Afr. Mat. (2016) 27:491–500
DOI 10.1007/s13370-015-0345-4

Strong convergence of some algorithms for λ-strict pseudo-contractions in Banach spaces

Meng Wen[1,3] · **Jigen Peng**[1,3] · **Changsong Hu**[2]

Received: 8 February 2014 / Accepted: 11 May 2015 / Published online: 31 May 2015

Abstract In this paper, we introduce a new iterative algorithm for finding a common element of the set of common fixed points of an infinite family of strictly pseudo-contractive mappings in a real 2-uniformly smooth Banach space. Then we proved a strong convergence theorem under some suitable conditions. Our results generalize and improve several recent results.

Keywords Strong convergence · λ-Strict pseudo-contractive mapping · Fixed point Banach space

Mathematics Subject Classification 47H09 · 47H10

1 Introduction

Throughout the paper unless otherwise stated, let E be a real Banach space and E^* the dual space of E. Let $\{x_n\}$ be any sequence in E, then $x_n \to x$(respectively, $x_n \rightharpoonup x$, $x_n \rightharpoondown x$) will denote strong (respectively, weak, $weak^*$) convergence of the sequence $\{x_n\}$. Let C be a nonempty, closed and convex subset of E and T be a self-mapping of C. We use $F(T)$ to denote the fixed points of T. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \quad and \quad \|f\| = \|x\|\}, \quad \forall x \in E,$$

✉ Meng Wen
wen5495688@163.com

Jigen Peng
jgpeng@mail.xjtu.edu.cn

1 School of Mathematics and Statistics, Xi'an Jiaotong University, 710049 Xi'an, People's Republic of China

2 Department of Mathematics, Hubei Normal University, 435002 Huangshi, People's Republic of China

3 Beijing Center for Mathematics and Information Interdisciplinary Sciences, Beijing, P.R. China

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel we shall donate single-valued duality mappings by j.

We recall that the modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) := \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

E is said to be uniformly smooth if $\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$.

Let $q > 1$. E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is well-known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable. If E is uniformly smooth, then the normalized duality map j is single-valued and norm to norm uniformly continuous.

If a Banach space E admits a sequentially continuous duality mapping J from weak topology to weak star topology, from Lemma 1 of [1], it follows that the duality mapping J is single-valued, and also E is smooth. In this case, duality mapping J is said to be weakly sequentially continuous, i.e., for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$; then $J(x_n) \rightharpoonup J(x)$ (see [1]).

A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x (n \to \infty)$ implies

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \quad \forall y \in E \quad with \quad x \neq y.$$

By Theorem 1 of [1], we know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition, and E is smooth; for the details, see [1].

Let C be a subset of a real Hilbert space H. Recall that a mapping $T : C \to C$ is said to be strictly pseudo-contractive if there exists a constant $0 < \lambda < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda \|(I - T)x - (I - T)y\|^2, \quad x, y \in C. \tag{1.1}$$

Let C be a subset of a real Banach space E. Recall that a mapping $T : C \to C$ is said to be strictly pseudo-contractive if there exists a constant $0 < \lambda < 1$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|(I - T)x - (I - T)y\|^2, \tag{1.2}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$.

From (1.2) we can prove that if T is λ-strict pseudo-contractive, then T is Lipschitz continuous with the Lipschitz constant $L = \frac{1+\lambda}{\lambda}$.

It is clear that the class of strictly pseudo-contractive mappings strictly includes the class of nonexpansive mappings, which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

Let C be a subset of E. Then $P_C : E \to C$ is called a retraction from E onto C if $P_C(x) = x$ for all $x \in C$. A retraction $P_C : E \to C$ is said to be sunny if $P_C(x + t(x - P_C(x))) = P_C(x)$ for all $x \in E$ and $t \geq 0$. A subset C of E is said to be a sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto C.

