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Modified general iterative algorithm for an infinite family of nonexpansive mappings in Banach spaces

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Abstract In this paper we introduce an iterative method for finding a common fixed point of an infinite family of nonexpansive mappings in q-uniformly real smooth Banach space which is also uniformly convex. We proved strong convergence of the proposed iterative algorithms to the unique solution of a variational inequality problem.

Keywords Fixed point · Strongly accretive mapping · Nonexpansive mapping · Uniformly smooth and convex Banach spaces

Mathematics Subject Classification 47H09 · 47J25

1 Introduction

Let E be a real Banach space and E^* be the dual space of E. A mapping $\varphi\colon [0,\infty) \to [0,\infty)$ is called a guage function if it is strictly increasing, continuous and $\varphi(0) = 0$. Let φ be a gauge function, a generalized duality mapping with respect to φ, $J_\varphi\colon E \to 2^{E^*}$ is defined by, $x \in E$,

$$J_\varphi x = \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\},$$

where $\langle ., .\rangle$ denotes the duality pairing between element of E and that of E^*. If $\varphi(t) = t$, then J_φ is simply called the normalized duality mapping and is denoted by J. For any $x \in E$, an element of $J_\varphi x$ is denoted by $j_\varphi x$.
If however $\varphi(t) = t^{q-1}$, for some $q > 1$, then J_φ is still called the generalized duality mapping and is denoted by J_q (see, for example [1,5]).

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Let $S(E) := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then space E is said to have *Gâteaux differentiable norm* if for any $x \in S(E)$ the limit

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} \tag{1.1}$$

exists $\forall y \in S(E)$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, the limit (1.1) is attained uniformly for $x \in S(E)$. If E has a uniformly Gâteaux differentiable norm, then $j : E \to E^*$ is uniformly continuous on bounded subsets of E to the weak* topology of E^*.

A mapping $G : D(G) \subset E \to E$ is said to be *accretive* if for all $x, y \in D(G)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Gx - Gy, j_q(x - y)\rangle \geq 0, \tag{1.2}$$

where $D(G)$ denote the domain of G. G is called $\eta - strongly\ accretive$ if for all $x, y \in D(G)$, there exists $j_q(x - y) \in J_q(x - y)$ and $\eta \in (0, 1)$ such that

$$\langle Gx - Gy, j_q(x - y)\rangle \geq \eta ||x - y||^q, \tag{1.3}$$

Let K be a nonempty, closed and convex subset of E and $G : K \to E$ be a nonlinear mapping. The variational inequality problem is to:

$$\text{find } u \in K \text{ such that } \langle Gu, j_q(v - u)\rangle \geq 0,\ \forall v \in K,$$

for some $j_q(v - u) \in J_q(v - u)$. The set of solution of variational inequality problem is denoted by $VI(K, G)$. If $E := H$, a real Hilbert space, the variational inequality problem reduces to:

$$\text{find } u \in K \text{ such that } \langle Gu, v - u\rangle \geq 0,\ \forall v \in K,$$

which was introduced and studied by Stampacchia [16].

Variational inequality theory has emerged as an important tool in studying a wide class of related problems in Mathematical, Physical, regional, engineering and nonlinear optimization sciences (see, for instance, [8,9,11,15,24–26]).

A mapping $T : E \to E$ is $L - Lipschitian$ if for some $L > 0$, $||Tx - Ty|| \leq L||x - y||\ \forall\ x, y \in E$. If $L \in [0, 1)$, then T is called contraction mapping, but if $L \leq 1$, then T is called nonexpansive mapping. A point $x \in E$ is called a fixed point of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in E : Tx = x\}$. In Hilbert spaces H, accretive operators are called monotone where inequality (1.2) and (1.3) hold with j_q replaced by the identity map on H.

In 2000, Moudafi [14] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n,\quad n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in (0,1). Xu [21] proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.4) strongly converges to the unique solution x^* in F of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0,\ \text{for}\ \ x \in F.$$

In [19], he proved, under some conditions on the real sequence $\{\alpha_n\}$, that the sequence $\{x_n\}$ defined by $x_0 \in H$ chosen arbitrary,

$$x_{n+1} = \alpha_n b + (1 - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.5}$$

converges strongly to $x^* \in F$ which is the unique solution of the minimization problem

$$\min_{x \in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where A is a strongly positive bounded linear operator. That is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}||x||^2, \quad \forall x \in H.$$

Combining the iterative method (1.4) and (1.5), Marino and Xu [13] consider the following general iterative method:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.6}$$

they proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $x^* \in F$ which solves the variational inequality

$$\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0 \;\; x \in F,$$

which is the optimality condition for the minimization problem

$$\min_{x \in F} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e. $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, Yamada [24] in 2001 introduced the following hybrid iterative method:

$$x_{n+1} = Tx_n - \lambda_n \mu GTx_n, \quad n \geq 0, \tag{1.7}$$

where G is a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0, \eta > 0$ and $0 < \mu < 2\eta/\kappa^2$. Under some appropriate conditions, he proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the variational inequality

$$\langle Gx^*, x - x^*\rangle \geq 0, \quad \forall x \in F.$$

Recently, combining (1.6) and (1.7), Tian [18] considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)T(x_n), \tag{1.8}$$

and proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution $x^* \in F$ of the variational inequality

$$\langle(\gamma f - \mu G)x^*, x - x^*\rangle \leq 0, \quad \forall x \in F.$$

Most recently, Ali et al [4], extended the result of Tian [18] to q-uniformly smooth Banach space whose duality mapping is weakly sequentially continuous. Under some assumptions on $\{\alpha_n\}$, γ, μ and G being η-accretive mapping in (1.8), they proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution $x^* \in F$ of the variational inequality

$$\langle(\gamma f - \mu G)x^*, j(x - x^*)\rangle \leq 0, \quad \forall x \in F.$$

Let $\{T_i\}$ be countable family of nonexpansive mapping. We denote by a set $N_I : = \{i \in \mathbb{N} : T_i \neq I\}$ (I being the identity mapping on E). Maingé [12] studied the Halpern-type scheme for approximation of a common fixed point of *countable infinite* family of nonexpansive mappings in a real Hilbert space. He proved the following theorems.

