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Some generalizations and refined Hardy type integral inequalities

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Abstract In this paper, by using Jensen's inequality and Chebyshev integral inequality, some generalizations and new refined Hardy type integral inequalities are obtained. In addition, the corresponding reverse relation are also proved.

Keywords Hardy integral inequality · Hölder inequality · Chebyshev integral inequality · Jensen inequality

Mathematics Subject Classification 26D15

1 Introduction

In 1920, Hardy [2] proved the following inequality. If $p > 1, f \geq 0,\ \ p-$integrable on $(0, \infty)$ and

$$F(x) = \int_0^x f(t)dt, \tag{1}$$

then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{2}$$

unless $f \equiv 0$. The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. Hardy's inequality plays an important role in analysis and applications.

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The previous inequality still holds for parameters a and b. That is, the inequality

$$\int_a^b \left(\frac{F(x)}{x}\right)^p < \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)dx,$$

is valid for $0 < a < b < \infty$, see [3].

In [4], the author gives a generalization and improvement for Hardy's inequality in the sens when f is non-decreasing. If $f \geq 0$, and non-decreasing, F is as defined by (1). $f \geq 0,\ g > 0,\ \frac{x}{g(x)}$ is non-increasing, $p > 1,\ 0 < a < 1$ then

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \leq \frac{1}{a(1-p)(1-a)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx. \tag{3}$$

The Hardy inequalities have applications in the theory of differential equations (ordinary or partial) and led to many interesting questions and connections between different areas of mathematical analysis. For example, Hardy inequalities are closely related to the quasiadditivity properties of capacities [1] and have recently been used to find the gaps between zeros of differential equations which appear in the bending of beams [8].

In this paper, by using Jensen's inequality and Chebyshev integral inequality, we first give a generalization of Hardy's inequality (2). Second, we prove some new refined Hardy type integral inequalities.

2 Some preliminary Lemmas

In this section, we state the following Lemmas, which are useful in the proofs of our results.

Lemma 1 [5] (Jensen inequality) *Let μ be a probability measure and let $\varphi \geq 0$ be a convex function. Then, for all f be a integrable function we have*

$$\int \varphi \circ f d\mu \geq \varphi\left(\int f d\mu\right).$$

Lemma 2 [6] (Chebyshev integral inequality) *If f, g: $[a, b] \longrightarrow \mathbb{R}$ are integrable functions, both increasing or both decreasing, and p: $[a, b] \longrightarrow \mathbb{R}$ is a positive integrable function, then*

$$\int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt \tag{4}$$

Note that if one of the functions f or g is decreasing and the other is increasing, then (4) is reversed.

A function φ is called submultiplicative, if $\varphi(xy) \leq \varphi(x)\varphi(y)$, for all $x, y > 0$. In particular, for all $n \geq 1$, we have $\varphi(x^n) \leq \varphi^n(x)\ x > 0$.

The next Lemma exist in [4].

Lemma 3 *Let $\varphi \geq 0$ is submultiplicative, and $\varphi(0) = 0$. If $\varphi^{'}(x)$ is non-decreasing (non-increasing), then $\frac{\varphi(x)}{x}$ is non-decreasing (non-increasing).*

Lemma 4 *Let $g > 0$, and G is as defined by (5). If the function $\frac{g(x)}{x}$ is non-decreasing (non-increasing) on $(0, \infty)$, then the function $\frac{G(x)}{x^2}$ is also non-decreasing (non-increasing) on $(0, \infty)$.*

Proof For all $x > 0$, and suppose that the function $\frac{g(x)}{x}$ is non-decreasing

$$\left(\frac{G(x)}{x^2}\right)' = \frac{xg(x) - 2G(x)}{x^3} = \frac{K(x)}{x^3},$$

and

$$K'(x) = xg'(x) - g(x) = x^2\left(\frac{g(x)}{x}\right)' \geq 0,$$

and consequently, the function K is non-decreasing. Since $K(0) = 0$, then $K(x) \geq 0$, which implies $\left(\frac{G(x)}{x^2}\right)' \geq 0$. Therefore $\frac{G(x)}{x^2}$ is non-decreasing. □

3 Main results

The following results gives a generalization of Hardy's inequality.

