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Fixed point theorems for α-integral type G-contraction mappings

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Abstract In this paper, we introduce the notion of α-integral type G-contraction mappings to generalize the notions of Banach G-contraction and integral G-contraction mappings. We also prove some fixed point theorems for α-integral type G-contraction mappings. By providing some example, we show that our results are real generalization of several results in literature.

Keywords α-subadmissible mappings · Graph-metric spaces · Banach G-contraction mappings · Integral G-contraction mappings · α-integral type G-contraction mappings

Mathematics Subject Classification 47H10 · 54H25

1 Introduction and preliminaries

Development of fixed point theory on a metric space endowed with graph has a lot of activities in last few year. Jachymski [1] introduced the notion of Banach G-contraction. Later on various authors proved many fixed point theorems for single-valued and multi-valued mappings on a metric space endowed with a graph, see, for example [2–10]. Recently, Asl

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et al. [3] defined a graph-metric space and proved fixed point theorems on it. In this paper we introduce the notion of α-integral type G-contraction to generalize the notions of Banach G-contraction and integral G-contraction. We also prove the fixed point theorems for such mappings and state some illustrative examples to claim that our results properly generalizes some results in literature.

Let (X, d) be a metric space and G be an undirected graph such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops in $V(G)$. Throughout this paper, we assume that G has no parallel edges. We also denote this space by G_d and call it a graph-metric space [3]. A mapping $T : G_d \to G_d$ is said to be G-continuous if for given sequence $\{x_n\}$ such that $x_n \to x$ as $n \to \infty$, where $x \in G_d$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $Tx_n \to Tx$ as $n \to \infty$.

In 2008, Jachymski [1] introduced the notion of Banach G-contraction mappings as follows:

Definition 1.1 [1] Let (X, d) be a metric space endowed with graph G. A mapping $T : X \to X$ is called a Banach G-contraction if T preserves the edges of G, i.e.,

$$\forall x, y \in X \quad ((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)) \tag{1}$$

and

$$\exists c \in (0, 1)\ \forall x, y \in X \quad ((x, y) \in E(G) \Rightarrow d(Tx, Ty) \leq cd(x, y)).$$

We denote by Φ the set of all Lebesgue integrable mappings $\phi : [0, \infty) \to [0, \infty)$ which are summable on each compact subset of $[0, \infty)$ and for each $\epsilon > 0$, we have

$$\int_0^{\epsilon} \phi(t)\, dt > 0. \tag{2}$$

In 2013, Samreen and Kamran [9] extended the notion of Banach G-contraction in the following way:

Definition 1.2 [9] Let (X, d) be a metric space endowed with graph G. A mapping $T : X \to X$ is called an integral G-contraction if T preserves the edges of G, i.e.,

$$\forall x, y \in X \quad ((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)) \tag{3}$$

and

$$\forall x, y \in X \quad \left((x, y) \in E(G) \Rightarrow \int_0^{d(Tx,Ty)} \phi(t)\, dt \leq c \int_0^{d(x,y)} \phi(t)\, dt \right),$$

where $c \in (0, 1)$ and $\phi \in \Phi$.

Recently, Kamran and Ali [11] introduced the notion of α-subadmissible mappings in the following way:

Definition 1.3 Let $\alpha : G_d \times G_d \to [0, \infty)$ be a mapping. A mapping $T : G_d \to G_d$ is said to be α-subadmissible if

(i) for $x, y \in G_d$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$;
(ii) for $x \in G_d$, $\alpha(T^n x, T^{n+1} x) \geq 1$ for all integers $n \geq 0$ implies $\alpha(T^m x, T^n x) \geq 1$ for all integers $m > n \geq 0$.

By Ξ, we mean the family of functions $\xi : [0, \infty) \to [0, \infty)$ such that ξ is nondecreasing, upper semicontinuous and $\lim_{n\to\infty} \xi^n(t) = 0$ for each $t \geq 0$. Note that if $\xi \in \Xi$, then $\xi(t) < t$ for each $t > 0$, and $\xi(0) = 0$.

2 Main results

We begin this section with the following definition.