Proposition 1.1 (See, e.g., Bruck [2], Reich [3], Goebel and Reich [4]) *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $P_C : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(a) *P_C is sunny and nonexpansive.*
(b) $\|P_C x - P_C y\|^2 \leq \langle x - y, J(P_C x - P_C y)\rangle, \forall x, y \in E.$
(c) $\langle x - P_C x, J(y - P_C x)\rangle \leq 0, \forall x \in E, y \in C.$

In 2011, Yao et al. [5] in a real Hilbert space, introduced the following iterative algorithm: for $x_0 = x \in C$,

$$x_{n+1} = P_C((1 - k - \alpha_n)x_n + kTx_n); \quad n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$ is a real sequence in (0; 1). He obtained that the sequence $\{x_n\}$ generated by (1.4) converges strongly to the minimum-norm fixed point of T.

In this paper, motivated and inspired by Yao et al. [5], we will introduce a new iterative scheme in a real 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping defined as:given $x_1 = x \in C$,

$$x_{n+1} = P_C\left((1 - k - \alpha_n)x_n + k\sum_{i=1}^{\infty} \eta_i^{(n)} T_i x_n\right); \quad n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}$ is a real sequence in (0; 1), $k \in (0; \frac{2\lambda}{C_2})$ and $\{\eta_i\}_{i=1}^{\infty}$ is a positive sequence such that $\sum_{i=1}^{\infty} \eta_i = 1$. We will prove that if the parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to a common element of the fixed points of an infinite family of λ_i-strictly pseudo-contractive mappings.

2 Preliminaries

In order to prove our main results, we need the following lemmas.

Lemma 2.1 [6] *Let E be a 2-uniformly smooth Banach space, then exists a constant $C_2 > 0$ such that*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + C_2\|y\|^2, \quad \forall x, y \in E.$$

Lemma 2.2 [7,8] *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions: (i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$, (ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n\delta_n < \infty$, (iii) $\gamma_n \geq 0 (n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$. *Then* $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (See [9, Lemma 1.3]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Suppose that the normalized duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero. Let $T : C \to E$ be a λ-strict pseudo-contraction with $0 < \lambda < 1$. Then for any $\{x_n\} \subset C$, if $x_n \rightharpoonup x$, and $x_n - Tx_n \to y \in E$, then $x - Tx = y$.*

Lemma 2.4 (See [10, Lemma 2.11]) *Let E be a 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping J from E to E^* and C be a nonempty convex subset of E. Assume that $T_i : C \to E$ is a countable family of λ_i-strict pseudocontraction for some $0 < \lambda_i < 1$ and $\inf\{\lambda_i : i \in \mathbb{N}\} > 0$ such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\{\eta_i\}_{i=1}^{\infty}$ is a positive sequence such that $\sum_{i=1}^{\infty} \eta_i = 1$. Then $\sum_{i=1}^{\infty} \eta_i T_i : C \to E$ is a λ-strict pseudocontraction with $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\}$ and $F\left(\sum_{i=1}^{\infty} \eta_i T_i\right) = F$.*

3 Main result

Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping J. Let C be also a sunny nonexpansive retraction of E and $T : C \to C$ be a λ-strict pseudo-contraction. Let $k \in (0; \frac{2\lambda}{C_2})$ be a constant. For each $t \in (0; 1)$, we consider the mapping T_t given by

$$T_t x = P_C((1-k-t)x + kTx), \quad \forall x \in C.$$

It is easy to check that $T_t : C \to C$ is a contraction for a small enough t. As a matter of fact, from Lemma 2.2 and (1.2),

$$\begin{aligned}
\|T_t x - T_t y\|^2 &= \|P_C((1-k-t)x + kTx) - P_C((1-k-t)y + kTy)\|^2 \\
&\le \|(1-t)(x-y) - k((x-Tx) - (y-Ty))\|^2 \\
&\le (1-t)^2\|x-y\|^2 - 2k(1-t)\langle (I-T)x - (I-T)y, J(x-y)\rangle \\
&\quad + C_2\|k[(I-T)x - (I-T)y]\|^2 \\
&\le (1-t)^2\|x-y\|^2 - 2k(1-t)\lambda\|(I-T)x - (I-T)y]\|^2 \\
&\quad + C_2k^2\|(I-T)x - (I-T)y]\|^2 \\
&= (1-t)^2\|x-y\|^2 - k[2(1-t)\lambda - C_2k]\|(I-T)x - (I-T)y]\|^2.
\end{aligned} \tag{3.1}$$

