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Iterative methods for convex proximal split feasibility problems and fixed point problems

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Abstract In this paper we prove strong convergence result for a problem of finding a point which minimizes a proper convex lower-semicontinuous function f which is also a fixed point of a total asymptotically strict pseudocontractive mapping such that its image under a bounded linear operator A minimizes another proper convex lower-semicontinuous function g in real Hilbert spaces. In our result in this work, our iterative scheme is proposed with a way of selecting the step-size such that its implementation does not need any prior information about the operator norm $||A||$ because the calculation or at least an estimate of the operator norm $||A||$ is very difficult, if it is not an impossible task. Our result complements many recent and important results in this direction.

Keywords Proximal split feasibility problems · Moreau-Yosida approximate · Total asymptotically strict pseudocontractive mapping · Strong convergence · Hilbert spaces

Mathematics Subject Classification 49J53 · 65K10 · 49M37 · 90C25

1 Introduction

In this paper, we shall assume that H is a real Hilbert space with inner product $\langle ., . \rangle$ and norm $||.||$. Let I denote the identity operator on H. Now, let us recall the definitions of some operators that will be used in this paper.

Let $T : H \to H$ be a mapping. A point $x \in H$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$. A point $x^* \in H$ is called a *minimum norm* fixed point of T if and only if $x^* \in F(T)$ and $||x^*|| = \min\{||x|| : x \in F(T)\}$.

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Definition 1.1 The mapping $T : H \to H$ is said to be

(a) *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \forall x, y \in H.$$

(b) *asymptotically nonexpansive* mapping if there exists a sequence $\{\mu_n\}$ of real positive numbers such that $\lim \mu_n = 0$ and

$$||T^n x - T^n y|| \leq (1 + \mu_n)||x - y||, \forall x, y \in H, \quad \forall n \geq 1.$$

(c) *k-strictly pseudocontractive* (see, [2]) if for $0 \leq k < 1$,

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \quad \forall \ x, y \in H. \tag{1.1}$$

(d) *asymptotically k-strict pseudo-contraction mapping in the intermediate sense* if there exist a constant $k \in [0, 1)$ and sequences $\{\mu_n\} \subset [0, \infty), \{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$ such that for all $n \geq 1$,

$$||T^n x - T^n y||^2 \leq (1 + \mu_n)||x - y||^2 + k||(x - y) - (Tx - Ty)||^2 + \xi_n, \forall x, y \in H.$$

(e) $(k, \{\mu_n\}, \{\xi_n\}, \phi)$*-total asymptotically strict pseudocontractive mapping* [6], if there exist a constant $k \in [0, 1)$ and sequences $\{\mu_n\} \subset [0, \infty), \{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, and a continuous and strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for all $n \geq 1$,

$$\begin{aligned} ||T^n x - T^n y||^2 \leq\ & ||x - y||^2 + k||(x - y) - (Tx - Ty)||^2 \\ & + \mu_n \phi(||x - y||) + \xi_n, \forall x, y \in H. \end{aligned}$$

For an example of a total asymptotically strict pseudocontractive mapping, we refer the reader to Chang *et al.* [6].

In this paper, we shall assume that H is a real Hilbert space with inner product $\langle ., . \rangle$ and norm $||.||$. Let I denote the identity operator on H. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (SFP) is to find a point

$$x \in C \ \text{ such that } \ Ax \in Q, \tag{1.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [5] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, for example, [4,12,15,17,19,21,23,26] and references therein).

Note that the split feasibility problem (1.2) can be formulated as a fixed-point equation by using the fact

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*, \tag{1.3}$$

where P_C is the metric projection from H_1 onto C [defined as $||x - P_C x|| = \min_{y \in C} ||x - y||$]. Hence, x^* solves the SFP (1.2) if and only if x^* solves the fixed point equation (1.3) (see [18] for the details). This implies that we can use fixed-point algorithms (see [24,25,27]) to solve SFP. A popular algorithm that solves the SFP (1.2) is due to Byrne's CQ algorithm [3] which is found to be a gradient-projection method (GPM) in convex minimization. Subsequently, Byrne [4] applied Krasnoselskii-Mann iteration to the CQ algorithm, and Zhao and Yang

[28] applied Krasnoselskii-Mann iteration to the perturbed CQ algorithm to solve the SFP. It is well known that the CQ algorithm and the Krasnoselskii-Mann algorithm for a split feasibility problem do not necessarily converge strongly in the infinite-dimensional Hilbert spaces.

Let $h : H \to \mathbb{R} \cup \{+\infty\}$ be a functional on a real Hilbert space H.

Definition 1.2 1. The *Moreau-Yosida approximate* of the function h of parameter $\lambda > 0$ is defined as $h_\lambda(y) := \min_{u\in H}\{h(u) + \frac{1}{2\lambda}||u - y||^2\}$.

2. $\operatorname{argmin} f := \{\bar{x} \in H : f(\bar{x}) \leq f(x), \forall x \in H\}$.

3. The *proximal mapping* of f is defined as $\operatorname{prox}_{\lambda f}(y) = \operatorname{argmin}_{u\in H}\{f(u) + \frac{1}{2\lambda}||u - y||^2\}$.

4. *The subdifferential* of f at x is the set

$$\partial f(x) := \{u \in H : f(y) \geq f(x) + \langle u, y - x\rangle, \forall y \in H\}.$$

Let us consider the following problem: find a solution $x^* \in H_1$ such that

$$\min_{x\in H_1}\{f(x) + g_\lambda(Ax)\}, \tag{1.4}$$

where H_1, H_2 are two real Hilbert spaces, $f : H_1 \to \mathbb{R} \cup \{+\infty\}$, $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ two proper, convex, lower semicontinuous functions and $A : H_1 \to H_2$ a bounded linear operator, $g_\lambda(y) = \min_{u\in H_2}\{g(u) + \frac{1}{2\lambda}||u - y||^2\}$ stands for the Moreau-Yosida approximate of the function g of parameter λ.

Observe that by taking $f = \delta_C$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise], $g = \delta_Q$ the indicator functions of two nonempty, closed and convex sets C, Q of H_1 and H_2 respectively, Problem (1.4) reduces to

$$\min_{x\in H_1}\{\delta_C(x) + (\delta_Q)_\lambda(Ax)\} \Leftrightarrow \min_{x\in C}\left\{\frac{1}{2\lambda}||(I - P_Q)(Ax)||^2\right\} \tag{1.5}$$

which, when $C \cap A^{-1}(Q) \neq \emptyset$, is equivalent to (1.2).

