudo almost automorphic process if it can be decomposed as follows

$$f = g + \varphi,$$

where $g \in SAA(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ and $\varphi \in \mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$.

We denote the space of all such stochastic processes by $SPAA(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$. Then we have

$$SPAA(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu) \subset SBC(\mathbb{R}, L^2(\Omega, \mathbb{H})).$$

Hence, together with Theorem 2.2 and Definition 2.9, we arrive at the following conclusion.

Theorem 2.9 *Let $\mu \in \mathcal{M}$ satisfy **(H)**. Then $SPAA(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ is translation invariant.*

Theorem 2.10 *Let $\mu \in \mathcal{M}$ and $f \in SPAA(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ be such that*

$$f = g + \varphi,$$

where $g \in SAA(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ *and* $\varphi \in \mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$. *If* $SPAA(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ *is translation invariant, then*

$$\overline{\{f(t)|t \in \mathbb{R}\}} \supset \{g(t)|t \in \mathbb{R}\}.$$

The proof of Theorem 2.10 is similar to the proof of Theorem 4.1 in [1].

Theorem 2.11 *Let* $\mu \in \mathcal{M}$. *Assume that* $SPAA(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ *is translation invariant. Then* $SPAA(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ *is a Banach space with the norm* $\| \cdot \|_\infty$.

The proof of the theorem above is similar to the proof of Theorem 4.9 in [1].

For $\mu \in \mathcal{M}$, we denote

$$SAA(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H})) = \{g(\cdot, x) \in SAA(\mathbb{R}, L^2(\Omega, \mathbb{H})) \text{ for any } x \in L^2(\Omega, \mathbb{H})\}.$$

Definition 2.10 Let $\mu \in \mathcal{M}$. A continuous function $f(t, x) : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ is said to be square-mean $\mu-$pseudo almost automorphic in $t \in \mathbb{R}$ for any $x \in L^2(\Omega, \mathbb{H})$ if it can be decomposed as $f = g + \varphi$, where $g \in SAA(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H})$ and $\varphi \in \mathcal{E}(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}), \mu)$.

Denote the set of all such stochastically continuous processes by $SPAA(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}), \mu)$.

Theorem 2.12 *Let* $\mu \in \mathcal{M}$ *satisfy* ***(H)***. *Suppose that* $f \in SPAA(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}), \mu)$ *and that there exists a positive number L such that, for any* $x, y \in L^2(\Omega, \mathbb{H})$,

$$\mathbb{E}\|f(t, x) - f(t, y)\|^2 \leq L \cdot \mathbb{E}\|x - y\|^2$$

for $t \in \mathbb{R}$. *Then* $t \mapsto f(t, x(t)) \in SPAA(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ *for any* $x \in SPAA(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$.

Proof Let $f = g + \varphi$ with $g \in SAA(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}))$ and $\varphi \in \mathcal{E}(\mathbb{R} \times L^2(\Omega, \mathbb{H}), L^2(\Omega, \mathbb{H}), \mu)$. Similarly, let $x = x_1 + x_2$ with $x_1 \in SAA(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ and $x_2 \in \mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$.

The function f can be decomposed as follows

$$\begin{aligned} f(t, x(t)) &= g(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + [f(t, x_1(t)) - g(t, x_1(t))] \\ &= g(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + \varphi(t, x_1(t)). \end{aligned}$$

From Theorem 2.10, we deduce that g is Lipschitz. Then using the theorem of composition of almost automorphic process (Theorem 2.6 in [21]), we obtain that $g(\cdot, x_1(\cdot)) \in SAA(\mathbb{R}, L^2(\Omega, \mathbb{H}))$. With the same argument used in the steps 2 and 3 of the proof of Theorem 5.7 in [15], gives as that $[f(\cdot, x(\cdot)) - f(\cdot, x_1(\cdot))], \varphi(\cdot, x_1(\cdot)) \in \mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$. □

3 Main results

To discuss the existence and uniqueness of the mild solution of the Eq. (1.2), we suppose that the following assumptions hold:

($\mathbf{H}_1$) The operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a uniformly exponentially stable semigroup $(T(t))_{t \geq 0}$ such that there exist constants $M \geq 1$ and $\delta > 0$ with

$$\|T(t)\| \leq Me^{-\delta t} \quad \text{for all} \quad t \geq 0.$$

($\mathbf{H}_2$) B_1 and B_2 are convolution-type kernels in $L^1(0,\infty)$ and $L^2(0,\infty)$ respectively.
($\mathbf{H}_3$) Let $f, g, h : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ be stochastic processes such that there existsonstant $L > 0$ such that

$$\mathbb{E} \parallel f(t,x) - f(t,y) \parallel^2 \leq L \cdot \mathbb{E} \parallel x - y \parallel^2,$$
$$\mathbb{E} \parallel g(t,x) - g(t,y) \parallel^2 \leq L \cdot \mathbb{E} \parallel x - y \parallel^2,$$
$$\mathbb{E} \parallel h(t,x) - h(t,y) \parallel^2 \leq L \cdot \mathbb{E} \parallel x - y \parallel^2$$

for all $t \in \mathbb{R}$ and for any $x, y \in L^2(\Omega, \mathbb{H})$.