We can choose a small enough t such that $2\lambda - C_2k > 0$. Then, from (3.1),

$$\|T_t x - T_t y\| \le (1-t)\|x-y\|, \quad \forall x, y \in C, \tag{3.2}$$

which implies that T_t is a contraction. Using the Banach contraction principle, there exists a unique fixed point x_t of T_t in C, that is,

$$x_t = P_C((1-k-t)x_t + kTx_t). \tag{3.3}$$

Theorem 3.1 *Suppose that $F(T) \neq \emptyset$. Then, as $t \to 0$, the net $\{x_t\}$ generated by (3.3) converges strongly to the minimum-norm fixed point of T.*

Proof First, we prove that $\{x_t\}$ is bounded. Take $p \in F(T)$. From (3.3) and (3.2),

$$\begin{aligned}
\|x_t - p\| &= \|P_C((1-k-t)x_t + kTx_t) - P_C p\| \\
&\le \|(1-k-t)(x_t - p) + k(Tx_t - p) - tp\| \\
&\le (1-t)\|x_t - p\| + t\|p\|,
\end{aligned}$$

that is, $\|x_t - p\| \le \|p\|$, which implies that $\{x_t\}$ is bounded and so is $\{Tx_t\}$.

From (3.3),

$$\begin{aligned}
\|x_t - Tx_t\| &= \|P_C((1-k-t)x_t + kTx_t) - P_C Tx_t\| \\
&\le \|(1-k)(x_t - Tx_t) - tx_t\| \\
&\le (1-k)\|x_t - Tx_t\| + t\|x_t\|.
\end{aligned}$$

It follows that

$$\|x_t - Tx_t\| \le \frac{t}{k}\|x_t\| \to 0. \tag{3.4}$$

Next we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0; 1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. It follows from (3.4) that

$$\|x_n - Tx_n\| \to 0 \quad as \quad n \to \infty. \tag{3.5}$$

Setting $y_t = (1-k-t)x_t + kTx_t$, we then have $x_t = P_C y_t$, and, for any $p \in F(T)$,

$$\begin{aligned} x_t - p &= x_t - y_t + y_t - p \\ &= x_t - y_t + (1-k-t)(x_t - p) + k(Tx_t - p) - tp. \end{aligned} \tag{3.6}$$

By Proposition 1.1, we have

$$\langle x_t - y_t, J(x_t - p)\rangle \le 0. \tag{3.7}$$

Combining (3.6) and (3.7),

$$\begin{aligned} \|x_t - p\|^2 &= \langle x_t - y_t, J(x_t - p)\rangle + (1-k-t)\langle x_t - p, J(x_t - p)\rangle \\ &\quad + k\langle Tx_t - p, J(x_t - p)\rangle - t\langle p, J(x_t - p)\rangle \\ &\le \|(1-k-t)(x_t - p) + k(Tx_t - p)\|\|x_t - p\| - t\langle p, J(x_t - p)\rangle \\ &\le (1-t)\|x_t - p\|^2 - t\langle p, J(x_t - p)\rangle, \end{aligned}$$

which implies that $\|x_t - p\|^2 \le \langle p, J(p - x_t)\rangle$. In particular,

$$\|x_n - p\|^2 \le \langle p, J(p - x_n)\rangle, \quad \forall p \in F(T). \tag{3.8}$$

