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c^*-Normal and s-semipermutable subgroups in finite groups

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Abstract A subgroup H of a group G is called c^*-normal in G if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N$ is S-quasinormally embedded in G. A subgroup K of G is said to be s-semipermutable if it is permutable with every Sylow p-subgroup of G with $(p, |K|) = 1$. In this article, we investigate the influence of c^*-normality and s-semipermutability of subgroups on the structure of finite groups and generalize some known results.

Keywords c^*-Normal subgoup · s-Semipermutable subgroup · p-Nilpotent · Saturated formation

Mathematics Subject Classification 20D10 · 20D20

1 Introduction

Throughout only finite groups are considered. Terminologies and notations employed agree with standard usage, as in Robinson [8].

Two subgroups H and K of a group G are said to be permutable if $HK = KH$. The subgroup H is said to be S-quasinormal in G if H permutes with every Sylow subgroups of G, i.e., $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [7] and has been studied widely by many authors, such as [2,9]. Recently, There is a generalization of S-quasinormality in [14]. The subgroup H is called s-semipermutable in G if H permutes with every Sylow p-subgroup of G with $(|H|, p) = 1$. An s-semipermutable subgroup is no need to be an S-quasinormal subgroup. S_3 is a counter-example. On the other hand, Wang [10] introduced the concept of c-normal subgroups. The subgroup H is said to be c-normal in G if there exists a normal subgroup U of G such that $G = HU$ and $H \cap U$ is contained in H_G, where H_G is the maximal normal subgroup of G which is contained in H. The c-normality is

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a generalization of the normality. Applying the c-normality of subgroups, Wang obtained new criteria for supersolvability of groups. In 2007, Wei and Wang [11] introduced the concept of c^*-normal subgroups which is both c-normality and S-quasinormal embedding and used the c^*-normality of maximal subgroups to give some necessary and sufficient conditions for a group to be p-nilpotent, p-supersolvable or supersolvable. Based on the observation above concepts, we note that c^*-normal subgroups and s-semipermutable subgroups are two different concepts. There are examples to show that s-semipermutable subgroups are not c^*-normal subgroups and in general the converse is also false. In this paper, we investigate s-semipermutable and c^*-normal subgroups of G and give criteria for a group belonging to $\mathscr{F}$. Some interesting results are obtained and known results on this topic are generalized.

2 Preliminaries

Lemma 2.1 [11, Lemma 2.3] *Let H be a subgroup of a group G.*

(1) *If H is c^*-normal in G and $H \leq M \leq G$, then H is c^*-normal in M.*
(2) *Let $N \lhd G$ and $N \leq H$. Then H is c^*-normal in G if and only if H/N is c^*-normal in G/N.*
(3) *Let π be a set of primes, H a π-subgroup of G and N a normal π'-subgroup of G. If H is c^*-normal in G, then HN/N is c^*-normal in G/N.*

Lemma 2.2 [14, Property] *Suppose that H is an s-semipermutable subgroup of G. Then*

(1) *If $H \leq K \leq G$, then H is s-semipermutable in K.*
(2) *Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*
(3) *If $H \leq O_p(G)$, then H is S-quasinormal in G.*

Lemma 2.3 [9,11] *Suppose that U is S-quasinormally embedded in a group G, and that $H \leq G$ and $K \unlhd G$.*

(1) *If $U \leq H$, then U is S-quasinormally embedded in H.*
(2) *UK is S-quasinormally embedded in G and UK/K is S-quasinormally embedded in G/K.*
(3) *If $K \leq H$ and H/K is S-quasinormally embedded in G/K, then H is S-quasinormally embedded in G.*
(4) *A p-subgroup H of G is S-quasinormal in G if and only if $N_G(H) \geq O^p(G)$ for some prime $p \in \pi(G)$.*

Lemma 2.4 [11, Lemma 2.8] *Let G be a group and let p be a prime number dividing $|G|$ with $(|G|, p-1) = 1$.Then*

(1) *If N is normal in G of order p, then N lies in $Z(G)$;*
(2) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent;*
(3) *If M is a subgroup of G with index p, then M is normal in G.*

Lemma 2.5 [1, A, Lemma 1.2] *Let U, V and W be subgroups of a group G. The following statements are equivalent.*

(1) $U \cap VW = (U \cap V)(U \cap W)$.
(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.6 [4, 6.4.8] *Let H, K be subgroups of the group G such that*

$$(|G : H|, |G : K|) = 1.$$

Then $G = HK$ and $|G : H \cap K| = |G : H||G : K|$.