By the differentiability of the Moreau-Yosida approximate g_λ, see for instance [16], we have the additivity of the subdifferentials and thus we can write

$$\partial(f(x) + g_\lambda(Ax)) = \partial f(x) + A^*\nabla g_\lambda(Ax) = \partial f(x) + A^*\Big(\frac{I - prox_{\lambda g}}{\lambda}\Big)(Ax).$$

This implies that the optimality condition of (1.4) can be then written as

$$0 \in \lambda\partial f(x) + A^*(I - prox_{\lambda g})(Ax). \tag{1.6}$$

Inclusion (1.6) in turn yields to the following equivalent fixed point formulation

$$\operatorname{prox}_{\mu\lambda f}\left(x^* - \mu A^*(I - \operatorname{prox}_{\lambda g})Ax^*\right) = x^*. \tag{1.7}$$

To solve (1.4), relation (1.7) suggests to consider the following split proximal algorithm

$$x_{n+1} = \operatorname{prox}_{\mu_n\lambda f}\left(x_n - \mu_n A^*(I - \operatorname{prox}_{\lambda g})Ax_n\right). \tag{1.8}$$

Based on an idea introduced in Lopez et al. [11], Moudafi and Thakur [13] recently proved weak convergence results for solving (1.4) in the case $\operatorname{argmin} f \cap A^{-1}(\operatorname{argmin} g) \neq \emptyset$, or in other words: in finding a minimizer x^* of f such that Ax^* minimizes g, namely

$$x^* \in \operatorname{argmin} f \;\text{ such that }\; Ax^* \in \operatorname{argmin}\; g, \tag{1.9}$$

f, g being two proper, lower semicontinuous convex functions. We will denote the solution set of (1.9) by Γ. Concerning problem (1.9), Moudafi and Thakur [13] introduced a new way of selecting the step-sizes: Set $\theta(x_n) := \sqrt{||\nabla h(x_n)||^2 + ||\nabla l(x_n)||^2}$ with $h(x_n) = \frac{1}{2}||(I - \text{prox}_{\lambda g})Ax_n||^2, l(x_n) = \frac{1}{2}||(I - \text{prox}_{\lambda\mu_n f})x_n||^2$ and introduced the following split proximal algorithm:

Split Proximal Algorithm: Given an initial point $x_1 \in H_1$. Assume that x_n has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} via the rule

$$x_{n+1} = \text{prox}_{\lambda\mu_n f}(x_n - \mu_n A^*(I - \text{prox}_{\lambda g})Ax_n), \quad n \geq 1, \tag{1.10}$$

where stepsize $\mu_n := \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.4) and the iterative process stops, otherwise, we set $n := n+1$ and go to (1.10).

Using the split proximal algorithm (1.10), Moudafi and Thakur [13] proved the following *weak convergence* theorem for approximating a solution of (1.9).

Theorem 1.3 *Assume that f and g are two proper convex lower-semicontinuous functions and that* (1.9) *is consistent (i.e., $\Gamma \neq \emptyset$). If the parameters satisfy the following conditions $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n)+l(x_n)} - \epsilon$ (for some $\epsilon > 0$ small enough), then the sequence $\{x_n\}$ generated by* (1.10) *weakly converges to a solution of* (1.9).

We remark here that it is quite usual to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. For instance, given a nonempty, closed and convex subset C of a Hilbert space H_1 and a bounded linear operator $A : H_1 \to H_2$, where H_2 is another Hilbert space. The C-constrained pseudoinverse of A, $A_C^{\dagger}$ is then defined as the minimum-norm solution of the constrained minimization problem

$$A_C^{\dagger}(b) := \text{argmin}_{x\in C}||Ax - b||$$

which is equivalent to the fixed point problem

$$x = P_C(x - \lambda A^*(Ax - b)),$$

where P_C is the metric projection from H_1 onto C, A^* is the adjoint of A, $\lambda > 0$ is a constant, and $b \in H_2$ is such that $P_{\overline{A(C)}}(b) \in A(C)$. It is therefore our aim in this paper to introduce an iterative algorithm that can generate sequence which converges strongly to the minimum-norm solution of a given convex proximal split feasibility problem and fixed point problems for total asymptotically strict pseudocontractive mapping in real Hilbert spaces.

2 Preliminaries

Definition 2.1 A mapping $T : H \to H$ is said to be *uniformly L-Lipschitzian continuous* if there exists a constant $L > 0$ such that $||T^n x - T^n y|| \leq L||x - y||, \forall x, y \in H, n \geq 1$.

Lemma 2.2 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and let $T : K \to K$ be a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping such that $F(T) \neq \emptyset$. Suppose there exist constants $M_0 > 0, K_1 > 0$ such that $\phi(\lambda) \leq M_0\lambda^2, \forall\lambda > K_1$. Then $F(T)$ is closed and convex.*

Proof Since T is a uniformly L-Lipschitzian continuous, $F(T)$ is closed. Next, we show that $F(T)$ is convex. Next, we show that $F(T)$ is convex. For $t \in [0, 1]$ and $x, y \in F(T)$, put

$z := tx + (1-t)y$, we show that $z = Tz$. Since ϕ is continuous, it follows that ϕ attains maximum (say M) in $[0, K_1]$ and by our assumption, $\phi(t) \leq M_0 t^2, \forall t > K_1$. In either case, we have that

$$\phi(t) \leq M + M_0 t^2, \forall t \in [0, \infty).$$

Using Lemma 2.3 (iii), we obtain

$$\begin{aligned}
||T^n z - z||^2 &= ||t(T^n z - x) + (1-t)(T^n z - y)||^2 \\
&= t||T^n z - x||^2 + (1-t)||T^n z - y||^2 - t(1-t)||x-y||^2 \\
&\leq t\Big(||z-x||^2 + k||T^n z - z||^2 + \mu_n \phi(||z-x||) + \xi_n\Big) \\
&\quad + (1-t)\Big(||z-y||^2 + k||T^n z - z||^2 + \mu_n \phi(||z-y||) + \xi_n\Big) \\
&\quad - t(1-t)||x-y||^2 \\
&\leq t\Big(||z-x||^2 + k||T^n z - z||^2 + \mu_n (M + M_0||z-x||^2) + \xi_n\Big) \\
&\quad + (1-t)\Big(||z-y||^2 + k||T^n z - z||^2 + \mu_n (M + M_0||z-y||^2) + \xi_n\Big) \\
&\quad - t(1-t)||x-y||^2 \\
&= t\Big((1-t)^2||x-y||^2 + k||T^n z - z||^2 + \mu_n (M + M_0||z-x||^2) + \xi_n\Big)(1-t) \\
&\quad \times \Big(t^2||x-y||^2 + k||T^n z - z||^2 + \mu_n (M + M_0||z-y||^2) + \xi_n\Big) \\
&\quad - t(1-t)||x-y||^2 \\
&= k||T^n z - z||^2 + \mu_n (M + M_0||z-y||^2) + \xi_n. \qquad (2.1)
\end{aligned}$$

This implies from (2.1) that

$$(1-k)||T^n z - z||^2 \leq \mu_n (M + M_0||z-y||^2) + \xi_n.$$

Thus, $\lim_{n\to\infty} ||T^n z - z|| = 0$, which implies that $\lim_{n\to\infty} T^n z = z$. By continuity of T, we obtain that

$$z = \lim_{n\to\infty} T^n z = \lim_{n\to\infty} T(T^{n-1} z) = T(\lim_{n\to\infty} T^{n-1} z) = Tz.$$

Hence, $z \in F(T)$, that $F(T)$ is convex. □

We state the following well-known lemmas which will be used in the sequel.

