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On the metric dimension of two families of convex polytopes

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Abstract A distance between two vertices of a connected graph is the shortest distance between them. The metric dimension of a connected graph G is the minimum cardinality of a subset W of vertices of G such that all vertices are uniquely determined by their distances from W. A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G) is finite and does not depend upon the choice of G in $\mathcal{G}$. In this paper we study the metric dimension of two classes of convex polytopes and show that these classes of convex polytopes have constant metric dimension.

1 Introduction and preliminaries

If G is a connected graph, the distance d(x,y) between two vertices $x, y \in V(G)$ is the length of a shortest distance between them. Let $W = \{w_1, w_2, w_3, \ldots, w_k\}$ be an ordered set of vertices of G and let x be a vertex of $G \mid W$. The representation $r(x \mid W)$ of x with respect to W is the k-tuple $(d(x, w_1), d(x, w_2), d(x, w_3), \ldots, d(x, w_k))$. If distinct vertices of G have distinct representations with respect to W then W is named as a resolving set for G. A resolving set containing a minimum number of vertices is called a metric basis for G and the number of vertices in a metric basis is called metric dimension of G denoted by dim(G). The metric dimension problem is NP-complete problem, to determine whether metric dimension is less than a given value. It was first introduced by Slater (1975) and Haray and Melter

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(1976). The concept of resolving set has proved to be useful to a variety of fields such as structure of chemical compounds [3], image processing, robotic navigation, combinatorial research and optimization etc. The join of two graphs G_1 and G_2 is the graph consisting of the union of the graphs G_1 and G_2 and all the edges joining $V(G_1)$ and $V(G_1)$. A fan is defined as $f_n = K_1 + P_n$ for $n \geq 1$ and Jahangir graph J_{2n} obtained from wheel graph W_{2n}. Tomescu and Javaid [8] determined the dimension of J_{2n}, Caceres et al. [10] the dimension of fan f_n and Imran et al. [4] the dimension of harary graphs $H_{5,n}$.

Theorem 1 *([8], [4], [10]) Let K_n be a complete graph, $K_{s,t}$ an n-vertex complete bipartite graph, J_{2n} be a jahangir graph and f_n a fan of order $n \geq 1$, then*

1. $dim(K_n) = n - 1$
2. $dim(K_{s,t}) = n - 2$
3. *For $n \geq 4$, $dim(J_{2n}) = \left\lfloor \frac{2n}{3} \right\rfloor$*
4. *For $n \geq 7$, $dim(f_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$*

The metric dimension of all these graphs depend upon the number of vertices in the graphs. We say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dimension of G is finite and does not depend upon the number of choices of G in $\mathcal{G}$. It is shown in [3] that the dimension of a path is 1, thus paths with n vertices constitute a family of graphs with constant metric dimension. Similarly cycles with $n(\geq 3)$ vertices is also a family of such graphs because their dimension is 2. Imran et al. [4] proved that metric dimension of flower snarks is 3 and dimension of an antiprism A_n is 3. Prisms are the 3-regular trivalent graphs obtained by the cross product of path P_2 and cycle C_n and

$$dim(D_n) = \begin{cases} 2, & \text{if } n \text{ is odd} \\ 3, & \text{if } n \text{ is even} \end{cases}$$

Ali et al. [1] proved that the dimension of extension of a prism is 3. Hence all these classes constitute a family of graphs with constant metric dimension. Also the 4-regular graphs obtained by the cartesian product of two cycles(also called torus grids) are a family of graphs with constant metric dimension as

$$dim(C_m \times C_n) = \begin{cases} 3, & \text{if } m \text{ or } n \text{ is odd} \\ 4, & \text{otherwise} \end{cases}$$

The generalized peterson graph $P(n, 3)$ has constant metric dimension [11]. Thus the problem of determining $dim(G) \leq k$ is an NP-complete problem. In this paper we extend the study of the metric dimension of convex polytopes and present two special types of convex polytopes which have constant metric dimension.

2 The graph of convex polytope R_n

The graph of convex polytope R_n (Fig. 1) consists of *2n* number of 3-sided faces, *n* number of 4-sided faces, *n* number of 6-sided faces and a pair of *n*-sided.

For our purpose we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner cycle, set of vertices $\{b_i : 1 \leq i \leq n\}$ the set of interior vertices, set of vertices $\{c_i : 1 \leq i \leq n\}$ the set of exterior vertices, cycle induced by $\{d_i : 1 \leq i \leq n\}$ the middle cycle and the cycle induced by $\{e_i : 1 \leq i \leq n\}$ the outer cycle.