Theorem 1.1 (Maingé [12]) *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}$ be countable family of nonexpansive self-mappings of K, $\{t_n\}$ and $\{\sigma_{i,t_n}\}$ be sequences in (0,1) satisfying the following conditions:* (i) $\lim t_n = 0$, (ii) $\sum_{i\geq 1} \sigma_{i,t_n} = 1 - t_n$, (iii) $\forall i \in N_I, \lim_{n\to\infty} \frac{t_n}{\sigma_{i,t_n}} = 0$. *Define a fixed point sequence $\{x_{t_n}\}$ by*

$$x_{t_n} = t_n C x_{t_n} + \sum_{i\geq 1} \sigma_{i,t_n} T_i x_{t_n}, \quad n \geq 1, \tag{1.9}$$

where $C : K \to K$ is a strict contraction. Assume $F : = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, the $\{x_{t_n}\}$ converges strongly to a unique fixed point of the contraction $P_F \circ C$, where P_F is a metric projection from H onto F.

Theorem 1.2 (Maingé [12]) *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}$ be countable family of nonexpansive self-mappings of K, $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ be sequences in (0,1) satisfying the following conditions:*

(i) $\sum \alpha_n = \infty$, $\sum_{i\geq 1} \sigma_{i,n} = 1 - \alpha_n$,

(ii)
$$\begin{cases} \frac{1}{\sigma_{i,n}}\left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right| \to 0, \quad or \quad \sum_n \frac{1}{\sigma_{i,n}} |\alpha_{n-1} - \alpha_n| < \infty \\ \frac{1}{\alpha_n}\left|\frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}}\right| \to 0, \quad or \quad \sum_n \left|\frac{1}{\sigma_{i,n}} - \frac{1}{\sigma_{i,n-1}}\right| < \infty \\ \frac{1}{\sigma_{i,n}\alpha_n} \sum_{k\geq 0} |\sigma_{k,n} - \sigma_{k,n-1}| \to 0, \quad or \quad \frac{1}{\sigma_{i,n}} \sum_{k\geq 0} |\sigma_{k,n} - \sigma_{k,n-1}| < \infty. \end{cases}$$

(iii) $\forall i \in N_I, \lim_{n\to\infty} \frac{\alpha_n}{\sigma_{i,n}} = 0$.

Then, the sequence $\{x_n\}$ define iteratively by $x_1 \in K$,

$$x_{n+1} = \alpha_n C x_n + \sum_{i\geq 1} \sigma_{i,n} T_i x_n, \quad n \geq 1, \tag{1.10}$$

where $C : K \to K$ is a strict contraction. Assume $F : = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, the $\{x_n\}$ converges strongly to a unique fixed point of the contraction $P_F \circ C$, where P_F is a metric projection from H onto F.

Motivated by the results above, we introduce an iterative method for finding a common fixed point of an infinite family of nonexpansive mappings in q-uniformly real smooth Banach space. We prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality problem.

2 Preliminaries

Let K be a nonempty, closed, convex and bounded subset of a Banach space E and let the diameter of K be defined by $d(K) : = \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (introduced in 1980 by Bynum [3], see also Lim [10] and the references contained therein) is

defined by $N(E):=\inf\{\frac{d(K)}{r(K)}$: K is a closed convex and bounded subset of E with $d(K) > 0\}$. A Banach space E such that $N(E) > 1$ is said to have *uniform normal structure* . It is known that every Banach space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see e.g., [5,23]).
Let E be a normed space with dimE ≥ 2. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup \left\{ \frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1; ||y|| = \tau \right\}.$$

The space E is called *uniformly smooth* if and only if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$. For some positive constant $q \in E$ is called $q-$ *uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q, t > 0$. It is well known that if E is smooth then the duality mapping is singled-valued, and if E is uniformly smooth then the duality mapping is norm-to-norm uniformly continuous on bounded subset of E.

Lemma 2.1 *Let E be a real normed space. Then*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle,$$

for all $x, y \in E$ *and for all* $j(x+y) \in J(x+y)$.

Lemma 2.2 *(*Xu, [22]*) Let E be a real q-uniformly smooth Banach space for some* $q > 1$, *then there exists some positive constant* d_q *such that*

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + d_q||y||^q \ \forall x, y \in E \text{ and } j_q \in J_q(x).$$

Lemma 2.3 *(*Xu, [21]*) Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \geq 0$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. *Then,* $a_n \to 0$ *as* $n \to \infty$.

Lemma 2.4 *(*Suzuki [17]*) Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequence in a Banach space E and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. *Suppose that* $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ *for all integer* $n \geq 1$ *and* $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. *Then,* $\lim_{n\to\infty} ||y_n - x_n|| = 0$.

Lemma 2.5 *(*See Lemma 2.1 of Ali [2]*) Let E be a real smooth and uniformly convex Banach space and let* $r > 0$. *Then there exists a strictly increasing, continuous and convex function* $g\colon [0, 2r] \to R$ *such that* $g(0) = 0$ *and* $g(||x-y) \leq ||x||^2 - 2\langle x, jy\rangle + ||y||^2$ *for all* $B_r = \{x \in E\colon ||x|| \leq r\}$.

Lemma 2.6 *Let E be a real Banach space,* $f\colon E \to E$ *be contraction mapping with a coefficient* $0 < \beta <$ *and let* $G\colon E \to E$ *be a* $\kappa-$*Lipschitzian and* $\eta-$*strongly accretive operator with* $\kappa > 0, \eta \in (0, 1)$. *Then for* $\gamma \in (0, \frac{\mu\eta}{\beta})$,

$$\langle(\mu G - \gamma f)x - (\mu G - \gamma f)y, j(x-y)\rangle \geq (\mu\eta - \gamma\beta)||x-y||^2,\ \forall x, y \in E.$$

That is, $(\mu G - \gamma f)$ *is strongly accretive with coefficient* $(\mu\eta - \gamma\beta)$.

