



Transient Analysis of Queueing-Based Congestion with Differentiated Vacations and Customer's Impatience Attributes

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Abstract

This research article studies the critical issue of the single-server congestion problem with prominent customer impatience attributes and server strategic differentiated vacation. Despite their apparent practical relevance, the proposed congestion problem has yet to be studied from a service/production perspective with transient analysis. The queue-theoretic approach is used for mathematical modeling. The transient queue-size distribution has been derived using a modified Bessel function and generating function technique. A time-dependent solution is advantageous for queueing systems' dynamic behavior over a planning phase and is predominantly valuable within the real-time design process for the state-of-the-art strategic system. The time-dependent explicit expression of variance and mean for the number of waiting customers in the system is also derived for quick statistical insights. Finally, numerical results are also exhibited to study the system's behavior in depth.

Keywords Modified Bessel function · Generating function · Differentiated vacations · Reneging · Balking

1 Introduction

The optimal service system emphasizes strategic congestion management to address the customer's traffic. Congestion management is an association between planning and operations. The research study's prime objective is to present a systematic process for managing customer congestion and provides critical information on the performance of the service system. The investigation identifies alternative strategies for alleviating congestion and enhancing customers' mobility to levels that attain a state of intended service. At the core, congestion management includes performance moni-

toring, alternative strategies for congestion, and norms for detecting when action is required. Studying congestion and its causes is used to develop more efficient and cost-effective services and systems. The critical goal of the studied service system is to prioritize strategies that would be most effective for congestion management. The queueing analysis is one of the most effective and practical mathematical tools for understanding and aiding decision-making in dealing with critical resources and managing congestion. The queueing theory aims to design efficient systems that render service competently to customers with a minimum delay but do not cost too much to be sustainable. Several queueing systems representing different service designs, regimes, and strategies, wherein the common feature is that customers arrive randomly at facilities to get service, need to investigate.

The queueing problems with customer impatience attribute and service provider strategic vacation interest many researchers in neoteric times due to their broad applicability in real-time congestion. Server vacation may occur for several reasons, including a low workload, maintenance time, the failure to repair, and many more. In recent years, there has been considerable research on customer impatience attributes in queueing systems with strategic server vacations/failures. Levy and Yechiali [1] were the ones who introduced the server vacation policy initially. A thorough, excellent, and exhaustive study of vacation queueing models is found in

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Doshi's survey [2], as well as in several publications on vacation queueing models (cf. [3–6]).

The strategic vacation policy variants include multi-vacation, single-vacation, working vacation, Bernoulli vacation, gated vacation, N -policy, etc. The server goes on vacation mode if found no waiting customer for service instead of continuing in an idle state and increasing the service cost. In a single-vacation policy, when the server returns from vacation, it serves any waiting customers in the system; otherwise, it stays idle. The server immediately takes another vacation in the multiple-vacation policy when it resumes from vacation and discovers no waiting customer in the system. In a working vacation policy, the server remotely offers the service at a slower rate instead of terminating it or removing itself from the system. In N -policy, the server remains on vacation until there is an accumulation of N customers. In the present study, we propose the multiple-vacation-based differentiated vacation queueing systems that are widely used strategies to control access to the service facility and simulate many energy-saving modes, such as wireless communications and flexible manufacturing systems. Isijola et al. [7] studied the variant of multiple vacations wherein two sorts of vacations, each with a different random duration, are analyzed. Vijayashree and Janani [8] analyzed the single-server queueing system incorporating differentiated vacations policy and obtained the transient probability using the modified Bessel function and Laplace transform techniques.

Kempa and Marjasz [9] derived the conditional probability distribution analytically for the queue size in a limited-buffer single-channel $M/G/1/N$ queueing model with batch arrivals operating under the multiple-vacation policy. They calculated the time to a first buffer overflow employing Korolyuk's potential, integral equations, and embedded Markov chain notions. Ayyappan and Deepa [10] analyzed a non-Markovian batch arrival bulk service $M^{[X]}/G(a, b)/1$ queueing system featuring multiple-vacation policies, service interruption & setup time with N -policy. In recent years, many researchers (cf [11–15]) opted for the multiple-vacation policy and several types of methodologies to analyze the performance characteristics and provided several numerical illustrations.