Theorem 2 *For $n \geq 6$, $dim(R_n) = 3$.*
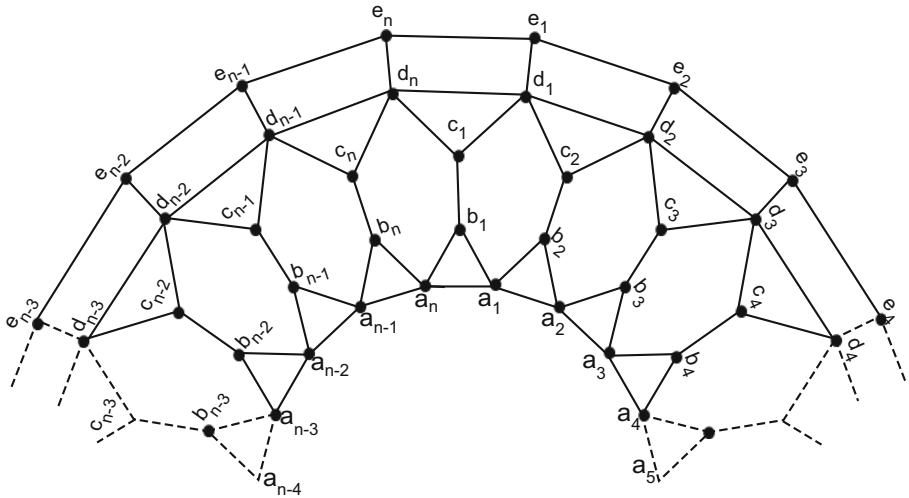
Fig. 1 The graph of convex polytope R_n

Proof We will prove the result by double inequalities. Here we consider two cases:

Case (i) When n is even.

In this case, we can write $n = 2k$, $k \geq 3$, $k \in Z^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(R_n)$, it is to be shown that W is a resolving set for R_n. For this we give representations of any vertex $V(S_n) \mid W$ with respect to W.

Representations of vertices on the inner cycle are:

$$r(a_i \mid W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k \end{cases}$$

Representations of the set of interior vertices are:

$$r(b_i \mid W) = \begin{cases} (1, 1, k), & i = 1 \\ (i, i-1, k-i+1), & 2 \leq i \leq k \\ (2k-i+1, i-1, 1), & i = k+1 \\ (2k-i+1, 2k-i+2, i-k), & k+2 \leq i \leq 2k \end{cases}$$

Representations of the set of exterior vertices are:

$$r(c_i \mid W) = \begin{cases} (2, 2, k+1), & i = 1 \\ (i+1, i, k-i+2), & 2 \leq i \leq k \\ (2k-i+2, i, 2), & i = k+1 \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \leq i \leq 2k \end{cases}$$

Representations of vertices on the middle cycle are:

$$r(d_i \mid W) = \begin{cases} (3, 3, k+1), & i = 1 \\ (i+2, i+1, k-i+2), & 2 \leq i \leq k-1 \\ (2k-i+2, i+1, 3), & i = k, k+1 \\ (2k-i+2, 2k-i+3, i-k+2), & k+2 \leq i \leq 2k-1 \\ (3, 3, k+2), & i = 2k \end{cases}$$

Representations of vertices on the outer cycle are:

$$r\left(e_i \mid W\right) = \begin{cases} (4, 4, k+2), & i = 1 \\ (i+3, i+2, k-i+3), & 2 \le i \le k-1 \\ (2k-i+3, i+2, 4), & i = k, k+1 \\ (2k-i+3, 2k-i+4, i-k+3), & k+2 \le i \le 2k-1 \\ (4, 4, k+3), & i = 2k \end{cases}$$

It is to be noted that there are no two vertices having same representations. This implies that $dim(R_n) \le 3$. We will now prove the opposite inequality by contradiction. Suppose that $dim(R_n) = 2$. Without loss of generality suppose that first resolving vertex is a_1, b_1, c_1, d_1 or e_1 and the other lies in inner cycle, set of interior and exterior vertices, middle and outer cycles then we have following diffrent possibilities:

Resolving vertices	Contradictions
$\{a_1, a_t\}, a_t \ (2 \le t \le n)$	For $2 \le t \le k$, $r(a_n \mid \{a_1, a_t\}) = r(b_n \mid \{a_1, a_t\}) = (1, t)$, for $t = k+1$, $r(a_n \mid \{a_1, a_{k+1}\}) = r(a_2 \mid \{a_1, a_{k+1}\}) = (1, k-1)$ and for $k+2 \le t \le n$, $r(a_2 \mid \{a_1, a_t\}) = r(b_1 \mid \{a_1, a_t\}) = (1, n-t+2)$, a contradiction
$\{b_1, b_t\}, b_t \ (2 \le t \le n)$	For $2 \le t \le k-1$, $r(a_n \mid \{b_1, b_t\}) = r(b_n \mid \{b_1, b_t\}) = (2, t+1)$, for $t = k$, $r(d_n \mid \{b_1, b_k\}) = r(b_n \mid \{b_1, b_k\}) = (2, k+1)$, for $t = k+1$, $r(a_1 \mid \{b_1, b_{k+1}\}) = r(a_2 \mid \{b_1, b_{k+1}\}) = (1, k)$ and for $k+2 \le t \le n$, $r(d_{t-1} \mid \{b_1, b_t\}) = r(b_{t-1} \mid \{b_1, b_t\}) = (n-t+3, 2)$, a contradiction
$\{c_1, c_t\}, c_t \ (2 \le t \le n)$	For $2 \le t \le k+1$, $r(a_t \mid \{c_1, c_t\}) = r(e_{t-1} \mid \{c_1, c_t\}) = (t, 2)$ and for $k+2 \le t \le n$, $r(a_t \mid \{c_1, c_t\}) = r(e_{t-1} \mid \{c_1, c_t\}) = (n-t+3, 2)$, a contradiction
$\{d_1, d_t\}, d_t \ (2 \le t \le n)$	For $2 \le t \le k$, $r(c_1 \mid \{d_1, d_t\}) = r(e_1 \mid \{d_1, d_t\}) = (1, t)$, for $t = k+1$, $r(c_t \mid \{d_1, d_{k+1}\}) = r(c_{t+1} \mid \{d_1, d_{k+1}\}) = (k, 1)$ and for $k+2 \le t \le n$, $r(c_t \mid \{d_1, d_t\}) = r(e_t \mid \{d_1, d_t\}) = (n-t+2, 1)$, a contradiction
$\{e_1, e_t\}, e_t \ (2 \le t \le n)$	For $2 \le t \le k$, $r(e_n \mid \{e_1, e_t\}) = r(d_1 \mid \{e_1, e_t\}) = (1, t)$, for $t = k+1$, $r(c_{k+1} \mid \{e_1, e_{k+1}\}) = r(c_{k+2} \mid \{e_1, e_{k+1}\}) = (k+1, 2)$ and for $k+2 \le t \le n$, $r(c_{k+2} \mid \{e_1, e_t\}) = r(d_{k+1} \mid \{e_1, e_t\}) = (k+1, t-k)$, a contradiction
$\{a_1, b_t\}, b_t \ (2 \le t \le n)$	For $1 \le t \le k-1$, $r(a_n \mid \{a_1, b_t\}) = r(b_n \mid \{a_1, b_t\}) = (1, t+1)$, for $t = k$, $r(a_n \mid \{a_1, b_k\}) = r(b_1 \mid \{a_1, b_k\}) = (1, k)$, for $t = k+1$, $r(a_2 \mid \{a_1, b_{k+1}\}) = r(b_n \mid \{a_1, b_{k+1}\}) = (1, k)$ and for $k+2 \le t \le n$, $r(a_2 \mid \{a_1, b_t\}) = r(b_1 \mid \{a_1, b_t\}) = (1, n-t+2)$, a contradiction
$\{a_1, c_t\}, c_t \ (2 \le t \le n)$	For $1 \le t \le k-1$, $r(a_n \mid \{a_1, c_t\}) = r(b_n \mid \{a_1, c_t\}) = (1, t+2)$, for $t = k$, $r(b_1 \mid \{a_1, c_k\}) = r(a_n \mid \{a_1, c_k\}) = (1, k+1)$, for $t = k+1$, $r(a_2 \mid \{a_1, c_{k+1}\}) = r(b_n \mid \{a_1, c_{k+1}\}) = (1, k+1)$ and for $k+2 \le t \le n$, $r(a_2 \mid \{a_1, b_t\}) = r(b_1 \mid \{a_1, b_t\}) = (1, n-t+3)$, a contradiction
$\{a_1, d_t\}, d_t \ (2 \le t \le n)$	For $t = 1$, $r(a_2 \mid \{a_1, d_1\}) = r(b_n \mid \{a_1, d_1\}) = (1, 3)$, for $2 \le t \le k$, $r(a_2 \mid \{a_1, d_t\}) = r(b_1 \mid \{a_1, d_t\}) = (1, t+1)$, for $k+1 \le t \le n-2$, $r(a_n \mid \{a_1, d_t\}) = r(b_n \mid \{a_1, d_t\}) = (1, n-t+1)$, for $t = n-1$, $r(b_1 \mid \{a_1, d_{n-1}\}) = r(a_n \mid \{a_1, d_{n-1}\}) = (1, 3)$ and for $t = n$, $r(b_1 \mid \{a_1, d_n\}) = r(b_n \mid \{a_1, d_n\}) = (2, 1)$, a contradiction
$\{a_1, e_t\}, e_t \ (1 \le t \le n)$	For $t = 1$, $r(a_2 \mid \{a_1, e_1\}) = r(b_n \mid \{a_1, e_1\}) = (1, 4)$, for $2 \le t \le k$, $r(a_2 \mid \{a_1, e_t\}) = r(b_1 \mid \{a_1, e_t\}) = (1, t+2)$, for $k+1 \le t \le n-2$, $r(a_n \mid \{a_1, e_t\}) = r(b_n \mid \{a_1, e_t\}) = (1, n-t+4)$, for $t = n-1$, $r(b_1 \mid \{a_1, e_{n-1}\}) = r(a_n \mid \{a_1, e_{n-1}\}) = (1, 4)$ and for $t = n$, $r(b_1 \mid \{a_1, e_n\}) = r(b_n \mid \{a_1, e_n\}) = (2, 2)$, a contradiction