Lemma 2.3 *Let H be a real Hilbert space. Then there holds the following well-known results:*

$$(i)\ \ ||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \forall x, y \in H.$$

$$(ii)\ \ ||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle, \forall x, y \in H.$$

$$(iii)\ \ ||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2, \forall x, y \in H, \forall t \in [0, 1].$$

Lemma 2.4 (Chang et al. [6]) *Let $T : H \to H$ be a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping, then $I - T$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x \in H$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.5 (Alber et al. [1]) *Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be nonnegative, $\{\alpha_n\}$ be positive real numbers such that*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \lambda_n + \gamma_n, n \geq 1.$$

Let for all $n > 1$*,*

$$\frac{\gamma_n}{\alpha_n} \leq c_1 \text{ and } \alpha_n \leq \alpha.$$

Then $\lambda_n \leq \max\{\lambda_1, K_*\}$*, where* $K_* = (1+\alpha)c_1$.

Lemma 2.6 (Xu, [20]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 1,$$

where

(i) $\{a_n\} \subset [0, 1], \quad \sum \alpha_n = \infty$;

(ii) $\limsup \sigma_n \leq 0$*;*

(iii) $\gamma_n \geq 0; \quad (n \geq 1), \quad \Sigma \gamma_n < \infty$.

Then, $a_n \to 0$ *as* $n \to \infty$.

Now, our interest is in studying the convergence properties of the following algorithm: Given an initial point $x_1 \in H_1$, then compute u_n using $u_n = (1-\alpha_n)x_n$. Set $\theta(u_n) := \sqrt{||\nabla h(u_n)||^2 + ||\nabla l(u_n)||^2}$ with $h(u_n) = \frac{1}{2}||(I - \text{prox}_{\lambda g})Au_n||^2, l(u_n) = \frac{1}{2}||(I - \text{prox}_{\lambda \tau_n f})u_n||^2$ and introduce the following algorithm:

Algorithm: Let $T : H_1 \to H_1$ is a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping such that $F(T) \neq \emptyset$. Given an initial point $x_1 \in H_1$, the compute u_n using $u_n = (1-\alpha_n)x_n$ and $\theta(u_n) \neq 0$, then compute x_{n+1} via the rule

$$\begin{cases} u_n = (1-\alpha_n)x_n, \\ y_n = \text{prox}_{\lambda \tau_n f}(u_n - \tau_n A^*(I - \text{prox}_{\lambda g})Au_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T^n y_n, \quad n \geq 1, \end{cases} \tag{2.2}$$

where stepsize $\tau_n := \rho_n \frac{h(u_n)+l(u_n)}{\theta^2(u_n)}$ with $0 < \rho_n < 4$. If $\theta(u_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.9) which is also a fixed point of a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping T and the iterative process stops, otherwise, we set $n := n+1$ and go to (2.2).

3 Main results

Theorem 3.1 *Let* $T : H_1 \to H_1$ *is a uniformly* L*-Lipschitzian continuous and* $(k, \{\mu_n\}, \{\xi_n\}, \phi)$*-total asymptotically strict pseudocontractive mapping such that* $F(T) \neq \emptyset$*. Assume that* $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ *is a proper convex lower-semicontinuous function,* $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ *is a proper convex lower-semicontinuous function such that* (1.9) *is consistent (i.e.,* $\Gamma \neq \emptyset$*) and* $\Gamma \cap F(T) \neq \emptyset$*. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $(0, 1)$*. If the parameters satisfy the following conditions* *(a)* $\sum_{n=1}^{\infty} \mu_n < \infty$; $\sum_{n=1}^{\infty} \xi_n < \infty$;

(b) $\mu_n = o(\alpha_n)$; $\xi_n = o(\alpha_n)$; $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\epsilon \leq \rho_n \leq \frac{4h(u_n)}{h(u_n)+l(u_n)} - \epsilon$ *for some* $\epsilon > 0$*;*

(d) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1 - k$*;*

(e) *there exist constants* $M_0 > 0$, $K_1 > 0$ *such that* $\phi(t) \leq M_0 t^2, \forall t > K_1$*;*

then sequence $\{x_n\}$ *generated by* (2.2) *converges strongly to* $x^* \in \Gamma \cap F(T)$ *which is also the minimum-norm solution (i.e.,* $x^* \in \Gamma \cap F(T)$ *and* $||x^*|| = \min\{||x|| : x \in \Gamma \cap F(T)\}$*).*

Proof Let $x^* \in \Gamma$. Observe that $\nabla h(u_n) = A^*(I - \text{prox}_{\lambda g})Au_n$, $\nabla l(u_n) = (I - \text{prox}_{\tau_n \lambda f})x$. Using the fact that $\text{prox}_{\tau_n \lambda f}$ is nonexpansive, x^* verifies (1.9) (since minimizers of any function are exactly fixed-points of its proximal mapping) and having in hand

$$\langle \nabla h(u_n), u_n - x^* \rangle = \langle (I - \text{prox}_{\lambda g})Au_n, Au_n - Ax^* \rangle \geq ||(I - \text{prox}_{\lambda g})Au_n||^2 = 2h(u_n),$$

thanks to the fact that $I - \text{prox}_{\lambda g}$ is firmly nonexpansive, we can write

$$\begin{aligned}
||y_n - x^*||^2 &= ||u_n - x^*||^2 + \tau_n^2 ||\nabla h(u_n)||^2 - 2\tau_n \langle \nabla h(u_n), u_n - x^* \rangle \\
&\leq ||u_n - x^*||^2 + \tau_n^2 ||\nabla h(u_n)||^2 - 4\tau_n h(u_n) \\
&= ||u_n - x^*||^2 + \rho_n^2 \frac{(h(u_n) + l(u_n))^2}{(\theta^2(u_n))^2} ||\nabla h(u_n)||^2 - 4\rho_n \frac{h(u_n) + l(u_n)}{\theta^2(u_n)} h(u_n) \\
&\leq ||u_n - x^*||^2 + \rho_n^2 \frac{(h(u_n) + l(u_n))^2}{\theta^2(u_n)} - 4\rho_n \frac{(h(u_n) + l(u_n))^2}{\theta^2(u_n)} \frac{h(u_n)}{h(u_n) + l(u_n)} \\
&= ||u_n - x^*||^2 - \rho_n \left(\frac{4h(u_n)}{h(u_n) + l(u_n)} - \rho_n \right) \frac{(h(u_n) + l(u_n))^2}{\theta^2(u_n)}. \qquad (3.1)
\end{aligned}$$