Definition 3.1 [5] An $\mathcal{F}_t$-progressively measurable process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution on $\mathbb{R}$ of the Eq. (1.2) provided that it satisfies the corresponding stochastic integral equation

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)g(s,x(s))ds + \int_a^t T(t-\sigma)\int_a^\sigma B_1(\sigma - s) f(s,x(s))dsd\sigma + \int_a^t T(t-\sigma)\int_a^\sigma B_2(\sigma - s)h(s,x(s))dW(s)d\sigma \tag{3.1}$$

for all $t \geq a$ and each $a \in \mathbb{R}$.

Remark 3.1 If we let $a \to -\infty$ in the stochastic integral equation (3.1), by the exponential dissipation condition of $(T(t))_{t\geq 0}$, then we obtain the stochastic process $x : \mathbb{R} \to L^2(\Omega, \mathbb{H})$ is a mild solution of the Eq. (1.2) if and only if x satisfies the stochastic integral equation

$$x(t) = \int_{-\infty}^t T(t-s)g(s,x(s))ds + \int_{-\infty}^t T(t-\sigma)\int_{-\infty}^\sigma B_1(\sigma - s) f(s,x(s))dsd\sigma + \int_{-\infty}^t T(t-\sigma)\int_{-\infty}^\sigma B_2(\sigma - s)h(s,x(s))dW(s)d\sigma. \tag{3.2}$$

Theorem 3.1 *Let $\mu \in \mathcal{M}$ satisfy **(H)** and ($\mathbf{H}_1$)–($\mathbf{H}_2$)–($\mathbf{H}_3$) hold. If f, g and h are square-mean μ-pseudo almost periodic, then the Eq.* (1.2) *has a unique square-mean μ-pseudo almost periodic mild solution on $\mathbb{R}$, whenever*

$$\Theta := 3\frac{M^2}{\delta^2}L(1 + \|B_1\|^2_{L^1(0,\infty)} + \|B_2\|^2_{L^2(0,\infty)}) < 1.$$

Proof We define the nonlinear operator Λ by

$$(\Lambda x)(t) = \int_{-\infty}^t T(t-s)g(s,x(s))ds + \int_{-\infty}^t T(t-\sigma)\int_{-\infty}^\sigma B_1(\sigma - s) f(s,x(s))dsd\sigma + \int_{-\infty}^t T(t-\sigma)\int_{-\infty}^\sigma B_2(\sigma - s)h(s,x(s))dW(s)d\sigma.$$

for any $x \in SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$.

We claim to prove that Λ is a strict contraction mapping from $SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ to itself.

Step 1 Firstly, we have to show that Λ is well defined. Let $x \in SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$, we have

$$\begin{aligned}
\mathbb{E}\|(\Lambda x)(t)\|^2 &= \mathbb{E}\Big\| \int_{-\infty}^{t} T(t-s)g(s,x(s))ds + \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_1(\sigma-s)f(s,x(s))dsd\sigma \\
&\quad + \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_2(\sigma-s)h(s,x(s))dW(s)d\sigma \Big\|^2 \\
&\le 3\mathbb{E}\Big\| \int_{-\infty}^{t} T(t-s)g(s,x(s))ds \Big\|^2 + 3\mathbb{E}\Big\| \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_1(\sigma-s)f(s,x(s))dsd\sigma \Big\|^2 \\
&\quad + 3\mathbb{E}\Big\| \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_2(\sigma-s)h(s,x(s))dW(s)d\sigma \Big\|^2.
\end{aligned}$$

Since **(H)** and $(\mathbf{H}_3)$ hold. Then by Theorem 2.7, we deduce that $g(\cdot,x(\cdot)),\ f(\cdot,x(\cdot)),\ h(\cdot,x(\cdot)) \in SPAP(\mathbb{R}, L^2(\Omega,\mathbb{H}),\mu)$. Using Cauchy-Schwartz's inequality and Ito's isometry property, we obtain that

$$\begin{aligned}
\mathbb{E}\|(\Lambda x)(t)\|^2 &\le 3M^2 \int_{-\infty}^{t} e^{-\delta(t-s)}ds \int_{-\infty}^{t} e^{-\delta(t-s)}\mathbb{E}\|g(s,x(s))\|^2 ds \\
&\quad + 3M^2\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}d\sigma\Big)\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\mathbb{E}\Big\|\int_{-\infty}^{\sigma} B_1(\sigma-s)f(s,x(s))ds\Big\|^2 d\sigma\Big) \\
&\quad + 3M^2\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}d\sigma\Big)\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\mathbb{E}\Big\|\int_{-\infty}^{\sigma} B_2(\sigma-s)h(s,x(s))dW(s)\Big\|^2 d\sigma\Big) \\
&\le \frac{3M^2}{\delta^2}\sup_{s\in\mathbb{R}}\mathbb{E}\|g(s,x(s))\|^2 \\
&\quad + \frac{3M^2}{\delta}\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\Big(\int_{-\infty}^{\sigma}\|B_1(\sigma-s)\|ds\Big) \\
&\quad \times\Big(\int_{-\infty}^{\sigma}\|B_1(\sigma-s)\|\mathbb{E}\Big\|f(s,x(s))\Big\|^2 ds\Big)d\sigma\Big) \\
&\quad + \frac{3M^2}{\delta}\Big(\int_{-\infty}^{t} e^{-\delta(t-\sigma)}\mathbb{E}\Big\|\int_{-\infty}^{\sigma}\|B_2(\sigma-s)\|^2\mathbb{E}\Big\|h(s,x(s))\Big\|^2 dsd\sigma\Big) \\
&\le \frac{3M^2}{\delta^2}\sup_{s\in\mathbb{R}}\mathbb{E}\|g(s,x(s))\|^2 + \frac{3M^2}{\delta^2}\|B_1\|^2_{L^1(0,\infty)}\sup_{s\in\mathbb{R}}\mathbb{E}\|f(s,x(s))\|^2 \\
&\quad + \frac{3M^2}{\delta^2}\|B_2\|^2_{L^1(0,\infty)}\sup_{s\in\mathbb{R}}\mathbb{E}\|h(s,x(s))\|^2 \\
&< \infty.
\end{aligned}$$

