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Sobolev-type inequalities and heat kernel bounds along the geometric flow

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Abstract This paper is concerned with Sobolev-type inequalities and upper bound for the fundamental solution to the heat-type equation defined on compact manifold whose metric evolves by the generalized geometric flow. It turns out that the pointwise estimates obtained in this paper depend on the constants in the uniform Sobolev inequalities for the flow or the best constants in the euclidean Sobolev embedding. We give various illustrations to show that our results are valid in many contexts of geometric flow, where we may not need explicit curvature constraint. Moreover, our approach here also demonstrates equivalence of Sobolev inequalities, log-Sobolev inequalities, ultracontractive estimates and heat kernel upper bounds.

Keywords Geometric flow · Heat-type equation · Sobolev-type inequalities · Heat kernel · Ricci flow

Mathematics Subject Classification 35K08 · 35K55 · 53C21 · 53C44

1 Introduction

Let M be an n-dimensional compact Riemannian manifold endowed with metric $g(x,t)$ evolving by the geometric flow in the interval $0 \leq t \leq T$, $T < T_\epsilon$, where T_ϵ is the time where there is (possibly) a blow-up of the curvature, so we do not need to deal with singularities. Let $u(x,t)$ be a positive solution to a heat-type equation on $M \times [0, T]$, we consider the following coupled system

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$$\begin{cases} \partial_t g_{ij}(x,t) = -2h_{ij}(x,t), & (x,t) \in M \times [0,T] \\ (-\partial_t - \Delta_g + \mathcal{H}(x,t))u(x,t) = 0, & (x,t) \in M \times [0,T], \end{cases} \tag{1.1}$$

where h_{ij} is a general time-dependent symmetric (0, 2)-tensor, $\mathcal{H} = g^{ij}h_{ij}$, the metric trace of 2-tensor h_{ij}, is a C^∞-function $\mathcal{H} : M \times [0,T] \to \mathbf{R}$ and Δ_g is the usual Laplace–Beltrami operator acting on functions in space with respect to metric $g(t)$ in time. The first in the system (1.1) is the generalized geometric flow equation, that is, the flow of metric by tensor with respect to abstract time. In practice, the geometric flow deforms and smoothens out irregularities in the metric to give a "nicer" form and thus, provides useful geometric and topological information on the manifold. For example if $h_{ij} = R_{ij}$, the Ricci curvature tensor, we then have the famous Hamilton Ricci flow [18], which has proven to be of fundamental importance in the global analysis on manifolds. The second equation is a heat-type-conjugate equation on M whose positive solution is a smooth function, at least C^2 in x and C^1 in t, $u(x,t) = u \in C^{2,1}(M \times [0,T])$. In this paper we study the behaviour of the fundamental (minimal positive) solution to the associated heat-type equation along the geometric flow. Let $U, V : M \times [0,T) \to (0,\infty)$ satisfy

$$\Box U = (\partial_t - \Delta_g)U = 0 \quad and \quad \Box^* V = (-\partial_t - \Delta_g + \mathcal{H})V = 0$$

with

$$\int_0^T \int_M \Box U V d\mu_g dt = \int_0^T \int_M U \Box^* V d\mu_g dt. \tag{1.2}$$

We say U and V are respectively solutions to the heat equation and heat-type-conjugate equation. An application of this is to solve geometric flow forward in time and solve the heat-type-conjugate equation backward in time. A very good example is the conjugate heat equation $\Box^* u = (\Delta_g - \partial_t + g^{ij}R_{ij})u(x,t) = 0$ (i.e., adjoint to the heat operator $\Box = (\Delta_g - \partial_t)$, where $g^{ij}R_{ij} = R$, the scalar curvature), which was introduced in a fundamental paper [25], there Perelman obtained Li–Yau Harnack estimates for the minimal positive solution to this equation among many results. The author also studied this in [1] under both forward and backward in time Ricci flow. The case $h_{ij} = -R_{ij}$ (resp. R_{ij}) is precisely when the manifold is being evolved with respect to forward (resp. backward) Ricci flow. In fact, one of the motivations to study this subject arises from the question; is there any merit or demerit of flowing Riemannian manifold by the Ricci flow? The readers who are interested more in this question can find any of the following books [9–11] and [30]. See also our short note [2] on Ricci flow on a closed manifold with positive Euler characteristics and [3] for further details.

Coupling geometric flow to the heat equation can be associated with some physical interpretation in terms of heat conduction process. Precisely, the manifold M with initial metric $g(x,0)$ can be thought of as having the temperature distribution $u(x,0)$ at $t = 0$. If we now allow the manifold to evolve under the geometric flow and simultaneously allow the heat to diffuse on M, then, the solution $u(x,t)$ will represent the space-time temperature on M. Moreover, if $u(x,t)$ approaches δ-function at the initial time, we know that $u(x,t) > 0$, this gives another physical interpretation that temperature is always positive, whence we can consider the potential $f = \log u$ as an entropy or unit mass of heat supplied and the local production entropy is given by $|\nabla f|^2 = \frac{|\nabla u|^2}{u^2}$. Suffice to say that heat kernel governs the evolution of temperature on a manifold with certain amount of heat energy prescribed at the initial time.

In this paper, we obtain Sobolev-type inequalities and some upper bounds for the fundamental solution (Heat kernel) to the heat-type equation defined on compact manifold whose metric is evolving by the geometric flow. Notice that the system (1.1) above is associated to Perelman's monotonicity formula [25], which has been a vital tool in the analysis of the Ricci flow. Perelman proved a lower bound for the heat kernel satisfying the conjugate heat equation with application of the maximum principle and his reduced distance, an outstanding feature of the estimate is that it does not require explicit assumption on metric curvature, the information is being embedded in the reduced distance. In the present too, the bound obtained in this paper needs no explicit curvature assumptions, it rather depends on the Sobolev-type constants similar to those of Zhang–Ricci–Sobolev [29], which in turn depends on the best constant in the usual Sobolev embedding controlled by the infimum of the Ricci curvature and the injectivity radius of the underlying manifold. The motivation for this was Zhang's result in [28], where he obtained upper bounds for conjugate heat kernel under backward Ricci flow, such bounds depend on Yamabe constant or Euclidean Sobolev embedding constant. He further showed that this type of heat kernel upper bounds are proper extension of an on-diagonal upper bound in the case of a fixed manifold, where one obtains a bound of the form

$$F(x,t;y,s) \le C(n)\max\left\{\frac{1}{(t-s)^{\frac{n}{2}}}, 1\right\} \tag{1.3}$$

with $C(n) > 0$ depending on n for all $t > s$ and $x, y \in M$. We also give a special case $\mathcal{H}(x,t)$ is nonnegative to support the above assertion. Recently, Băileşteanu [6] has adopted Zhang's approach to obtain similar estimate for the fundamental solution of the heat equation coupled to Ricci flow. Our calculation is based on the ideas of both papers [6] and [28] cited above, (see also [7]). We remark that the similarity in our results is a justification of the fact that heat diffusion on a bounded geometry with either static or evolving metric behaves like heat diffusion in Euclidean space, many a times, their estimates even coincide. A result of Cheeger and Yau [8] has revealed that the heat kernel of a complete manifold with bounded Ricci curvature can be compared with that of the space form whose curvature determines the lower bound for the manifold's Ricci curvature.