Let μ be a linear continuous functional on l^∞ and let $a = (a_1, a_2, \cdots) \in l^\infty$. We will sometimes write $\mu_n(a_n)$ in place of the value $\mu(a)$. A linear continuous functional μ such that $||\mu|| = 1 = \mu(1)$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2, \cdots) \in l^\infty$ is called a *Banach limit*. It is known that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \le \mu(a_n) \le \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$ (see, for example, [5,6])

3 Main results

In the sequel we assume for each $\alpha \in (0, 1)$, the sequence $\{\sigma_{i,\alpha}\}$ satisfies $\sum_{i\ge1} \sigma_{i,\alpha} = 1-\alpha$ and for the sequence $\{\alpha_n\} \subset (0, 1)$, $\{\sigma_{i,n}\} = 1 - \alpha_n$.

Lemma 3.1 *Let E be a $q-$uniformly smooth real Banach space with constant $d_q, q > 1$. Let $f : E \to E$ be a $\beta-$contraction mapping with a coefficient $\beta \in (0, 1)$. Let $T_i : E \to E$ $i \in N$, be a family of nonexpansive maps such that $F :\ = \cap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and $G : E \to E$ be an $\eta-$strongly accretive mapping which is also $\kappa-$Lipschizian. Let $\mu \in \left(0, \min\left\{1, (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{q-1}}\right\}\right)$ and $\tau :\ = \mu\left(\eta - \frac{\mu^{q-1}d_q\kappa^q}{q}\right)$. For each $t, \alpha \in (0, 1)$ with $\alpha < t$ and $\beta \in (0, \frac{\tau}{\gamma})$. Assume that $S :\ = \alpha u + (1-\delta)(1-\alpha)I + \delta\sum_{i\ge1}\sigma_{i,\alpha}T_i$, where δ is some fixed number in $(0, 1)$ and $u \in E$, $\alpha, \sigma_{i,t}$ are in $(0, 1)$. Define the following mapping W_t on E by*

$$W_t x := t\gamma f(x) + (I - \mu t G)Sx.$$

where t is in (0,1). Then W_t is a strict contraction mapping. Furthermore, for any $x, y \in E$,

$$||W_t x - W_t y|| \le [1 - t(\tau - \beta\gamma)]||x - y||$$

Proof Observe that, for any $x, y \in E$

$$\begin{aligned} ||Sx - Sy|| &\le (1-\delta)(1-\alpha)||x-y|| + \delta\sum_{i\ge1}\sigma_{i,\alpha}||T_i x - T_i y|| \\ &\le (1-\delta)(1-\alpha)||x-y|| + \delta(1-\alpha)||x-y|| \\ &\le ||x-y||. \end{aligned} \tag{3.1}$$

Without loss of generality, assume $\eta < \frac{1}{q}$. Then, as $\mu < (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{q-1}}$, we have $0 < q\eta - \mu^{q-1} d_q\kappa^q$. Furthermore, from $\eta < \frac{1}{q}$ we have $q\eta - \mu^{q-1}d_q\kappa^q < 1$ so that $0 < q\eta - \mu^{q-1}d_q\kappa^q < 1$. Also as $\mu < 1$ and $t \in (0, 1)$, we obtain that $0 < t\mu(q\eta - \mu^{q-1}d_q\kappa^q) < 1$.
For each $t, \alpha \in (0, 1)$, then for any $x, y \in E$, define $K_t x = (1 - t\mu G)Sx$, then from (3.1), we obtain

$$\begin{aligned} ||K_t x - K_t y||^q &= ||(1-t\mu G)Sx - (1-t\mu G)Sy||^q \\ &= ||(Sx - Sy) - t\mu(G(Sx) - G(Sy))||^q \\ &\le ||G(Sx) - G(Sy)||^q - qt\mu\langle G(Sx) - G(Sy), j_q(Sx - Sy)\rangle \\ &\quad + t^q\mu^q d_q||Sx - Sy||^q \\ &\le [1 - t\mu(q\eta - t^{q-1}\mu^{q-1}\kappa^q d_q)]||x-y||^q \\ &\le [1 - qt\mu(\eta - \frac{\mu^{q-1}\kappa^q d_q}{q})]||x-y||^q \end{aligned}$$

$$\leq [1 - t\mu(\eta - \frac{\mu^{q-1}\kappa^q d_q}{q})]^q ||x - y||^q$$
$$= (1 - t\tau)^q ||x - y||^q,$$

therefore

$$\begin{aligned}||W_t x - W_t y|| &= ||t\gamma(f(x) - f(y)) + (K_t(Sx) - K_t(Sy))|| \\ &\leq t\gamma||f(x) - f(y)|| + ||K_t(Sx) - K_t(S_t y)|| \\ &\leq t\beta\gamma||x - y|| + (1 - t\tau)||x - y|| \\ &= [1 - t(\tau - \beta\gamma)]||x - y||.\end{aligned}$$

Hence

$$||W_t x - W_t y|| \leq [1 - t(\tau - \beta\gamma)]||x - y||, \tag{3.2}$$

which implies that W_t is a strict contraction, by Banach contraction mapping principle, there exists a unique fixed point x_t of W_t in E. That is,

$$x_t = t\gamma f(x_t) + (1 - t\mu G)Sx_t. \tag{3.3}$$

□

Theorem 3.2 *Let E be a q-uniformly real smooth Banach space which is also uniformly convex. Let T_i, $f, G, \mu, \tau, \beta, \gamma, S$ and F be as in Lemma 3.1. Let $\{t_n\}$, $\{\alpha_n\}$ be sequences in (0,1), such that $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Let $\{x_{t_n}\}$ be a sequence satisfying (3.3), then*