Definition 2.1 A mapping $T : G_d \to G_d$ is said to be an α-integral type G-contraction mapping if there exist three functions $\alpha : G_d \times G_d \to [0, \infty)$, $\phi \in \Phi$ and $\xi \in \Xi$ such that

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \phi(t)\, dt \le \xi\left(\int_0^{d(x,y)} \phi(t)\, dt\right), \tag{4}$$

for each $(x, y) \in E(G)$.

Remark 2.2 Let (X, d) be a metric space. Define graph G by $V(G) = X$ and $E(G) = X \times X$, $\alpha(x, y) = 1$ for each $x, y \in X$ and $\xi(t) = ct$ for all $t \ge 0$, where $c \in [0, 1)$. Then (4) reduces to

$$\int_0^{d(Tx,Ty)} \phi(t)\, dt \le c \int_0^{d(x,y)} \phi(t)\, dt,$$

for each $x, y \in X$, where $\phi \in \Phi$, which is the contractive condition considered by Branciari [12].

Note that every integral G-contraction mapping is an α-integral type G-contraction mapping. The following example shows that the converse is not true in general.

Example 2.3 Let $X = [0, \infty)$ with the usual metric d and X endowed with graph G is defined by $V(G) = X$ and $E(G) = \{(x, y) : x \ge y\}$. Define $T : G_d \to G_d$ and $\alpha : G_d \times G_d \to [0, \infty)$ by

$$Tx = \begin{cases} \frac{x}{4} & \text{if } x \ge 4, \\ x^2 & \text{if } 0 \le x < 4, \end{cases} \tag{5}$$

and

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \ge 4, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Take $\xi(t) = \frac{t}{2}$ for each $t \ge 0$ and $\phi(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{t^{-1/2}}{2} & \text{if } t > 0. \end{cases}$ If $x, y \ge 4$, then we have

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \phi(t)\, dt = \sqrt{\frac{|x - y|}{4}} = \frac{1}{2} \int_0^{d(x,y)} \phi(t)\, dt,$$

and for otherwise, we have

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \phi(t)\, dt = 0 \le \frac{1}{2} \int_0^{d(x,y)} \phi(t)\, dt.$$

Thus, (4) holds for each $(x, y) \in E(G)$. Therefore, T is an α-integral type G-contraction mapping. But T is not an integral G-contraction, since $(4, 2) \in E(G) \nRightarrow (T4, T2) \in E(G)$.

Theorem 2.4 *Let G_d be a complete graph-metric space and $T : G_d \to G_d$ be an α-integral type G-contraction mapping satisfying the following assumptions*:

(i) *T is α-subadmissible*;

(ii) *there exists* $x_0 \in G_d$ *such that* $\alpha(x_0, Tx_0) \geq 1$;
(iii) *if* $x, y \in G_d$ *and* $\alpha(x, y) \geq 1$, *then* $(x, y) \in E(G)$;
(iv) T *is* G*-continuous*;

Then T *has a fixed point.*

Proof Starting from $x_0 \in G_d$ in (ii). Define a sequence $\{x_n\}$ in G_d such that $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$. Since T is α-subadmissible, by induction we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for each } n \in \mathbb{N} \cup \{0\}. \tag{7}$$

From (4), we have

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})} \phi(t)\, dt &= \int_0^{d(Tx_{n-1},Tx_n)} \phi(t)\, dt \\ &\leq \alpha(x_{n-1}, x_n) \int_0^{d(Tx_{n-1},Tx_n)} \phi(t)\, dt \\ &\leq \xi\left(\int_0^{d(x_{n-1},x_n)} \phi(t)\, dt\right),\end{aligned}$$

for all $n \in \mathbb{N}$. By induction, we have

$$\int_0^{d(x_n,x_{n+1})} \phi(t)\, dt \leq \xi^n\left(\int_0^{d(x_0,x_1)} \phi(t)\, dt\right), \tag{8}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (8), we have

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \phi(t)\, dt = 0. \tag{9}$$

From (2) and (9), we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{10}$$

Now, using a similar argument as given by authors in [13,14], we will show that $\{x_n\}$ is a Cauchy sequence in G_d. Assume on contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two sequences of positive integers $\{n_k\}$ and $\{m_k\}$ such that for each $k \in \mathbb{N}$, we have

$$n_k > m_k > k, \quad d(x_{n_k}, x_{m_k}) \geq \epsilon, \quad d(x_{n_k-1}, x_{m_k}) < \epsilon. \tag{11}$$