Using (3.1) in (2.2), we obtain that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||(1 - \beta_n)y_n + \beta_n T^n y_n - x^*||^2 \\
&= ||(1 - \beta_n)(y_n - x^*) + \beta_n (T^n y_n - x^*)||^2 \\
&= (1 - \beta_n)||y_n - x^*||^2 + \beta_n ||T^n y_n - x^*||^2 - \beta_n (1 - \beta_n)||T^n y_n - y_n||^2 \\
&\leq (1 - \beta_n)||y_n - x^*||^2 + \beta_n \Big[||y_n - x^*||^2 + k||T^n y_n - y_n||^2 \\
&\quad + \mu_n \phi(||y_n - x^*||) + \xi_n \Big] - \beta_n (1 - \beta_n)||T^n y_n - y_n||^2 \\
&= ||y_n - x^*||^2 + \beta_n (\beta_n - (1 - k))||T^n y_n - y_n||^2 + \beta_n \mu_n \phi(||y_n - x^*||) \\
&\quad + \beta_n \xi_n. \qquad (3.2)
\end{aligned}$$

Since ϕ is continuous, it follows that ϕ attains maximum (say M) in $[0, K_1]$ and by our assumption, $\phi(t) \leq M_0 t^2, \forall t > K_1$. In either case, we have that

$$\phi(t) \leq M + M_0 t^2, \forall t \in [0, \infty). \qquad (3.3)$$

Using (3.2) and (3.3), we have that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||y_n - x^*||^2 + \beta_n(\beta_n - (1-k))||T^n y_n - y_n||^2 + \beta_n\mu_n M_0||y_n - x^*||^2 \\
&\quad + \beta_n\xi_n + \beta_n\mu_n M_1 \\
&\leq ||y_n - x^*||^2 + \beta_n\mu_n M_0||y_n - x^*||^2 + \beta_n\xi_n + \beta_n\mu_n M_1 \\
&\leq ||u_n - x^*||^2 + \beta_n\mu_n M_0||u_n - x^*||^2 + \beta_n\xi_n + \beta_n\mu_n M_1 \\
&\leq \Big[(1-\alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2\Big] \\
&\quad + \beta_n\mu_n M_0\Big[(1-\alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2\Big] + \beta_n\xi_n + \beta_n\mu_n M_1 \\
&\leq \Big[(1-\alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2\Big] + \beta_n\mu_n M_0||x_n - x^*||^2 \\
&\quad + \beta_n\mu_n M_0||x^*||^2 + \beta_n\xi_n + \beta_n\mu_n M_1 \\
&= ||x_n - x^*||^2 - \Big[\alpha_n - \beta_n\mu_n M_0\Big]||x_n - x^*||^2 + \sigma_n, \qquad (3.4)
\end{aligned}$$

where $\sigma_n = \alpha_n||x^*||^2 + \beta_n\mu_n M_0||x^*||^2 + \beta_n\xi_n + \beta_n\mu_n M_1, \ \ \forall n \geq 1$. From (3.4), we have

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - \Big[\alpha_n - \mu_n M_0\Big]||x_n - x^*||^2 + \sigma_n. \qquad (3.5)$$

Since $\mu_n = o(\alpha_n), \ \ \lambda_n = o(\alpha_n)$ and $\xi_n = o(\alpha_n)$, we may assume without loss of generality that there exist constants $k_0 \in (0, 1)$ and $M_2 > 0$ such that for all $n \geq 1$,

$$\frac{\mu_n}{\alpha_n} M_0 \leq 1 - k_0, \ \ \text{and} \ \ \frac{\sigma_n}{\alpha_n} \leq M_2.$$

Thus, we obtain from (3.5) that

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - \alpha_n k_0||x_n - x^*||^2 + \sigma_n.$$

By Lemma 2.5, we have that

$$||x_n - x^*||^2 \leq \max\{||x_1 - x^*||^2, (1 + k_0)M_2\}.$$

Therefore, $\{x_n\}$ is bounded. Furthermore, the sequences $\{y_n\}$ and $\{u_n\}$ are bounded.

Observe that since T is a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive, then

$$\begin{aligned}
&||T^n x - x^*||^2 \leq ||x - x^*||^2 + k||x - T^n x||^2 + \mu_n\phi(||x - x^*||) + \xi_n \\
&\quad \Rightarrow \langle T^n x - x^*, T^n x - x^*\rangle \leq \langle x - x^*, x - T^n x\rangle + \langle x - x^*, T^n x - x^*\rangle + k||x - T^n x||^2 \\
&\qquad + \mu_n\phi(||x - x^*||) + \xi_n \\
&\quad \Rightarrow \langle T^n x - x^*, T^n x - x\rangle \leq \langle x - x^*, x - T^n x\rangle + k||x - T^n x||^2 + \mu_n\phi(||x - x^*||) + \xi_n \\
&\quad \Rightarrow \langle T^n x - x, T^n x - x\rangle + \langle x - x^*, T^n x - x\rangle \leq \langle x - x^*, x - T^n x\rangle + k||x - T^n x||^2 \\
&\qquad + \mu_n\phi(||x - x^*||) + \xi_n \\
&\quad \Rightarrow (1-k)||x - T^n x||^2 \leq 2\langle x - x^*, x - T^n x\rangle + \mu_n\phi(||x - x^*||) + \xi_n. \qquad (3.6)
\end{aligned}$$