In everyday life, numerous queueing circumstances happen, and a long queue may deter customers. As a response, customers either elect not to join the line (i.e., balk) or leave after waiting due to impatience (i.e., renege). The dissatisfaction level of customers increases due to long waiting for service and deciding to leave the system without getting served at random times. Haight [16] first conceptualized customers' balking attribute in a single-server queueing model. Later, Haight [17] again proposed the renege attribute of customers for the $M/M/1$ queueing model. Many service systems originating in real-world applications may have intermittently inaccessible servers, impacting a customer's

sojourn duration and willingness to join. Naor [18] pioneered the research of queueing systems concerned with customers' reluctance behavior from an economic perspective. The economic assessment of customer-balking behavior is significant. Indeed, the approach and findings are also more important (cf. [19–21]). The decision of a waiting customer to stay or renege is continually offered till his departure from the system. The waiting time before renege depends on the service type [22]. For instance, if a customer is waiting for a mode of transportation and an unexpected event occurs that might cause more delays, the customer may opt to renege and utilize one of the available alternative service options instead (cf. [23, 24]). Al-Seedy et al. [25] provided a technique for evaluating transient probabilities of the queueing model $M/M/c$ incorporating renege and balking. Hassin [26] presumed to renege as a crucial component for the realistic modeling of customers' strategic behavior in queueing models involving vacations. Due to their adaptability and applicability, these models with impatient customers have been thoroughly evaluated (cf. [27–31]). Customers' impatience attributes are comprehended as a possible loss of customers, resulting in a loss of total income owing to their insurmountable influence on a system's intended financial situation from a cost perspective.

Kumar [32] is the first researcher who introduced the efficient notion of retention of the renege customer. Later, many researchers (cf. [33–37]) investigated retention of the renege customer in the service sector in economic perspective. Bouchentouf and Guendouzi [38] studied the $M^X/M/C$ queueing model, including multi-working vacation variants in modeling and computed the steady-state solution and henceforth performance measure for economic analysis using the probability generating function (PGF).

To the best of our surveys, no studies have been undertaken on customers' impatience attributes: balking and renege in queueing systems with differentiated-multiple vacations. The research gap makes a broader platform for our study. Customers may opt to be reluctant to service when a server goes on vacation and system congestion grows. In comparison with earlier research, the importance of our analysis is that we concentrate on the impact of balking and renege options in systems with differentiated-multiple vacations.

The structure of the remaining article is organized in the following order. We describe the proposed queueing-based congestion model along with its states and notations in Sect. 2. Section 3 explains the proposed methodology: modified Bessel function and generating function. In Sect. 4, we discuss the transient analysis employing the Laplace transformation and derive the state probabilities of the studied model. The system's performance measures are derived in Sect. 5 with the help of transient probabilities computed in the previous section. Section 6 contains different experimental results, numerical findings, and significant qualitative insights. In the



end, we conclude and offer potential study prospects for the future in Sect. 7.

2 Problem Statements and Associated Equations

In this article, we have considered a single-server queueing system with the following assumptions and notations:

Notations

- λ : the arrival rate of the customers
- β : the balking probability
- ξ : the renegeing rate
- μ : the service rate
- θ_1 : the type-1 vacation parameter
- θ_2 : the type-2 vacation parameter
- $N(t)$: number of customers in the system at time t
- $J(t)$: the state of the service provider at time t
- $\pi_{n,j}$: the probability that there are n customers in the system and service provider is in state j
- $m(t)$: expected number of the customers in the system at time t
- $V(t)$: variance of the number of the customers in the system at time t

- The customers are generated randomly from the population of prospective customers of size infinite.
- The inter-time between arrivals of customers for the intended service in the system is assumed exponentially with mean arrival rate λ .
- Upon arrival, the prospective customer gets the intended service immediately if the service provider is idle; otherwise, the customer joins the queue and waits for service.
- The customer may be impatient at the arrival epoch if the server is on vacation or busy. Each arrived customer may decide whether to balk or join the system with probability β or complementary probability $1 - \beta$, respectively.
- After waiting for some subsequent time interval, the customer may renege from the system. The random waiting time before renegeing is exponentially distributed with a mean time of $1/\xi$.
- There is one reliable server to serve the customer waiting in the system with finite capacity.
- The waiting customer is chosen for service following first-come-first-serve (FCFS) queue discipline.
- The continuous random variable, time-to-serve a customer, follows exponential distribution (memoryless distribution) with parameter μ .
- Under the strategic policy, we assume that there are two types of vacations: type-1 vacation and type-2 vacation.
- The type-1 vacation is initiated after a nonzero-length busy period and is independent of the busy period. The

vacation time for type-1 is exponentially distributed with parameter θ_1 .

- The type-2 vacation is initiated when no customer is queued for the service when the service provider returns from vacation. The duration of type-2 vacation follows an exponential distribution with parameter θ_2 .

All events' arrival/service, balking/renegeing, and vacation are independent of each other.