By using triangular inequality and (11), we have

$$\begin{aligned}\epsilon \leq d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) \\ &< d(x_{n_k}, x_{n_k-1}) + \epsilon.\end{aligned}$$

Letting $k \to \infty$ in above inequality and using (10), we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon. \tag{12}$$

Again by using triangular inequality, we have

$$d(x_{n_k-1}, x_{m_k-1}) \leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1})$$

and

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k})$$

for all $k \in \mathbb{N}$. Using (10) and (12) in above two inequalities, we get

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{13}$$

As T is α-subadmissible, from (7), we have $\alpha(x_{n_k-1}, x_{m_k-1}) \geq 1$ for all $k \in \mathbb{N}$. Then by Condition (iii) we get $(x_{n_k-1}, x_{m_k-1}) \in E(G)$ for all $k \in \mathbb{N}$. From (4), we have

$$\begin{aligned}\int_0^{d(x_{n_k},x_{m_k})} \phi(t)\,dt &= \int_0^{d(Tx_{n_k-1},Tx_{m_k-1})} \phi(t)\,dt\\ &\leq \alpha(x_{n_k-1}, x_{m_k-1}) \int_0^{d(Tx_{n_k-1},Tx_{m_k-1})} \phi(t)\,dt\\ &\leq \xi\left(\int_0^{d(x_{n_k-1},x_{m_k-1})} \phi(t)\,dt\right) \end{aligned}\tag{14}$$

for all $k \in \mathbb{N}$. Using properties of ξ and let $k \to \infty$ in (14), we have

$$\int_0^{\epsilon} \phi(t)\,dt \leq \xi\left(\int_0^{\epsilon} \phi(t)\,dt\right) < \int_0^{\epsilon} \phi(t)\,dt.$$

A contradiction to our assumption. Hence $\{x_n\}$ is a Cauchy sequence in G_d. Since G_d is complete, there exists $x^* \in G_d$ such that $x_n \to x^*$ as $n \to \infty$. G-continuity of T implies $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. By uniqueness of limit, we have $Tx^* = x^*$. □

Theorem 2.5 *Let G_d be a complete graph-metric space and $T : G_d \to G_d$ be an α-integral type G-contraction mapping satisfying the following assumptions*:

(i) *T is α-subadmissible*;
(ii) *there exists $x_0 \in G_d$ such that $\alpha(x_0, Tx_0) \geq 1$*;
(iii) *if $x, y \in G_d$ and $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$*;
(iv) *if $\{x_n\}$ is a sequence in G_d such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Proof According to the proof of Theorem 2.4, we know that $\{x_n\}$ be a Cauchy sequence in G_d. Since G_d is complete, there exists $x^* \in G_d$ such that $x_n \to x^*$ as $n \to \infty$. From (7) and Conditions (iv) and (iii), we have

$$\alpha(x_n, x^*) \geq 1 \text{ and } (x_n, x^*) \in E(G) \quad \text{for each } n \in \mathbb{N} \cup \{0\}. \tag{15}$$

By the triangular inequality, we have

$$|d(Tx^*, x^*) - d(Tx_n, x^*)| \leq d(Tx^*, Tx_n) \tag{16}$$

for all $n \in \mathbb{N} \cup \{0\}$. Thus, we have

$$\begin{aligned}\int_0^{|d(Tx^*,x^*)-d(Tx_n,x^*)|} \phi(t)\,dt &\leq \int_0^{d(Tx^*,Tx_n)} \phi(t)\,dt\\ &\leq \alpha(x_n, x^*) \int_0^{d(Tx^*,Tx_n)} \phi(t)\,dt\\ &\leq \xi\left(\int_0^{d(x^*,x_n)} \phi(t)\,dt\right)\end{aligned}\tag{17}$$

for all $n \in \mathbb{N} \cup \{0\}$. Letting $n \to \infty$ in (17), we have

$$\lim_{n\to\infty} \int_0^{|d(Tx^*,x^*)-d(Tx_n,x^*)|} \phi(t)\, dt = 0. \tag{18}$$

From (18), we have

$$\lim_{n\to\infty} |d(Tx^*, x^*) - d(Tx_n, x^*)| = 0. \tag{19}$$

This implies that

$$d(Tx^*, x^*) = \lim_{n\to\infty} d(Tx_n, x^*) = \lim_{n\to\infty} d(x_{n+1}, x^*) = 0.$$

Hence $Tx^* = x^*$. □

We use the following condition to obtain the uniqueness of the fixed point.
(A) For each fixed points x and y of T, we have $\alpha(x, y) \geq 1$.