Resolving vertices	Contradictions
$\{b_1, c_t\}, c_t \ (1 \le t \le n)$	For $1 \le t \le k - 1$, $r(a_n \mid \{b_1, c_t\}) = r(b_n \mid \{b_1, c_t\}) = (2, t + 1)$, for $t = k$, $r(b_2 \mid \{b_1, c_k\}) = r(d_n \mid \{b_1, c_k\}) = (2, k)$ and for $k + 1 \le t \le n$, $r(a_1 \mid \{b_1, c_t\}) = r(c_1 \mid \{b_1, c_t\}) = (1, n - t + 2)$, a contradiction
$\{b_1, d_t\}, d_t \ (1 \le t \le n)$	For $1 \le t \le k - 1$, $r(c_n \mid \{b_1, d_t\}) = r(e_n \mid \{b_1, d_t\}) = (3, t + 1)$, for $t = k$, $r(e_1 \mid \{b_1, d_k\}) = r(c_n \mid \{b_1, d_k\}) = (3, k)$, for $t = k + 1$, $r(c_{k+2} \mid \{b_1, d_{k+1}\}) = r(d_k \mid \{b_1, d_{k+1}\}) = (k + 1, 1)$ and for $k + 2 \le t \le n$, $r(e_t \mid \{b_1, d_t\}) = r(c_t \mid \{b_1, d_t\}) = (n - t + 3, 1)$, a contradiction
$\{b_1, e_t\}, e_t \ (1 \le t \le n)$	For $t = 1$, $r(a_1 \mid \{b_1, e_1\}) = r(a_2 \mid \{b_1, e_1\}) = (1, 4)$, for $2 \le t \le k$, $r(d_n \mid \{b_1, e_t\}) = r(b_2 \mid \{b_1, e_t\}) = (2, t + 1)$, for $t = k + 1$, $r(a_n \mid \{b_1, e_{k+1}\}) = r(d_1 \mid \{b_1, e_{k+1}\}) = (2, k + 1)$, for $k + 2 \le t \le n - 2$, $r(a_n \mid \{b_1, e_t\}) = r(b_n \mid \{b_1, e_t\}) = (2, n - t + 2)$, for $t = n - 1$, $r(c_n \mid \{b_1, e_{n-1}\}) = r(e_1 \mid \{b_1, e_{n-1}\}) = (3, 2)$ and for $t = n$, $r(a_1 \mid \{b_1, e_n\}) = r(a_2 \mid \{b_1, e_n\}) = (1, 4)$, a contradiction
$\{c_1, d_t\}, d_t \ (1 \le t \le n)$	For $t = 1$, $r(a_1 \mid \{c_1, d_1\}) = r(a_2 \mid \{c_1, d_1\}) = (2, 3)$, for $2 \le t \le k$, $r(c_n \mid \{c_1, d_t\}) = r(e_n \mid \{c_1, d_t\}) = (2, t + 1)$, for $t = k$, $r(c_n \mid \{c_1, d_k\}) = r(e_1 \mid \{c_1, d_k\}) = (2, k)$, for $t = k + 1$, $r(c_2 \mid \{c_1, d_{k+1}\}) = r(e_n \mid \{c_1, d_{k+1}\}) = (2, k)$. For $k + 2 \le t \le n - 1$, $r(a_1 \mid \{c_1, d_t\}) = r(e_1 \mid \{c_1, d_t\}) = (2, n - t + 2)$ and for $t = n$, $r(c_n \mid \{c_1, d_n\}) = r(e_n \mid \{c_1, d_n\}) = (2, 1)$, a contradiction
$\{c_1, e_t\}, e_t \ (1 \le t \le n)$	For $t \in \{1, n\}$, $r(a_1 \mid \{c_1, e_t\}) = r(a_2 \mid \{c_1, e_t\}) = (2, 4)$, for $2 \le t \le k$, $r(c_2 \mid \{c_1, e_t\}) = r(e_n \mid \{c_1, e_t\}) = (2, t)$, for $k + 1 \le t \le n - 2$, $r(a_n \mid \{c_1, e_t\}) = r(b_n \mid \{c_1, e_t\}) = (3, n - t + 2)$ and for $t = n - 1$, $r(c_n \mid \{c_1, e_{n-1}\}) = r(e_1 \mid \{c_1, e_{n-1}\}) = (2, 2)$, a contradiction
$\{d_1, e_t\}, e_t \ (1 \le t \le n)$	For $t = 1$, $r(c_1 \mid \{d_1, e_1\}) = r(c_2 \mid \{d_1, e_1\}) = (1, 2)$, for $2 \le t \le k - 1$, $r(a_1 \mid \{d_1, e_t\}) = r(b_n \mid \{d_1, e_t\}) = (3, t + 3)$, for $t = k$, $r(d_2 \mid \{d_1, e_k\}) = r(e_1 \mid \{d_1, e_k\}) = (1, k - 1)$ and for $k + 1 \le t \le n$, $r(e_1 \mid \{d_1, e_t\}) = r(d_n \mid \{d_1, e_t\}) = (1, n - t + 1)$, a contradiction

From all the possibilities it is shown that there is no resolving set with two vertices which proves that $dim(R_n) \ge 3$. Thus we have $dim(R_n) = 3$ when n is even.

