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Semicontinuity of bifunctions and applications to regularization methods for equilibrium problems

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Abstract In this paper, we introduce a concept of semicontinuity on a subset with respect to the whole space and obtain that upper and lower semicontinuity are not needed in the whole space when solving equilibrium problems. The well-known Ky Fan's minimax inequality theorem is extended and applications to regularization methods for pseudomonotone bilevel equilibrium problems are given.

Keywords Bilevel equilibrium problem · Penalized equilibrium problem · Quasiconvexity · Semicontinuity · Pseudomonotonicity

Mathematics Subject Classification 26A15 · 65K10 · 90C33

1 Introduction

Let C be a nonempty closed and convex subset of $\mathbb{R}^n$ and let $\phi, \psi : C \times C \longrightarrow \mathbb{R}$ be two bifunctions satisfying $\phi(x, x) = \psi(x, x) = 0$, for every $x \in C$. Such bifunctions ϕ and ψ are called equilibrium bifunctions. We consider the following bilevel equilibrium problem

$$\text{find } x^* \in S(C, \psi) \text{ such that } \phi\left(x^*, y\right) \geq 0 \quad \forall y \in S(C, \psi) \qquad \text{BEP}$$

where $S(C, \psi)$ is the solution set of the following equilibrium problem

$$\text{find } \overline{x} \in C \text{ such that } \psi(\overline{x}, y) \geq 0 \quad \forall y \in C \qquad \text{EP}(C, \psi)$$

This problem encompasses several problems including variational inequalities, mathematical programming, Nash equilibrium, optimization and many other problems in nonlinear analysis.

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Regularization methods which are widely used in convex optimization and variational inequalities have been recently applied to equilibrium problems. The proximal point method which is a fundamental regularization technique for handling ill-posed problems has been recently applied in [22] to monotone bilevel equilibrium problems. Auxiliary problem principle and penalty function method as well as many other methods have also been considered for solving bilevel equilibrium problems, see for example, [8,10,12,16,19,21–23] and the references therein.

In this paper, we apply a penalty function method for solving pseudomonotone bilevel equilibrium problems under weak conditions of semicontinuity. First, we deal with results on existence of solutions for equilibrium problems. Some approaches introduced in [1–3] concerning continuity of bifunctions will be developed here and adapted to the notion of semicontinuity. We introduce the notions of upper and lowed semicontinuity on a subset with respect to the whole space. These notions allow us to state that both upper and lower semicontinuity in the first and second variable of equilibrium bifunctions are not needed in the whole space when solving equilibrium problems. This yields generalizations of some results on the existence of solutions of equilibrium problems including the well-known Ky Fan's minimax inequality theorem and the theorem of Bianchi and Schaible for pseudomonotone equilibrium problems.

In last part of this paper, we give applications of our approach to a penalty function method for bilevel equilibrium problems and provide an example of a bilevel equilibrium problem involving a strictly pseudomonotone bifunction which is not strongly monotone and not upper semicontinuous in the second variable on the whole space and a pseudomonotone bifunction which is not strictly pseudomonotone and not upper semicontinuous in the first variable on the whole space.

2 On semicontinuity of equilibrium bifunctions

In this section, we introduce the notions of lower and upper semicontinuity of bifunctions on subsets and apply them to obtain generalizations of some old existing results on the existence of solutions of equilibrium problems.

Let A be a subset of C. We say that a function $f : C \longrightarrow \mathbb{R}$ is

1. *Upper semicontinuous on A with respect to C* if for every $x \in A$ and every sequence $(x_n)_n$ in C converging to x, we have
$$f(x) \geq \limsup_{n\to+\infty} f(x_n)$$
where $\limsup_{n\to+\infty} f(x_n) = \inf_n \sup_{k\geq n} f(x_k)$.
2. *Lower semicontinuous on A with respect to C* if for every $x \in A$ and every sequence $(x_n)_n$ of C converging to x, we have
$$f(x) \leq \liminf_{n\to+\infty} f(x_n)$$
where $\liminf_{n\to+\infty} f(x_n) = \sup_n \inf_{k\geq n} f(x_k)$.

The following result provides us with sufficient conditions for semicontinuity on a subset with respect to the whole space and brings to light some tools for constructing examples, see Example 1 in next section.