Lemma 2.7 [11, Lemma 2.5] *Let G be a group, K an S-quasinormal subgroup of G and P a Sylow p-subgroup of K, where p is a prime. If $K_G = 1$, then P is S-quasinormal in G.*

Lemma 2.8 [11, Theorem 4.1] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of any Sylow subgroup of H are c^*-normal in G, then $G \in \mathscr{F}$.*

Lemma 2.9 [11, Theorem 4.3] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all maxim al subgroups of any Sylow subgroup of $\mathscr{F}^*(H)$ are c^*-normal in G, then $G \in \mathscr{F}$.*

Lemma 2.10 [5, X, 13] *Let G be a group. If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

Lemma 2.11 [5, IV, Satz 4.7] *If P is a Sylow p-subgroup of G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

3 Main results

Theorem 3.1 *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P is either s-semipermutable or c^*-normal in G, then G is p-nilpotent.*

Proof Asuume that the theorem is false and G is a counterexample with minimal order. We will consider the following steps.

(1) G has a unique minimal normal subgroup N such that G/N is p-nilpotent and $\Phi(G) = 1$. Let N be a minimal normal subgroup of G. We have to show G/N satisfies the hypotheses of the theorem. Let M/N be a maximal subgroup of PN/N. We can see $M = P_1N$ for some maximal subgroup P_1 of P. By the hypotheses, P_1 is either s-semipermutable or c^*-normal in G. If P_1 is c^*-normal in G, then there is a normal subgroup K_1 of G such that $G = P_1K_1$ and $P_1 \cap K_1$ is S-quasinormally embedded in G. Then $G/N = M/N \cdot K_1N/N = P_1N/N \cdot K_1N/N$. It is easy to see that K_1N/N is normal in G/N. Since $(|N : P_1 \cap N|, |N : K_1 \cap N|) = 1$, $(P_1 \cap N)(K_1 \cap N) = N = N \cap G = N \cap (P_1K_1)$ by Lemma 2.6. Now using Lemma 2.5, $(P_1N) \cap (K_1N) = (P_1 \cap K_1)N$. It follows that $(P_1N)/N \cap (K_1N)/N = (P_1 \cap K_1)N/N$ is S-quasinormally embedded in G/N by Lemma 2.3. Thus M/N is c^*-normal in G/N. If P_1 is s-semipermutable in G, then $M/N = P_1N/N$ is s-semipermutable in G/N by Lemma 2.2. Consequently, G/N satisfies the hypotheses of the theorem. The choice of G yields that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ are obvious.

(2) $O_{p'}(G) = 1$.
If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by (1). By Lemmas 2.1 and 2.2, G/N satisfies the hypotheses, hence G/N is p-nilpotent. Now the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction.