Case (ii) When n is odd.

In this case, we can write $n = 2k + 1$, $k \ge 3, k \in Z^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(R_n)$, we show that W is a resolving set for R_n. For this we give representations of vertices of $V(S_n) \mid W$ with respect to W.

Representations of vertices on the inner cycle are:

$$r(a_i \mid W) = \begin{cases} (i - 1, i - 2, k - i + 1), & 3 \le i \le k \\ (n - i + 1, n - i + 2, i - k - 1), & k + 2 \le i \le 2k + 1 \end{cases}$$

Representations of the set of interior vertices are:

$$r(b_i \mid W) = \begin{cases} (1, 1, k), & i = 1 \\ (i, i - 1, k - i + 1), & 2 \le i \le k \\ (2k - i + 2, i - 1, 1), & i = k + 1 \\ (2k - i + 2, 2k - i + 3, i - k), & k + 2 \le i \le 2k + 1 \end{cases}$$

Representations of the set of exterior vertices are:

$$r(c_i \mid W) = \begin{cases} (2, 2, k+1), & i = 1 \\ (i+1, i, k-i+2), & 2 \le i \le k \\ (2k-i+3, i, 2), & i = k+1 \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \le i \le 2k+1 \end{cases}$$

Representations of vertices on the middle cycle are:

$$r(d_i \mid W) = \begin{cases} (3, 3, k+1), & i = 1 \\ (i+2, i+1, k-i+2), & 2 \le i \le k-1 \\ (2k-i+2, i+1, 3), & i = k \\ (2k-i+3, i+1, 3), & i = k+1 \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \le i \le 2k \\ (3, 3, k+2), & i = 2k+1 \end{cases}$$

Representations of vertices on the outer cycle are:

$$r(e_i \mid W) = \begin{cases} (4, 4, k+2), & i = 1 \\ (i+3, i+2, k-i+3), & 2 \le i \le k-1 \\ (2k-i+3, i+2, 4), & i = k \\ (2k-i+4, i+2, 4), & i = k+1 \\ (2k-i+4, 2k-i+5, i-k+3), & k+2 \le i \le 2k \\ (4, 4, k+3), & i = 2k+1 \end{cases}$$

Again we see that there are no two vertices having same representations implying that $dim(R_n) \le 3$.

Now we are to prove the opposite inequality $dim(R_n) \ge 3$ by contradiction. Therefore we suppose that $dim(R_n) = 2$ then there the same possibilities as discussed in case (i) and contradiction can be obtained similarly. It follows that $dim(R_n) = 3$ which completes the proof. □

3 The graph of convex polytope T_n

The graph of convex polytope T_n; n is even (Fig. 2) consists of $n/2$ number of 3-sided faces, $n/2$ number of 4-sided faces, $2n$ number of 5-sided faces and a pair of n-sided faces and is obtained by the adding the new edges $\{a_{2i}b_{2i+1}; 1 \le i \le n/2-1, a_n b_1\}$ in the graph of convex polytope U_n in [2]. Here $n = 2k$; $k \ge 3$, $k \in Z^+$, n an even integer because parallelograms and triangles occur alternatively. For odd n two parallelograms occur consecutively and we cannot have a convex polytope of this type.

For our purpose we call the cycle induced by $\{a_i : 1 \le i \le n\}$, the inner cycle, cycle induced by $\{b_i : 1 \le i \le n\}$ the interior cycle, cycle induced by $\{c_i : 1 \le i \le n\} \cup \{d_i : 1 \le i \le n\}$ the exterior cycle and the cycle induced by $\{e_i : 1 \le i \le n\}$ the outer cycle.

Theorem 3 *For $n \ge 6$, the metric dimension of convex polytope T_n is exactly 3 i,e. $dim(T_n) = 3$.*