Theorem 1 *Let $f \geq 0,\ g > 0,\ 0 < a < 1,\ p > 1,\ q > \frac{p-a(p-1)}{2}$ and*

$$G(x) = \int_0^x g(t)dt. \tag{5}$$

If the function $\frac{x}{g(x)}$ is non-increasing function. Then the following inequality

$$\int_0^\infty \frac{F^p(x)}{G^q(x)}dx \leq \frac{1}{((a-1)(p-1)+2q-1)(1-a)^{p-1}}\int_0^\infty \frac{(tf(t))^p}{G^q(t)}dt, \tag{6}$$

is valid. In particular, if we put $a = \frac{1}{p}$, $q = \frac{p}{2}$ and $G(x) = x^2$ we obtain (2).

Proof From Lemma 4, we obtain that the function $\frac{x^2}{G(x)}$ is non-increasing. By using the Hölder inequality we get

$$\begin{aligned}
\int_0^\infty \frac{F^p(x)}{G^q(x)}dx &= \int_0^\infty G^{-q}(x)\left(\int_0^x t^{\frac{-a(p-1)}{p}} t^{\frac{a(p-1)}{p}} f(t)dt\right)^p dx \\
&\leq \int_0^\infty G^{-q}(x)\left[\left(\int_0^x t^{-a}dt\right)^{\frac{p-1}{p}}\left(\int_0^x t^{a(p-1)} f^p(t)dt\right)^{\frac{1}{p}}\right]^p dx \\
&= \int_0^\infty G^{-q}(x)\left(\int_0^x t^{-a}dt\right)^{p-1}\int_0^x t^{a(p-1)} f^p(t)dtdx \\
&= \frac{1}{(1-a)^{p-1}}\int_0^\infty t^{a(p-1)} f^p(t)\int_t^\infty x^{(1-a)(p-1)}G^{-q}(x)dxdt \\
&\leq \frac{1}{(1-a)^{p-1}}\int_0^\infty t^{a(p-1)} f^p(t)\left(\frac{t^2}{G(t)}\right)^q\int_t^\infty x^{(1-a)(p-1)-2q}dxdt \\
&= \frac{1}{((a-1)(p-1)+2q-1)(1-a)^{p-1}}\int_0^\infty \frac{(tf(t))^p}{G^q(t)}dt. \qquad (7)
\end{aligned}$$

□

The following result concerns the converse inequality.

Theorem 2 *Let $f \geq 0,\ g > 0,\ 0 < p < 1,\ a > 0$, and $q > \frac{p+a(p-1)}{2}$. If the function $\frac{x}{g(x)}$ is non-increasing function. Then the following inequality*

$$\int_0^\infty \frac{F^p(x)}{G^q(x)}dx \geq \frac{1}{((a+1)(p-1)+2q-1)(1+a)^{p-1}} \int_0^\infty \frac{(tf(t))^p}{G^q(t)}dt. \tag{8}$$

is valid. In particular, if we put $a = \frac{1}{p},\ 0 < p < 1,\ q = \frac{p}{2}$ and $G(x) = x^2$, we obtain [4]

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \geq \frac{1+p}{1-p}\left(\frac{p}{1+p}\right)^p \int_0^\infty \left(\frac{f(x)}{x}\right)^p dx. \tag{9}$$