Proposition 1 *Let $f : C \longrightarrow \mathbb{R}$ be a function and let A be a subset of C. If every point of A has a neighborhood in C on which the restriction of f is upper* (resp. *lower*) *semicontinuous, then f is upper* (resp. *lower*) *semicontinuous on A with respect to X.*

Corollary 1 *Let $f : C \longrightarrow \mathbb{R}$ be a function and let A be a subset of C. If the restriction of f to an open set in C containing A is upper* (resp. *lower*) *semicontinuous, then any extension of f to the space C is a upper* (resp. *lower*) *semicontinuous function on A with respect to C.*

Corollary 2 *Let $f : C \longrightarrow \mathbb{R}$ be a function and let A be a subset of C. If f is a upper* (resp. *lower*) *semicontinuous function on A with respect to C, then it is upper* (resp. *lower*) *semicontinuous on every subset of A with respect to C.*

The following lemma provides us with some properties of semicontinuous functions on a subset with respect to the whole space. Also, it shows the role played by this concept of semicontinuity and it will be used later for solving equilibrium problems.

Lemma 1 *Let $f : C \longrightarrow \mathbb{R}$ be a function, A a subset of C and $a \in \mathbb{R}$.*

1. *If f is a upper semicontinuous function on A with respect to C, then*
$$\overline{\{x \in C \mid f(x) \geq a\}} \cap A = \{x \in A \mid f(x) \geq a\}.$$
In particular, the traces on A of upper level sets of f are closed in A.
2. *If f is a lower semicontinuous function on A with respect to C, then*
$$\overline{\{x \in C \mid f(x) \leq a\}} \cap A = \{x \in A \mid f(x) \leq a\}.$$
In particular, the traces on A of lower level sets of f are closed in A.

Proof The second statement being similar to the first, we prove only the case of upper semicontinuity. Let
$$x^* \in \overline{\{x \in C \mid f(x) \geq a\}} \cap A.$$
Let $(x_n)_n$ be a sequence in the set $\{x \in C \mid f(x) \geq a\}$ converging to x^*. Since $x^* \in A$, then by upper semicontinuity of f on A with respect to C, we have
$$f\left(x^*\right) \geq \limsup_{n \to +\infty} f(x_n) \geq a.$$
Thus, $x^* \in \{x \in A \mid f(x) \geq a\}$. The converse holds from the fact that
$$\{x \in A \mid f(x) \geq a\} = \{x \in C \mid f(x) \geq a\} \cap A$$
which is obvious.

In the sequel, for a bifunction $\theta : C \times C \longrightarrow \mathbb{R}$ and $y \in C$, we define the following sets:
$$\theta^+(y) = \{x \in C \mid \theta(x, y) \geq 0\} \quad \text{and} \quad \theta^-(y) = \{x \in C \mid \theta(y, x) \leq 0\}.$$
Clearly, $x^* \in C$ is a solution of the equilibrium problem
$$\text{find } x^* \in C \text{ such that } \theta\left(x^*, y\right) \geq 0 \quad \forall y \in C \qquad \text{EP}(C, \theta)$$
if and only if $x^* \in \bigcap_{y \in C} \theta^+(y)$.

The following result extends the well-known Ky Fan's minimax inequality theorem (see [15,18,20]) for upper semicontinuous bifunctions on the compact subset of coercivity property with respect to C.

Theorem 1 *Let $\theta : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction and suppose the following assumptions hold:*

1. $\theta(x, x) = 0$, *for every $x \in C$;*
2. θ *is quasiconvex in its second variable on C;*
3. *there exists a compact subset K of C and $y_0 \in K$ such that*

$$\theta(x, y_0) < 0 \quad \forall x \in C \backslash K;$$

4. θ *is upper semicontinuous in its first variable on K with respect to C.*

Then, the equilibrium problem

$$\text{find } x^* \in C \text{ such that } \theta\left(x^*, y\right) \geq 0 \quad \forall y \in C$$

has a solution.