Step 2 Now, we verify that Λ is a self-mapping. Let $x \in SPAP(\mathbb{R}, L^2(\Omega,\mathbb{H}),\mu)$. Then

$$x = x_1 + x_2$$

where $x_1 \in AP(\mathbb{R}, L^2(\Omega,\mathbb{H}))$ and $x_2 \in \mathcal{E}(\mathbb{R}, L^2(\Omega,\mathbb{H}),\mu)$. Since

- $f \in SPAP(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}),\mu)$ then there exist $f_1 \in AP(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}))$ and $f_2 \in \mathcal{E}(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}),\mu)$ such that $f = f_1 + f_2$.
- $g \in SPAP(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}),\mu)$ then there exist $g_1 \in AP(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}))$ and $g_2 \in \mathcal{E}(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}),\mu)$ such that $g = g_1 + g_2$.
- $h \in PAP(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}),\mu)$ then there exist $h_1 \in AP(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H})$ and $h_2 \in \mathcal{E}(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}),\mu)$ such that $h = h_1 + h_2$.

Hence the functions f, g and h are decomposed as follows:

$$\begin{aligned}
f(t,x(t)) &= f_1(t,x_1(t)) + [f(t,x(t)) - f(t,x_1(t))] + [f(t,x_1(t)) - f_1(t,x_1(t))] \\
&= f_1(t,x_1(t)) + [f(t,x(t)) - f(t,x_1(t))] + f_2(t,x_1(t)),
\end{aligned}$$

$$\begin{aligned} g(t,x(t)) &= g_1(t,x_1(t)) + [g(t,x(t)) - g(t,x_1(t))] + [g(t,x_1(t)) - g_1(t,x_1(t))] \\ &= g_1(t,x_1(t)) + [g(t,x(t)) - g(t,x_1(t))] + g_2(t,x_1(t)), \end{aligned}$$

and

$$\begin{aligned} h(t,x(t)) &= h_1(t,x_1(t)) + [h(t,x(t)) - h(t,x_1(t))] + [h(t,x_1(t)) - h_1(t,x_1(t))] \\ &= h_1(t,x_1(t)) + [h(t,x(t)) - h(t,x_1(t))] + h_2(t,x_1(t)). \end{aligned}$$

It follows that

$$(\Lambda x)(t) = (\Lambda_1 x_1)(t) + \Psi(t),$$

where

$$\begin{aligned} (\Lambda_1 x_1)(t) = &\int_{-\infty}^{t} T(t-s) g_1(s,x_1(s))ds + \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_1(\sigma-s) f_1(s,x_1(s))dsd\sigma \\ &+ \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_2(\sigma-s) h_1(s,x_1(s))dW(s)d\sigma \end{aligned}$$

and

$$\begin{aligned} \Psi(t) = &\int_{-\infty}^{t} T(t-s)[[g(s,x(s)) - g(s,x_1(s))] + g_2(s,x_1(s))]ds \\ &+ \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_1(\sigma-s)[[f(s,x(s)) - f(s,x_1(s))] + f_2(s,x_1(s))]dsd\sigma \\ &+ \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_2(\sigma-s)[[h(s,x(s)) - h(s,x_1(s))] + h_2(s,x_1(s))]dW(s)d\sigma. \end{aligned}$$

Using Theorem 2.5 and ($\mathbf{H}_3$), we obtain that

$f_1, g_1, h_1 \in AP(\mathbb{R}\times L^2(\Omega,\mathbb{H}), L^2(\Omega,\mathbb{H}))$ are Lipschitzian in the following sense

$$\mathbb{E} \parallel f_1(t,x) - f_1(t,y) \parallel^2 \leq L\cdot\mathbb{E} \parallel x-y \parallel^2, \tag{3.3}$$

$$\mathbb{E} \parallel g_1(t,x) - g_1(t,y) \parallel^2 \leq L\cdot\mathbb{E} \parallel x-y \parallel^2 \tag{3.4}$$

and

$$\mathbb{E} \parallel h_1(t,x) - h_1(t,y) \parallel^2 \leq L\cdot\mathbb{E} \parallel x-y \parallel^2 . \tag{3.5}$$

Hence, using (3.3), (3.4) and (3.5), it follows from Theorem 3.2 in [5] that $\Lambda_1 x_1$ is almost periodic in the square-mean sense.