It follows from (2.2) and (3.6) that

$$\begin{aligned}
||x_{n+1}-x^*||^2 &= ||(1-\beta_n)y_n+\beta_n T^n y_n - x^*||^2\\
&= ||(y_n-x^*)+\beta_n(Ty_n-y_n)||^2\\
&= ||y_n-x^*||^2+\beta_n^2||y_n-T^ny_n||^2-2\beta_n\langle y_n-x^*, y_n-Ty_n\rangle\\
&\le ||y_n-x^*||^2+\beta_n\Big[\beta_n-(1-k))||y_n-T^ny_n||^2+\mu_n\phi(||y_n-x^*||)+\xi_n\Big]\\
&\le ||u_n-x^*||^2+\beta_n(\beta_n-(1-k))||y_n-T^ny_n||^2+\beta_n\Big[\mu_n\phi(||y_n-x^*||)+\xi_n\Big]\\
&= ||(1-\alpha_n)x_n-x^*||^2+\beta_n(\beta_n-(1-k))||y_n-T^ny_n||^2\\
&\quad+\beta_n\Big[\mu_n\phi(||y_n-x^*||)+\xi_n\Big]\\
&= (1-\alpha_n)^2||x_n-x^*||^2+\alpha_n^2||x^*||^2-\alpha_n(1-\alpha_n)\langle x_n-x^*, x^*\rangle\\
&\quad+\beta_n(\beta_n-(1-k))||y_n-T^ny_n||^2+\beta_n\Big[\mu_n\phi(||y_n-x^*||)+\xi_n\Big]\\
&\le ||x_n-x^*||^2+\alpha_n^2||x^*||^2-\alpha_n(1-\alpha_n)\langle x_n-x^*, x^*\rangle\\
&\quad+\beta_n(\beta_n-(1-k))||y_n-T^ny_n||^2+\beta_n\Big[\mu_n\phi(||y_n-x^*||)+\xi_n\Big]\\
&\le ||x_n-x^*||^2+\alpha_n^2||x^*||^2-\alpha_n(1-\alpha_n)\langle x_n-x^*, x^*\rangle\\
&\quad+\beta_n(\beta_n-(1-k))||y_n-T^ny_n||^2+\beta_n\Big[\mu_n(M_1+M_0||y_n-x^*||^2)+\xi_n\Big].
\end{aligned} \tag{3.7}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, $\exists M>0$ such that

$$-(1-\alpha_n)\langle x_n-x^*, x^*\rangle \le M, \quad \forall n\ge 1.$$

Therefore,

$$\begin{aligned}
\beta_n((1-k)-\beta_n)||y_n-T^ny_n||^2 \le \alpha_n^2||x^*||^2+\alpha_n M-||x_{n+1}-x^*||^2+||x_n-x^*||^2\\
+\mu_n(M_1+M_0||y_n-x^*||^2)+\xi_n.
\end{aligned} \tag{3.8}$$

Now we divide the rest of the proof into two cases.

Case1
Assume that $\{||x_n-x^*||\}$ is monotonically decreasing sequence. Then $\{||x_n-x^*||\}$ is convergent, obviously

$$||x_{n+1}-x^*||^2-||x_n-x^*||^2 \to 0, n\to\infty. \tag{3.9}$$

This together with (3.8) and the conditions that $\alpha_n\to 0$, $\mu_n\to 0$ and $\xi_n\to 0$ imply that

$$||y_n-T^ny_n||\to 0, n\to\infty.$$

From (3.1) and (2.2), we have that

$$\begin{aligned}
&\rho_n\Big(\frac{4h(u_n)}{h(u_n)+l(u_n)}-\rho_n\Big)\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)} \le (1+M^*\lambda_n)||u_n-x^*||^2-||y_n-x^*||^2\\
&\le (1-\alpha_n)||x_n-x^*||^2+\alpha_n||x^*||^2-||x_{n+1}-x^*||^2+M^*\lambda_n||u_n-x^*||^2\\
&\le ||x_n-x^*||^2-||x_{n+1}-x^*||^2+\alpha_n||x^*||^2+M^*\lambda_n||u_n-x^*||^2.
\end{aligned}$$

Using conditions that $\alpha_n \to 0$ and $\lambda_n \to 0$, we have that

$$\rho_n\Big(\frac{4h(u_n)}{h(u_n)+l(u_n)} - \rho_n\Big)\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)} \to 0, \quad n \to \infty.$$

Hence, we obtain

$$\frac{(h(u_n)+l(u_n))^2}{\theta^2(u_n)} \to 0, \quad n \to \infty. \tag{3.10}$$

Consequently, we have

$$\lim_{n\to\infty}(h(u_n)+l(u_n)) = 0 \Leftrightarrow \lim_{n\to\infty} h(u_n) = 0 \text{ and } \lim_{n\to\infty} l(u_n) = 0,$$

because $\theta^2(u_n) = ||\nabla h(u_n)||^2 + ||\nabla l(u_n)||^2$ is bounded. This follows from the fact that ∇h is Lipschitz continuous with constant $||A||^2$, ∇l is nonexpansive and $\{u_n\}$ is bounded. More precisely, for any x^* which solves (1.9), we have

$$||\nabla h(u_n)|| = ||\nabla h(u_n) - \nabla x^*|| \le ||A||^2||u_n - x^*|| \text{ and}$$
$$||\nabla l(u_n)|| = ||\nabla l(u_n) - \nabla x^*|| \le ||u_n - x^*||.$$

Now, let z be a weak cluster point of $\{u_n\}$, there exists a subsequence $\{u_{n_j}\}$ which weakly converges to z. The lower-semicontinuity of h then implies that

$$0 \le h(z) \le \liminf_{j\to\infty} h(u_{n_j}) = \lim_{n\to\infty} h(u_n) = 0.$$

That is, $h(z) = \frac{1}{2}||(I - \text{prox}_{\lambda g})Az|| = 0$, i.e., Az is a fixed point of the proximal mapping of g or equivalently, $0 \in \partial_{pg}(Az)$. In other words, Az is a minimizer of g.

Likewise, the lower-semicontinuity of l implies that

$$0 \le l(z) \le \liminf_{j\to\infty} l(u_{n_j}) = \lim_{n\to\infty} l(u_n) = 0.$$

That is, $l(z) = \frac{1}{2}||(I - \text{prox}_{\tau\lambda f})z|| = 0$, i.e., z is a fixed point of the proximal mapping of f or equivalently, $0 \in \partial f(z)$. In other words, z is a minimizer of f. Hence, $z \in \Gamma$.

Next, we show that $z \in F(T)$. Since $x^* = \text{prox}_{\lambda\tau_n f}(x^* - \mu_n A^*(I - \text{prox}_{\lambda g})Ax^*)$ and $A^*(I - \text{prox}_{\lambda g})A$ is Lipschitz with constant $||A||^2$, we have from (2.2) that