(i) $\{x_{t_n}\}$ *is bounded for* $t_n \in (0, \frac{1}{\tau})$.
(ii) $\lim_{n\to\infty} ||x_{t_n} - T_i x_{t_n}|| = 0, \quad \forall i \in \mathbb{N}$.
(iii) *then $\{x_{t_0}\}$ converges strongly to a common fixed point p in F which is a unique solution of the variational inequality*

$$\langle(\mu G - \gamma f)p, j(p - x)\rangle \leq 0, \quad \forall x \in F. \tag{3.4}$$

Proof Let $p \in F$ and $\alpha_n \leq t_n$, then

$$\begin{aligned}||x_{t_n} - p||^2 &= \langle t_n\gamma f(x_{t_n}) + (I - \mu t_n G)Sx_{t_n} - p, j(x_{t_n} - p)\rangle \\ &= t_n\langle\gamma f(p) - \mu G(p), j(x_{t_n} - p)\rangle + t_n\gamma\langle f(x_{t_n}) - f(p), j(x_{t_n} - p)\rangle \\ &\quad +\langle(I - t_n\mu G)Sx_{t_n} - (I - t_n\mu G)p, j(x_{t_n} - p)\rangle \\ &\leq t_n\langle\gamma f(p) - \mu G(p), j(x_{t_n} - p)\rangle + \beta\gamma t_n||x_{t_n} - p||^2 \\ &\quad +(1 - \tau t_n)||Sx_{t_n} - p||||x_{t_n} - p|| \\ &\leq t_n\langle\gamma f(p) - \mu G(p), j(x_{t_n} - p)\rangle + \beta\gamma t_n||x_{t_n} - p||^2 \\ &\quad +(1 - \tau t_n)[\alpha_n||u - p|| + (1 - \alpha_n)||x_{t_n} - p||]||x_{t_n} - p|| \\ &\leq t_n\langle\gamma f(p) - \mu G(p), j(x_{t_n} - p)\rangle + [1 - t_n(\tau - \gamma\beta)||x_{t_n} - p||^2 \\ &\quad +(1 - \tau t_n)\alpha_n||u - p||||x_{t_n} - p||.\end{aligned}$$

Since $(1-\tau t_n)(\alpha_n/t_n) \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $(1-\tau t_n)(\alpha_n/t_n) < (\tau - \gamma\beta)/2$ for $n \geq n_0$. Furthermore

$$||x_{t_n} - p||^2 \leq \frac{\langle(\gamma f - \mu G)p, j(x_{t_n} - p)\rangle}{\tau - \gamma\beta}$$

$$+\frac{(1-\tau t_n)\alpha_n}{t_n} \times \frac{||u-p||||x_{t_n}-p||}{\tau-\gamma\beta} \tag{3.5}$$

for all $n \geq n_0$. That is, $||x_{t_n} - p|| \leq (\frac{||\gamma f(p)-\mu G(p)||}{\tau-\gamma\beta} + \frac{||u-p||}{2})$ for all $n \geq n_0$. Thus $\{x_{t_n}\}$ is bounded, so are $\{f(x_{t_n})\}$, $\{G(x_{t_n})\}$, $\{T_i(x_{t_n})\}$ and $\{G(T_i x_{t_n})\}$.
(ii) From (3.3), we have

$$||x_{t_n} - Sx_{t_n}|| = t_n||\gamma f(x_{t_n}) - \mu G(Sx_{t_n})|| \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Using Lemma 2.5, we have the following estimate

$$\begin{aligned} g(||T_i x_{t_n} - x_{t_n}||) &= g[||(p - T_i x_{t_n}) - (p - x_{t_n})||] \\ &\leq ||p - T_i x_{t_n}||^2 - 2\langle p - T_i x_{t_n}, j(p - x_{t_n})\rangle + ||p - x_{t_n}||^2 \\ &\leq ||p - T_i x_{t_n}||^2 - 2\langle p - x_{t_n} + x_{t_n} - T_i x_{t_n}, j(p - x_{t_n})\rangle + ||p - x_{t_n}||^2 \\ &\leq 2||p - T_i x_{t_n}||^2 - 2\langle p - T_i x_{t_n}, j(p - x_{t_n})\rangle + 2\langle x_{t_n} - T_i x_{t_n}, j(x_{t_n} - p)\rangle \\ &\leq 2\langle x_{t_n} - T_i x_{t_n}, j(x_{t_n} - p)\rangle \end{aligned} \tag{3.7}$$

Therefore

$$\begin{aligned} \frac{\delta}{2}\sum_{i\geq 1}\sigma_{i,n} g(||T_i x_{t_n} - x_{t_n}||) &\leq \langle \delta(1-\alpha_n)x_{t_n} - \delta\sum_{i\geq 1}\sigma_{i,n} T_i x_{t_n}, j(x_{t_n} - p)\rangle \\ &= \langle \alpha_n(u - x_{t_n}) + x_{t_n} - Sx_{t_n}, j(x_{t_n} - p)\rangle \\ &\leq [\alpha_n||u - x_{t_n}|| + ||x_{t_n} - Sx_{t_n}||]||x_{t_n} - p||. \end{aligned}$$

Then we immediately obtain $\lim_{n\to\infty} \sum_{i\geq 1} \sigma_{i,n} g(||T_i x_{t_n} - x_{t_n}||) = 0$, it follows that $\lim_{n\to\infty} g(||T_i x_{t_n} - x_{t_n}||) = 0 \ \forall i \in \mathbb{N}$. By the property of g we have that

$$\lim_{n\to\infty} ||T_i x_{t_n} - x_{t_n}|| = 0 \ \forall i \in \mathbb{N}. \tag{3.8}$$

(iii) By Lemma 2.6, $(\mu G - \gamma f)$ is strongly accretive, so the variational inequality (3.4) has a unique solution in F. Below we use $q \in F$ to denote the unique solution of (3.4). Next, we prove that $x_t \to q$ $(t \to 0)$.
Let $\{t_n\}$ be a sequence in (0, 1) such that $\{x_{t_n}\}$ satisfies (3.3). By writing $\{x_n\}$ instead of $\{x_{t_n}\}$, define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) := \mu_n||x_n - y||^2, \quad \forall y \in E.$$