Next, we have to check that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])}\int_{-r}^{r} \mathbb{E}\|\Psi(t)\|^2 d\mu(t) = 0.$$

Let

$$\begin{aligned} \Psi(t) = &\int_{-\infty}^{t} T(t-s)G(s)ds + \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_1(\sigma-s)F(s)dsd\sigma \\ &+ \int_{-\infty}^{t} T(t-\sigma)\int_{-\infty}^{\sigma} B_2(\sigma-s)H(s)dW(s)d\sigma, \end{aligned}$$

where

$$\begin{aligned} G(s) &= [[g(s,x(s)) - g(s,x_1(s))] + g_2(s,x_1(s))], \\ F(s) &= [[f(s,x(s)) - f(s,x_1(s))] + f_2(s,x_1(s))] \end{aligned}$$

and

$$H(s) = [[h(s, x(s)) - h(s, x_1(s))] + h_2(s, x_1(s))].$$

Using the similar arguments performs in the steps 2 and 3 of the proof of Theorem 5.7 in [15], we obtain that $G, F, H \in \mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$. Then, we have

$$\mathbb{E}\|\Psi(t)\|^2 \leq 3\mathbb{E}\left\| \int_{-\infty}^{t} T(t-s)G(s)ds \right\|^2 + 3\mathbb{E}\left\| \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_1(\sigma - s)F(s)dsd\sigma \right\|^2 + 3\mathbb{E}\left\| \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_2(\sigma - s)H(s)dW(s)d\sigma \right\|^2.$$

That implies for $r > 0$, that

$$\begin{aligned} &\frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|\Psi(t)\|^2 d\mu(t) \\ &\leq \frac{3}{\mu([-r,r])} \left\{ \int_{-r}^{r} \mathbb{E}\left\| \int_{-\infty}^{t} T(t-s)G(s)ds \right\|^2 d\mu(t) \right. \\ &\quad + \int_{-r}^{r} \mathbb{E}\left\| \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_1(\sigma - s)F(s)dsd\sigma \right\|^2 d\mu(t) \\ &\quad \left. + \int_{-r}^{r} \mathbb{E}\left\| \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_2(\sigma - s)H(s)dW(s)d\sigma \right\|^2 d\mu(t) \right\} \\ &\leq \frac{3}{\mu([-r,r])} \left\{ I_1 + I_2 + I_3 \right\}. \end{aligned}$$

Firstly, using Cauchy–Schwartz's inequality and Fubini's theorem, we obtain the following estimation of I_1

$$\begin{aligned} I_1 &\leq M^2 \int_{-r}^{r} d\mu(t) \int_{-\infty}^{t} e^{-\delta(t-s)}\, ds \int_{-\infty}^{t} e^{-\delta(t-s)} \mathbb{E}\|G(s)\|^2\, ds \\ &\leq \frac{M^2}{\delta} \int_{-r}^{r} d\mu(t) \int_{0}^{\infty} e^{-\delta u} \mathbb{E}\|G(t-u)\|^2\, du \quad (\text{setting} \quad u = t - s) \\ &\leq \frac{M^2}{\delta} \int_{0}^{\infty} \left\{ e^{-\delta u} \int_{-r}^{r} \mathbb{E}\|G(t-u)\|^2\, d\mu(t) \right\} du. \end{aligned}$$

It follows that

$$\frac{1}{\mu([-r,r])} I_1 \leq \frac{M^2}{\delta} \int_{0}^{\infty} \left\{ \frac{e^{-\delta u}}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|G(t-u)\|^2\, d\mu(t) \right\} du.$$

Since

$$\left| \frac{e^{-\delta u}}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|G(s)\|^2\, d\mu(t) \right| \leq e^{-\delta u} \|F\|_{\infty}^2.$$

Then, using the Lebesgue dominated convergence theorem and the fact that $\mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ is translation invariant, we get that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} I_1 \leq \frac{M^2}{\delta} \int_{0}^{\infty} \left\{ e^{-\delta u} \lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|G(t-u)\|^2\, d\mu(t) \right\} du = 0. \tag{3.6}$$

Next, we have to estimate I_2. We have

$$I_2 = \int_{-r}^{r} \mathbb{E}\left\| \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_1(\sigma - s)F(s)dsd\sigma \right\|^2 d\mu(t)$$
$$\leq M^2 \int_{-r}^{r} \mathbb{E}\bigg(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \left\| \int_{-\infty}^{\sigma} B_1(\sigma - s)F(s)ds \right\| d\sigma \bigg)^2 d\mu(t).$$

Using Cauchy–Schwartz's inequality and Fubini's theorem, we obtain that

$$I_2 \leq M^2 \int_{-r}^{r} \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d\sigma \right) \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \mathbb{E} \left\| \int_{-\infty}^{\sigma} B_1(\sigma-s)F(s)ds \right\|^2 d\sigma \right) d\mu(t)$$
$$\leq \frac{M^2}{\delta} \int_{-r}^{r} \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \mathbb{E} \left\| \int_{-\infty}^{\sigma} B_1(\sigma-s)F(s)ds \right\|^2 d\sigma \right) d\mu(t)$$
$$\leq \frac{M^2}{\delta} \int_{-r}^{r} \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \left\{ \int_{-\infty}^{\sigma} \left\| B_1(\sigma-s) \right\| ds \right\} \left\{ \int_{-\infty}^{\sigma} \left\| B_1(\sigma-s) \right\| \mathbb{E} \left\| F(s) \right\|^2 ds \right\} d\sigma \right) d\mu(t)$$
$$\leq \frac{M^2}{\delta} \|B_1\|_{L^1(0,\infty)} \int_{-r}^{r} \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \int_{-\infty}^{\sigma} \left\| B_1(\sigma-s) \right\| \mathbb{E} \left\| F(s) \right\|^2 dsd\sigma \right) d\mu(t)$$
$$\leq \frac{M^2}{\delta} \|B_1\|_{L^1(0,\infty)} \int_{-r}^{r} \left(\int_{0}^{\infty} e^{-\delta u} \int_{0}^{\infty} \left\| B_1(v) \right\| \mathbb{E} \left\| F(t-u-v) \right\|^2 dvdu \right) d\mu(t) \quad \text{(setting } u = t-\sigma,\ v = \sigma - s)$$
$$\leq \frac{M^2}{\delta} \|B_1\|_{L^1(0,\infty)} \int_{0}^{\infty} e^{-\delta u} \int_{0}^{\infty} \left\| B_1(v) \right\| \left(\int_{-r}^{r} \mathbb{E} \left\| F(t-u-v) \right\|^2 d\mu(t) \right) dvdu.$$