Proof By quasiconvexity of θ in its second variable and since $\theta^+(y_0)$ is contained in the compact K, then the conditions of Ky Fan Lemma are satisfied for the family $\left(\overline{\theta^+(y)}\right)_{y\in C}$ (see for example, [2,3,6,14,15,17,18]). That is, for every $y \in C$, $\overline{\theta^+(y)}$ are nonempty and closed, $\overline{\theta^+(y)}$ is compact and the convex hull of every finite subset $\{y_1, \ldots, y_n\}$ of C is contained in $\bigcup_{n=1}^{n} \overline{\theta^+(y)}$. Then, we have

$$\bigcap_{y\in C} \overline{\theta^+(y)} \neq \emptyset.$$

In the other hand, we have

$$\bigcap_{y\in C} \overline{\theta^+(y)} = \left(\bigcap_{y\in C} \overline{\theta^+(y)}\right) \cap K = \bigcap_{y\in C} \left(\overline{\theta^+(y)} \cap K\right).$$

By Lemma 1, we have

$$\overline{\theta^+(y)} \cap K = \theta^+(y) \cap K \quad \forall y \in C.$$

Thus,

$$\bigcap_{y\in C} \theta^+(y) = \bigcap_{y\in C} \overline{\theta^+(y)} \neq \emptyset$$

which completes the proof.

Remark 1 The condition (3) in the above theorem is known in the literature under the name of coercivity property.

The Minty Lemma for equilibrium problems deals in particular with properties such as compactness and convexity of the solution sets of equilibrium problems (see for example, [20]).

It is easily seen that under assumptions of Theorem 1, the solution set $S(C, \theta)$ of the equilibrium problem $(EP(C, \theta))$ is a nonempty compact set. For additional properties, let us recall the following concepts of monotonicity for bifunctions.

A bifunction $\theta : C \times C \longrightarrow \mathbb{R}$ is called

1. *strongly monotone* on C with modulus β if

$$\theta(x, y) + \theta(y, x) \leq -\beta\|x - y\|^2, \quad \forall x, y \in C,$$

2. *monotone* on C if

$$\theta(x,y)+\theta(y,x)\le 0,\quad \forall x,y\in C,$$

3. *strictly pseudomonotone* on C if

$$\theta(x,y)\ge 0 \Longrightarrow \theta(y,x)<0,\quad \forall x,y\in C, x\neq y,$$

4. *pseudomonotone* on C if

$$\theta(x,y)\ge 0 \Longrightarrow \theta(y,x)\le 0,\quad \forall x,y\in C.$$

Every strongly monotone bifunction is both monotone and strictly pseudomonotone and every strictly pseudomonotone bifunction θ is pseudomonotone provided that $\theta(x,x)=0, \forall x\in C$.

Remark 2 If $\theta : C\times C \longrightarrow \mathbb{R}$ is a strictly pseudomonotone bifunction, then for every subset A of C, the following equilibrium problem

$$\text{find } x^*\in A \text{ such that } \theta\left(x^*,y\right)\ge 0 \quad \forall y\in A$$

has at most one solution (see [6, Theorem 4.2]).

Proposition 2 *Under assumptions of Theorem 1, and if in addition*

1. *θ is pseudomonotone on C and*
2. *θ is explicitly quasiconvex in its second variable on C,*

then

$$\bigcap_{y\in C}\theta^+(y)=\left(\bigcap_{y\in C}\theta^-(y)\right)\cap K.$$

Thus, if in addition K is convex, then the solution set $S(C,\theta)$ is convex.

Proof By pseudomonotonicity, we have

$$\theta^+(y)\subset\theta^-(y).$$

Since $\bigcap_{y\in C}\theta^+(y)\subset K$, then

$$\bigcap_{y\in C}\theta^+(y)\subset\left(\bigcap_{y\in C}\theta^-(y)\right)\cap K.$$

Now, by explicit quasiconvexity (see [3]), we obtain

$$\left(\bigcap_{y\in C}\theta^-(y)\right)\cap K\subset\bigcap_{y\in C}\theta^+(y).$$

Finally, by quasiconvexity, the set $\theta^-(y)$ is convex, for every y. Thus, the solution set $S(C,\theta)$ is convex whenever K is convex.

As well-known in the literature, the equilibrium problem $(EP\,(C,\theta))$ can be also solved when the bifunction θ is not upper semicontinuous on its first variable. In this case some additional conditions are needed. The following result extends (under the settings of $\mathbb{R}^n$) some results of [3,6] on existence of solutions for pseudomontone equilibrium problems. Our restriction to $\mathbb{R}^n$ is motivated by the fact that the proof of Lemma 1 is based on the Fréchet-Urysohn property which is of course verified by $\mathbb{R}^n$. A space verifies the Fréchet-Urysohn property if a point is in the closure of a subset if and only if it is a limit of a sequence in the subset (see [4,13]).