Since $\{x_n\}$ is bounded we may assume, without loss of generality, that $\{x_n\}$ converges weakly to a point $x^* \in C$. From (3.5) and Lemma 2.4, we have that $x^* \in F(T)$. Hence it follows from (3.8) that

$$\|x_n - x^*\|^2 \le \langle x^*, J(x^* - x_n)\rangle.$$

Since J is weak sequentially continuous and $x_n \rightharpoonup x^*$, we have that $x_n \to x^*$. So we prove that the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

To show that the entire net $\{x_t\}$ converges to x^*, assume $x_{t_m} \to \bar{x} \in F(T)$, where $t_m \to 0$. Put $x_m = x_{t_m}$. Similarly, we obtain

$$\|x_m - x^*\|^2 \le \langle x^*, J(x^* - x_m)\rangle$$

and hence

$$\|\bar{x} - x^*\|^2 \le \langle x^*, J(x^* - \bar{x})\rangle. \tag{3.9}$$

Interchanging x^* and $\bar{x}$, we have

$$\|x^* - \bar{x}\|^2 \le \langle \bar{x}, J(\bar{x} - x^*)\rangle. \tag{3.10}$$

Adding (3.9) and (3.10), we obtain

$$2\|x^* - \bar{x}\|^2 \le \|x^* - \bar{x}\|^2,$$

which implies that $\bar{x} = x^*$.

Finally, we return to (3.8) and take the limit as $n \to \infty$ to get

$$\|x^* - p\|^2 \le \langle p, J(p - x^*)\rangle, \quad \forall p \in F(T).$$

Equivalently,

$$\|x^*\|^2 \le \langle x^*, J(p)\rangle, \quad \forall p \in F(T).$$

This clearly implies that

$$\|x^*\| \le \|p\| \quad \forall p \in F(T).$$

Therefore, x^* is a minimum-norm fixed point of T. This completes the proof. □

Corollary 3.2 *Suppose that $F(T) \neq \emptyset$ and the origin 0 belongs to C. Then, as $t \to 0_+$, the net $\{x_t\}$ generated by the algorithm*

$$x_t = (1-k-t)x_t + kTx_t$$

converges strongly to the minimum-norm fixed point of T.

Now we propose the following iterative algorithm which is the discretisation of the implicit method (3.3).

Theorem 3.3 *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be also a sunny nonexpansive retraction of E, $T_i : C \to C$ be λ_i-strictly pseudo-contractive mapping such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\} > 0$. Assume for each n, $\{\eta_i^n\}_{i=1}^{\infty}$ be an infinity sequence of positive number such that $\sum_{i=1}^{\infty} \eta_i^n = 1$ and for all n, $\eta_i^n > 0$. For given $x_1 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = P_C((1-k-\alpha_n)x_n + k\sum_{i=1}^{\infty} \eta_i^n T_i x_n), \quad \forall n \ge 1, \tag{3.11}$$

where $\{\alpha_n\}$ is a real sequence in $(0; 1)$ and $k \in (0, \frac{2\lambda}{C_2})$. The following control conditions are satisfied:

(A1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(A2) $\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} |\eta_i^{n+1} - \eta_i^n| < \infty$, $\eta_i = \lim_{n\to\infty} \eta_i^n > 0$.

Then the sequence $\{x_n\}$ generated by (3.11) strongly converges to the minimum-norm fixed point $x^* \in F$.

Proof For each $n \ge 1$, put $B_n = \sum_{i=1}^{\infty} \eta_i^n T_i$. By Lemma 2.4, each B_n is a λ-strict pseudo-contraction on C and $F(B_n) = F$ for all n.

First, we show that the sequence $\{x_n\}$ is bounded. Take $p \in F$, it follows from (3.11) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|P_C((1-k-\alpha_n)x_n + kB_n x_n) - p\| \\ &\le \|(1-k-\alpha_n)(x_n - p) + k(B_n x_n - p)\| + \alpha_n \|p\|. \end{aligned} \tag{3.12}$$

From (3.2), we note that

$$\|(1-k-\alpha_n)(x_n - p) + k(B_n x_n - p)\| \le (1-\alpha_n)\|x_n - p\|. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\begin{aligned} \|x_{n+1} - p\| &\le (1-\alpha_n)\|x_n - p\| + \alpha_n \|p\| \\ &\le \max\{\|x_n - p\|, \|p\|\} \\ &\le \max\{\|x_1 - p\|, \|p\|\}. \end{aligned}$$

Hence, $\{x_n\}$ is bounded and so is $\{B_n x_n\}$.