$$\begin{aligned}
&||y_n - x^*||^2\\
&= ||\text{prox}_{\lambda\tau_n f}(u_n - \mu_n A^*(I - \text{prox}_{\lambda g})Au_n) - \text{prox}_{\lambda\tau_n f}\left(x^* - \tau_n A^*(I - \text{prox}_{\lambda g})Ax^*\right)||^2\\
&\le \langle (u_n - \tau_n A^*(I - \text{prox}_{\lambda g})Au_n) - \left(x^* - \tau_n A^*(I - \text{prox}_{\lambda g})Ax^*\right), y_n - x^*\rangle\\
&= \frac{1}{2}\Big[||(u_n - \tau_n A^*(I - \text{prox}_{\lambda g})Au_n) - \left(x^* - \tau_n A^*(I - \text{prox}_{\lambda g})Ax^*\right)||^2 + ||y_n - x^*||^2\\
&\quad - ||(u_n - \tau_n A^*(I - \text{prox}_{\lambda g})Au_n) - \left(x^* - \tau_n A^*(I - \text{prox}_{\lambda g})Ax^*\right) - (y_n - x^*)||^2\Big]\\
&\le \frac{1}{2}\Big[(1 + \tau_n||A||^2)^2||u_n - x^*||^2 + ||y_n - x^*||^2 - ||u_n - y_n - \tau_n(A^*(I - \text{prox}_{\lambda g})Au_n\\
&\quad - A^*(I - \text{prox}_{\lambda g})Ax^*)||^2\Big]\\
&= \frac{1}{2}\Big[(1 + \tau_n||A||^2)^2||u_n - x^*||^2 + ||y_n - x^*||^2\\
&\quad - ||u_n - y_n||^2 + 2\tau_n\langle u_n - y_n, A^*(I - \text{prox}_{\lambda g})Au_n - A^*(I - \text{prox}_{\lambda g})Ax^*)\rangle\\
&\quad - \tau_n^2||A^*(I - \text{prox}_{\lambda g})Au_n - A^*(I - \text{prox}_{\lambda g})Ax^*)||^2\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}||y_n - x^*||^2 &\le (1+\tau_n||A||^2)^2||x_n - x^*||^2 - ||u_n - y_n||^2\\ &\quad + 2\tau_n\langle u_n - y_n, A^*(I-\text{prox}_{\lambda g})Au_n - A^*(I-\text{prox}_{\lambda g})Ax^*\rangle\\ &\quad - \tau_n^2||A^*(I-\text{prox}_{\lambda g})Au_n - A^*(I-\text{prox}_{\lambda g})Ax^*)||^2. \qquad (3.11)\end{aligned}$$

We observe that

$$0 < \tau_n < 4\frac{h(u_n)+l(u_n)}{\theta^2(u_n)} \to 0, \quad n \to \infty$$

implies that $\tau_n \to 0, \quad n \to \infty$. Furthermore, we obtain from (3.11) and (2.2) that

$$\begin{aligned}||u_n - y_n||^2 &\le (1+\tau_n||A||^2)^2||u_n - x^*||^2 - ||y_n - x^*||^2\\ &\quad + 2\tau_n\langle u_n - y_n, A^*(I-\text{prox}_{\lambda g})Au_n - A^*(I-\text{prox}_{\lambda g})Ax^*)\rangle\\ &= ||u_n - x^*||^2 + \tau_n||A||^2(2+\tau_n||A||^2)||u_n - x^*||^2 - ||y_n - x^*||^2\\ &\quad + 2\tau_n\langle u_n - y_n, A^*(I-\text{prox}_{\lambda g})Au_n - A^*(I-\text{prox}_{\lambda g})Ax^*)\rangle\\ &\le (1-\alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2 - ||x_{n+1} - x^*||^2\\ &\quad + \tau_n||A||^2(2+\tau_n||A||^2)||u_n - x^*||^2 + 2\tau_n\langle u_n - y_n, A^*(I-\text{prox}_{\lambda g})Au_n\\ &\quad - A^*(I-\text{prox}_{\lambda g})Ax^*)\rangle\\ &\le ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + \alpha_n||x^*||^2 + \tau_n||A||^2(2+\tau_n||A||^2)||u_n - x^*||^2\\ &\quad + 2\tau_n\langle u_n - y_n, A^*(I-\text{prox}_{\lambda g})Au_n - A^*(I-\text{prox}_{\lambda g})Ax^*)\rangle. \qquad (3.12)\end{aligned}$$

Since $\tau_n \to 0, \quad n \to \infty$ and $\alpha_n \to 0, \quad n \to \infty$, we obtain that

$$\lim_{n\to\infty} ||u_n - y_n|| = 0.$$

We observe that $||u_n - x_n|| \le \alpha_n||x_n|| \to 0, \quad n \to \infty$ and

$$||x_n - y_n|| \le ||u_n - x_n|| + ||u_n - y_n|| \to 0, \quad n \to \infty.$$

Using Lemma 2.3 (i), we have that

$$\begin{aligned}||x_{n+1} - x_n||^2 &= ||x_n - y_n + y_n - x_{n+1}||^2\\ &\le ||y_n - x_{n+1}||^2 + 2\langle x_n - y_n, x_n - x_{n+1}\rangle\\ &\le ||y_n - x_{n+1}||^2 + 2||x_n - y_n||||x_n - x_{n+1}||. \qquad (3.13)\end{aligned}$$

Using (2.2) and (3.13), we have

$$\begin{aligned}||y_n - T^n y_n||^2 &= \frac{1}{\beta_n^2}||y_n - x_{n+1}||^2\\ &\ge \frac{1}{\beta_n^2}\Big[||y_n - x_{n+1}||^2 + 2||x_n - y_n||||x_n - x_{n+1}||\Big]. \qquad (3.14)\end{aligned}$$

Since $\lim_{n\to\infty} ||y_n - T^n y_n|| = 0$ and $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1-k$, then we have that

$$\lim_{n\to\infty}\Big[||y_n - x_{n+1}||^2 + 2||x_n - y_n||||x_n - x_{n+1}||\Big] = 0,$$

from which we have

$$\lim_{n\to\infty} ||x_n - x_{n+1}|| = 0.$$

Consequently,

$$\begin{aligned}||u_{n+1}-u_n|| &= ||(1-\alpha_{n+1})x_{n+1}-(1-\alpha_n)x_n||\\ &\leq |\alpha_{n+1}-\alpha_n|||x_{n+1}||+(1-\alpha_n)||x_{n+1}-x_n|| \to 0, \quad n\to\infty. \end{aligned} \tag{3.15}$$

Now,

$$||y_{n+1}-y_n|| \leq ||y_{n+1}-u_{n+1}||+||u_n-y_n||+||u_{n+1}-u_n||. \tag{3.16}$$

Using (3.15) and the fact that $\lim_{n\to\infty}||u_n-y_n||=0$ in (3.16), we obtain

$$\lim_{n\to\infty}||y_{n+1}-y_n||=0.$$

Using the fact that T is uniformly L-Lipschitzian, we have

$$\begin{aligned}||Ty_n-y_n|| &\leq ||Ty_n-T^{n+1}y_n||+||T^{n+1}y_n-T^{n+1}y_{n+1}||\\ &\quad +||T^{n+1}y_{n+1}-y_{n+1}||+||y_{n+1}-y_n||\\ &\leq L||y_n-T^ny_n||+(L+1)||y_{n+1}-y_n||+||T^{n+1}y_{n+1}-y_{n+1}||\\ &\leq L||y_n-T^ny_n||+(L+1)||A||||y_{n+1}-y_n||+||T^{n+1}y_{n+1}-y_{n+1}||.\end{aligned} \tag{3.17}$$

By using $\lim_{n\to\infty}||y_{n+1}-y_n||=0$ and $\lim_{n\to\infty}||y_n-T^ny_n||=0$ in (3.17), we obtain

$$||y_n-Ty_n|| \to 0, n\to\infty. \tag{3.18}$$

Using the fact that $u_{n_j} \rightharpoonup z \in H_1$ and $||u_n-y_n|| \to 0, \ n\to\infty$, we have that $y_{n_j} \rightharpoonup z \in H_1$. Similarly, $u_{n_j} \rightharpoonup z \in H_1$ since $||u_n-x_n|| \to 0, \ n\to\infty$. Using Lemma 2.4 and (3.18), we have that $z\in F(T)$. Therefore, $z\in\Gamma\cap F(T)$.