Proof By using Lemma 4, we obtain that the function $\frac{x^2}{G(x)}$ is non-decreasing. From the reverse Hölder inequality we get

$$\begin{aligned}
\int_0^\infty \frac{F^p(x)}{G^q(x)}dx &= \int_0^\infty G^{-q}(x)\left(\int_0^x t^{\frac{a(p-1)}{p}} t^{\frac{-a(p-1)}{p}} f(t)dt\right)^p dx \\
&\geq \int_0^\infty G^{-q}(x)\left[\left(\int_0^x t^a dt\right)^{\frac{p-1}{p}} \left(\int_0^x t^{-a(p-1)} f^p(t)dt\right)^{\frac{1}{p}}\right]^p dx \\
&= \int_0^\infty G^{-q}(x)\left(\int_0^x t^a dt\right)^{p-1} \int_0^x t^{-a(p-1)} f^p(t)dtdx \\
&= \frac{1}{(1+a)^{p-1}} \int_0^\infty t^{-a(p-1)} f^p(t) \int_t^\infty x^{(1+a)(p-1)} G^{-q}(x)dxdt \\
&\leq \frac{1}{(1+a)^{p-1}} \int_0^\infty t^{-a(p-1)} f^p(t)\left(\frac{t^2}{G(t)}\right)^q \int_t^\infty x^{(1+a)(p-1)-2q}dxdt \\
&= \frac{1}{((a+1)(1-p)+2q-1)(1+a)^{p-1}} \int_0^\infty \frac{(tf(t))^p}{G^q(t)}dt.
\end{aligned} \tag{10}$$

□

Remark 1 Theorems 1 and 2 are an answer to an open problem proposed by Sroysang [7].

The other type is given by the following.

Theorem 3 *Let $f, g > 0$ and non-decreasing on $(0,\infty)$. F be as defined by (1) and G defined by (5). Let $\varphi,\ \psi$ are non-decreasing ,submultiplicative and convex. If $\varphi(f(x)\psi(g(x))$ is integrable, then the following inequality*

$$\int_0^\infty \varphi(F(x))\psi(G(x))x^{1-p}dx \leq \frac{1}{p-1}\int_0^\infty \varphi(f(x))\psi(g(x))x^{1-p}dx.$$

holds for all $p > 1$.

Proof Since the functions φ and ψ are convex and submultiplicative, we obtain

$$\begin{aligned}
\int_0^\infty \frac{\varphi(F(x))\psi(G(x))x^{1-p}}{\varphi(x)\psi(x)}dx &= \int_0^\infty \frac{x^{1-p}}{\varphi(x)\psi(x)}\varphi\left(\int_0^x f(t)dt\right)\psi\left(\int_0^x f(t)dt\right)dx \\
&= \int_0^\infty \frac{x^{1-p}}{\varphi(x)\psi(x)}\varphi\left(\int_0^x f(t)dt\right)\psi\left(\int_0^x f(t)dt\right)dx \\
&\leq \int_0^\infty x^{1-p}\varphi\left(\frac{1}{x}\int_0^x f(t)dt\right)\psi\left(\frac{1}{x}\int_0^x g(t)dt\right)dx.
\end{aligned}$$

So, by Lemma 1 and the previous inequality we get

$$\int_0^\infty \frac{\varphi(F(x))\psi(G(x))x^{1-p}}{\varphi(x)\psi(x)}dx \leq \int_0^\infty x^{-1-p}\left(\int_0^x \varphi(f(t))dt\right)\left(\int_0^x \psi(g(t))dt\right)dx. \tag{11}$$

Since the functions φ, ψ, f and g are non-decreasing then the function $\varphi \circ f$ and $\psi \circ g$ are also non-decreasing and consequently we can applied Lemma 2 where $p(x) = 1$, and by the inequality (11) we have

$$\begin{aligned}\int_0^\infty \frac{\varphi(F(x))\psi(G(x))x^{1-p}}{\varphi(x)\psi(x)}dx &\leq \int_0^\infty x^{-p}\left(\int_0^x \varphi(f(t))\psi(g(t))dt\right)dx \\ &= \int_0^\infty \varphi(f(t))\psi(g(t))\left(\int_t^\infty x^{-p}\varphi(x)\psi(x)dx\right)dt \\ &= \int_0^\infty \varphi(f(t))\psi(g(t))\left(\int_t^\infty x^{-p}dx\right)dt \\ &= \frac{1}{p-1}\int_0^\infty x^{1-p}\varphi(f(x)\psi(g(x))dx. \end{aligned} \tag{12}$$