(3) $O_p(G) = 1$ and G is not solvable.
If $O_p(G) \neq 1$, (1) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Hence, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. Since $O_p(G) \cap M$ is normalized by N and M, $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Obviously $P = NM_p = N(P \cap M)$. Since $P \cap M < P$, we can take a maximal subgroup P_1 of P such that $P \cap M \leq P_1 < \cdot P$. Then $P = NP_1$ and $P \cap M = P_1 \cap M$. By the hypotheses of theorem, P_1 is either s-semipermutable or c^*-normal in G. If P_1 is c^*-normal in G, then there is a normal subgroup K_1 such that $G = P_1K_1$ and $P_1 \cap K_1$ is S-quasinormally embedded in G. Thus $P_1 \cap K_1$ is a Sylow p-subgroup of some S-quasinormal subgroup K of G. If $K_G \neq 1$, by (1) we have that $N \leq K_G$. Hence, $P = NP_1 \leq P_1$, a contradiction. If $K_G = 1$, by Lemma 2.7 we have that $P_1 \cap K_1$ is S-quasinormal in G. It follows that $P_1 \cap K_1$ is normalized by P and $O^p(G)$. Now we know $P_1 \cap K_1 \lhd G$. If $(P_1 \cap K_1)M = G$, then $P_1M = PM = G$ and so $P_1 = P$, a contradiction. Thus $(P_1 \cap K_1)M = M$ and so $(P_1 \cap K_1) \leq M$. On the other hand, $P_1 \cap K_1 \leq N$. We know $P_1 \cap K_1 \leq N \cap M$. Hence, $P_1 \cap K_1 = 1$ and so $|P \cap K_1| = p$. So $|K|_p = p$. By the uniqueness of N, we have that $N \leq K_1$, of course, N is a cyclic group of order p. By Lemma 2.4, $N \leq Z(G)$. Since G/N is p-nilpotent, G is also p-nilpotent, a contradiction. Now suppose that P_1 is s-semipermutable in G. Then P_1M_q is a group for $q \neq p$. Therefore, $P_1 < M_p, M_q | q \in \pi(M), q \neq p >= P_1M$ is a group. Then $P_1M = M$ or G by maximality of M. If $P_1M = G$, then $P = P \cap P_1M = P_1(P \cap M) = P_1$, which is a contradiction. If $P_1M = M$, then $P_1 \leq M$. Hence, $P_1 \cap N = 1$ and N is of prime order. Then the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction. Combining this with (2), it is easy to see that G is not solvable, now thus (3) holds.

(4) For any $q \neq p$, $PG_q < G$, where G_q is a Sylow q-subgroup of G. That is to say, PG_q is p-nilpotent.
At first, we have $NP = G$. In fact, if $NP < G$, then NP is p-nilpotent since NP satisfies the hypotheses of theorem. Hence, N is p-nilpotent and by (1) we know N is a nontrivial p-group, but this is a contradiction with (3). So we have that $NP = G$. If for all $P_1 < \cdot P$, we have that $NP_1 < G$. Then $(P \cap N)P_1 < P$ and so $P \cap N \leq P_1$. Hence, $P \cap N \leq \Phi(P)$ and N is p-nilpotent by Lemma 2.11, a contradiction. So there exists $P_1 < \cdot P$ such that $G = NP_1$. By the hypotheses, if P_1 is c^*-normal in G, then there is a normal subgroup K such that $G = P_1K_1$ and $P_1 \cap K_1$ is S-quasinormally embedded in G. So $P_1 \cap K_1 \in Syl_p(K)$, where K is S-quasinormal in G. If $K_G \neq 1$, then $N \leq K_G$. It follows that $P_1 \cap K_1 \cap N \in Syl_p(N)$. Now by $G = NP_1$ we get $P_1 \in Syl_p(G)$, a contradiction. So $K_G = 1$. By Lemma 2.7 we have that $P_1 \cap K_1$ is S-quasinormal in G, so $P_1 \cap K_1 \leq O_p(G) = 1$ and $P_1 \cap K_1 = 1$. Moreover, $|P \cap K_1| = p$ and so $|K_1|_p = p$. By Lemma 2.4 we know K_1 is p-nilpotent. Of course, N is also p-nilpotent, a contradiction. From [5, IV, Satz 2.8], it follows that P is non-cyclic. We could take a maximal subgroup P_2 of P satisfying $G = NP_2$. By the same argument, we know that P_2 cannot be c^*-normal in G. Now suppose that P_i is s-semipermutable in G and P_iG_q is a group, $i = 1, 2$, where G_q is a Sylow q-subgroup of G. Thus we have P_1, P_2 such that $P = P_1P_2$. Hence PG_q is a group. By (3) and the famous p^aq^b-theorem we infer PG_q is a proper subgroup of G. Therefore, PG_q is p-nilpotent by the minimality of G.