2 Preliminaries and examples of geometric flow

2.1 Technical details

Throughout this paper, M is assumed to be a compact Riemannian manifold without boundary. We denote the fundamental solution (heat kernel) to the heat-type equation by $F(x,t;y,s) \in (M \times [0,T] \times M \times [0,T])$ and partial differential operator with respect to time by ∂_t. We now give a formal definition and some important properties of heat kernel.

Definition 1 We say that $F(x,t;y,s)$ is a fundamental solution to the heat-type equation centred at (y,σ) for $x, y \in M$, $s < t \in [0,T]$, if it satisfies the following system

$$\begin{cases} (-\partial_t - \Delta_{(x,t)} + \mathcal{H}(x,t))F(x,t;y,s) = 0 \\ \lim_{t\to s} F(x,t;y,s) = \delta_y(x) \end{cases} \tag{2.1}$$

for any $x \in M$, where $\delta_y(\cdot)$ is the dirac-delta function concentrated at some point y.

Thus, $F(x,t;y,s)$ is the unique minimal positive solution to the equation which from henceforth we refer to as the heat kernel.

Lemma 1 *The heat kernel satisfies the following properties.*

1. $\int_M F(x,t;y,s)d\mu(g(x,t)) = 1$
2. $F(x,t;y,0) = \int_M F(x,t;z,\frac{t}{2})F(z,\frac{t}{2};y,0)d\mu(g(z,\frac{t}{2}))$ *(Semigroup property)*
3. $F(x,t;y,s)$ *is also the fundamental solution to the heat equation in* (y,s)*-variables i.e,*

$$\begin{cases}(\partial_s - \Delta_{(y,s)})F(x,t;y,s) = 0\\ \lim_{s\to t} F(x,t;y,s) = \delta_x(y).\end{cases} \tag{2.2}$$

4. $\int_M F(x,t;y,s)d\mu(g(y,s)) \le 1.$

Other important properties of heat kernel such as existence, uniqueness, smoothness, symmetry have been studied by many authors, Guenther in [17] and Garofalo and Lanconelli in [14] for examples.

Interestingly, when a manifold is being evolved under a geometric flow all the associated quantities also evolve along the flow. For examples, the Riemannian volume measure $d\mu$ of (M,g) evolves by

$$\partial_t d\mu = -\mathcal{H}d\mu$$

and $\mathcal{H}$ by

$$\partial_t\mathcal{H} = g^{ij}\partial_t h_{ij} + 2|h_{ij}|^2$$

where g^{ij} is the inverse of the metric g_{ij} and $|h_{ij}|^2 = g^{ik}g^{jl}h_{ij}h_{kl}$. Denote $\beta := g^{ij}\partial_t h_{ij}$, in particular, under the Ricci flow, where $h_{ij} = R_{ij}$ and $\mathcal{H} = R$, we have $\beta = \Delta R$. Here in this paper we will assume that

$$\beta - \Delta\mathcal{H} \ge 0. \tag{2.3}$$

This is motivated by an error term appearing in a result of Müller [22, Lemma 1.6]. For our case the error term reads; for any time-dependent vector field X on M

$$\mathcal{D}(X) := 2(R_{ij} - h_{ij})(X,X) + 2\langle 2div\, h - \nabla\mathcal{H}, X\rangle + \partial_t\mathcal{H} - \Delta\mathcal{H} - 2|h_{ij}|^2, \tag{2.4}$$

where div is the divergence operator, i.e., $(div\, h)_k = g^{ij}\nabla_i h_{jk}$. Clearly the last three terms in (2.4) above is the same as the quantity $\beta - \Delta\mathcal{H}$. It does make sense to assume (2.3) whenever $\mathcal{D}(X)$ is nonnegative. The application of this is that we are on a steady or shrinking soliton (self-similar solution to the geometric flow) if the equality in (2.3) holds. Note that we can also express $|h_{ij}|^2 \ge \frac{1}{n}\mathcal{H}^2$ since $|g^{ij}h_{ij}|^2 = \mathcal{H}^2$. Using the condition that $\beta - \Delta\mathcal{H} \ge 0$, we have a governing differential inequality for the evolution of $\mathcal{H}$ as follows

$$\frac{\partial}{\partial t}\mathcal{H} \ge \Delta\mathcal{H} + \frac{2}{n}\mathcal{H}^2. \tag{2.5}$$

Suppose $\mathcal{H} \ge \mathcal{H}_{min}$, we can apply the maximum principle by comparing the solution of the differential inequality with that of the following ordinary differential equation

$$\begin{cases}\dfrac{d\psi(t)}{dt} = \dfrac{2}{n}(\psi(t))^2\\ \psi(0) = \mathcal{H}_{min}(0),\end{cases} \tag{2.6}$$

which is solved to

$$\psi(t) = \frac{\mathcal{H}_{min}(0)}{1 - \frac{2}{n}\mathcal{H}_{min}(0)t}.$$

Therefore

$$\mathcal{H}_{g(t)} \ge \psi(t) = \frac{\mathcal{H}_{min}(0)}{1 - \frac{2}{n}\mathcal{H}_{min}(0)t} \tag{2.7}$$

for all $t \ge 0$ as long as the flow exists.

2.2 Examples of geometric flow

In the following, we give some examples of geometric flows where our results are valid. We remark that in these cases the error term $\mathcal{D}$ and the quantity $\beta - \Delta\mathcal{H}$ are nonnegative. More examples can be found in [22, Section 2].

2.2.1 Hamilton's Ricci flow [18]

Let $(M, g(t))$ be a solution to the Ricci flow. This is the case where $h_{ij} = R_{ij}$ is the Ricci tensor and $\mathcal{H} = R$ is the scalar curvature on M. Here, the scalar curvature evolves by

$$\partial_t R = \Delta R + 2|R_{ij}|^2.$$

By twice contracted second Bianchi identity $g^{ij}\nabla_i R_{jk} = \frac{1}{2}\nabla_k R$, the quantity $\mathcal{D}(X)$ vanishes identically and $\beta - \Delta R \equiv 0$.

2.2.2 Ricci-harmonic map flow [23]

Let (M, g) and (N, ξ) be compact (without boundary) Riemannian manifolds of dimensions m and n respectively. Let a smooth map $\varphi : M \to N$ be a critical point of the Dirichlet energy integral $E(\varphi) = \int_M |\nabla\varphi|^2 d\mu_g$, where N is isometrically embedded in $\mathbb{R}^d$, $d \geq n$, by the Nash embedding theorem. The configuration $(g(x, t), \varphi(x, t))$, $t \in [0, T)$ of a one parameter family of Riemannian metrics $g(x, t)$ and a family of smooth maps $\varphi(x, t)$ is defined to be Ricci-harmonic map flow if it satisfies the coupled system of nonlinear parabolic equations

$$\begin{cases} \dfrac{\partial}{\partial t} g(x, t) = -2Rc(x, t) + 2\alpha\nabla\varphi(x, t) \otimes \nabla\varphi(x, t) \\ \dfrac{\partial}{\partial t}\varphi(x, t) = \tau_g\varphi(x, t), \end{cases} \tag{2.8}$$

where $Rc(x, t)$ is the Ricci curvature tensor for the metric g, $\alpha(t) \equiv \alpha > 0$ is a time-dependent coupling constant, $\tau_g\varphi$ is the intrinsic Laplacian of φ, which denotes the tension field of map φ and $\nabla\varphi \otimes \nabla\varphi = \varphi^*\xi$ is the pullback of the metric ξ on N via the map φ. See List [21] when the target manifold is one dimensional. Here $h_{ij} = R_{ij} - \alpha\nabla_i\varphi \otimes \nabla_j\varphi =: \mathcal{S}_{ij}$, $\mathcal{H} = R - \alpha|\nabla\varphi|^2 =: \mathbf{S}$ and

$$\partial_t \mathbf{S} = \Delta\mathbf{S} + 2|\mathcal{S}_{ij}|^2 + 2\alpha|\tau_g\varphi|^2 - 2\dot{\alpha}|\nabla\varphi|^2. \tag{2.9}$$