Then, $\phi(y) \to \infty$ as $||y|| \to \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $q \in E$ such that $\phi(q) = \min_{u\in E}\phi(u)$. Hence, the set

$$K^* := \{y \in E : \phi(y) = \min_{u\in E}\phi(u)\} \neq \emptyset.$$

Since $\lim_{n\to\infty} ||x_n - T_i x_n|| = 0$, $\lim_{n\to\infty} ||x_n - T_i^m x_n|| = 0$, for any $m \geq 1$ and $i \in \mathbb{N}$ by induction. Now let $v \in K^*$, we have for any $i \in \mathbb{N}$

$$\begin{aligned} \phi(T_i v) = \mu_n||x_n - T_i v||^2 &= \mu_n||x_n - T_i x_n + T_i x_n - T_i v||^2 \\ &\leq \mu_n||x_n - v||^2 = \phi(v), \end{aligned}$$

and hence $T_i v \in K^*$.
Now let $z \in F$, then $z = T_i z$, for any $i \in \mathbb{N}$. Since K^* is a closed convex set, there exists a unique $v^* \in K^*$ such that

$$||z - v^*|| = \min_{u\in K^*}||z - u||.$$

But, for any $i \in \mathbb{N}$

$$||z - T_i v^*|| = ||T_i z - T_i v^*|| \leq ||z - v^*||,$$

which implies $v^* = T_i v^*$ and so $K^* \cap F \neq \emptyset$.
Let $p \in K^* \cap F$ and $\epsilon \in (0, 1)$. Then, it follows that $\phi(p) \leq \phi(p - \epsilon(G - \gamma f)p)$ and using Lemma 2.1, we obtain that

$$||x_n - p + \epsilon(G - \gamma f)p||^2 \leq ||x_n - p||^2 + 2\epsilon\langle (G - \gamma f)p, j(x_n - p + \epsilon(G - \gamma f)p)\rangle$$

which implies

$$\mu_n\langle(\gamma f - G)p, j(x_n - p + \epsilon(G - \gamma f)p)\rangle \leq 0.$$

Moreover,

$$\begin{aligned}\mu_n\langle(\gamma f - G)p, j(x_n - p)\rangle &= \mu_n\langle(\gamma f - G)p, j(x_n - p) - j(x_n - p + \epsilon(G - \gamma f)p)\rangle \\ &\quad + \mu_n\langle(\gamma f - G)p, j(x_n - p + \epsilon(G - \gamma f)p)\rangle \\ &\leq \mu_n\langle(\gamma f - G)p, j(x_n - p) - j(x_n - p + \epsilon(G - \gamma f)p)\rangle.\end{aligned}$$

Since j is norm-to-norm uniformly continuous on bounded subsets of E, and $\epsilon \to 0$ we have that

$$\mu_n\langle(\gamma f - G)p, j(x_n - p)\rangle \leq 0.$$

Now from (3.5) and since $\lim\limits_{n\to\infty} \frac{\alpha_n}{t_n} = 0$, we have

$$\begin{aligned}\mu_n||x_n - p||^2 &\leq \mu_n\Big(\frac{\langle(\gamma f - \mu G)p, j(x_n - p)\rangle}{\tau - \gamma\beta}\Big) \\ &\quad + \mu_n\Big(\frac{(1 - \tau t_n)\alpha_n}{t_n} \times \frac{||u - p||||x_n - p||}{\tau - \gamma\beta}\Big)\end{aligned}$$

and so

$$\mu_n\|x_n - p\|^2 \leq 0.$$

Thus there exist a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ such that $lim_{j\to\infty}x_{n_j} = p$.
By definition of S_α as $S_{\alpha_n}x_n := \alpha_n u + (1 - \delta)(1 - \alpha_n)x_n + \delta\sum_{i\geq1}\sigma_{i,n}T_i x_n$, which implies $S_{\alpha_{n_j}}x_{n_j} := x_{n_j} + \alpha_{n_j}(u - x_{n_j}) + \delta\sum_{i\geq1}\sigma_{i,n_j}(T_i x_{n_j} - x_{n_j})$, then $lim_{j\to\infty}Sx_{n_j} = lim_{j\to\infty}x_{n_j} = p$ and $S_\alpha p = p$. Thus for any $z \in F$, using (3.3) we have

$$\begin{aligned}\langle\mu G(x_{n_j}) - \gamma f(x_{n_j}), j(x_{n_j} - z)\rangle &= \frac{-1}{t_{n_j}}\langle(I - S)x_{n_j} - (I - S)p, j(x_{n_j} - z)\rangle \\ &\quad + \mu\langle Gx_{n_j} - GSx_{n_j}, j(x_{n_j} - z)\rangle \\ &\leq \mu\langle Gx_{n_j} - GSx_{n_j}, j(x_{n_j} - z)\rangle, \end{aligned} \tag{3.9}$$

since $\langle(I - S)x_{n_j} - (I - S)p, j(x_{n_j} - z)\rangle \geq 0$. As G is Lipschitzian and the fact that $\|x_{n_j} - Sx_{n_j}\| \leq \alpha_{n_j}\|u - x_{n_j}\| + \delta\sum_{i\geq1}\sigma_{i,n_j}\|(T_i x_{n_j} - x_{n_j})\| \to 0$ as $j \to \infty$, we have $Gx_{n_j} - GSx_{n_j} \to 0$ as $j \to \infty$. From this and (3.9), taking limit as $j \to \infty$ we obtain

$$\langle(\mu G - \gamma f)p, j(p - z)\rangle \leq 0.$$

Hence p is the unique solution of the variational inequality (3.4). Now assume there exists another subsequence of $\{z_n\}$ say $\{x_{n_k}\}$ such that $lim_{k\to\infty}x_{n_k} = p^*$. Then, using (3.8) we have $p^* \in F$. Repeating the above argument with p replaced by p^* we can easily obtain that p^*

also solved the variational inequality (3.4). By uniqueness of the solution of the variational inequality, we obtained that $p = p^*$ and this completes the proof. □