It follows that

$$\frac{1}{\mu([-r,r])} I_1 \leq \frac{M^2}{\delta} \|B_1\|_{L^1(0,\infty)} \int_0^\infty e^{-\delta u} \int_0^\infty \|B_1(v)\|$$
$$\times \left(\frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|F(t-u-v)\|^2 d\mu(t) \right) dvdu.$$

Since

$$\left| e^{-\delta u} \int_0^\infty \|B_1(v)\| \left(\frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|F(t-u-v)\|^2 d\mu(t) \right) dv \right| \leq e^{-\delta u} \|B_1\|_{L^1(0,\infty)} \|F\|_\infty^2$$

and

$$\left| \|B_1(v)\| \left(\frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|F(t-u-v)\|^2 d\mu(t) \right) \right| \leq \|B_1(v)\| \|F\|_\infty^2.$$

Then, using the Lebesgue dominated convergence theorem and the fact that $\mathcal{E}(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ is translation invariant, we get that

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} I_1 \leq \frac{M^2}{\delta} \|B_1\|_{L^1(0,\infty)} \int_0^\infty e^{-\delta u} \int_0^\infty \|B_1(v)\|$$
$$\times \left(\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \mathbb{E}\|F(t-u-v)\|^2 d\mu(t) \right) dvdu = 0. \tag{3.7}$$

Next, we need to estimate I_3. In fact using Cauchy–Schwartz's inequality and Ito's isometry property, we obtain that

$$
\begin{aligned}
I_3 &= \int_{-r}^{r} \mathbb{E}\left\| \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_2(\sigma - s)H(s)dW(s)d\sigma \right\|^2 d\mu(t) \Big\} \\
&\leq M^2 \int_{-r}^{r} \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} d\sigma \right) \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \mathbb{E}\left\| \int_{-\infty}^{\sigma} B_2(\sigma - s)H(s)dW(s) \right\|^2 d\sigma \right) d\mu(t) \\
&\leq \frac{M^2}{\delta} \int_{-r}^{r} \left(\int_{-\infty}^{t} e^{-\delta(t-\sigma)} \int_{-\infty}^{\sigma} \|B_2(\sigma - s)\|^2 \mathbb{E}\|H(s)\|^2 ds d\sigma \right) d\mu(t).
\end{aligned}
$$

Using Fubini's theorem, Lebesgue dominated convergence theorem and the same calculus techniques used above, one can obtain that

$$
\lim_{r\to\infty} \frac{1}{\mu([-r, r])} I_3 = 0. \tag{3.8}
$$

Consequently, combining (3.6)–(3.8), we obtain that

$$
\lim_{r\to\infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \mathbb{E}\|\Psi(t)\|^2 d\mu(t) = 0.
$$

Hence Λ is a self-mapping from $SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ to itself. Next, we have to check that Λ is a strict contraction. For $x_1, x_2 \in SPAP(\mathbb{R}, L^2(\Omega, \mathbb{H}), \mu)$ and each $t \in \mathbb{R}$, we have

$$
\begin{aligned}
&\mathbb{E}\|(\Lambda x_1)(t) - (\Lambda x_2)(t)\|^2 \\
&\quad = \mathbb{E}\Bigg\| \int_{-\infty}^{t} T(t-s)[g(s, x_1(s)) - g(s, x_2(s))]ds \\
&\qquad + \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_1(\sigma - s)[f(s, x_1(s)) - f(s, x_2(s))]dsd\sigma \\
&\qquad + \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_2(\sigma - s)[h(s, x_1(s)) - h(s, x_2(s))]dW(s)d\sigma \Bigg\|^2 \\
&\quad \leq 3\mathbb{E}\left\| \int_{-\infty}^{t} T(t-s)[g(s, x_1(s)) - g(s, x_2(s))]ds \right\|^2 \\
&\qquad + 3\mathbb{E}\left\| \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_1(\sigma - s)[f(s, x_1(s)) - f(s, x_2(s))]dsd\sigma \right\|^2 \\
&\qquad + 3\mathbb{E}\left\| \int_{-\infty}^{t} T(t-\sigma) \int_{-\infty}^{\sigma} B_2(\sigma - s)[h(s, x_1(s)) - h(s, x_2(s))]dW(s)d\sigma \right\|^2
\end{aligned}
$$