Following [2,3], recall that a function $f : C \longrightarrow \mathbb{R}$ is said to be *upper hemicontinuous* on A with respect to C if for every $x \in C$ and $\overline{x} \in A$, there exists a sequence $(x_n)_n$ in the segment between x and $\overline{x}$ such that $\lim_{n\to+\infty} x_n = \overline{x}$ and

$$f\left(\overline{x}\right) \geq \limsup_{n\to+\infty} f\left(x_n\right).$$

Theorem 2 *Let $\theta : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction and suppose the following assumptions hold:*

1. $\theta\,(x,x) = 0$*, for every $x \in C$;*
2. *θ is pseudomonotone on C;*
3. *there exists a compact subset K of C and $y_0 \in K$ such that*

$$\theta\,(x, y_0) < 0 \quad \forall x \in C \backslash K;$$

4. *θ is upper hemicontinuous in its first variable on K with respect to C;*
5. *θ is explicitly quasiconvex in its second variable on C;*
6. *θ is lower semicontinuous in its second variable on K with respect to C.*

Then, the equilibrium problem

$$\textit{find } x^* \in C \textit{ such that } \theta\left(x^*, y\right) \geq 0 \quad \forall y \in C$$

has a solution.

Proof By Ky Fan Lemma, we have

$$\bigcap_{y\in C} \overline{\theta^+\,(y)} \neq \emptyset$$

and since $\theta^+\,(y_0)$ is contained in the compact K, then

$$\bigcap_{y\in C} \overline{\theta^+\,(y)} = \bigcap_{y\in C} \left(\overline{\theta^+\,(y)} \cap K\right).$$

Since θ is lower semicontinuous in its second variable on K with respect to C, then by applying Lemma 1, the set $\theta^-\,(y) \cap K$ is closed, for every $y \in C$. From pseudomonotonicity, we have $\theta^+\,(y) \subset \theta^-\,(y)$, for every $y \in C$. It follows that

$$\bigcap_{y\in C} \left(\overline{\theta^+\,(y)} \cap K\right) \subset \bigcap_{y\in C} \left(\theta^-\,(y) \cap K\right).$$

By [3, Lemma 1.3], we have

$$\bigcap_{y\in C} \left(\theta^-\,(y) \cap K\right) \subset \bigcap_{y\in C} \theta^+\,(y)\,.$$

This yields,

$$\bigcap_{y \in C} \overline{\theta^{+}(y)} = \bigcap_{y \in C} \theta^{+}(y)$$

and completes the proof.

As in Proposition 2, let us point out the following result concerning the solution sets of equilibrium problems.

Proposition 3 *Under assumptions of Theorem 2, the solution set $S(C, \theta)$ of the equilibrium problem $(EP(C, \theta))$ is a nonempty compact set. If in addition K is convex, then $S(C, \theta)$ is convex.*

Before closing this section, let us point out that among various generalizations of the notion of semicontinuity, there exists the notion of transfer semicontinuity introduced in [24] and used by several authors. The notion of transfer semicontinuity plays an important role in the existence of solutions of different kinds of equilibrium problems, see for example [3,17,24] and the references therein.

Here is a discussion in order to compare our notion of semicontinuity on a subset with respect to the whole space with that of transfer semicontinuity.

Let X and Y be two Hausdorff topological spaces. Recall that a bifunction $\theta : X \times Y \longrightarrow \mathbb{R}$ is said to be *transfer lower semicontinuous* in its second variable if, for every $x, y \in X \times Y$ with $\theta(x, y) > 0$, there exist $x' \in X$ and a neighborhood $V(y)$ of y in Y such that

$$\Phi\left(x', z\right) > 0 \quad \forall z \in V(y).$$

A bifunction $\theta : X \times Y \longrightarrow \mathbb{R}$ is said to be *transfer upper semicontinuous* in its second variable if $-\theta$ is transfer lower semicontinuous in its second variable.

It is well-known and easy to see that every upper (*resp.*. lower) semicontinuous bifunction in the second variable is a transfer upper (*resp.*. lower) semicontinuous bifunction in the second variable.