Theorem 3.3 *Let E be a q-uniformly real smooth Banach space which is also uniformly convex. Let $T_i : E \to E$ $i \in \{1, 2, \ldots\}$ be a family of nonexpansive mappings with $F :=\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $G : E \to E$ be an η-strongly accretive map which is also κ-Lipschitzian. Let $f : E \to E$ be a contraction map with coefficient $0 < \beta < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in (0,1) satisfying:*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(ii) $\sum \alpha_n < \infty$ *and* $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$;

(iii) $\lim_{n\to\infty} \sum_{i\geq 1} |\sigma_{i,n+1} - \sigma_{i,n}| = 0$ *and* $\sum_{i\geq 1} \sigma_{i,n} = 1 - \alpha_n$.

Let $\mu, \gamma,$ and τ be as in Lemma 3.1 and $\delta \in (0, 1)$ be fixed. define a sequence $\{x_n\}_{n=1}^{\infty}$ iteratively in E by $x_0 \in E$

$$\begin{cases} y_n = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta \sum_{i\geq 1} \sigma_{i,n} T_i x_n \\ x_{n+1} = \beta_n \gamma f(x_n) + (I - \beta_n \mu G) y_n. \end{cases} \tag{3.10}$$

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F$ which is also a solution to the following variational inequality

$$\langle (\gamma f - \mu G)x^*, j(y - x^*) \rangle \leq 0, \quad \forall y \in F. \tag{3.11}$$

Proof Since $(\mu G - \gamma f)$ is strongly accretive, then the variational inequality (3.11) has a unique solution in F. Now we show that $\{x_n\}_{n=1}^{\infty}$ is bounded. Let $p \in F$ then, for every $i \in \mathbb{N}$, $T_i p = p$. From (3.10), we obtain

$$\begin{aligned} ||y_n - p|| &= ||\alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta \sum_{i\geq 1} \sigma_{i,n} T_i x_n - p|| \\ &\leq \alpha_n ||u - p|| + (1-\delta)(1-\alpha_n)||x_n - p|| \\ &\quad + \delta \sum_{i\geq 1} \sigma_{i,n} ||T_i x_n - p|| \\ &\leq \alpha_n ||u - p|| + (1-\alpha_n)||x_n - p|| \\ &\leq \alpha_n ||u - p|| + ||x_n - p||. \end{aligned} \tag{3.12}$$

Also from (3.10) and (3.12), we obtain

$$\begin{aligned} ||x_{n+1} - p|| &= ||\beta_n \gamma f(x_n) + (I - \beta_n \mu G) y_n - p|| \\ &\leq \beta \gamma \beta_n ||x_n - p|| + \beta_n ||\gamma f(p) - \mu G(p)|| \\ &\quad + ||(I - \beta_n \mu G) y_n - (I - \beta_n \mu G) p|| \\ &\leq \beta \gamma \beta_n ||x_n - p|| + \beta_n ||\gamma f(p) - \mu G(p)|| \\ &\quad + (1 - \tau \beta_n) ||y_n - p|| \\ &\leq \beta \gamma \beta_n ||x_n - p|| + \beta_n ||\gamma f(p) - \mu G(p)|| \\ &\quad + (1 - \tau \beta_n)[\alpha_n ||u - p|| + ||x_n - p||] \\ &\leq [1 - \beta_n(\tau - \gamma\beta)] ||x_n - p|| \\ &\quad + \beta_n [||\gamma f(p) - \mu G(p)|| + ||u - p||] \\ &\leq \max \left\{ ||x_n - p||, \frac{||\gamma f(p) - \mu G(p)|| + ||u - p||}{\tau - \gamma\beta} \right\} \end{aligned}$$

$$\leq \cdots \leq \max\left\{||x_1 - p||, \frac{||\gamma f(p) - \mu G(p)|| + ||u - p||}{\tau - \gamma\beta}\right\}.$$

Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}, \{T_i x_n\}, \{G y_n\}, \{G T_i y_n\}$ and $\{f(y_n)\}$ are also bounded.
Next, we show that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. Define two sequences $\{\lambda_n\}$ and $\{z_n\}$ by $\lambda_n := (1-\delta)\alpha_n + \delta$ and

$$z_n := \frac{x_{n+1} - x_n + \lambda_n x_n}{\lambda_n}.$$

Observe that $\{z_n\}$ is bounded and that

$$\begin{aligned} ||z_{n+1} - z_n|| - ||x_{n+1} - x_n|| \leq & \left(\frac{\beta_{n+1}}{\lambda_{n+1}} + \frac{\beta_n}{\lambda_n}\right) M + \left|\frac{\alpha_{n+1}}{\lambda_{n+1}} - \frac{\alpha_n}{\lambda_n}\right| ||u|| \\ & + \left[\frac{\delta(1-\alpha_{n+1})}{\lambda_{n+1}} - 1\right] ||x_{n+1} - x_n|| \\ & + \frac{\delta M}{\lambda_{n+1}} \sum_{i\geq 1} |\sigma_{i,n+1} - \sigma_{i,n}| \\ & + \frac{\delta M}{\lambda_{n+1}} \sum_{i\geq 1} \sigma_{i,n} |\lambda_{n+1} - \lambda_n| \end{aligned}$$

for some real number $M := \sup_{n\geq 1}\{||\gamma f(x_n) - \mu G(y_n)||, ||T_i x_n||, i = 1, 2, ...\}$.
This implies

$$\limsup_{n\to\infty} (||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0,$$

and by Lemma 2.4, we obtain

$$\lim_{n\to\infty} ||z_n - x_n|| = 0.$$

Hence

$$||x_{n+1} - x_n|| = \lambda_n ||z_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{3.13}$$