From (2.2), we have

$$\begin{aligned}||x_{n+1}-x^*||^2 &= ||(1-\beta_n)y_n+\beta_nT^ny_n-x^*||^2\\ &= ||(1-\beta_n)(y_n-x^*)+\beta_n(T^ny_n-x^*)||^2\\ &= (1-\beta_n)^2||y_n-x^*||^2+\beta_n^2||T^ny_n-x^*||^2\\ &\quad +2\beta_n(1-\beta_n)\langle y_n-x^*, T^ny_n-x^*\rangle\\ &\leq (1-\beta_n)^2||y_n-x^*||^2+\beta_n^2[||y_n-x^*||^2+k||y_n-T^ny_n||^2]\\ &\quad +\beta_n^2\mu_n\phi(||y_n-x^*||)+\beta_n^2\xi_n+2\beta_n(1-\beta_n)[||y_n-x^*||^2\\ &\quad -\frac{1-k}{2}||y_n-T^ny_n||^2]\\ &= (1-2\beta_n+\beta_n^2)||y_n-x^*||^2+\beta_n^2[||y_n-x^*||^2+k||y_n-T^ny_n||^2]\\ &\quad +2\beta_n||y_n-x^*||^2-2\beta_n^2||y_n-x^*||^2-\beta_n(1-\beta_n)(1-k)||y_n-T^ny_n||^2\\ &\quad +\beta_n^2\mu_n(M+M_0||y_n-x^*||^2)+\beta_n^2\xi_n\\ &= ||y_n-x^*||^2+[\beta_n^2k-\beta_n(1-\beta_n)(1-k)]||y_n-T^ny_n||^2\\ &\quad +\beta_n^2\mu_n(M+M_0||y_n-x^*||^2)+\beta_n^2\xi_n\\ &= ||y_n-x^*||^2+\beta_n[k+\beta_n-1]||y_n-T^ny_n||^2\\ &\leq ||y_n-x^*||^2+\beta_n^2\mu_n(M+M_0||y_n-x^*||^2)+\beta_n^2\xi_n\\ &= ||y_n-x^*||^2+a_n,\end{aligned} \tag{3.19}$$

where $a_n=\beta_n^2\mu_n(M+M_0||y_n-x^*||^2)+\beta_n^2\xi_n$.

Now, from (3.1), (3.19) and Lemma 2.3 (ii), we have

$$\begin{aligned} ||x_{n+1}-x^*||^2 &\le ||y_n-x^*||^2+a_n \le ||u_n-x^*||^2+a_n \\ &= ||(1-\alpha_n)(x_n-x^*)-\alpha_n x^*||^2+a_n \\ &\le (1-\alpha_n)||x_n-x^*||^2-2\alpha_n\langle u_n-x^*, x^*\rangle + a_n. \end{aligned} \tag{3.20}$$

It is clear that

$$\limsup_{n\to\infty} -2\langle u_n-x^*, x^*\rangle = \lim_{j\to\infty} -2\langle u_{n_j}-x^*, x^*\rangle - 2\langle z-x^*, x^*\rangle \le 0,$$

since $\{u_n\}$ converges weakly to z and x^* is the minimum-norm solution (i.e., $x^* = P_{\Gamma\cap F(T)}0$). Also, we observe that $\sum_{n=1}^{\infty} a_n < \infty$. Now, using Lemma 2.6 in (3.20), we have $||x_n - x^*|| \to 0$. That is, $x_n \to x^*, n \to \infty$.

Case 2

Assume that $\{||x_n - x^*||\}$ is not monotonically decreasing sequence. Set $\Gamma_n = ||x_n - x^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough)by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k < \Gamma_{k+1}\}$$

Clearly, τ is a non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \ge 0, \quad \forall n \ge n_0.$$

From (3.18), it is easy to see that

$$\lim_{n\to\infty} ||y_{\tau(n)} - Ty_{\tau(n)}|| = 0.$$

Furthermore, we can show that

$$\lim_{n\to\infty} h(x_{\tau(n)}) = 0 \text{ and } \lim_{n\to\infty} l(x_{\tau(n)}) = 0.$$

By similar argument as above in Case 1, we conclude immediately that $x_{\tau(n)}, y_{\tau(n)}$ and $u_{\tau(n)}$ weakly converge to z as $\tau(n) \to \infty$. At the same time, from (3.20), we note that, for all $n \ge n_0$,

$$\begin{aligned} 0 &\le ||x_{\tau(n)+1} - x^*||^2 - ||x_{\tau(n)} - x^*||^2 \\ &\le \alpha_{\tau(n)}[-2\langle u_{\tau(n)} - x^*, x^*\rangle - ||x_{\tau(n)} - x^*||^2] + a_{\tau(n)}, \end{aligned}$$

which implies

$$||x_{\tau(n)} - x^*||^2 \le -2\langle u_{\tau(n)} - x^*, x^*\rangle + a_{\tau(n)}.$$

Hence, we deduce that

$$\lim_{n\to\infty} ||x_{\tau(n)} - x^*|| = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \ge n_0$, it is easy to see that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ if $n \ne \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j \ge \Gamma_{j+1}$ for $\tau(n) + 1 \le j \le n$. As a consequence, we obtain for all $n \ge n_0$,

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim \Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to x^*. This completes the proof. □

4 Applications

Applying Theorem 3.1 to the case where $f = \delta_C$, $g = \delta_Q$ the indicator functions of two nonempty, closed and convex sets C, Q of H_1 and H_2 respectively, we have the following result.