As a punctual definition, we say that $\theta : X \times Y \longrightarrow \mathbb{R}$ is a *transfer lower semicontinuous* bifunction in its second variable at a point $y \in Y$ if for every $x \in X$ with $\theta(x, y) > 0$, there exist $x' \in X$ and a neighborhood $V(y)$ of y in Y such that

$$\Phi\left(x', z\right) > 0 \quad \forall z \in V(y).$$

We say that a bifunction $\theta : X \times Y \longrightarrow \mathbb{R}$ is *transfer upper semicontinuous* in its second variable at a point $y \in Y$ if $-\theta$ is transfer lower semicontinuous in its second variable at $y \in Y$.

It is easily seen that $\theta : X \times Y \longrightarrow \mathbb{R}$ is a transfer upper (*resp.*. lower) semicontinuous bifunction in its second variable if and only if it is a transfer upper (*resp.*. lower) semicontinuous bifunction in its second variable at every point $y \in Y$.

Now, we consider our settings of $X = Y = C$, a nonempty closed and convex subset of $\mathbb{R}^n$, and let A be a subset of C. By analogy, we say that $\theta : C \times C \longrightarrow \mathbb{R}$ is a transfer upper (*resp.*. lower) semicontinuous bifunction in its second variable on A with respect to C if it is a transfer upper (*resp.*. lower) semicontinuous bifunction in its second variable at every point $y \in A$.

By using standard arguments as in Lemma 1, we can easily prove that if $\theta : C \times C \longrightarrow \mathbb{R}$ is a upper (*resp.*. lower) semicontinuous bifunction in its second variable on A with respect to C, then θ is a transfer upper (*resp.*. lower) semicontinuous bifunction in its second variable on A with respect to C.

Note that there exists in the literature a notion of transfer semicontinuity on a subset which has been introduced in [17] (see also [3, Theorem 2.4]) as follows. A bifunction $\theta : C \times C \longrightarrow \mathbb{R}$ is said to be a transfer upper (*resp.*. lower) semicontinuous bifunction in its second variable on a subset A of C if the restriction bifunction $\theta : A \times A \longrightarrow \mathbb{R}$ is transfer upper (*resp.*. lower) semicontinuous in the second variable.

Curiously, the notion of transfer semicontinuity seems to be not hereditary by restriction, that is, if $\theta : C \times C \longrightarrow \mathbb{R}$ is a transfer upper (*resp.*. lower) semicontinuous bifunction in its second variable, then nothing can guarantee that $\theta : A \times A \longrightarrow \mathbb{R}$ is transfer upper (*resp.*. lower) semicontinuous in its second variable. This is because x' which is required in A may exist only in X. It follows that the notions of being transfer upper (*resp.*. lower) semicontinuous in the second variable on a subset with respect to the whole space is different from that of being a transfer upper (*resp.*. lower) semicontinuous bifunction in the second variable as a restriction bifunction. Also, the notions of being a upper (*resp.*. lower) semicontinuous bifunction in the second variable on a subset with respect to the whole space and that of being a transfer upper (*resp.*. lower) semicontinuous bifunction in the second variable as a restriction bifunction are different and both are important when dealing with existence of solutions of equilibrium problems, see [3,17,24]. See also [9, Lemma 2] and [11, Lemma 2.3] for further related results.

To our opinion, more investigations are necessary in order to bring to light the exact role played in the various results concerning existence of solutions of equilibrium problems by each one of these two notions of semicontinuity on a subset with respect to the whole space and that of transfer semicontinuity on a subset. Although, these two notions seem to be similar but different, maybe they include a common property which is the key tool for existence of solutions of equilibrium problems.

3 A penalty function method for bilevel equilibrium problems

In this section, we use a penalty function method for solving bilevel equilibrium problems involving upper semicontinuous bifunctions in their first variable on a subset of coercivity property with respect to C.