and from (3.10), we also obtain

$$||x_{n+1} - y_n|| = \beta_n ||\gamma_n f(x_n) - \mu G(y_n)|| \to 0 \text{ as } n \to \infty. \tag{3.14}$$

from (3.13) and (3.14), we have

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{3.15}$$

Next we show that $\lim_{n\to\infty} ||T_i x_n - x_n|| = 0$ for all $i \in \mathbb{N}$. Since $p \in F$, using the same argument in (3.7), we obtain

$$\begin{aligned} \frac{\delta}{2} \sum_{i\geq 1} \sigma_{i,n} g(||T_i x_n - x_n||) &\leq \delta \sum_{i\geq 1} \sigma_{i,n} \langle x_n - T_i x_n, j(x_n - p)\rangle \\ &\leq \langle \delta(1-\alpha_n) x_n - \delta \sum_{i\geq 1} \sigma_{i,n} T_i x_n, j(x_n - p)\rangle \\ &\leq \langle \alpha_n(u - x_n) + x_n - y_n, j(x_n - p)\rangle \\ &\leq [\alpha_n ||u - x_n|| + ||x_n - y_n||] ||x_n - p||. \end{aligned}$$

From (3.15) and $\lim_{n\to\infty}\alpha_n = 0$, we obtain

$$\lim_{n\to\infty}\sum_{i\geq 1}\sigma_{i,n}||T_i x_n - x_n|| = 0,$$

it follows that for every $i \in \mathbb{N}$,

$$\lim_{n\to\infty}||T_i x_n - x_n|| = 0. \tag{3.16}$$

Let $z_t = t\gamma f(z_t) + (1 - t\mu G)Sz_t$, where $S: = \alpha u + (1-\delta)(1-\alpha)I + \delta\sum_{i\geq 1}\sigma_{i,\alpha}T_i$, as in Theorem 3.1. Then,

$$z_t - x_n = t(\gamma f(z_t) - Gz_t) + t\mu(Gz_t - G(Sz_t)) + Sz_t - x_n$$

Hence

$$\begin{aligned}
||z_t - x_n||^2 &= \langle t(\gamma f(z_t) - Gz_t) + t\mu(Gz_t - G(Sz_t)) + Sz_t - x_n, j(z_t - x_n)\rangle \\
&= t\langle\gamma f(z_t) - \mu G(z_t), j(z_t - x_t)\rangle + t\mu\langle Gz_t - G(Sz_t), j(z_t - x_n)\rangle \\
&\quad + \langle Sz_t - x_n, j(z_t - x_n)\rangle \\
&\leq t\langle\gamma f(z_t) - \mu Gz_t, j(z_t - x_n)\rangle + t\mu\kappa||z_t - Sz_t||||z_t - x_n|| \\
&\quad + ||Sz_t - x_n||||z_t - x_n|| \\
&\leq t\langle\gamma f(z_t) - \mu Gz_t, j(z_t - x_n)\rangle + t(1+\mu)||z_t - Sz_t||||z_t - x_n|| \\
&\quad + ||z_t - x_n||^2 + ||Sx_n - x_n||||z_t - x_n||.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle\gamma f(z_t) - \mu Gz_t, j(x_n - z_t)\rangle &\leq (1+\mu\kappa)||z_t - Sz_t||||z_t - x_n|| \\
&\quad + \frac{1}{t}||Sx_n - x_n||||z_t - x_n||
\end{aligned}$$

Now, taking limit superior as $n \to \infty$ firstly, and then as $t \to 0$, we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle\gamma f(z_t) - \mu Gz_t, j(x_n - z_t)\rangle \leq 0 \tag{3.17}$$

Moreover, we note that

$$\begin{aligned}
\langle\gamma f(p) - \mu Gp, j(x_n - p)\rangle &= \langle\gamma f(p) - \mu Gp, j(x_n - p)\rangle - \langle\gamma f(p) - \mu Gp, j(x_n - z_t)\rangle \\
&\quad + \langle\gamma f(p) - \mu Gp, j(x_n - z_t)\rangle - \langle\gamma f(p) - \mu Gz_t, j(x_n - z_t)\rangle \\
&\quad + \langle\gamma f(p) - \mu Gz_t, j(x_n - z_t)\rangle - \langle\gamma f(z_t) - \mu Gz_t, j(x_n - z_t)\rangle \\
&\quad + \langle\gamma f(z_t) - \mu Gz_t, j(x_n - z_t)\rangle \\
&= \langle\gamma f(p) - \mu Gp, j(x_n - p) - j(x_n - z_t)\rangle \\
&\quad + \mu\langle Gz_t - Gp, j(x_n - z_t)\rangle \\
&\quad + \langle\gamma f(z_t) - \gamma f(p), j(x_n - z_t)\rangle \\
&\quad + \langle\gamma f(z_t) - \mu Gz_t, j(x_n - z_t)\rangle
\end{aligned} \tag{3.18}$$

Taking limit superior as $n \to \infty$ in (3.18), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle\gamma f(p) - \mu Gp, j(x_n - p)\rangle &\leq \limsup_{n\to\infty}\langle\gamma f(p) - \mu Gp, j(x_n - p) - j(x_n - z_t)\rangle \\
&\quad + \mu||Gz_t - Gp||\limsup_{n\to\infty}||x_n - z_t|| \\
&\quad + ||\gamma f(z_t) - \gamma f(p)||\limsup_{n\to\infty}||x_n - z_t||
\end{aligned}$$

$$
\begin{aligned}
&+\limsup_{n\to\infty}\langle \gamma f(z_t)-\mu G z_t, j(x_n-z_t)\rangle\\
&\le \limsup_{n\to\infty}\langle \gamma f(p)-\mu G p, j(x_n-p)-j(x_n-z_t)\rangle\\
&\quad +((\mu+1)+\alpha\gamma)\|z_t-p\|\limsup_{n\to\infty}||x_n-z_t||\\
&\quad +\limsup_{n\to\infty}\langle \gamma f(z_t)-\mu G z_t, j(x_n-z_t)\rangle \qquad (3.19)
\end{aligned}
$$

since E has a uniformly Gâteaux differentiable norm, so j is norm-to-norm* uniformly continuous on bounded subset of E. Then, from Theorem 3.1 (i.e., $z_t \to p \ (t\to 0^+)$), we obtain