We now estimate $\|x_{n+1} - x_n\|$. From (3.11),

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|P_C((1-k-\alpha_n)x_n + kB_nx_n) - P_C((1-k-\alpha_{n-1})x_{n-1} + kB_{n-1}x_{n-1})\| \\
&\le \|(1-k-\alpha_n)(x_n - x_{n-1}) + k(B_nx_n - B_nx_{n-1}) + k(B_nx_{n-1} - B_{n-1}x_{n-1}) \\
&\quad + (\alpha_{n-1} - \alpha_n)x_{n-1}\| \\
&\le \|(1-k-\alpha_n)(x_n - x_{n-1}) + k(B_nx_n - B_nx_{n-1})\| + k\|B_nx_{n-1} - B_{n-1}x_{n-1}\| \\
&\quad + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + k\|B_nx_{n-1} - B_{n-1}x_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + k\sum_{i=1}^{\infty} |\eta_i^n - \eta_i^{n-1}|\|T_ix_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|x_{n-1}\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + M\left[\sum_{i=1}^{\infty} |\eta_i^n - \eta_i^{n-1}| + |\alpha_{n-1} - \alpha_n|\right],
\end{aligned}$$

where $M = \max\{\sup_{i\ge1}\sup_{n\ge1}\|T_ix_{n-1}\|, \sup_{n\ge1}\|x_{n-1}\|\}$. By Lemma 2.3, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

On the other hand, we note that

$$\begin{aligned}
\|x_n - B_nx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - B_nx_n\| \\
&\le \|x_n - x_{n+1}\| + (1-k)\|x_n - B_nx_n\| + \alpha_n\|x_n\|,
\end{aligned}$$

which implies

$$\|x_n - B_nx_n\| \le \frac{1}{k}(\|x_n - x_{n+1}\| + \alpha_n\|x_n\|).$$

Noticing conditions (A1) and (3.14), we have

$$\lim_{n\to\infty} \|x_n - B_nx_n\| = 0. \tag{3.15}$$

Define $B = \sum_{i=1}^{\infty} \eta_iT_i$, then $B : C \to C$ is a λ-strict pseudocontraction such that $F(B) = \bigcap_{i=1}^{\infty} F(T_i) = F$ by Lemma 2.4, furthermore $B_nx \to Bx$ as $n \to \infty$ for all $x \in C$. We observe that

$$\|x_n - Bx_n\| \le \|x_n - B_nx_n\| + \|B_nx_n - Bx_n\|.$$

By (3.15) and (A2), we obtain

$$\lim_{n\to\infty} \|x_n - Bx_n\| = 0.$$

Let the net $\{x_t\}$ be defined by (3.3). By Theorem 3.1, $x_t \to x^*$ as $t \to 0$. Next we prove that $\limsup_{n\to\infty}\langle x^*, J(x^* - x_n)\rangle \le 0$.

Set $y_t = (1-k-t)x_t + kBx_t$. It follows that

$$\begin{aligned}
\|x_t - x_n\|^2 &= \langle x_t - y_t, J(x_t - x_n)\rangle + \langle y_t - x_n, J(x_t - x_n)\rangle \\
&\le \langle y_t - x_n, J(x_t - x_n)\rangle \\
&= \langle (1-k-t)(x_t - x_n) + k(Bx_t - Bx_n), J(x_t - x_n)\rangle \\
&\quad + k\langle Bx_n - x_n, J(x_t - x_n)\rangle - t\langle x_n, J(x_t - x_n)\rangle
\end{aligned}$$