□

Theorem 4 *Let $f \geq 0$, non-decreasing, F is as defined by (1). Let $g > 0$, be a continuous on $(0, \infty)$, G is as defined by (5). Let $\phi \geq 0$, and non-decreasing, and $0 < b \leq \infty$. If g is non-increasing, and $\phi\left(\frac{f(x)}{g(x)}\right)$ is integrable on $0 < b \leq \infty$, the following inequality is valid*

$$\int_0^b \phi\left(\frac{F(x)}{G(x)}\right)dx \leq \int_0^b \phi\left(\frac{f(x)}{g(x)}\right)dx. \tag{13}$$

In particular, by putting $\phi(x) = x^p,\ p \geq 1$, we obtain

$$\int_0^b \left(\frac{F(x)}{G(x)}\right)^p dx \leq \int_0^b \left(\frac{f(x)}{g(x)}\right)^p dx. \tag{14}$$

Proof Since g is non-increasing, f and ϕ are non-decreasing, we get

$$\begin{aligned}\int_0^b \phi\left(\frac{F(x)}{G(x)}\right)dx &= \int_0^b \phi\left(\frac{1}{G(x)}\int_0^x f(t)dt\right) \\ &\leq \int_0^b \phi\left(\frac{xf(x)}{G(x)}\right)dx \\ &= \int_0^b \phi\left(\frac{xf(x)}{\int_0^x g(t)dt}\right)dx \\ &\leq \int_0^b \phi\left(\frac{xf(x)}{xg(x)}\right)dx.\end{aligned}$$

□

The next theorem is a generalization of Theorem 2.5 in [4].

Theorem 5 *Let $\varphi(x) \geq 0$ be a twice differentiable function on $(0, \infty)$, convex, submultiplicative and $\varphi(0) = 0$. Let $q \in \mathbb{N}$ and $p > 1$. If $x^{2-p}\frac{\varphi(f(x))}{\varphi(x)}$ is integrable, then the following inequality*

$$\int_0^\infty x^{2-p} \frac{\varphi\left(x^q F(x)\right)}{\varphi^{q+2}(x)} dx \leq \frac{1}{p-1} \int_0^\infty x^{2-p} \frac{\varphi(f(x))}{\varphi(x)} dx. \tag{15}$$

holds.

Proof Since the function $\varphi(x)$ is convex, we have $\frac{\varphi(x)}{x}$ is non-decreasing by Lemma 3. Then, by the Jensen's inequality, Fubini theorem and since φ is submultiplicative (In particular $\varphi(x^n) \leq \varphi^n(x)\ n \geq 1$), we get

$$\begin{aligned}
\int_0^\infty x^{2-p} \frac{\varphi\left(x^q F(x)\right)}{\varphi^{q+2}(x)} dx &= \int_0^\infty \frac{x^{2-p}}{\varphi^{q+2}(x)} \varphi\left(x^{q+1} \frac{F(x)}{x}\right) dx \\
&\leq \int_0^\infty \frac{x^{2-p}}{\varphi^{q+2}(x)} \varphi(x^{q+1}) \varphi\left(\frac{F(x)}{x}\right) dx \\
&\leq \int_0^\infty \frac{x^{2-p}}{\varphi^{q+2}(x)} \varphi^{q+1}(x) \varphi\left(\frac{F(x)}{x}\right) dx \\
&= \int_0^\infty \frac{x^{2-p}}{\varphi(x)} \varphi\left(\frac{F(x)}{x}\right) dx \\
&= \int_0^\infty \frac{x^{2-p}}{\varphi(x)} \varphi\left(\frac{1}{x} \int_0^x f(t) dt\right) dx \\
&\leq \int_0^\infty \frac{x^{2-p}}{\varphi(x)} \left(\frac{1}{x} \int_0^x \varphi(f(t)) dt\right) dx \\
&= \int_0^\infty \varphi(f(t)) \left(\int_t^\infty x^{-p} \frac{x}{\varphi(x)} dx\right) dt \\
&\leq \int_0^\infty \varphi(f(t)) \frac{t}{\varphi(t)} \left(\int_t^\infty x^{-p} dx\right) dt \\
&= \frac{1}{p-1} \int_0^\infty x^{2-p} \frac{\varphi(f(x))}{\varphi(x)} dx.
\end{aligned}$$