Let $(N(t), J(t))$ define a two-tuple continuous-time Markov chain (CTMC) with two-dimensional state space $S = \{(n, j) : n = 0, 1, 2, \dots \& j = 0, 1, 2\}$, where

- $N(t) \equiv$ number of customers present in the system at instant t
- $J(t) \equiv$ state of the service provider (SP) at instant t

where

$$J(t) = \begin{cases} 0; & \text{The SP is in active busy mode at instant } t \\ 1; & \text{The SP is on a type-1 vacation at instant } t \\ 2; & \text{The SP is on a type-2 vacation at instant } t \end{cases}$$

For modeling purposes, we define the joint probability distribution as

$$\pi_{n,j}(t) = Prob[N(t) = n, J(t) = j]; (n, j) \in S$$

The Chapman–Kolmogorov differential-difference equations for the studied model are derived using the assumptions and notations stated above. We start the analysis with the formation of equations for rate of change of joint probabilities $\pi_{n,j}; \forall n, j$ (state probabilities) for different states by balancing the inflow–outflow rates, i.e., outflow rate with negative sign and inflow with positive sign along with state probabilities.

$$\frac{d\pi_{1,0}(t)}{dt} = -(\lambda\beta + \mu)\pi_{1,0}(t) + \theta_1\pi_{1,1}(t) + \theta_2\pi_{1,2}(t) + (\mu + \xi)\pi_{2,0}(t) \tag{1}$$

$$\frac{d\pi_{n,0}(t)}{dt} = -(\lambda\beta + \mu + (n - 1)\xi)\pi_{n,0}(t) + \lambda\beta\pi_{n-1,0}(t) + \theta_1\pi_{n,1}(t) + \theta_2\pi_{n,2}(t) + (\mu + n\xi)\pi_{n+1,0}(t); \quad n = 2, 3, 4, \dots \tag{2}$$

$$\frac{d\pi_{0,1}(t)}{dt} = -(\lambda + \theta_1)\pi_{0,1}(t) + \mu\pi_{1,0}(t) \tag{3}$$

$$\frac{d\pi_{1,1}(t)}{dt} = -(\lambda\beta + \theta_1)\pi_{1,1}(t) + \lambda\pi_{0,1}(t) \tag{4}$$

$$\frac{d\pi_{n,1}(t)}{dt} = -(\lambda\beta + \theta_1)\pi_{n,1}(t) + \lambda\beta\pi_{n-1,1}(t); \quad n=2, 3, 4, \dots \tag{5}$$

$$\frac{d\pi_{0,2}(t)}{dt} = -\lambda\pi_{0,2}(t) + \theta_1\pi_{0,1}(t) \tag{6}$$

$$\frac{d\pi_{1,2}(t)}{dt} = -(\lambda\beta + \theta_2)\pi_{1,2}(t) + \lambda\pi_{0,2}(t) \tag{7}$$

$$\frac{d\pi_{n,2}(t)}{dt} = -(\lambda\beta + \theta_2)\pi_{n,2}(t) + \lambda\beta\pi_{n-1,2}(t); \quad n=2, 3, 4, \dots \tag{8}$$

The system of differential-difference equations (1)–(8) dependent on the initial conditions

$$\pi_{n,j}(0) = \begin{cases} 1; & n = 0, j = 1 \\ 0; & \text{otherwise} \end{cases}$$

is solved to obtain state probabilities employing mathematical notions of hypergeometric Laplace transform, modified Bessel’s function, generating function in the forthcoming section.

3 Mathematical Preliminaries

This section introduces some basic principles of modified Bessel functions and generating functions that the fellow researcher will need to comprehend this article better.

3.1 Modified Bessel Function

Bessel’s modified equation is given by:

$$t^2y''(t) + ty'(t) - (t^2 + r^2)y(t) = 0, \quad r \geq 0$$

The solution of the above equation is the first kind of modified Bessel function of order r , indicated by B_r , defined as

$$B_r(t) = \sum_{m=0}^{\infty} \frac{(t/2)^{2m+r}}{m!\Gamma(m+r+1)}, \quad r > 0$$

In particular, $B_r(t) = B_{-r}(t)$ for $r \geq 0$.