(5) The final contradiction.

By (4) we have $[P, G_q] \leq G_q$ for any $q \neq p$. Suppose that S_1 is an arbitrary subgroup of P. Let $N_G(S_1) = N_1$. Since $[S_1, (N_1)_q] \leq S_1 \cap G_q = 1$, S_1 is centralized by $(N_1)_{p'}$.

Thus G is p-nilpotent by the famous Frobenius Theorem [8, 10.3.2], which is the final contradiction. □

Corollary 3.2 *Suppose that G is a group and P a Sylow subgroup of G. If every maximal subgroup of P is either s-semipermutable or c^*-normal in G, then G has a Sylow tower of supersolvable type.*

Proof Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. By hypothesis, every maximal subgroup of P is either s-semipermutable or c^*-normal in G. In particular, G satisfies the condition of Theorem 3.1, so G is p-nilpotent. Let U be the normal p-complement of G. By Lemmas 2.1 and 2.2, U satisfies the hypothesis. It follows by induction that U, and hence G possesses the Sylow tower property of supersolvable type. □

Corollary 3.3 [6, Theorem 3.1] *Let G be a group and $P = G_p$ a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P is either s-semipermutable or c-normal in G, then G is p-nilpotent.*

Corollary 3.4 *Suppose that G is a group and P a Sylow subgroup of G. If every maximal subgroup of P is either s-semipermutable or c-normal in G, then G has a Sylow tower of supersolvable type.*

We are now in a position to unify and generalize Theorem 4.1 and Theorem 4.3 in [11].

Theorem 3.5 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of any Sylow subgroup of H are either c^*-normal or s-semipermutable in G, then $G \in \mathscr{F}$.*

Proof Suppose that P is a Sylow p-subgroup of H , $\forall p \in \pi(H)$. Since every maximal subgroups of P are either c^*-normal or s-semipermutable in G, thus in H by Lemmas 2.1 and 2.2. By Corollary 3.2 we konw that H has a Sylow tower of supersolvable type. Let q be the maximal prime divisor of $|H|$ and $Q \in Syl_q(H)$. Then Q char $H \trianglelefteq G$. Since $(G/Q, H/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathscr{F}$. Since $Q \leq O_q(G)$, every maximal subgroups of Q are either S-quasinormal or c^*-normal in G by Lemma 2.2, in particular, c^*-normal in G. So $G \in \mathscr{F}$ by Lemma 2.8. □

Theorem 3.6 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all maxim al subgroups of any Sylow subgroup of $F^*(H)$ are either c^*-normal or s-semipermutable in G, then $G \in \mathscr{F}$.*

Proof Suppose that P is a Sylow p-subgroup of $F^*(H)$, $\forall p \in \pi(F^*(H))$. Since every maximal subgroups of P are either c^*-normal or s-semipermutable in G, thus in $F^*(H)$ by Lemmas 2.1 and 2.2. By Corollary 3.2 we konw that $F^*(H)$ has a Sylow tower of supersolvable type, in particular, $F^*(H)$ is Solvable. By Lemma 2.10 we have that $F^*(H) = F(H)$. Since $F^*(H)_p = F(H)_p$, $\forall p \in (H)$, every maximal subgroups of P are either c^*-normal or S-quasinormal in G by Lemma 2.2, in particular, c^*-normal in G. Applying Lemma 2.9 we get that $G \in \mathscr{F}$. □

Corollary 3.7 [12, Theorem 3.1] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of every Sylow subgroup of $F^*(H)$ are c-normal in G, then $G \in \mathscr{F}$.*

Corollary 3.8 [14, Theorem 1] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of any Sylow subgroup of H are s-semipermutable in G, then $G \in \mathscr{F}$.*

Corollary 3.9 [14, Theorem 2] *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are s-semipermutable in G, then $G \in \mathscr{F}$.*

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