So, the proof of Theorem 5 is complete. □

Theorem 6 *Let* $f \geq 0$, *non-decreasing on* $(0, \infty)$ *and* F *is as defined by* (1). *Let* $g > 0$, *be a non-decreasing on* $(0, \infty)$ *and* G *is as defined by* (5). *Let* $\phi \geq 0$, *and non-decreasing, and* $0 < a < b < \infty$. *If the function* $\phi(f(x)g(x))$ *is integrable on* $[a, b]$, *then*

$$\int_a^b \phi\left(\frac{F(x)G(x)}{x^2}\right) dx \leq \int_a^b \phi\left(f(x)g(x)\right) dx. \tag{16}$$

In particular, by putting $\phi(x) = x^p,\ \ p \geq 1$ *and* $g(x) = 1$, *we obtain*

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \int_a^b f^p(x) dx.$$

Proof By the Chebyshev integral inequality and the assumption of the functions f, g and ϕ, and we consider the function $p(x) = 1$ for all $x \in [a, b]$ we obtain

$$\begin{aligned}
\int_a^b \phi\left(\frac{F(x)G(x)}{x^2}\right) dx &= \int_a^b \phi\left[\frac{1}{x^2}\left(\int_0^x f(t) dt\right)\left(\int_0^x g(t) dt\right)\right] \\
&\leq \int_a^b \phi\left[\frac{1}{x^2} \int_0^x 1 dt \int_0^x f(t) g(t) dt\right] dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \phi \left[\frac{1}{x} \int_0^x f(t)g(t)dt \right] dx \\
&\le \int_a^b \phi \left[\frac{f(x)}{x} \int_0^x g(t)dt \right] dx \\
&\le \int_a^b \phi \left[\frac{f(x)g(x)}{x} \int_0^x dt \right] dx \\
&= \int_a^b \phi \left(f(x)g(x) \right) dx.
\end{aligned}
$$

□

The following result concerns the converse inequality (16).

Theorem 7 *Let* $f \ge 0$*, non-decreasing,* F *is as defined by* (1)*. Let* $g > 0$*, non-decreasing,* G *is as defined by* (5)*. Let* $\phi \ge 0$*, and non-increasing, and* $0 < a < b < \infty$*, then*

$$
\int_a^b \phi \left(\frac{F(x)G(x)}{x^2} \right) dx \ge \int_a^b \phi \left(f(x)g(x) \right) dx. \tag{17}
$$

Proof By the Chebyshev integral inequality and the assumption of the functions f, g and ϕ, and we consider the function $p(x) = 1$ for all $x \in [a, b]$ we obtain

$$
\begin{aligned}
\int_a^b \phi \left(\frac{F(x)G(x)}{x^2} \right) dx &= \int_a^b \phi \left[\frac{1}{x^2} \left(\int_0^x f(t)dt \right) \left(\int_0^x g(t)dt \right) \right] \\
&\ge \int_a^b \phi \left[\frac{1}{x^2} \int_0^x 1\, dt \int_0^x f(t)g(t)dt \right] dx \\
&= \int_a^b \phi \left[\frac{1}{x} \int_0^x f(t)g(t)dt \right] dx \\
&\ge \int_a^b \phi \left[\frac{f(x)}{x} \int_0^x g(t)dt \right] dx \\
&\ge \int_a^b \phi \left[\frac{f(x)g(x)}{x} \int_0^x dt \right] dx \\
&= \int_a^b \phi \left(f(x)g(x) \right) dx.
\end{aligned}
$$

□

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