Using the twice contracted second Bianchi identity, we have

$$\left(g^{ij}\nabla_i\mathcal{S}_{jk} - \frac{1}{2}\nabla_k\mathbf{S}\right)X_j = -\alpha\tau_g\varphi\nabla_j\varphi X^j. \tag{2.10}$$

Then, $\mathcal{D}(\mathcal{S}_{ij}, X) = 2|\tau_g\varphi - \langle\nabla\varphi, X\rangle|^2 - 2\dot{\alpha}|\nabla\varphi|^2$ and $\beta - \Delta\mathbf{S} = 2\alpha|\tau_g\varphi|^2 - \dot{\alpha}|\nabla\varphi|^2$ for all X on M. Thus both $\mathcal{D}$ and $\beta - \Delta\mathbf{S}$ are nonnegative as long as $\alpha(t)$ is nonincreasing in time. (See [4] for more results in this direction.)

2.3 Lorentzian mean curvature flow

Let $M^n(t) \subset L^{n+1}$ be a family of space-like hypersurfaces in ambient Lorentzian manifold evolving by Lorentzian mean curvature flow. Then, the induced metric evolves by

$$\partial_t g_{ij} = 2H\Pi_{ij},$$

where Π_{ij} denotes the components of the second fundamental form Π on M and $H = g^{ij}\Pi_{ij}$ denotes the mean curvature of M. In this case $h_{ij} = -H\Pi_{ij}$ and $\mathcal{H} = -H^2$. Also $\partial_t H = \Delta H - H(|\Pi|^2 + \widetilde{Rc}(\nu,\nu)$, $\beta - \Delta H = 2H^2|\Pi|^2 + |\nabla H|^2 + 2H\widetilde{Rc}(\nu,\nu)$ and

$$\mathcal{D}X = 2|\nabla H - \Pi(X,\cdot)|^2 + 2\widetilde{(}H\nu - X, H\nu - X) + 2\langle\widetilde{Rm}(X,\nu)\nu, X\rangle, \tag{2.11}$$

where $\widetilde{Rc}$ and $\widetilde{Rm}$ denote the Ricci and Riemman curvature tensor of L^{n+1} respectively. ν denotes future-oriented timelike unit normal vector on M. Obviously both $\mathcal{D}(X)$ and $\beta - \Delta H$ are nonnegative when assuming nonnegativity on sectional curvature of L^{n+1}.

3 Sobolev-type inequalities along the flow

In this section, we give a brief discussion on the version of Sobolev embedding that will be used in the proof of the main theorems. The main ingredients used here are logarithmic Sobolev inequalities and ultracontractivity property of the heat semigroup. It is well known that Gross logarithmic Sobolev inequality [15] is equivalent to Nelson's hypercontractive inequality [24], both of which may imply ultracontractivity of the heat semigroup (see also [12,13]).

3.1 The Sobolev embedding

Let (M, g) be an n-dimensional ($n \geq 3$) Riemannian manifold without boundary, it is well known that when M is compact the Sobolev space $H_1^q(M)$ is continuously embedded in $L^{q^*}(M)$ for any $1 \leq q < n$ and $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$. Here $H_1^q(M)$ is the completion of $C^\infty(M)$ with respect to the standard norm

$$\|u\|_q = \left(\int_M |\nabla u|^q d\mu(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q d\mu(g)\right)^{\frac{1}{q}} \tag{3.1}$$

and the embedding of $H_1^q(M)$ in $L^{q^*}(M)$ is critical. Similarly, the following Sobolev embedding inequality holds true; there exists a positive constant B_q depending on q such that for any $u \in H_1^q(M)$

$$\left(\int_M |u|^{q^*} d\mu(g)\right)^{\frac{1}{q^*}} \leq K(n,q)\left(\int_M |\nabla u|^q d\mu(g)\right)^{\frac{1}{q}} + B_q)\left(\int_M |u|^q d\mu(g)\right)^{\frac{1}{q}}, \tag{3.2}$$

where $K(n, q)$, an explicit constant depending on n and q is the smallest constant having this property, $K(n, q)$ is the best constant in the Sobolev embedding for $\mathbb{R}^n$). See Aubin [5] and Hebey [19] and Talenti [26]. In other words, there exist positive constants A and B such that for all $u \in W^{1,2}(M, g)$, we have

$$\left(\int_M u^{\frac{2n}{n-2}} d\mu(g)\right)^{\frac{n-2}{n}} \leq A\int_M |\nabla u|^2 d\mu(g) + B\int_M u^2 d\mu(g). \tag{3.3}$$

On the compact manifold whose metric evolves along the Ricci flow, Zhang, [29], Hsu [20] and Ye [27] have adopted Perelman $\mathcal{W}$-entropy monotonicity formula to derive various Sobolev embedding that holds for the case $n \geq 3$. In this section we shall make use of Zhang's version to prove the following:

Theorem 1 *Let (M, g) be a compact Riemannian manifold with dimension $n \geq 3$ whose metric evolves by the geometric flow in the interval $t \in [0, T]$. Let there exist positive* *constants A and B for the initial metric g_0 such that the following Sobolev inequality holds for any $u \in W^{1,2}(M, g_0)$*

$$\left(\int_M u^{\frac{2n}{n-2}} d\mu(g_0)\right)^{\frac{n-2}{n}} \leq A \int_M |\nabla u|^2 d\mu(g_0) + B \int_M u^2 d\mu(g_0). \tag{3.4}$$

Then, there exist positive functions of time $A(t)$ and $B(t)$ depending only on the initial metric g_0 and t such that for $u \in W^{1,2}(M, g(t))$, $t > 0$, it holds that

$$\left(\int_M u^{\frac{2n}{n-2}} d\mu(g(t))\right)^{\frac{n-2}{n}} \leq A(t) \int_M \left(|\nabla u|^2 + \frac{1}{4}\mathcal{H}u^2\right) d\mu(g(t)) + B(t) \int_M u^2 d\mu(g(t)), \tag{3.5}$$

Moreover, if $\mathcal{H}(x, 0) > 0$, then $A(t)$ and $B(t)$ are in independent of t.