For $\epsilon > 0$, define the bifunction $\varphi_\epsilon : C \times C \longrightarrow \mathbb{R}$ by

$$\varphi_\epsilon (x, y) = \psi (x, y) + \epsilon\phi (x, y) .$$

Let $(\epsilon_n)_n$ be a sequence of positive numbers. For every n, we consider the penalized equilibrium problem $\text{PEP}\big(C, \varphi_{\epsilon_n}\big)$ defined by

$$\text{find } x_n^* \in C \text{ such that } \varphi_{\epsilon_n} \left(x_n^*, y\right) \geq 0 \quad \forall y \in C \qquad \left(\text{PEP}\left(C, \varphi_{\epsilon_n}\right)\right)$$

where its set of solutions is denoted by $S\left(C, \varphi_{\epsilon_n}\right)$. Note that when ϕ or ψ is pseudomonotone, the penalized equilibrium problem $\left(\text{PEP}\left(C, \varphi_{\epsilon_n}\right)\right)$, in general, does not inherit any monotonicity property from ϕ and ψ. Thus, Theorem 1 is more adapted for solving penalized equilibrium problems.

The following result has been proved in [12] under some additional conditions including strong monotonicity of the bifunction ϕ, upper semicontinuity of ϕ and ψ in their first variable on C and lower semicontinuity of ϕ and ψ in their second variable on C.

Theorem 3 *Let K be a subset of C and $(\epsilon_n)_n$ is a sequence of positive numbers such that* $\lim_{n\to+\infty} \epsilon_n = 0$. *Suppose that the following assumptions hold:*

1. *ϕ and ψ are pseudomonotone on C;*

2. ϕ *and* ψ *are convex in the second variable on* C*;*
3. ϕ *and* ψ *are upper semicontinuous in the first variable on* K *with respect to* C.

Then any cluster point $x^* \in K$ *of a sequence* $(x_n)_n$ *with* $x_n \in S\left(C, \varphi_{\epsilon_n}\right)$ *for every n, is a solution to the original bilevel equilibrium problem* (BEP)*. If in addition, the following assumptions hold:*

1. ϕ *is strictly pseudomonotone;*
2. K *is compact and there exists* $y_0 \in C$ *such that* $\phi(x, y_0) < 0$*, for every* $x \in C \backslash K$*;*
3. *there exists* $A \subset K$ *such that* $\psi(x, y_0) < 0$*, for every* $x \in C \backslash A$.

Then for every n, the penalized problem $\left(\text{PEP}\left(C, \varphi_{\epsilon_n}\right)\right)$ *is solvable, and any sequence* $(x_n)_n$ *with* $x_n \in S\left(C, \varphi_{\epsilon_n}\right)$ *for every n, converges to the unique solution of the bilevel equilibrium problem* (BEP).

Proof Let $(x_n)_n$ be a sequence such that $x_n \in S\left(C, \varphi_{\epsilon_n}\right)$ admitting $x^* \in K$ as a cluster point. Without loss of generality, we may assume that $(x_n)_n$ converges to x^*. Then for every n, we have

$$\psi(x_n, y) + \epsilon_n \phi(x_n, y) \geq 0 \quad \forall y \in C.$$

Let $z \in S(C, \psi)$. By pseudomonotonicity of ψ, we have $\psi(x_n, z) \leq 0$. Since

$$\psi(x_n, z) + \epsilon_n \phi(x_n, z) \geq 0,$$

we have,

$$\epsilon_n \phi(x_n, z) \geq -\psi(x_n, z) \geq 0$$

which implies that $\phi(x_n, z) \geq 0$. Letting n go to $+\infty$, we obtain by upper semicontinuity of ϕ in its first variable on K with respect to C that $\phi(x^*, z) \geq 0$. Thus,

$$\phi\left(x^*, z\right) \geq 0 \quad \forall z \in S(C, \psi).$$

To complete the first part of the theorem, it remains to prove that $x^* \in S(C, \psi)$. Again by upper semicontinuity of ϕ and ψ in their first variable on K with respect to C and since

$$\psi(x_n, y) + \epsilon_n \phi(x_n, y) \geq 0 \quad \forall y \in C,$$

we have

$$\psi\left(x^*, y\right) \geq 0 \quad \forall y \in C.$$

Now, we prove the second part of the theorem. Note that for every n,

$$\psi(x_n, y_0) + \epsilon_n \phi(x_n, y_0) < 0 \quad \forall x \in C \backslash K.$$

Thus, by Theorem 1, the problem $\left(\text{PEP}\left(C, \varphi_{\epsilon_n}\right)\right)$ is solvable and its set of solutions $S\left(C, \varphi_{\epsilon_n}\right)$ is contained in K, for every n. Let $(x_n)_n$ be a sequence such that $x_n \in S\left(C, \varphi_{\epsilon_n}\right)$, for every n. Then, the sequence $(x_n)_n$ has a cluster point $x^* \in K$ and by the first part of the theorem, x^* is a solution to the bilevel equilibrium problem (BEP). Since ϕ is strictly pseudomonotone, the bilevel equilibrium problem (BEP) has a unique solution. It follows that every subsequence of the sequence $(x_n)_n$ admits x^* as a cluster point. Thus, the sequence $(x_n)_n$ converges to the unique solution x^* of the bilevel equilibrium problem (BEP).