$$\begin{aligned}
&\le (1-t)\|x_t - x_n\|^2 + k\|Bx_n - x_n\|\|x_t - x_n\| \\
&\quad - t\langle x_n - x_t, J(x_t - x_n)\rangle - t\langle x_t, J(x_t - x_n)\rangle \\
&= \|x_t - x_n\|^2 + k\|Bx_n - x_n\|\|x_t - x_n\| - t\langle x_t, J(x_t - x_n)\rangle,
\end{aligned}$$

and hence that

$$\langle x_t, J(x_t - x_n)\rangle \le \frac{k}{t}\|Bx_n - x_n\|\|x_t - x_n\|.$$

Therefore,

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t, J(x_t - x_n)\rangle \le 0. \tag{3.16}$$

From $x_t \to x^*$ as $t \to 0$, we have $x_t - x_n \to x^* - x_n$ as $t \to 0$. Noticing that J is single valued and norm to norm uniformly continuous on bounded sets of a uniformly smooth Banach space E, we obtain

$$\begin{aligned}
&|\langle x^*, J(x^* - x_n)\rangle - \langle x_t, J(x_t - x_n)\rangle| \\
&\quad = |\langle x^*, J(x^* - x_n) - J(x_t - x_n)\rangle + \langle x^* - x_t, J(x_t - x_n)\rangle| \\
&\quad \le |\langle x^*, J(x^* - x_n) - J(x_t - x_n)\rangle| + \|x^* - x_t\|\|x_t - x_n\| \to 0 \quad as \quad t \to 0.
\end{aligned}$$

Hence, $\forall\epsilon > 0$; $\exists\, \delta > 0$ such that $\forall t \in (0; \delta)$, for all $n \ge 1$, we have

$$\langle x^*, J(x^* - x_n)\rangle \le \langle x_t, J(x_t - x_n)\rangle + \epsilon.$$

By (3.16), we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\langle x^*, J(x^* - x_n)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle x^*, J(x^* - x_n)\rangle \\
&\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t, J(x_t - x_n)\rangle + \epsilon \\
&\le \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we have

$$\limsup_{n\to\infty}\langle x^*, J(x^* - x_n)\rangle \le 0. \tag{3.17}$$

Finally, we show that $x_n \to x^*$. Set $y_n = (1-k-\alpha_n)x_n + kB_n x_n$ for all $n \ge 1$. From (3.11), we observe

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle x_{n+1} - y_n, J(x_{n+1} - x^*)\rangle + \langle y_n - x^*, J(x_{n+1} - x^*)\rangle \\
&\le \langle y_n - x^*, J(x_{n+1} - x^*)\rangle \\
&= \langle (1-k-\alpha_n)(x_n - x^*) + k(B_n x_n - x^*), J(x_{n+1} - x^*)\rangle \\
&\quad + \alpha_n\langle x^*, J(x^* - x_{n+1})\rangle \\
&\le \|(1-k-\alpha_n)(x_n - x^*) + k(B_n x_n - x^*)\|\|x_{n+1} - x^*\| \\
&\quad + \alpha_n\langle x^*, J(x^* - x_{n+1})\rangle \\
&\le (1-\alpha_n)\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle x^*, J(x^* - x_{n+1})\rangle \\
&\le \frac{1-\alpha_n}{2}(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n\langle x^*, J(x^* - x_{n+1})\rangle,
\end{aligned}$$

which implies

$$\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + \frac{2\alpha_n}{1+\alpha_n}\langle x^*, J(x^* - x_{n+1})\rangle. \tag{3.18}$$

Apply Lemma 2.3 to (3.18), we obtain $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Corollary 3.4 *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be also a sunny nonexpansive retraction of E. Let T_i, λ, k and η_i^n be as in Theorem 3.3. Suppose that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and the origin 0 belongs to C. Assume that the conditions (A1) and (A2) are satisfied. Then the sequence $\{x_n\}$ generated by the algorithm*

$$x_{n+1} = (1 - k - \alpha_n)x_n + k \sum_{i=1}^{\infty} \eta_i^n T_i x_n, \quad \forall n \geq 1,$$

converges strongly to the minimum-norm fixed point x^* of F.