Here we take $u = u(x, t)$ as an $L^2(M)$-solution of the heat type equation and then prove the above theorem using ultracontractive estimates for heat kernel semigroup. Note that we have from the Sobolev embedding for $1 \leq q < n$ that $W^{1,q}(M)$ can be continuously embedded in $L^{q^*}(M)$, i.e, there exists a constant $C = C(n, q)$, such that

$$\|u\|_{L^{q^*}(M)} \leq C(n, q)\|u\|_{W^{1,q}(M)}$$

for all $u \in W^{1,q}(M)$. So by Holder's inequality we have $(p \geq q)$

$$\int_M |u|^p d\mu = \int_M |u|^q |u|^{p-q} \leq \left(\int_M |u|^{\frac{qn}{n-q}} d\mu\right)^{\frac{n-q}{n}} \left(\int_M |u|^{\frac{n}{q}(p-q)} d\mu\right)^{\frac{q}{n}}. \tag{3.6}$$

This is reduced to (from the interpolation inequality)

$$\int_M u^2 d\mu \leq \left(\int_M u^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n+2}} \left(\int_M u d\mu\right)^{\frac{4}{n+2}} \tag{3.7}$$

for the case $q = 2$ and $n \geq 3$. Then by Sobolev inequality for manifold evolving by the Ricci flow we have

$$\int_M u^2 d\mu \leq \left(A(t) \int_M \left(|\nabla u|^2 + \frac{1}{4}\mathcal{H}u^2\right) d\mu(g(t)) + B(t) \int_M u^2 d\mu(g(t))\right)^{\frac{n}{n+2}} \times \left(\int_M u d\mu(g(t))\right)^{\frac{4}{n+2}}.$$

Let $h(t) := \left(\int_M u d\mu(g(t))\right)^{\frac{4}{n+2}}$, the last inequality becomes

$$\int_M |\nabla u|^2 d\mu(g(t)) \geq \frac{1}{A(t)}\left(h^{-1}(t) \int_M u^2 d\mu(g(t))\right)^{\frac{n+2}{n}} - \frac{B(t)}{A(t)} \int_M u^2 \mu(g(t)) - \frac{1}{4} \int_M \mathcal{H}u^2 \mu(g(t)). \tag{3.8}$$

Thus, we have proved the following by using the bound on the scalar curvature (2.7) as discussed in the Sect. 1.

Lemma 2 *With the hypothesis of Theorem 1 the following inequality holds*

$$\left.\begin{aligned}\int_M |\nabla u|^2 d\mu(g(t)) \geq \tfrac{1}{A(t)}\Big(h^{-1}(t)\textstyle\int_M u^2 d\mu(g(t))\Big)^{\frac{n+2}{n}}\\ -\left(\frac{B(t)}{A(t)}+\frac{\psi(t)}{4}\right)\int_M u^2 d\mu(g(t))\end{aligned}\right\}, \tag{3.9}$$

where $\psi(t)$ is as defined in (2.7).

3.2 Log-Sobolev inequalities and ultracontractive estimates

By the results cited above. Let there exist positive constants $A_0, B_0 < \infty$ such that for all $u \in W^{1,2}(M, g_0)$,

$$\|u\|_{\frac{2n}{n-2}} \leq A_0\|\nabla u\|_2 + B_0\|u\|_2, \tag{3.10}$$

where A_0 and B_0 depends only on n, g_0, lower bound for the Ricci curvature and injectivity radius. We can then write (3.10) as

$$\left(\int_M u^{\frac{2n}{n-2}} d\mu_{g_0}\right)^{\frac{n-2}{n}} \leq A\int_M (4|\nabla u|^2 + \mathcal{H}_g u^2) d\mu_{g_0} + B\int_M u^2 d\mu_{g_0}, \tag{3.11}$$

where

$$A = \frac{1}{4}A_0, \quad and \quad B = \frac{1}{4}A_0 \sup \mathcal{H}_g^-(\cdot, 0) + B_0$$

since $\mathcal{H}_g(x, 0) + \sup \mathcal{H}_g^-(\cdot, 0) = \mathcal{H}_g^+(x, 0) - \mathcal{H}_g^-(x, 0)$. We will assume that (3.11) holds uniformly for $g(t)$, $t > 0$ and different A and B in order to prove the logarithmic Sobolev inequalities.

The usual way of deriving logarithmic Sobolev inequality follows from a careful application of Hölder's and Jensen's inequalities, since $\log v$ is a concave function in which case

$$\int u^2 \ln u^{q-2} d\mu \leq \ln \int u^q d\mu$$

with the assumption that $\int u^2 d\mu = 1$, then

$$\int u^2 \ln u \, d\mu \leq \frac{q}{q-2} \ln\left(\int u^q d\mu\right)^{\frac{1}{q}}.$$

Taking $q = \frac{2n}{n-2}$, we have

$$\int u^2 \ln u \, d\mu \leq \frac{n}{2} \ln\left(\int u^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{2n}},$$

multiplying both sides by 2 we obtain the following

Lemma 3 *For any $u \in W^{1,2}(M, g_0)$ with $\|u\|_2 = 1$*

$$\int_M u^2 \ln u^2 d\mu_{g_0} \leq \frac{n}{2} \ln\left(A\int_M (4|\nabla u|^2 + \mathcal{H}_g u^2) d\mu_{g_0} + B\right). \tag{3.12}$$

See [20,27,29] for similar proofs. Inequalities in (3.12) are usually estimated further by the application of an elementary inequality of the form $\ln y \leq \theta y - \ln\theta - 1,\ \ \theta, y, \geq 0$. Precisely, taking $y = A\int_M (4|\nabla u|^2 + \mathcal{H}_g u^2)d\mu_{g_0} + B$ in (3.12) gives us

$$\int_M u^2 \ln u^2 d\mu_{g_0} \leq \frac{n\theta}{2}\left\{A\int_M (4|\nabla u|^2 + \mathcal{H}_g u^2)d\mu_{g_0} + B\right\} - \frac{n}{2}(1+\ln\theta) \tag{3.13}$$

It is well known that log Sobolev inequalities and ultracontractivity are equivalent and both may imply sharp upper bound for the heat kernel, see Gross [15,16] and Davies [12]. We now use the ultracontractive estimates on heat semigroup to prove Theorem 1.

Proof of Theorem 1

This section discusses how one obtains a uniform Sobolev-type inequality form global bounds on the heat kernel along the geometric flow. The proof of this type is standard as contained in [12, Chapter 2], the same procedures have been adapted in [29] for Kähler–Ricci flow, See also [27] and [20]. For completeness we give the summary of the approach.

For any $t \in [0, T)$ we define the operator

$$\mathcal{A} := -\Delta_g + \frac{\mathcal{H}_g + \sup_M \mathcal{H}_g^-}{4}. \tag{3.14}$$

Since $\mathcal{H}_g(\cdot,\tau) \geq -\sup_M \mathcal{H}_g(\cdot,\tau)$, we know that $\Phi = \frac{1}{4}(\mathcal{H}_g + \sup_M \mathcal{H}_g^-) \geq 0$, $\Phi \in L^\infty(M)$, then $\mathcal{A} \geq 0$ and essentially a self-adjoint operator on $L^2(M)$ with the associated quadratic form

$$\mathbf{Q}(u) = \int_M (|\nabla u|^2 + \Phi u^2)d\mu_g, \quad \forall u \in W^{1,2}(M). \tag{3.15}$$

By the heat kernel convolution property we have

$$e^{-t\mathcal{A}}w_0 = \int_M F(x,t;y)w_0(y)d\mu_g(y), \tag{3.16}$$

where $e^{-t\mathcal{A}}$ is a self-adjoint positivity preserving semigroup for all $t \geq 0$. It is also a contraction on $L^\infty(M)$ and $L^1(M)$ for all $t \geq 0$, then

$$\|e^{-t\mathcal{A}}w_0\|_\infty \leq C_0 t^{-\frac{n}{2}}\|w_0\|_1. \tag{3.17}$$

The next is to apply a theorem in [12] which we state below as a lemma.