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(p)-\mu G p, j(x_n-p)-j(x_n-z_t)\rangle = 0,$$

hence, using (3.17) in (3.19), we obtain

$$
\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(p)-\mu G p, j(x_n-p)\rangle &\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(z_t)-\mu G z_t, j(x_n-p)\rangle\\
&\le 0
\end{aligned}
$$

Finally, we show that $x_n \to p$. From the recursion formula (3.10), by using (2.1) and taking $n \ge N$ where $N \in \mathbb{N}$ is large enough, we obtain

$$
\begin{aligned}
||x_{n+1}-p||^2 &= ||\beta_n\gamma f(x_n)-\beta_n\mu G(p)+(I-\beta_n\mu G)y_n-(I-\beta_n\mu G)p||^2\\
&\le ||(I-\beta_n\mu G)y_n-(I-\beta_n\mu G)p||^2+2\beta_n\langle \gamma f(x_n)-\mu G(p), j(x_{n+1}-p)\rangle\\
&\le (1-\beta_n\tau)^2||y_n-p||^2+2\beta_n\langle \gamma f(x_n)-\gamma f(p), j(x_{n+1}-p)\rangle\\
&\quad +2\beta_n\langle \gamma f(p)-\mu G(p), j(x_{n+1}-p)\rangle\\
&\le \alpha_n||u-p||^2+(1-\beta_n\tau)^2||x_n-p||^2\\
&\quad +2\beta_n\langle \gamma f(x_n)-\gamma f(p), j(x_{n+1}-p)\rangle\\
&\quad +2\beta_n\langle \gamma f(p)-\mu G(p), j(x_{n+1}-p)\rangle
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\langle \gamma f(x_n)-\gamma f(p), j(x_{n+1}-p)\rangle &\le \gamma\beta||x_n-p||||x_{n+1}-p||\\
&\le \gamma\beta||u-p||||x_n-p||\sqrt{\alpha_n}+\gamma\beta(1-\beta_n\tau)||x_n-p||^2\\
&\quad +\gamma\beta||x_n-p||\sqrt{2|\langle \gamma f(x_n)-\gamma f(p), j(x_{n+1}-p)\rangle|}\sqrt{\beta_n}\\
&\quad +\gamma\beta||x_n-p||\sqrt{2|\langle \gamma f(p)-\mu G(p), j(x_{n+1}-p)\rangle|}\sqrt{\beta_n}.
\end{aligned}
$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, we pick a constant $G_0 > 0$ such that

$$
\begin{aligned}
&\sup\Big\{\gamma\beta||x_n-p||||u-p||, \gamma\beta||x_n-p||\Big(\sqrt{2|\langle \gamma f(x_n)-\gamma f(p), j(x_{n+1}-p)\rangle|}\\
&+\gamma\beta\sqrt{2|\langle \gamma f(p)-\mu G(p), j(x_{n+1}-p)\rangle|}\Big)\Big\} < G_0, \forall n\in\mathbb{N}.
\end{aligned}
$$

Therefore

$$\langle \gamma f(x_n)-\gamma f(p), j(x_{n+1}-p)\rangle \le \gamma\beta(1-\beta_n\tau)||x_n-p||^2+G_0(\sqrt{\alpha_n}+\sqrt{\beta_n})$$

Hence

$$||x_{n+1}-p||^2 \le \alpha_n||u-p||^2+(1-\beta_n\tau)^2||x_n-p||^2$$

$$
\begin{aligned}
&+2\beta_n\gamma\beta(1-\beta_n\tau)||x_n-p||^2+2\beta_n G_0(\sqrt{\alpha_n}+\sqrt{\beta_n})\\
&+2\beta_n\langle\gamma f(p)-\mu G(p), j(x_{n+1}-p)\rangle\\
=&\Big[1-2\beta_n(1-\beta_n\tau)(\tau-\gamma\beta)\Big]||x_n-p||^2+\alpha_n||u-p||^2\\
&+2\beta_n G_0(\sqrt{\alpha_n}+\sqrt{\beta_n})+2\beta_n\langle\gamma f(p)-\mu G(p), j(x_{n+1}-p)\rangle\\
\le&\Big[1-\beta_n(1-\beta_n\tau)(\tau-\gamma\beta)\Big]||x_n-p||^2+\theta_n
\end{aligned}
$$

where $\theta_n := \beta_n\Big(\alpha_n/\beta_n||u-p||^2+2G_0(\sqrt{\alpha_n}+\sqrt{\beta_n}).+2\langle\gamma f(p)-\mu G(p), j(x_{n+1}-p)\rangle\Big)$
By using Lemma 2.3 we obtain $x_n \to p$ as $n \to \infty$. This complete the proof. □

Corollary 3.4 *Let H be a real Hilbert space, $\{z_t\}_{t\in(0,1)}$, be as in Theorem 3.2. Then $\{z_t\}$ converges strongly to a common fixed points of the family $\{T_i\}_{i=1}^{\infty}$ say p which is a unique solution of the variational inequality*

$$\langle(\mu G-\gamma f)p, q-p\rangle \ge 0,\quad \forall q\in F.$$

Corollary 3.5 *Let H be a real Hilbert space and let C a nonempty closed convex subset of H. Let $G\colon H\to H$, $f\colon E\to E$, $\{T_i\}_{i=1}^{\infty}$ F, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be as in Theorem (3.1), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p\in F$, which is also the unique solution of the variational inequality*

$$\langle\gamma f(p)-\mu Gp, q-p\rangle \le 0,\ \forall q\in F$$

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