Corollary 3.5 *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be also a sunny nonexpansive retraction of E. Let $T : C \to C$ be a λ-strict pseudo-contraction. Suppose that $F = F(T) \neq \emptyset$. Assume that the condition (A1) is satisfied. Then the sequence $\{x_n\}$ generated by the algorithm*

$$x_{n+1} = (1 - k - \alpha_n)x_n + kTx_n, \quad \forall n \geq 1,$$

converges strongly to the minimum-norm fixed point x^* of F.

Corollary 3.6 *Let C be a nonempty closed convex subset of a Hilbert space H, $T_i : C \to C$ be λ_i-strictly pseudo-contractive mapping such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\lambda = \inf\{\lambda_i : i \in \mathbb{N}\} > 0$. Assume for each n, $\{\eta_i^n\}_{i=1}^{\infty}$ be an infinity sequence of positive number such that $\sum_{i=1}^{\infty} \eta_i^n = 1$ and for all n, $\eta_i^n > 0$. For given $x_1 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = P_C\left((1 - k - \alpha_n)x_n + k \sum_{i=1}^{\infty} \eta_i^n T_i x_n\right), \quad \forall n \geq 1, \tag{3.19}$$

where $\{\alpha_n\}$ is a real sequence in $(0; 1)$ and $k \in (0; 1 - \lambda)$. The following control conditions are satisfied:

(A1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(A2) $\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} |\eta_i^{n+1} - \eta_i^n| < \infty$, $\eta_i = \lim_{n\to\infty} \eta_i^n > 0$.

Then the sequence $\{x_n\}$ generated by (3.19) strongly converges to the minimum-norm fixed point $x^* \in F$.

Remark 3.7 Our results improve and extend the results of Yao et al. [5] in the following aspects:

(i) Hilbert space is replaced by a 2-uniformly smooth Banach space E which admits a weakly sequentially continuous duality mapping;
(ii) Theorem 3.3 extends Theorem 3.3 of Yao et al. [5] from one strictly pseudo-contractive mapping to an infinite family of strictly pseudo-contractive mappings.

Acknowledgments This work was supported by the National Natural Science Foundation of China (11131006, 41390450, 91330204, 11401293), the National Basic Research Program of China (2013CB 329404).

References

1. Gossez, J.P., Dozo, E.L.: Some geomethic properties related to the fixed point theory for nonexpansive mappings. Pac. J. Math. **40**, 565–573 (1972)
2. Bruck, R.E.: Nonexpansive projections on subsets of Banach spaces. Pac. J. Math. **47**, 341–355 (1973)
3. Reich, S.: Asymptotic behavior of contractions in Banach spaces. J. Math. Anal. Appl. **44**, 57–70 (1973)
4. Goebel, K., Reich, S.: Uniform convexity. In: Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker, New York (1984)
5. Yao, Y.H., Liou, Y.C., Marino, G.: Strong convergence of some algorithms for λ-strict pseudo-contractions in Hilbert space. Bull. Aust. Math. Soc. **85**, 232–240 (2012). doi:10.1017/S000497271100311X
6. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**(12), 1127–1138 (1991)
7. Liu, L.S.: Iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**, 114–125 (1995)
8. Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Sov. **66**, 240–256 (2002)
9. Zhou, H.: Convergence theorems for λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces. Nonlinear Anal. **69**, 3160–3173 (2008)
10. Song Y.L.: Strong convergence theorems of a new general iterative process with Meir–Keeler contractions for a countable family of λ_i-strict pseudocontractions in q-uniformly smooth banach spaces. Fixed Point Theory Appl. Vol. Article ID 354202, p. 19 (2010). doi:10.1155/2010/354202