Theorem 2.6 *Adding the Condition* (A) *to the hypotheses of Theorem* 2.4 (*resp. Theorem* 2.5), *we get the uniqueness of the fixed point of* T.

Proof Suppose that x^* and y^* are two distinct fixed points of T. From Condition (A), we get $\alpha(x^*, y^*) \geq 1$ and hence

$$\begin{aligned}\int_0^{d(x^*,y^*)} \phi(t)\, dt &\leq \alpha(x^*, y^*) \int_0^{d(Tx^*,Ty^*)} \phi(t)\, dt \\ &\leq \xi\left(\int_0^{d(x^*,y^*)} \phi(t)\, dt\right) \\ &< \int_0^{d(x^*,y^*)} \phi(t)\, dt. \end{aligned} \tag{20}$$

A contradiction to our assumption. Hence T has unique fixed point. □

Example 2.7 Let $X = \mathbb{R}$ with the usual metric d and X endowed with the graph G such that $V(G) = X$ and $E(G) = \{(x, y) : x, y \geq -1\}$. Define $T : G_d \to G_d$ and $\alpha : G_d \times G_d \to [0, \infty)$ by

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \geq 0, \\ x & \text{if } x < 0, \end{cases} \tag{21}$$

and

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Take $\xi(t) = \frac{t}{4}$ and $\phi(t) = t$ for each $t \geq 0$. If $x, y \geq 0$, then we have

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \phi(t)\, dt = \frac{1}{2}\left(\frac{|x-y|}{2}\right)^2 = \frac{1}{4}\int_0^{d(x,y)} \phi(t)\, dt = \xi\left(\int_0^{d(x,y)} \phi(t)\, dt\right),$$

and for otherwise, we have

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \phi(t)\, dt = 0 \leq \frac{1}{4}\int_0^{d(x,y)} \phi(t)\, dt = \xi\left(\int_0^{d(x,y)} \phi(t)\, dt\right).$$

This shows that (4) holds for each $(x, y) \in E(G)$. Therefore, T is an α-integral type G-contraction mapping. If $x, y \in G_d$ and $\alpha(x, y) = 1$, then $x, y \geq 0$. By definition of T and α, we have $\alpha(Tx, Ty) = 1$. For each $x \in G_d$ such that $\alpha(T^n x, T^{n+1} x) \geq 1$ for all integers $n \geq 0$, we have $\alpha(T^m x, T^n x) = 1$ for all integers $m > n \geq 0$. Hence T is α-subadmissible. Also, we have $x_0 = 1 \in G_d$ such that $\alpha(1, T1) = \alpha(1, 1/2) = 1$. Moreover, if $x, y \in G_d$ such that $\alpha(x, y) \geq 1$, then $x, y \geq 0$ which implies that $(x, y) \in E(G)$. Finally, it is easy to see that the continuity of T implies that T is G-continuous. Therefore, all the hypotheses of Theorem 2.4 hold. Hence T has a fixed point, that is, a point 0.

Note that T is neither a Banach G-contraction mapping nor an integral G-contraction mapping. Indeed, if we take $(x, y) = (-1, -1/2) \in E(G)$, we get

$$d(Tx, Ty) = d(T(-1), T(-1/2)) > cd(-1, -1/2) = cd(x, y)$$

and

$$\int_0^{d(Tx,Ty)} \phi(t)\, dt = \int_0^{d(T(-1),T(-1/2))} \phi(t)\, dt > c \int_0^{d(-1,-1/2)} \phi(t)\, dt = c \int_0^{d(x,y)} \phi(t)\, dt$$

for all $c \in (0, 1)$ and $\phi \in \Phi$.

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