Using Cauchy–Schwartz's inequality and Ito's isometry property, we obtain that

$$
\begin{aligned}
&\mathbb{E}\|(\Lambda x_1)(t) - (\Lambda x_2)(t)\|^2 \\
&\quad \leq \frac{3M^2}{\delta} \int_{-\infty}^{t} e^{-\delta(t-s)} \mathbb{E}\|g(s, x_1(s)) - g(s, x_2(s))\|^2 ds \\
&\qquad + \frac{3M^2}{\delta} \|B_1\|_{L^1(0,\infty)} \int_{-\infty}^{t} e^{-\delta(t-\sigma)} \left(\int_{-\infty}^{\sigma} \|B_1(\sigma - s)\| \mathbb{E}\|f(s, x_1(s)) - f(s, x_2(s))\|^2 ds \right) d\sigma \\
&\qquad + \frac{3M^2}{\delta} \int_{-\infty}^{t} e^{-\delta(t-\sigma)} \left(\int_{-\infty}^{\sigma} \|B_2(\sigma - s)\|^2 \mathbb{E}\|h(s, x_1(s)) - h(s, x_2(s))\|^2 ds \right) d\sigma \\
&\quad \leq \frac{3M^2 L}{\delta} \int_{-\infty}^{t} e^{-\delta(t-s)} \mathbb{E}\|x_1(s) - x_2(s)\|^2 ds
\end{aligned}
$$

$$+\frac{3M^2L}{\delta}\|B_1\|_{L^1(0,\infty)}\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\left(\int_{-\infty}^{\sigma}\|B_1(\sigma-s)\|\mathbb{E}\|x_1(s)-x_2(s)\|^2ds\right)d\sigma$$
$$+\frac{3M^2L}{\delta}\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\left(\int_{-\infty}^{\sigma}\|B_2(\sigma-s)\|^2\mathbb{E}\|x_1(s)-x_2(s)\|^2ds\right)d\sigma$$
$$\leq\frac{3M^2L}{\delta^2}\sup_{s\in\mathbb{R}}\mathbb{E}\|x_1(s)-x_2(s)\|^2+\frac{3M^2L}{\delta^2}\|B_1\|^2_{L^1(0,\infty)}\sup_{s\in\mathbb{R}}\mathbb{E}\|x_1(s)-x_2(s)\|^2$$
$$+\frac{3M^2L}{\delta^2}\|B_2\|^2_{L^2(0,\infty)}\sup_{s\in\mathbb{R}}\mathbb{E}\|x_1(s)-x_2(s)\|^2$$
$$\leq\Theta\sup_{s\in\mathbb{R}}\mathbb{E}\|x_1(s)-x_2(s)\|^2.$$

Thus, it follows that for each $t\in\mathbb{R}$

$$\|(\Lambda x_1)(t)-(\Lambda x_2)(t)\|^2_{L^2}\leq\Theta\sup_{s\in\mathbb{R}}\|x_1(s)-x_2(s)\|^2_{L^2}\leq\Theta(\sup_{s\in\mathbb{R}}\|x_1(s)-x_2(s)\|_{L^2})^2.$$

Hence

$$\|(\Lambda x_1)(t)-(\Lambda x_2)(t)\|_\infty=\sup_{t\in\mathbb{R}}\|(\Lambda x_1)(t)-(\Lambda x_2)(t)\|_{L^2}\leq\sqrt{\Theta}\|x_1-x_2\|_\infty.$$

Since $\Theta<1$, it follows Λ is a strict contraction mapping on $SPAP(\mathbb{R},L^2(\Omega,\mathbb{H}),\mu)$. We deduce that Λ has a unique fixed point, which gives a unique $\mu-$pseudo almost periodic mild solution on $\mathbb{R}$ of Eq. (1.2). The proof is complete. □

Theorem 3.2 *Let $\mu\in\mathcal{M}$ satisfy **(H)** and **(H$_1$)**–**(H$_2$)**–**(H$_3$)** hold. If f, g and h are square-mean $\mu-$pseudo almost automorphic, then the Eq. (1.2) has a unique square-mean $\mu-$pseudo almost automorphic mild solution on $\mathbb{R}$, whenever*

$$3\frac{M^2}{\delta^2}L(1+\|B_1\|^2_{L^1(0,\infty)}+\|B_2\|^2_{L^2(0,\infty)})<1.$$

Proof We reproduce the proof of Theorem 3.1 in what we take $x_1,f_1,g_1,h_1\in SAA(\mathbb{R},L^2(\Omega,\mathbb{H}))$. Then, we only need to verify that $\Lambda_1x_1\in SAA(\mathbb{R},L^2(\Omega,\mathbb{H}))$. We have

$$(\Lambda_1x_1)(t)=\int_{-\infty}^{t}T(t-s)g_1(s,x_1(s))ds+\int_{-\infty}^{t}T(t-\sigma)\int_{-\infty}^{\sigma}B_1(\sigma-s)f_1(s,x_1(s))dsd\sigma$$
$$+\int_{-\infty}^{t}T(t-\sigma)\int_{-\infty}^{\sigma}B_2(\sigma-s)h_1(s,x_1(s))dW(s)d\sigma$$
$$:=(\Gamma_1x_1)(t)+(\Gamma_2x_1)(t)+(\Gamma_3x_1)(t).$$