Lemma 4 *If $n \geq 2$, then a bound of the form*

$$\|e^{-t\mathcal{A}}w_0\|_\infty \leq C_1 t^{-\frac{n}{4}}\|w_0\|_2. \tag{3.18}$$

for all $t > 0$ and all $w_0 \in L^2(M)$ is equivalent to a bound of the form

$$\|w_0\|^2_{\frac{2n}{n-2}} \leq C_2\mathbf{Q}(w_0) \ \ \forall \ \ w_0 \in W^{1,2}(M). \tag{3.19}$$

By Lemma 4 we can prove that

$$\left(\int_M u^{\frac{2n}{n-2}}d\mu_g\right)^{\frac{n-2}{2}} \leq A_0\int_M \left(|\nabla u|^2 + \frac{1}{4}\left(\mathcal{H}_g + \sup_M \mathcal{H}_g^-\right)u^2\right)d\mu_g \tag{3.20}$$

using the estimate of the form (3.22) below. The only thing remaining for us to show is that estimate (3.17) and (3.18) are equivalent. We do this via the following lemma and Hölder inequality.

Lemma 5 *Suppose $n \geq 2$ and $T < \infty$. Let $C_1 > 0$ be the same as C_1 in (3.18), then we have*

$$\|e^{-t\mathcal{A}}w_0\|_2 \leq C_1 t^{-\frac{n}{4}}\|w_0\|_1 \quad \forall\ w_0 \in L^1(M). \tag{3.21}$$

Now write $e^{-t\mathcal{A}}w_0 = e^{-\frac{1}{2}t\mathcal{A}}e^{-\frac{1}{2}t\mathcal{A}}w_0$ and by assuming (3.18) we have

$$\|e^{-t\mathcal{A}}w_0\|_\infty \leq C_1 t^{-\frac{n}{4}}\|e^{-\frac{1}{2}t\mathcal{A}}w_0\|_2 \leq C_1^2 t^{-\frac{n}{2}}\|w_0\|_1.$$

Similarly, combining the fact that $e^{-t\mathcal{A}}$ is a contraction on $L^\infty(M)$ with bound (3.17) gives us (3.18). Indeed,

$$\begin{aligned}
\|e^{-t\mathcal{A}}w_0\|_\infty &= \left|\int_M F(x,t;y)w_0(y)d\mu_g(y)\right| \\
&\leq \left(\int_M F^{q'}(x,t;y)d\mu_g(y)\right)^{\frac{1}{q'}}\left(\int_M w_0^q d\mu_g(y)\right)^{\frac{1}{q}} \\
&= \left(\int_M F^{q'-1}F d\mu_g(y)\right)^{\frac{1}{q'}}\left(\int_M w_0^q d\mu_g(y)\right)^{\frac{1}{q}} \\
&\leq \left(C_0 t^{-\frac{n}{2}(q'-1)}\int_M F d\mu_g(y)\right)^{\frac{1}{q'}}\left(\int_M w_0^q d\mu_g(y)\right)^{\frac{1}{q}} \\
&\leq C t^{-\frac{n}{2q}}\|w_0\|_q,
\end{aligned}$$

$\forall\ w_0 \in L^q(M)$ with $1/q = 1 - 1/q'$ and $\int_M F(x,t:y)d\mu_g \leq 1$. Here, we take q to satisfy $1 \leq q < n$ for obvious reason (though, by Riez–Thorin interpolation theorem, the above holds for any $1 \leq q < \infty$ since $e^{-t\mathcal{A}}$ is a contraction on $L^1(M)$ and $L^\infty(M)$).

The main point here is the following

Theorem 2 *With the condition of Theorem 1 we claim that estimate of the form*

$$F(x,T;y) \leq CT^{-\frac{n}{2}} \tag{3.22}$$

where C depends on n, t, T, A_0, B_0 and $\sup \mathcal{H}_g(\cdot,0)$, implies the uniform Sobolev inequality (3.11) which is essential the same as (3.5) which we wanted to proof.

Proof Based on the previous argument and modification of the calculation in [29] we define the operator $\widetilde{\mathcal{A}} = \mathcal{A} + 1$, which also has all the properties of $\mathcal{A}$ ($\widetilde{\mathcal{A}} \geq 0$ and generates a symmetric Markov semigroup). Then for any positive constant c depending on n, T, a lower bound for $\mathcal{H}_{g_0}$ and upper bound for A_0, such that for all $t \in [0,T)$ and $v \in Dom(\tilde{\mathcal{A}}) \subseteq W^{1,q}(M)$, there holds for $n \geq 3$

$$\|\widetilde{\mathcal{A}}^{-\frac{1}{2}}w\|_{\frac{nq}{n-q}} \leq c\|w\|_q \quad \forall\ w \in W_0^{1,2}(M). \tag{3.23}$$

Since $\widetilde{\mathcal{A}}^{-\frac{1}{2}}$ is of weak type (p,q), $p = \frac{nq}{n-q}$ for any $1 < q < n$, a simple analysis and the Marcinkiewicz interpolation theorem tell us that $\widetilde{\mathcal{A}}^{-\frac{1}{2}}$ is a bounded operator from L^q to L^p and that (3.23) holds true.

Define $u(x,t) = \widetilde{\mathcal{A}}^{-\frac{1}{2}}w(x,t)$ which implies $w(x,t) = \widetilde{\mathcal{A}}^{\frac{1}{2}}u(x,t)$. Taking $q = 2$ we have

$$\|w\|_2^2 = \int_M \widetilde{\mathcal{A}}^{\frac{1}{2}}u\, \widetilde{\mathcal{A}}^{\frac{1}{2}}u\, d\mu_g = \int_M (\widetilde{\mathcal{A}}u)u\, d\mu_g = \int_M ((\mathcal{A}+1)u)u\, d\mu_g.$$

Combining with (3.23) and (3.19) we obtain the Sobolev inequality

$$\|u\|^2_{\frac{2n}{n-2}} \leq c \cdot C_2 \left(\mathbf{Q}(u) + \int_M u^2 \mu_g \right), \tag{3.24}$$

whereby (3.5) follows with $A = c \cdot C_2$ and $B = \frac{1}{4} c \cdot C_2 (\sup_M \mathcal{H}_g + 1)$.