Remark 3 When ϕ is strongly monotone on C, and convex and lower semicontinuous in its second variable, then by [12, Lemma 2.6], there exists a compact subset A such that

$$\varphi_{\epsilon_n}(x, y_0) < 0, \quad \forall x \in C \backslash (K \cup A), \forall n.$$

In this case, we need ϕ and ψ in the above theorem to be upper semicontinuous on $K \cup A$ with respect to C.

Let us point out in this section that techniques related to different kinds of regularization methods for solving equilibrium problems abound in the literature. In this direction, we mention that the approach of penalty function method for solving bilevel equilibrium problems used in Theorem 3 has been considered earlier in [8, Sect. 6] under the name of *Viscosity Principle for Equilibrium Problems* with ψ is monotone and ϕ is not necessarily strictly pseudomonotone. In this case, the bilevel equilibrium problem (BEP) may not be uniquely solvable.

As well-known, the monotonicity is an important concept which is stronger than pseudomontonicity and then, the other conditions involved for existence of solutions of equilibrium problems can be weaker in the presence of monotonicity than those in the presence of pseudomonotonicity. As a comparison with [8, Theorem 6.1 and Theorem 6.2], not only Theorem 3 is obtained under weakened conditions of semicontinuity on subsets with respect to the whole space, but also we avoid some conditions such as the lower semicontinuity in the second variable. However, the intriguing presence of upper hemicontinuity in the first variable of the bifunction ψ in [8, Theorem 6.1] rather than upper semicontinuity is very interesting to investigate. By similar techniques as in Theorem 2, it could be possible to replace upper hemicontinuity in the first variable of the bifunction ψ in [8, Theorem 6.1] by upper hemicontinuity in the first variable on the set of coercivity property which is introduced in [8, Theorem 6.1] with respect to the whole space. On the other hand, it is not clear whether upper hemicontinuity can be replaced by upper hemicontinuity on the subset of coercivity property with respect to the whole space in [8, Theorem 4.4] which is useful to obtain existence of solutions of the associated penalized equilibrium problems called *approximate problems* in [8].

As a conclusion of this short discussion, it seems that our approach based on the notion of semicontinuity on a subset with respect to the whole space can be used with different techniques employed in the literature for solving equilibrium problems. Further investigations are necessary to know all these techniques where our approach of weakening semicontinuity can be applied jointly. The Viscosity Principle for Equilibrium Problems studied in [8] reveals a priority interest to these investigations since it is stated under the more general setting of real topological Hausdorff vector spaces.

Now, we turn to constructing examples in relation with our techniques of weakening semicontinuity. There are in the literature some well-known examples of pseudomonotone equilibrium bifunctions, see for example, [3,5–7] and the references therein. With slight modification of them, we construct here the following bilevel equilibrium problem which provides us with two bifunctions ϕ and ψ satisfying all the conditions of Theorem 3 without being upper semicontinuous in their first variable on the whole space C.

Example 1 Let $C = \mathbb{R}$, $K = [-1, +1]$ and $y_0 = 0$.

– *Define the bifunction* $\phi : C \times C \longrightarrow \mathbb{R}$ *by*

$$\phi(x, y) = \begin{cases} \frac{y^4 - x^4}{65} & \text{if} \quad x = 2, \\ y^4 - x^4 & \text{otherwise.} \end{cases}$$

Clearly ϕ is strictly pseudomonotone on C, $\phi(x, x) = 0$, for every $x \in C$ and $\phi(x, 0) < 0$, for every $x \notin [-1, +1]$.

To see that ϕ is convex in its second variable, let $x \in C$ be fixed.

1. If $x = 2$, then $\phi(2, y) = \frac{y^4-16}{65}$, for every $y \in C$. The function $y \mapsto \frac{y^4-16}{65}$ is convex on C.
2. If $x \neq 2$, then $\phi(x, y) = y^4 - x^4$, for every $y \in C$. The function $y \mapsto y^4 - x^4$ is convex on C.