Proof We will prove the result by double inequalities. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(R_n)$, we show that W is a resolving set for T_n. For this we give representations of any vertex $V(T_n) \mid W$ with respect to W.
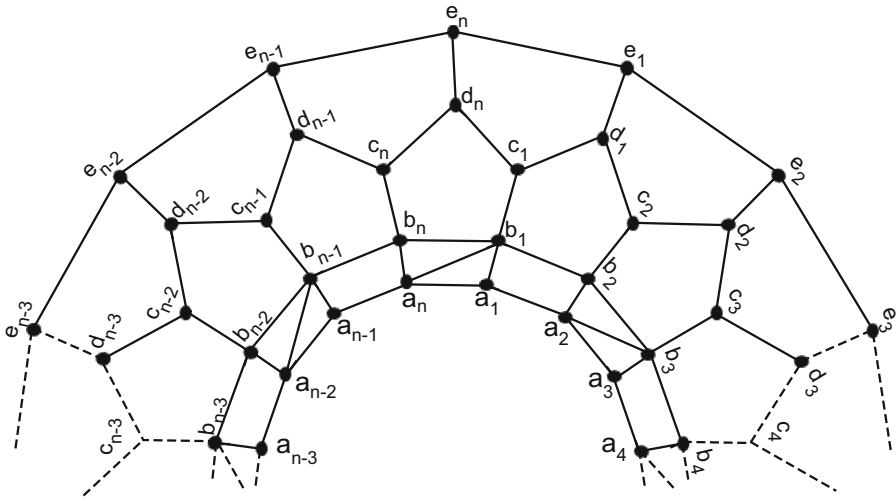
Fig. 2 The graph of convex polytope T_n

Representations of vertices on the inner cycle are:

$$r(a_i \mid W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \le i \le 2k \end{cases}$$

Representations of vertices on the interior cycle are:

$$r(b_i \mid W) = \begin{cases} (1, 2, k+1), & i = 1 \\ (2, 1, k), & i = 2 \\ (i-1, i-2, k-i+2), & 3 \le i \le k+1 \\ (k, k, 2 - k(mod 2)), & i = k+2 \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \le i \le 2k \end{cases}$$

Representations of vertices on exterior cycle are:

$$r(c_i \mid W) = \begin{cases} (2, 3, k+2), & i = 1 \\ (3, 2, k+1), & i = 2 \\ (i, i-1, k-i+3), & 3 \le i \le k+1 \\ (k+1, k+1, 3 - k(mod 2)), & i = k+2 \\ (2k-i+3, 2k-i+4, i-k), & k+3 \le i \le 2k \end{cases}$$

$$r(d_i \mid W) = \begin{cases} (3, 3, k+2), & i = 1 \\ (4, 3, k+1), & i = 2 \\ (i+1, i, k-i+3), & 3 \le i \le k \\ (k+2, k+1, 3), & i = k+1 \\ (k+1, k+2, 4 - k(mod 2)), & i = k+2 \\ (2k-i+3, 2k-i+4, i-k+1), & k+3 \le i \le 2k \end{cases}$$

Representations of vertices on the outer cycle are:

$$
r(e_i \mid W) = \begin{cases}
(4, 4, k + 3), & i = 1 \\
(5, 4, k + 2), & i = 2 \\
(i + 2, i + 1, k - i + 4), & 3 \le i \le k \\
(k + 3, k + 2, 4), & i = k + 1 \\
(k + 2, k + 3, 5 - k(mod 2)), & i = k + 2 \\
(2k - i + 4, 2k - i + 5, i - k + 2), & k + 3 \le i \le 2k
\end{cases}
$$

Note that there are no two vertices having same representations. This implies that $dim(T_n) \le 3$. Now on the other hand we show that $dim(T_n) \ge 3$. On contrary suppose that $dim(T_n) = 2$. We consider two cases:

Case (i) When k is even then without loss of generality we assume that one resolving vertex is a_1, b_1, c_1, d_1 or e_1 and the other lies in inner cycle, interior cycle, exterior and outer cycles then there are following possibilities to be discussed:

Resolving vertices	Contradictions
$\{a_1, a_t\}, a_t (2 \le t \le n)$	For $2 \le t \le k, r(a_n \mid \{a_1, a_t\}) = r(b_1 \mid \{a_1, a_t\}) = (1, t)$ for $t = k + 1$, $r(a_n \mid \{a_1, a_{k+1}\}) = r(b_n \mid \{a_1, a_{k+1}\}) = (1, k - 1)$ and for $k + 2 \le t \le n, r(b_2 \mid \{a_1, a_t\}) = r(c_1 \mid \{a_1, a_t\}) = (2, n - t + 2)$, a contradiction
$\{b_1, b_t\}, b_t (2 \le t \le n)$	for $t = 2, r(a_1 \mid \{b_1, b_2\}) = r(c_1 \mid \{b_1, b_2\}) = (1, 2)$, For $3 \le t \le k$, $r(a_n \mid \{b_1, b_t\}) = r(c_1 \mid \{b_1, b_t\}) = (1, t)$, for $t = k + 1$, $r(a_1 \mid \{b_1, b_t\}) = r(a_n \mid \{b_1, b_t\}) = (1, k - 1)$ and for $k + 2 \le t \le n$, $r(a_1 \mid \{b_1, b_t\}) = r(b_2 \mid \{b_1, b_t\}) = (1, n - t + 2)$
$\{c_1, c_t\}, c_t (2 \le t \le n)$	For $t = 2, r(a_n \mid \{c_1, c_2\}) = r(e_n \mid \{c_1, c_2\}) = (2, 3)$, for $3 \le t \le k + 1$, $r(e_1 \mid \{c_1, c_t\}) = r(a_1 \mid \{c_1, c_t\}) = (2, t)$ and for $k + 2 \le t \le n, r(a_n \mid \{c_1, c_t\}) = r(e_n \mid \{c_1, c_t\}) = (2, n - t + 2)$, a contradiction
$\{d_1, d_t\}, d_t (2 \le t \le n)$	For $2 \le t \le k, r(b_1 \mid \{d_1, d_t\}) = r(e_n \mid \{d_1, d_t\}) = (2, t + 1)$, for $k + 2 \le t \le n - 2, r(b_1 \mid \{d_1, d_t\}) = r(d_n \mid \{d_1, d_t\}) = (2, n - t + 2)$, for $t = n - 1, r(c_1 \mid \{d_1, d_{n-1}\}) = r(e_1 \mid \{d_1, d_{n-1}\}) = (1, 3)$ and for $t = n, r(b_2 \mid \{d_1, d_n\}) = r(e_2 \mid \{d_1, d_n\}) = (2, 3)$, a contradiction
$\{e_1, e_t\}, e_t (2 \le t \le n)$	For $2 \le t \le k$ $r(e_n \mid \{e_1, e_t\}) = r(d_1 \mid \{e_1, e_t\}) = (1, t)$, for $t = k + 1$, $r(e_2 \mid \{e_1, e_{k+1}\}) = r(e_n \mid \{e_1, e_{k+1}\}) = (1, k - 1)$ and for $k + 2 \le t \le n, r(e_2 \mid \{e_1, e_t\}) = r(d_1 \mid \{e_1, e_t\}) = (1, n - t + 2)$, a contradiction
$\{a_1, b_t\}, b_t (1 \le t \le n)$	For $1 \le t \le k, r(c_n \mid \{a_1, b_t\}) = r(d_n \mid \{a_1, b_t\}) = (3, t + 1)$, for $t = k + 1, r(b_1 \mid \{a_1, b_{k+1}\}) = r(a_n \mid \{a_1, b_{k+1}\}) = (1, k)$ and for $k + 2 \le t \le n, r(a_n \mid \{a_1, b_t\}) = r(b_1 \mid \{a_1, b_t\}) = (1, n - t + 1)$, a contradiction
$\{a_1, c_t\}, c_t (1 \le t \le n)$	For $t = 1, r(d_1 \mid \{a_1, c_1\}) = r(d_n \mid \{a_1, c_1\}) = (3, 1)$, for $t = 2$, $r(b_3 \mid \{a_1, c_2\}) = r(c_1 \mid \{a_1, c_2\}) = (2, 2)$, for $t = 3$, $r(e_2 \mid \{a_1, c_3\}) = r(e_3 \mid \{a_1, c_3\}) = (5, 2)$, for $4 \le t \le k + 1$, $r(a_t \mid \{a_1, c_t\}) = r(c_{t-1} \mid \{a_1, c_t\}) = (t - 1, 2)$ and for $k + 2 \le t \le n$, $r(a_n \mid \{a_1, c_t\}) = r(b_1 \mid \{a_1, c_t\}) = (1, n - t + 2)$, a contradiction
$\{a_1, d_t\}, d_t (1 \le t \le n)$	For $t = 1, r(a_2 \mid \{a_1, d_1\}) = r(a_n \mid \{a_1, d_1\}) = (1, 3)$, for $t = 2$, $r(a_2 \mid \{a_1, d_2\}) = r(b_1 \mid \{a_1, d_2\}) = (1, 3)$, for $3 \le t \le k - 1$, $r(b_n \mid \{a_1, d_t\}) = r(c_1 \mid \{a_1, d_t\}) = (2, t + 2)$, for $t = k$, $r(d_{k-1} \mid \{a_1, d_k\}) = r(b_{k+1} \mid \{a_1, d_k\}) = (k, 2)$, for $t = k + 1$, $r(a_2 \mid \{a_1, d_{k+1}\}) = r(a_n \mid \{a_1, d_{k+1}\}) = (1, k + 1)$, for $k + 2 \le t \le n - 1, r(a_n \mid \{a_1, d_t\}) = r(b_1 \mid \{a_1, d_t\}) = (1, n - t + 2)$ and for $t = n, r(d_{n-1} \mid \{a_1, d_n\}) = r(e_1 \mid \{a_1, d_n\}) = (4, 2)$, a contradiction