Remark 1 Fixing t_0 during geometric flow, it is clear that $\widetilde{H} = e^{-1} H$ is the heat kernel generated by $\widetilde{\mathcal{A}}$ and that

$$\int_M \widetilde{F}(x, t; y) d\mu_g(y) \leq \int_M F(x, t; y) d\mu_g(y) \leq 1.$$

By the upper bound for F, we are sure that $\widetilde{F}$ obeys global upper bound

$$\widetilde{F}(x, t; y) d\mu_g(y) \leq \tilde{C} t^{-\frac{n}{2}}, \quad t > 0,$$

where $\tilde{C}$ depends on n, A_0, B_0, t_0 and T. Similarly

$$\|e^{-t\widetilde{\mathcal{A}}} w\|_\infty = \|e^{-t} e^{-t\mathcal{A}} w\|_\infty \leq e^{-t} C t^{-\frac{n}{2}} \|w\|_1 = \widetilde{C} t^{-\frac{n}{2}} \|w\|_1.$$

4 Pointwise upper bound with Sobolev inequality

In this section, we prove an upper estimate on the heat kernel of the manifold evolving by the geometric flow, it turns out that the estimate depends on the best constants in Sobolev-type inequalities (3.5) for the geometric flow and the bound on the metric trace of h_{ij}. The main result of this section is the following

Theorem 3 *Let $(M, g(x, t))$, $t \in [0, T]$ be a solution to the geometric flow with $n \geq 3$ and $F(x, t; y, s)$ be the fundamental solution to the heat-type equation. Then for a constant C_n depending on n only, the following estimate holds*

$$F(x, t; y, s) \leq \frac{C_n}{\left(\int_s^{\frac{t+s}{2}} \frac{e^{\frac{2}{n} P(\tau)}}{\alpha(\tau) A(\tau)} d\tau \cdot \int_{\frac{t+s}{2}}^{s} \frac{e^{-\frac{2}{n} P(\tau)}}{A(\tau)} d\tau \right)^{\frac{n}{4}}} \tag{4.1}$$

for $0 \leq s < t \leq T$, where $\alpha(\tau) = \frac{\rho^{-1} - \frac{2}{n}\tau}{\rho^{-1}}$, $\mathcal{H}(g_0) \geq \rho$ being the infimum of the metric trace of h_{ij} at the initial time, $P(\tau) = \int_s^t (B(\tau) A^{-1}(\tau) - \frac{1}{2}\phi(\tau)) d\tau$, with $A(t)$ and $B(t)$ being positive constants in the Zhang–Ricci–Sobolev inequality and $\phi(t)$ is the lower bound for the scalar curvature.

4.1 Proof of Theorem 3

Proof We suppose here and thereafter that $s = 0$ without loss of generality. Since $F(x, t; y, s)$ is the fundamental solution, it then follows from its semigroup property and Cauchy–Schwarz inequality that

$$F(x, t; y, 0) = \int_M F\left(x, t; z, \frac{t}{2}\right) F\left(z, \frac{t}{2}; y, 0\right) d\mu(g(z, t))$$

$$\leq \left(\int_M F^2 \left(x, t; z, \frac{t}{2}\right) d\mu \left(g\left(z, \frac{t}{2}\right)\right) \right)^{\frac{1}{2}} \left(\int_M F^2 \left(z, \frac{t}{2}; y, 0\right) d\mu \left(g\left(z, \frac{1}{2}\right)\right) \right)^{\frac{1}{2}}.$$

Traditionally, deriving an upper bound for each of the terms in the right hand side of the last inequality suffices to settle the proof, the nature of the bound to obtain depends largely upon the ingredient. In the present, we rely on estimates from Sobolev embedding theorems on the manifold evolving by the geometric flow. Now denote, say

$$V(t) = \int_M F^2(x,t;y,s)d\mu(g(x,t))$$
$$W(t) = \int_M F^2(x,t;y,s)d\mu(g(y,s)).$$

Thus, the pointwise estimate on the quantities $V(t)$ and $W(t)$ will determine an upper bound for the fundamental solution $F(x,t;y,s)$. Approaches to obtaining bound for each of the quantities $V(t)$ and $W(t)$ differ slightly due to the interpolation of the heat kernel between the heat-type equation in the variables (x,t) and the heat equation in the variables (y,s), i.e.,

$$(-\partial_t - \Delta_x + \mathcal{H}(x,t))F(x,t;\cdot,\cdot) = 0$$
$$(\partial_s - \Delta)F(\cdot,\cdot;y,s) = 0.$$

We first treat the case when $F(x,t;y,s)$ solves the heat-type equation, that is, we want to estimate $V(t)$. The idea is to find an inequality involving $V(t)$. Hence

$$\begin{aligned} V'(t) &= \int_M (2F\partial_t F - \mathcal{H}F^2)d\mu(x,t) \\ &= \int_M 2F(-\Delta F + \mathcal{H}F)d\mu(x,t) - \int_M \mathcal{H}F^2 d\mu(x,t) \\ &= 2\int_M |\nabla F|^2 d\mu(x,t) + \int_M \mathcal{H}F^2 d\mu(x,t). \end{aligned}$$

Using Lemma 2, we arrive at

$$\begin{aligned} V'(t) &\geq 2A^{-1}(t)\left(h^{-1}(t)\int_M F^2 d\mu(x,t)\right)^{\frac{n+2}{n}} - 2\left(B(t)A^{-1}(t) + \frac{1}{4}\psi(t)\right)\int_M F^2 d\mu(x,t) \\ &\quad + \psi(t)\int_M F^2 d\mu(x,t) \\ &= 2A^{-1}(t)\left(h^{-1}(t)\int_M F^2 d\mu(x,t)\right)^{\frac{n+2}{n}} - \left(2B(t)A^{-1}(t) - \frac{1}{2}\psi(t)\right)\int_M F^2 d\mu(x,t). \end{aligned}$$

The problem is reduced to solving the following ODE

$$V'(t) + q(t)V(t) \geq 2A^{-1}(t)V(t)^{\frac{n+2}{n}}, \tag{4.2}$$

where $q(t) = 2B(t)A^{-1}(t) - \frac{1}{2}\psi(t)$. Equation (4.2) is due to the fact that under variables (x,t), the fundamental solution F satisfies

$$\int_M F(x,t;y,s)d\mu(x,t) = 1$$

and consequently then

$$h(t) = \left(\int_M F d\mu\right)^{\frac{4}{n+2}} = 1.$$

Notice that the resulting ODE (4.2) is true for any $\tau \in [s,t]$, we then solve it by using integrating factor method. Denote $Q(\tau) = \int q(\tau)d\tau$, the integrating factor is $e^{Q(\tau)}$, therefore we have

$$(e^{Q(\tau)}V(\tau))' \geq 2A^{-1}(\tau)(e^{Q(\tau)}V(\tau))^{\frac{n+2}{n}}e^{-\frac{2}{n}Q(\tau)}$$

integrating from s to t since it is true for all $\tau \in [s,t]$, with the facts that

$$\int_s^t \frac{(e^{Q(\tau)}V(\tau))'}{(e^{Q(\tau)}V(\tau))^{\frac{n+2}{n}}}d\tau = -\frac{n}{2}(e^{Q(\tau)}V(\tau))^{-\frac{2}{n}}\Big|_s^t$$

and

$$\lim_{\tau \searrow s} V(t) = \int_M \lim_{\tau \searrow s} F^2(x,t;y,s)d\mu(x,t) = \int_M \delta_y^2(x)d\mu(x,t) = 0$$

we obtain the bound as follows

$$V(t) \leq \frac{\left(\frac{2}{n}\right)^{\frac{n}{2}}e^{-Q(t)}}{\left(2\int_s^t \frac{e^{-\frac{2}{n}Q(\tau)}}{A(\tau)}d\tau\right)^{\frac{n}{2}}} = \frac{\left(\frac{1}{n}\right)^{\frac{n}{2}}e^{-Q(t)}}{\left(\int_s^t \frac{e^{-\frac{2}{n}Q(\tau)}}{A(\tau)}d\tau\right)^{\frac{n}{2}}}.$$