To see that ϕ is upper semicontinuous in its first variable on $[-1, +1]$ with respect to C, let $y \in C$ be fixed and denote by $f : C \longrightarrow \mathbb{R}$ the function defined by

$$f(x) = \phi(x, y).$$

The restriction $f_{|U}$ of f to the open set $U =]-\infty, 2[$ containing $[-1, +1]$ is defined by $f_{|U}(x) = y^4 - x^4$ which is continuous on U and then by Proposition 1, f is upper semicontinuous on $[-1, +1]$ with respect to C.

Finally, the bifunction ϕ is not upper semicontinuous in its first variable on C. Indeed, consider $y = 3$ for example. Let $(x_n)_n$ be a converging sequence to 2 such that $x_n \neq 2$, for every n. We have

$$\phi(2, 3) = 1 < 65 = \limsup_{n\to+\infty} \phi(x_n, 3).$$

– *Now, define the bifunction* $\psi : C \times C \longrightarrow \mathbb{R}$ by

$$\psi(x, y) = \begin{cases} (x+2)(y-x) & \text{if } \; x \in]-\infty, -2[, \\ (x+1)(y-x) & \text{if } \; x \in [-2, -1[, \\ \max(x, 0)(y-x) & \text{otherwise.} \end{cases}$$

Clearly, $\psi(x, x) = 0$, for every $x \in C$ and $\psi(x, 0) < 0$, for every $x \notin [-1, +1]$.

To verify that ψ is pseudomonotone on C, let $x, y \in C$ be such that $\psi(x, y) \geq 0$.

1. If $x \in]-\infty, -2[$, then $\psi(x, y) = (x+2)(y-x)$. It follows that $y - x \leq 0$ and then $y < -2$. Thus $\psi(y, x) = (y+2)(x-y) \leq 0$.
2. If $x \in [-2, -1[$, then $y - x \leq 0$ and then $y < -1$. If $y \in [-2, -1[$, then $\psi(y, x) = (y+1)(x-y) \leq 0$, and if $y \in]-\infty, -2[$, then $\psi(y, x) = (y+2)(x-y) \leq 0$.
3. If $x \geq -1$, then $y \geq x$. It follows that $y \geq -1$ and then $\psi(y, x) = \max(y, 0)(x-y) \leq 0$.

Clearly ψ is convex in its second variable on C and upper semicontinuous in its first variable on $[-1, +1]$ with respect to C.

To see that ψ is not upper semicontinuous in its first variable on C, consider $y > -2$ and take a sequence $(x_n)_n$ in $]-\infty, -2[$ converging to -2. We have

$$\psi(-2, y) = -(y+2) < 0 = \limsup_{n\to+\infty} (x_n+2)(y+2) = \limsup_{n\to+\infty} \psi(x_n, y).$$

Note that ψ is not lower semicontinuous in its first variable on C too. To see this fact, consider $y < -2$ and take a sequence $(x_n)_n$ in $]-\infty, -2[$ converging to -2. We have

$$\psi(-2, y) = -(y+2) > 0 = \liminf_{n\to+\infty} (x_n+2)(y+2) = \liminf_{n\to+\infty} \psi(x_n, y).$$

Finally, let us point out that ψ is not strictly pseudomonotone on C since $\psi(x, y) = \psi(y, x) = 0$ whenever $x, y \in [-1, 0]$.

In conclusion, the results and characterizations obtained in this paper about the notion of semicontinuity on a subset with respect to the whole space provides us with some tools to verify and construct examples of bifunctions which are upper (*resp.*. lower) semicontinuous on a subset with respect to the whole space without being upper (*resp.*. lower) on the whole space. On the other hand, the last example shows explicitly that even under additional conditions such as monotonicity and convexity, we can construct bifunctions which are semicontinuous on a subset with respect to the whole space without being semicontinuous on the whole space.

In this paper, we have also investigated equilibrium problems and obtained generalizations of some old existing results of Ky Fan, and Bianchi and Schaible about existence of solutions of equilibrium problems. As an application, we have stated in the last section of this paper that our techniques of weakening semicontinuity to the subset of coercivity property with respect to the whole space can be applied to regularization methods for solving bilevel equilibrium problems.

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