Theorem 4.1 *Let C, Q be nonempty, closed and convex subset of real Hilbert spaces H_1 and H_2 respectively. Let $T : C \to C$ is a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping such that $F(T) \neq \emptyset$. Assume that problem (1.2) is consistent (i.e., $\Gamma \neq \emptyset$) and $\Gamma \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$. If the parameters satisfy the following conditions*

(a) $\sum_{n=1}^{\infty} \mu_n < \infty$; $\sum_{n=1}^{\infty} \xi_n < \infty$;

(b) $\mu_n = o(\alpha_n)$; $\xi_n = o(\alpha_n)$; $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\epsilon \leq \rho_n \leq \frac{4h(u_n)}{h(u_n)+l(u_n)} - \epsilon$ *for some* $\epsilon > 0$;

(d) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1 - k$;

(e) there exist constants $M_0 > 0$, $K_1 > 0$ *such that* $\phi(t) \leq M_0 t^2, \forall t > K_1$;

then sequence $\{x_n\}$ generated by

$$\begin{cases} u_n = (1 - \alpha_n)x_n, \\ y_n = P_C(u_n - \tau_n A^*(I - P_Q)Au_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n T^n y_n, \quad n \geq 1, \end{cases} \tag{4.1}$$

converges strongly to $x^* \in \Gamma \cap F(T)$ which is also the minimum-norm solution (i.e., $x^* \in \Gamma \cap F(T)$ and $||x^*|| = \min\{||x|| : x \in \Gamma \cap F(T)\}$).

Proof Take $f = \delta_C$ and $g = \delta_Q$ in Theorem 3.1. Then we have $\text{prox}_{\lambda f} = P_C$ and $\text{prox}_{\lambda g} = P_Q$. Furthermore, we see that algorithm reduces to (2.2) reduces to (4.1) and the conclusion of Theorem 4.1. □

We next apply our results to split equilibrium problem and fixed point problem. Let C, Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let f be a bifunction of $C \times C$ into $\mathbb{R}$ and g a bifunction of $Q \times Q$ into $\mathbb{R}$. Suppose $A : H_1 \to H_2$ is a bounded linear operator. Let us consider the following split equilibrium problem. The split equilibrium problem is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C \tag{4.2}$$

and

$$g(Ax^*, y) \geq 0, \quad \forall y \in Q. \tag{4.3}$$

We shall denote the solutions set of (4.2)-(4.3) by Γ. In order to solve the split equilibrium problem for a bifunctions $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$, let us assume that f and g satisfy the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$ and $g(x, x) = 0$ for all $x \in Q$;
(A2) f and g are monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y, \in C$ and $g(x, y) + g(y, x) \leq 0$ for all $x, y, \in Q$;

(A3) for each $x, y \in C$, $\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x, y)$ and for each $x, y \in Q$, $\lim_{t\to 0} g(tz + (1-t)x, y) \leq g(x, y)$;
(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous and for each $x \in Q$, $y \mapsto g(x, y)$ is convex and lower semicontinuous.

The following lemma follows from [8].

Lemma 4.2 (Combettes and Hirstoaga, [8]) *Let C, Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Assume that f, g satisfy (A1)-(A4). For $r > 0$, define mappings $T_r^f : H_1 \to C$ and $T_r^g : H_2 \to Q$ as follows:*

$$T_r^f(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

and

$$T_r^g(x) = \{z \in Q : g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}.$$

Then, the following assertions:

1. *T_r^f and T_r^g are single-valued;*
2. *T_r^f and T_r^g are firmly nonexpansive-type mapping, i.e., for any $x, y \in H_1$,*
$$||T_r^f x - T_r^f y||^2 \leq \langle T_r^f x - T_r^f y, x - y\rangle;$$
3. *$F(T_r^f) = EP(f)$ and $F(T_r^g) = EP(g)$;*
4. *$EP(f)$ and $EP(g)$ are closed and convex.*

Now, we prove the following theorem.

Theorem 4.3 *Let $T : C \to C$ is a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping such that $F(T) \neq \emptyset$. Let f be a bifunction from $C \times C$ and g a bifunction from $Q \times Q$ both satisfying $(A1) - (A4)$ such that $\Gamma \neq \emptyset$) and $\Gamma \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$. If the parameters satisfy the following conditions*

(a) $\sum_{n=1}^{\infty} \mu_n < \infty; \sum_{n=1}^{\infty} \xi_n < \infty;$

(b) $\mu_n = o(\alpha_n); \xi_n = o(\alpha_n); \lim_{n\to\infty} \alpha_n = 0; \sum_{n=1}^{\infty} \alpha_n = \infty;$

(c) $\epsilon \leq \rho_n \leq \frac{4h(u_n)}{h(u_n)+l(u_n)} - \epsilon$ *for some* $\epsilon > 0$*;*

(d) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1 - k$*;*

(e) there exist constants $M_0 > 0$, $K_1 > 0$ *such that* $\phi(t) \leq M_0 t^2, \forall t > K_1$*;*

then sequence $\{x_n\}$ generated by $x_1 \in C$;

$$\begin{cases} u_n = (1-\alpha_n)x_n, \\ y_n = T_r^f(u_n - \tau_n A^*(I - T_r^g)Au_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T^n y_n, \quad n \geq 1, \end{cases} \tag{4.4}$$

converges strongly to $x^* \in \Gamma \cap F(T)$ which is also the minimum-norm solution (i.e., $x^* \in \Gamma \cap F(T)$ and $||x^*|| = \min\{||x|| : x \in \Gamma \cap F(T)\}$).

Proof Replace the proximal mappings of the convex functions f and g in Theorem 3.1 by the resolvent operators associated to the two monotone equilibrium bifunctions T_r^f and T_r^g. Hence, we have the desired result. □

Remark 1 The following are our contributions in this paper.

1. We obtain strong convergence result concerning convex split feasibility problem and fixed point problem in real Hilbert spaces. We recall that Moudafi and Thakur [13] obtained weak convergence result for split feasibility problem alone and thus our result improves on and extends the results of Moudafi and Thakur [13].
2. It is worth mentioning here that our result in this paper is more applicable than the result of Moudafi and Thakur [13] in the sense that our result can be applied to finding an approximate common solution to proximal split feasibility problem and fixed point problem for a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping.
3. In all our results in this paper, our iterative scheme is proposed with a way of selecting the step-size such that the implementation of our algorithm does not need any prior information about the operator norm $||A||$ because the calculation or at least an estimate of the operator norm $||A||$ is very difficult, if not an impossible task. Therefore, improved on the results of Chang et al. [6], Yang et al. [22], Cholamjiak and Shehu [7] and other related works.
4. Our iterative algorithm in this paper appears more efficient and implementable. Our algorithm appears simpler than the "CQ" algorithm used [10] and other related papers for similar problems. Furthermore, our iterative scheme gives strong convergence without imposing any extra compactness type condition (like semi-compactness) on the mapping T. This compactness condition *appears strong* as only few mappings are semi-compact. Therefore, we improved on the results of Chang et al. [6], Qin et al. [14], Ding and Quan [9] and other related results.

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