Taking $C_n := \left(\frac{1}{n}\right)^{\frac{n}{2}}$, we arrive at

$$\int_M F^2(x,t;y,s)d\mu(x,t) = V(t) \leq \frac{C_n e^{-Q(t)}}{\left(\int_s^t A^{-1}(\tau)e^{-\frac{2}{n}Q(\tau)}d\tau\right)^{\frac{n}{2}}}. \tag{4.3}$$

The next is to estimate

$$W(s) = \int_M F^2(x,t;y,s)d\mu(y,s).$$

Due to the asymmetry of the equation, the computation is slightly different. We recall that $F(x,t;y,s)$ satisfies the heat equation in the variables (y,s), then we similarly have

$$\begin{aligned} W'(s) &= \int_M (2F\partial_s F - \mathcal{H}F^2)d\mu(y,s) \\ &= \int_M 2F(\Delta F) - \mathcal{H}F^2 d\mu(y,s) \\ &= -2\int_M |\nabla F|^2 d\mu(y,s) - \int_M \mathcal{H}F^2 d\mu(y,s). \end{aligned}$$

Using Lemma 2 again we arrive at

$$\begin{aligned}W'(s) &\leq -2A^{-1}(s)\Big(h^{-1}(s)\int_M F^2 d\mu(y,s)\Big)^{\frac{n+2}{n}} + 2\Big(B(s)A^{-1}(s)+\frac{1}{4}\psi(s)\Big)\\ &\quad \int_M F^2 d\mu(y,s) - \psi(t)\int_M F^2 d\mu(y,s)\\ &= -2A^{-1}(s)\Big(h^{-1}(s)\int_M F^2 d\mu(y,s)\Big)^{\frac{n+2}{n}}\\ &\quad + \Big(2B(s)A^{-1}(t) - \frac{1}{2}\psi(s)\Big)\int_M F^2 d\mu(y,s). \end{aligned} \tag{4.4}$$

We can further estimate the quantity $h(s) = (\int_M F d\mu)^{\frac{4}{n+2}}$. Notice that contrary to what was obtainable in the variable (x,t), $\int_M F(x,t;y,s)d\mu(y,s) \neq 1$, since the coordinate (x,t) are kept fixed here and we only integrate in (y,s). Therefore

$$\begin{aligned}\lambda'(s) &= \frac{d}{ds}\Big(\int_M F(x,t;y,s)d\mu(y,s)\Big)\\ &= \int_M \partial_s F(x,t;y,s)d\mu(y,s) - \int_M \mathcal{H}(y,s)F d\mu(y,s)\\ &= \int_M \Delta_{y,s}F(x,t;y,s)d\mu(y,s) - \int_M \mathcal{H}(y,s)F(x,t;y,s)d\mu(y,s)\\ &\leq -\psi(s)\int_M F(x,t;y,s)d\mu(y,s).\end{aligned}$$

The last inequality is due to the fact that we are on compact manifold, where $\int_M \Delta F d\mu = 0$ and by the lower bound on quantity $\mathcal{H}$ due to the maximum principle. Now for any $\tau \in [s,t]$ and by lower bound (2.7)

$$\begin{aligned}\lambda'(\tau) &\leq -\psi(\tau)\lambda(\tau)\\ \frac{\lambda'(\tau)}{\lambda(\tau)} &\leq -\psi(\tau) = -\frac{1}{\rho^{-1}-\frac{2}{n}\tau},\end{aligned}$$

integrating this from s to t we get

$$\ln\lambda(t) - \ln\lambda(s) \leq \frac{n}{2}\ln(\rho^{-1}-\frac{2}{n}\tau)\Big|_s^t$$

$$\frac{\lambda(t)}{\lambda(s)} \leq \left(\frac{\rho^{-1}-\frac{2}{n}t}{\rho^{-1}-\frac{2}{n}s}\right)^{\frac{n}{2}} \implies \lambda(t) \leq \left(\frac{\rho^{-1}-\frac{2}{n}t}{\rho^{-1}-\frac{2}{n}s}\right)^{\frac{n}{2}}\lambda(s),$$

we can show that $\lambda(s) \equiv 1$ as follows

$$\begin{aligned}\lambda(s) = \lim_{t\to s}\int_M F(x,t;y,s)d\mu(y,s) &= \int_M \lim_{t\to s} F(x,t;y,s)d\mu(y,s)\\ &= \int_M \delta_x(y) = 1,\end{aligned}$$

combining these we have

$$h(t) = \left(\frac{\rho^{-1} - \frac{2}{n}t}{\rho^{-1} - \frac{2}{n}s}\right)^{\frac{n}{2}\cdot\frac{4}{n+2}} = \left(\frac{\rho^{-1} - \frac{2}{n}t}{\rho^{-1} - \frac{2}{n}s}\right)^{\frac{2n}{n+2}} =: \alpha^{\frac{2n}{n+2}}.$$

By this (4.4) is now reduced to the following

$$W(s) \leq -2A^{-1}(s)\alpha^{-2}(s)\left(\int_M F^2 d\mu(y,s)\right)^{\frac{n+2}{n}} + \left(2B(s)A^{-1}(s) - \frac{1}{2}\psi(s)\right)\int_M F^2 d\mu(y,s), \tag{4.5}$$

we are then to solve the following ODE

$$W'(s) \leq -2A^{-1}\alpha^{-2}W(s)^{\frac{n+2}{n}} + r(s)W(s), \tag{4.6}$$

where $r(s) = 2B(s)A^{-1}(s) - \frac{1}{2}\psi(s)$. In the similar vein to the previous estimate, we also solve (4.6) using integrating factor method. Denote $R(\tau) = \int r(\tau)d\tau$, the integrating factor is $e^{-R(\tau)}$. Therefore we have

$$(e^{-R(\tau)}W(\tau))' \leq -2A^{-1}\alpha^{-2}(e^{-R(\tau)}W(\tau))^{\frac{n+2}{n}}e^{\frac{2}{n}R(\tau)},$$

integrating from s to t since it is true for any $\tau \in [s,t]$ we have immediately

$$W'(s) \leq \frac{\left(\frac{2}{n}\right)^{\frac{n}{2}} e^{R(s)}}{\left(2\int_s^t \frac{e^{\frac{2}{n}R(\tau)}}{\alpha^2(\tau)A(\tau)}d\tau\right)^{\frac{n}{2}}} = \frac{\left(\frac{1}{n}\right)^{\frac{n}{2}} e^{R(s)}}{\left(\int_s^t \alpha^{-2}(\tau)A^{-1}(\tau)e^{\frac{2}{n}R(\tau)}d\tau\right)^{\frac{n}{2}}},$$

hence

$$\int_M F^2(x,t;y,s)d\mu(y,s) = W(s) \leq \frac{C_n e^{R(s)}}{\left(\int_s^t \alpha^{-2}(\tau)A^{-1}(\tau)e^{\frac{2}{n}R(\tau)}d\tau\right)^{\frac{n}{2}}}. \tag{4.7}$$