Similarly as the proof of Theorem 4.2 in [21], $\Gamma_1x_1\in SAA(\mathbb{R},L^2(\Omega,\mathbb{H}))$. Next, we have to check that $\Gamma_2x_1,\Gamma_3x_1\in SAA(\mathbb{R},L^2(\Omega,\mathbb{H}))$. Since $f_1(\cdot,x_1(\cdot))\in SAA(\mathbb{R},L^2(\Omega,\mathbb{H}))$, then for every sequence of real numbers $(t_n^{'})_n$ we can extract a subsequence $(t_n)_n$ such that, for some stochastic process $\tilde{f}_1\in L^2(\Omega,\mathbb{H})$, we have

$$\lim_{n\to+\infty}\mathbb{E}\|f_1(s-t_n,x_1(s-t_n))-\tilde{f}_1(s)\|^2=0\quad\text{for all}\quad s\in\mathbb{R}$$

and

$$\lim_{n\to+\infty}\mathbb{E}\|\tilde{f}_1(s-t_n)-f_1(s,x_1(s))\|^2=0\quad\text{for all}\quad s\in\mathbb{R}.$$

For $t\in\mathbb{R}$, we define $\widetilde{\Gamma_2}x_1(t)=\int_{-\infty}^{t}T(t-\sigma)\int_{-\infty}^{\sigma}B_1(\sigma-s)\tilde{f}_1(s)dsd\sigma$. Using Cauchy–Schwartz's inequality, we have

$$
\begin{aligned}
&\mathbb{E}\|\Gamma_2x_1(t+t_n)-\widetilde{\Gamma_2}x_1(t)\|^2\\
&\quad=\mathbb{E}\left\|\int_{-\infty}^{t}T(t-\sigma)\int_{-\infty}^{\sigma}B_1(\sigma-s)(f_1(s-t_n,x_1(s-t_n))-\tilde{f}_1(s))dsd\sigma\right\|^2\\
&\quad\le\frac{M^2}{\delta}\left(\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\mathbb{E}\left\|\int_{-\infty}^{\sigma}B_1(\sigma-s)(f_1(s-t_n,x_1(s-t_n))-\tilde{f}_1(s))ds\right\|^2d\sigma\right)\\
&\quad\le\frac{M^2}{\delta}\|B_1\|_{L^1(0,\infty)}\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\int_{-\infty}^{\sigma}\|B_1(\sigma-s)\|\mathbb{E}\|f_1(s-t_n,x_1(s-t_n))-\tilde{f}_1(s)\|^2dsd\sigma.
\end{aligned}
$$

Using the Lebesgue dominated convergence theorem, one has

$$\lim_{n\to+\infty}\mathbb{E}\|\Gamma_2x_1(t+t_n)-\widetilde{\Gamma_2}x_1(t)\|^2=0.$$

Similarly, $\lim_{n\to+\infty}\mathbb{E}\|\widetilde{\Gamma_2}x_1(t-t_n)-\Gamma_2x_1(t)\|^2=0$. Hence $\Gamma_2x_1\in SAA(\mathbb{R},L^2(\Omega,\mathbb{H}))$.

To end the proof, we need check that $\Gamma_3x_1\in SAA(\mathbb{R},L^2(\Omega,\mathbb{H}))$. Since $h_1(\cdot,x_1(\cdot))\in SAA(\mathbb{R},L^2(\Omega,\mathbb{H}))$, then for every sequence of real numbers $(t_n^{'})_n$ we can extract a subsequence $(t_n)_n$ such that, for some stochastic process $\tilde{h_1}\in L^2(\Omega,\mathbb{H})$, we have

$$\lim_{n\to+\infty}\mathbb{E}\|h_1(s-t_n,x_1(s-t_n))-\tilde{h_1}(s)\|^2=0\quad\text{for all}\quad s\in\mathbb{R}$$

and

$$\lim_{n\to+\infty}\mathbb{E}\|\tilde{h_1}(s-t_n)-h_1(s,x_1(s))\|^2=0\quad\text{for all}\quad s\in\mathbb{R}.$$

For $t\in\mathbb{R}$, we define $\widetilde{\Gamma_3}x_1(t)=\int_{-\infty}^{t}T(t-\sigma)\int_{-\infty}^{\sigma}B_2(\sigma-s)\tilde{h_1}(s)dW(s)d\sigma$. Let $\tilde{W}(s)=W(s+t_n)-W(t_n)$ for each $s\in\mathbb{R}$. Then $\tilde{W}$ is also a Wiener process having the same distribution as W. Using Cauchy–Schwartz's inequality and the Ito's isometry, we obtain the following estimation

$$
\begin{aligned}
&\mathbb{E}\|\Gamma_3x_1(t+t_n)-\widetilde{\Gamma_3}x_1(t)\|^2\\
&\quad=\mathbb{E}\left\|\int_{-\infty}^{t}T(t-\sigma)\int_{-\infty}^{\sigma}B_2(\sigma-s)(h_1(s-t_n,x_1(s-t_n))-\tilde{h_1}(s))dW(s)d\sigma\right\|^2\\
&\quad\le\frac{M^2}{\delta}\left(\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\mathbb{E}\left\|\int_{-\infty}^{\sigma}B_2(\sigma-s)(h_1(s-t_n,x_1(s-t_n))-\tilde{h_1}(s))dW(s)\right\|^2d\sigma\right)\\
&\quad\le\frac{M^2}{\delta}\left(\int_{-\infty}^{t}e^{-\delta(t-\sigma)}\int_{-\infty}^{\sigma}\|B_2(\sigma-s)\|^2\mathbb{E}\|h_1(s-t_n,x_1(s-t_n))-\tilde{h_1}(s)\|^2dsd\sigma\right)
\end{aligned}
$$