Resolving vertices	Contradictions
$\{a_1, e_t\}, e_t \ (1 \leq t \leq n)$	For $t = 1$, $r(c_2 \mid \{a_1, e_1\}) = r(d_n \mid \{a_1, e_1\}) = (3, 2)$, for $t = 2$, $r(a_2 \mid \{a_1, e_2\}) = r(b_1 \mid \{a_1, e_2\}) = (1, 4)$, for $3 \leq t \leq k$, $r(b_2 \mid \{a_1, e_t\}) = r(c_1 \mid \{a_1, e_t\}) = (2, t + 1)$, for $t = k + 1$, $r(a_2 \mid \{a_1, e_{k+1}\}) = r(a_n \mid \{a_1, e_{k+1}\}) = (1, k + 2)$, for $k + 2 \leq t \leq n - 1$, $r(a_n \mid \{a_1, e_t\}) = r(b_1 \mid \{a_1, e_t\}) = (1, n - t + 3)$ and for $t = n$, $r(c_n \mid \{a_1, e_n\} = r(d_1 \mid \{a_1, e_n\}) = (3, 2)$, a contradiction
$\{b_1, c_t\}, c_t \ (1 \leq t \leq n)$	For $1 \leq t \leq k$, $r(a_n \mid \{b_1, c_t\} = r(b_n \mid \{b_1, c_t\}) = (1, t + 1)$, for $k + 1 \leq t \leq n - 1$, $r(a_1 \mid \{b_1, c_t\}) = r(c_1 \mid \{b_1, c_t\}) = (1, n - t + 3)$ and for $t = n$, $r(a_n \mid \{b_1, c_n\}) = r(c_1 \mid \{b_1, c_n\}) = (1, 2)$, a contradiction
$\{b_1, d_t\}, d_t \ (1 \leq t \leq n)$	For $1 \leq t \leq k$, $r(a_n \mid \{b_1, d_t\}) = r(b_n \mid \{b_1, d_t\}) = (1, t + 2)$, for $t = k + 1$, $r(c_1 \mid \{b_1, d_{k+1}\}) = r(a_1 \mid \{b_1, d_{k+1}\}) = (1, k + 2)$, for $k + 2 \leq t \leq n - 2$, $r(a_1 \mid \{b_1, d_t\}) = r(c_1 \mid \{b_1, d_t\}) = (1, n - t + 3)$ and for $t = n - 1, n$, $r(c_1 \mid \{b_1, d_t\}) = r(a_n \mid \{b_1, d_t\}) = (1, n - t + 2)$, a contradiction
$\{b_1, e_t\}, e_t \ (1 \leq t \leq n)$	For $t = 1, 2$, $r(a_1 \mid \{b_1, e_t\}) = r(a_n \mid \{b_1, e_t\}) = (1, t + 3)$, for $3 \leq t \leq k$, $r(b_2 \mid \{b_1, e_t\} = r(c_1 \mid \{b_1, e_t\}) = (1, t + 1)$, for $t = k + 1$, $r(a_n \mid \{b_1, e_{k+1}\}) = r(b_2 \mid \{b_1, e_{k+1}\}) = (1, k + 2)$, for $k + 2 \leq t \leq n - 1$, $r(d_n \mid \{b_1, e_t\}) = r(c_n \mid \{b_1, e_t\}) = (2, n - t + 1)$ and for $t = n$, $r(d_1 \mid \{b_1, e_n\}) = r(c_n \mid \{b_1, e_n\}) = (2, 2)$, a contradiction
$\{c_1, d_t\}, d_t \ (1 \leq t \leq n)$	For $t = 1$, $r(b_1 \mid \{c_1, d_1\}) = r(d_n \mid \{c_1, d_1\} = (1, 2)$, for $2 \leq t \leq k + 1$, $r(b_2 \mid \{c_1, d_t\}) = r(e_1 \mid \{c_1, d_t\}) = (2, t)$, for $k + 2 \leq t \leq n - 1$, $r(b_n \mid \{c_1, d_t\}) = r(e_n \mid \{c_1, d_t\}) = (2, n - t + 1)$ and for $t = n$, $r(b_n \mid \{c_1, d_n\}) = r(e_1 \mid \{c_1, d_n\}) = (2, 2)$, a contradiction
$\{c_1, e_t\}, e_t \ (1 \leq t \leq n)$	Due to the symmetry of the graph this case is analogue to the case when one resolving vertex is in the set $\{b_t : 1 \leq t \leq n\}$ and the other is in the set $\{d_t : 1 \leq t \leq n\}$
$\{d_1, e_t\}, e_t \ (1 \leq t \leq n)$	Due to the symmetry of the graph this case is analogue to the case when one resolving vertex is in the set $\{b_t : 1 \leq t \leq n\}$ and the other is in the set $\{c_t : 1 \leq t \leq n\}$

Hence from all above contradictions it follows that there is no resolving set with two vertices. This implies that $dim(T_n) \geq 3$. Hence $dim(T_n) = 3$ when k is even.

Case (ii) When k is odd then there are the same possibilities as discussed in case (i) and contradictions can be obtained analogously. Hence $dim(T_n) = 3$ in this case and this completes the proof. $\qquad\square$

4 Concluding remarks

In this paper, we have studied the metric dimension of two special classes of convex polytopes and proved that these classes have constant metric dimension that does not depend upon the number of vertices in these graphs and only three vertices choosen sufficiently resolve all the vertices of these classes of convex polytopes.

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