We can then see from the computation above that

$$V\left(\frac{t}{2}\right) = \int_M F^2\left(x,t;z,\frac{t}{2}\right)d\mu\left(z,\frac{t}{2}\right) = \frac{C_n e^{-Q(\frac{t}{2})}}{\left(\int_s^t A^{-1}(\tau)e^{-\frac{2}{n}Q(\tau)}d\tau\right)^{\frac{n}{2}}}$$

and

$$W\left(\frac{t}{2}\right) = \int_M F^2\left(z,\frac{t}{2};y,0\right)d\mu\left(z,\frac{t}{2}\right) = \frac{C_n e^{R(\frac{t}{2})}}{\left(\int_s^t \left(\frac{\rho^{-1}-\frac{2}{n}\tau}{\rho^{-1}}\right)^{-2} A^{-1}(\tau)e^{\frac{2}{n}R(\tau)}d\tau\right)^{\frac{n}{2}}}.$$

Here we choose

$$P\left(\frac{t}{2}\right) = \int_0^{\frac{t}{2}} \left[B(\tau)A^{-1}(\tau) - \frac{1}{2}\phi(\tau)\right]d\tau = Q\left(\frac{t}{2}\right) = R\left(\frac{t}{2}\right)$$

$$with \quad \phi(t) := \frac{1}{\rho^{-1} - \frac{2}{n}t}.$$

Finally we obtain the bound

$$F(x,t;y,s) \leq \frac{C_n}{\left(\int_s^{\frac{t+s}{2}} \left(\frac{\rho^{-1}-\frac{2}{n}\tau}{\rho^{-1}}\right)^{-2} A^{-1}(\tau)e^{\frac{2}{n}P(\tau)} \cdot \int_{\frac{t+s}{2}}^{s} A^{-1}(\tau)e^{-\frac{2}{n}P(\tau)}d\tau\right)^{\frac{n}{4}}}. \tag{4.8}$$

The required estimate follows immediately.

4.2 The special case of nonnegative $\mathcal{H}(x,t)$

Note that if $\mathcal{H}(x,0) \geq 0$, the maximum principle shows that it remains so as long as the geometric flow exists. For this case we obtain a Sobolev type embedding from Lemma 2

$$\int_M |\nabla u|^2 d\mu(g(t)) \geq \frac{1}{A}\left(\int_M u^2 d\mu(g(t))\right)^{\frac{n+2}{2}} - \frac{B}{A}\int_M u^2 d\mu(g(t)), \tag{4.9}$$

where A and B are absolute constant independent of time, in fact $A = K(n,2)^2$ is the best constant in Euclidean Sobolev embedding and B can be taken to be equivalent to zero when $\mathcal{H}(x,0) = 0$.

In the case $\mathcal{H}(x,0) > 0$, we have $\lambda'(s) \leq 0$ showing that $\lambda(s)$ is decreasing, that is $\lambda(s) \leq \lambda(t)$. This implies that $h(s) \leq h(t) = 1$, then (4.6) becomes

$$W'(s) \leq -2A^{-1}W(s)^{\frac{n+2}{n}} + r(s)W(s)$$

with $\tilde{r} = \frac{2B}{A}$ and we obtain the estimate

$$W(s) \leq \frac{C_n e^{\tilde{R}(s)}}{A^{-1}\left(\int_s^t (\tau)e^{\frac{2}{n}R(\tau)}d\tau\right)^{\frac{n}{2}}},$$

similarly

$$V(t) \leq \frac{C_n e^{-\tilde{R}(t)}}{A^{-1}\left(\int_s^t (\tau)e^{-\frac{2}{n}R(\tau)}d\tau\right)^{\frac{n}{2}}}.$$

Putting these together we have a counterpart estimate to (4.8) as follows

$$F(x,t;y,s) \leq \frac{C_n}{\left[A^{-2}\left(\int_s^{\frac{t+s}{2}} e^{\frac{2}{n}\tilde{R}(\tau)}d\tau \cdot \int_{\frac{t+s}{2}}^{s} e^{-\frac{2}{n}\tilde{R}(\tau)}d\tau\right)\right]^{\frac{n}{4}}}. \tag{4.10}$$

Here, the denominator in the right hand side of the inequality (4.10) is simplified to

$$\begin{aligned}
&\left[A^{-2}\left(\int_s^{\frac{t+s}{2}} e^{\frac{2}{n}\tilde{R}(\tau)}d\tau \cdot \int_{\frac{t+s}{2}}^{s} e^{-\frac{2}{n}\tilde{R}(\tau)}d\tau\right)\right]^{\frac{n}{4}} \\
&= \left[\frac{n^2}{16B^2}\left(e^{\frac{4B}{nA}\cdot\frac{t+s}{2}} - e^{\frac{4B}{nA}\cdot s}\right)\left(e^{-\frac{4B}{nA}\cdot\frac{t+s}{2}} - e^{-\frac{4B}{nA}\cdot t}\right)\right]^{\frac{n}{4}} \\
&= \left[\frac{n^2}{16B^2}\left(1 - e^{-\frac{4B}{nA}\cdot\frac{t-s}{2}}\right)^2\right]^{\frac{n}{4}}.
\end{aligned}$$

Therefore

$$F(x,t;y,s) \leq \frac{C_n}{\left[\frac{n}{4B}\left(1 - e^{-\frac{4B}{nA}\cdot\frac{t-s}{2}}\right)\right]^{\frac{n}{2}}} \leq \frac{\tilde{C}_n}{(t-s)^{\frac{n}{2}}}$$

by Taylor series expansion (i.e., $1 - e^{-z} \lesssim z$), where $\tilde{C}_n = C_n \cdot (2A)^{\frac{n}{2}}$.

In the case $\mathcal{H}(x,0) = 0$, $B(t) \equiv 0$, $\tilde{R}(t) = \frac{B}{A}t \equiv 0$ and

$$F(x,t;y,s) \leq \frac{C_n}{\left[A_0^{-2}\left(\int_s^{\frac{t+s}{2}} d\tau \cdot \int_{\frac{t+s}{2}}^{s} d\tau\right)\right]^{\frac{n}{4}}} = \frac{C_n}{\left[A_0^{-1}\left(\frac{t-s}{2}\right)\right]^{\frac{n}{2}}} = \frac{\tilde{C}_n}{(t-s)^{\frac{n}{2}}}, \quad (4.11)$$

where $\tilde{C}_n = C_n \cdot (2A_0)^{\frac{n}{2}} = (\frac{2}{n}k(n,2))^{\frac{n}{2}}$.

5 Conclusion

We have obtained uniform Sobolev-type inequalities which are valid on a compact Riemannian manifold whose metric solves the abstract geometric flow. The input of these inequalities are taken to be $W^{1,2}(M)$-solutions of the heat-type equation and as a consequence we derive upper bound for the minimal positive solution (heat kernel) without any explicit restriction on the curvature of the underlying manifold. The semigroup property of the heat kernel plays a crucial role in this result. However, the estimates hold for any positive solution with some normalization condition. In application, the heat-type equation could be an adjoint heat equation which needs to be solved backward in time as the geometric flow is being solved forward in time. This will allow us gain some control on singularities as we reach maximum time T, since the fundamental solution will tend to δ-function as t tends to T. The examples of geometric flows mentioned in Sect. 2 show that our results are more general and valid in various cases. As a by-product of our approach in this paper, we have established the equivalence of Sobolev inequalities, log-Sobolev inequalities, contractive estimates and heat kernel upper bounds.

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