By the Lebesgue dominated convergence theorem, one can obtain that

$$\lim_{n\to+\infty}\mathbb{E}\|\Gamma_3x_1(t+t_n)-\widetilde{\Gamma_3}x_1(t)\|^2=0.$$

Similarly, $\lim_{n\to+\infty}\mathbb{E}\|\widetilde{\Gamma_3}x_1(t-t_n)-\Gamma_3x_1(t)\|^2=0$. Hence $\Gamma_3x_1\in SAA(\mathbb{R},L^2(\Omega,\mathbb{H}))$. This completes the proof. □

4 Example

Consider the following stochastic integro-differential equations:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + g(t,u(t,x)) + \int_{-\infty}^{t} e^{-\omega(t-s)} f(s,u(s,x))ds \\ \quad + \int_{-\infty}^{t} e^{-\omega(t-s)} h(s,u(s,x))dW(s), \\ (t,x) \in \mathbb{R} \times (0,1), \\ u(t,0) = u(t,1) = 0, \qquad t \in \mathbb{R}, \end{cases} \tag{4.1}$$

where W(t) is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(w) - W(v) | w, v \leq t\}$ and $\omega > 0$.

To apply our theoretical results, we consider the measure μ where its Radon–Nikodym derivative is

$$\rho(s) = \begin{cases} \exp(s) & \text{if} \quad s \leq 0 \\ 1 \quad \text{if} \quad s > 0. \end{cases}$$

Then μ satisfies **(H)** (cf. Example 2.2). The forcing terms as follows:

$$g(s,u) = \frac{L}{2} u \sin \frac{1}{2 + \cos s + \cos \sqrt{2} s} + e^{-|s|} \cos u,$$

$$f(s,u) = \frac{L}{2} u \cos \frac{1}{2 + \sin s + \sin \pi s} + e^{-2|s|} \sin u,$$

$$h(s,u) = \frac{L}{2} u \sin \frac{1}{2 + \cos s + \cos \sqrt{3} s} + e^{-3|s|} \sin u.$$

In order to write the system (4.1) on the abstract form (1.2), we consider the linear operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$, given by

$$D(A) = H^2(0,1) \cap H_0^1(0,1),$$
$$Ax(\xi) = x''(\xi) \quad \text{for} \quad \xi \in (0,1) \quad \text{and} \quad x \in D(A).$$

It is well-known that A generates a C_0-semigroup $(T(t))_{t \geq 0}$ on $L^2(0,1)$ defined by

$$(T(t)x)(r) = \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \langle x, e_n \rangle_{L^2} e_n(r),$$

where $e_n(r) = \sqrt{2} \sin(n\pi r)$ for $n = 1, 2, ...$, and $\|T(t)\| \leq e^{-\pi^2 t}$ for all $t \geq 0$. Hence $(\mathbf{H}_1)$ hold.

Then the system (4.1) takes the following abstract form

$$u'(t) = Au(t) + g(t,u(t)) + \int_{-\infty}^{t} e^{-\omega(t-s)} f(s,u(s))ds$$
$$+ \int_{-\infty}^{t} e^{-\omega(t-s)} h(s,u(s))dW(s) \quad \text{for all} \quad t \in \mathbb{R}. \tag{4.2}$$

$u \sin \frac{1}{2+\cos s + \cos \sqrt{2} s} + e^{-|s|} \cos u$ belongs to $SPAA(\mathbb{R} \times L^2(\Omega, L^2[0,1]), L^2(\Omega, L^2[0,1]), \mu)$ where $u \sin \frac{1}{2+\cos s + \cos \sqrt{2} s}$ is the almost automorphic component and $e^{-|s|} \cos u$ is the μ-ergodic perturbation, since

$$\frac{1}{\mu([-r,r])}\int_{-r}^{0}\mathbb{E}\|e^{-|s|}\cos u\|^2 d\mu(s) \le \frac{1}{1-e^{-r}+r}\int_{-r}^{0} e^{-2|s|}e^{s}ds = \frac{1}{1-e^{-r}+r}\int_{-r}^{0} e^{3s}ds$$
$$= \frac{1}{3}\frac{1-e^{-3r}}{1-e^{-r}+r} \to 0 \quad \text{as} \quad r \to +\infty$$

and

$$\frac{1}{\mu([-r,r])}\int_{0}^{r}\mathbb{E}\|e^{-|s|}\cos u\|^2 d\mu(s) \le \frac{1}{1-e^{-r}+r}\int_{0}^{r} e^{-2|s|}dt = \frac{1}{1-e^{-r}+r}\int_{0}^{r} e^{-2s}ds$$
$$= \frac{1}{2}\frac{1-e^{-2r}}{1-e^{-r}+r} \to 0 \quad \text{as} \quad r \to +\infty.$$

By analogous argument performed above, we have also that $f, h \in SPAA(\mathbb{R} \times L^2(\Omega, L^2[0,1]), L^2(\Omega, L^2[0,1]), \mu)$.

Clearly, g, f and h satisfy the Lipschitz conditions ($\mathbf{H}_3$). Moreover, it is easy to see that ($\mathbf{H}_2$) hold. Therefore, by Theorem 3.2, the Eq. (4.2) has a unique square-mean μ-pseudo almost automorphic mild solution on $\mathbb{R}$ whenever L is small enough.

Acknowledgements The author would like to extend their thanks to the referee for his careful reading of the manuscript and insightful comments.

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