3.2 Generating Function

The following is a definition of a generating function $G(z, t)$ in powers of t for a collection of functions $\{f_m(z)\}$.

$$G(z, t) = \sum_{m=1}^{\infty} c_m f_m(z)t^m \tag{9}$$

where c_m is a parameter coefficient function of m of the set $\{f_m(z)\}$ and independent to z and t . The symbol $\{f_m(z)\}$ is used to indicate the infinite set $\{f_0(z), f_1(z), \dots, f_m(z), \dots\}$. If $f_m(z)$ is also defined for negative, function $H(z, t)$ having

a Laurent series expansion is of the form:

$$H(z, t) = \sum_{-\infty}^{\infty} c_m f_m(z)t^m \tag{10}$$

If $f_m(z)$ is the point probability function of a $drv z$, then the generating function is called a probability-generating function (cf. [39, 40]).

4 Transient Analysis

Using pre-stated mathematical notions of the Bessel function and generating function, we obtain the explicit formula for time-dependent queue-size distribution for the studied queueing-based congestion system in this section. We employ the following sequel for this purpose.

4.1 Laplace Transform

The following is the definition of the Laplace transform L of state probabilities $\pi_{n,j} \forall n, j$ and corresponding derivatives

$$\pi_{n,j}^*(s) = L(\pi_{n,j}(t)) = \int_0^{\infty} e^{-st} \pi_{n,j}(t) dt; \quad \forall n, j$$

$$L\left(\frac{d\pi_{n,j}(t)}{dt}\right) = s\pi_{n,j}^*(s) - \pi_{n,j}(0); \quad \forall n, j$$

The system of differential-difference equations from Eqs. (1)–(8) is converted as system of linear equations from Eqs.(11)–(18) on applying predefined Laplace transform as follows:

$$s\pi_{1,0}^*(s) - \pi_{1,0}(0) = -(\lambda\beta + \mu)\pi_{1,0}^*(s) + \theta_1\pi_{1,1}^*(s) + \theta_2\pi_{1,2}^*(s) + (\mu + \xi)\pi_{2,0}^*(s) \tag{11}$$

$$s\pi_{n,0}^*(s) - \pi_{n,0}(0) = -(\lambda\beta + \mu + (n-1)\xi)\pi_{n,0}^*(s) + \lambda\beta\pi_{n-1,0}^*(s) + \theta_1\pi_{n,1}^*(s) + \theta_2\pi_{n,2}^*(s) + (\mu + n\xi)\pi_{n+1,0}^*(s); \quad n = 2, 3, 4, \dots \tag{12}$$

$$s\pi_{0,1}^*(s) - \pi_{0,1}(0) = -(\lambda + \theta_1)\pi_{0,1}^*(s) + \mu\pi_{1,0}^*(s) \tag{13}$$

$$s\pi_{1,1}^*(s) - \pi_{1,1}(0) = -(\lambda\beta + \theta_1)\pi_{1,1}^*(s) + \lambda\pi_{0,1}^*(s) \tag{14}$$

$$s\pi_{n,1}^*(s) - \pi_{n,1}(0) = -(\lambda\beta + \theta_1)\pi_{n,1}^*(s) + \lambda\beta\pi_{n-1,1}^*(s); \quad n = 2, 3, 4, \dots \tag{15}$$

$$s\pi_{0,2}^*(s) - \pi_{0,2}(0) = -\lambda\pi_{0,2}^*(s) + \theta_1\pi_{0,1}^*(s) \tag{16}$$

$$s\pi_{1,2}^*(s) - \pi_{1,2}(0) = -(\lambda\beta + \theta_2)\pi_{1,2}^*(s) + \lambda\pi_{0,2}^*(s) \tag{17}$$

$$s\pi_{n,2}^*(s) - \pi_{n,2}(0) = -(\lambda\beta + \theta_2)\pi_{n,2}^*(s) + \lambda\beta\pi_{n-1,2}^*(s); \quad n = 2, 3, 4, \dots \tag{18}$$

Analytical solutions, even if approximate, give a straightforward method for decision-makers to estimate congestion and waiting time more quickly. They also typically lower the calculation time of traditional models by introducing better initial parameters into their optimization search space.

On applying initial condition $\pi_{0,1}(0) = 1$, from Eq. (13) we have

$$\begin{aligned} s\pi_{0,1}^*(s) &= 1 - (\lambda + \theta_1)\pi_{0,1}^*(s) + \mu\pi_{1,0}^*(s) \\ (s + \lambda + \theta_1)\pi_{0,1}^*(s) &= 1 + \mu\pi_{1,0}^*(s) \\ \pi_{0,1}^*(s) &= \frac{1}{s + \lambda + \theta_1} + \frac{\mu}{s + \lambda + \theta_1}\pi_{1,0}^*(s) \end{aligned} \tag{19}$$

Similarly on applying the initial condition $\pi_{1,1}(0) = 0$, from Eq.(14) we get

$$\begin{aligned} s\pi_{1,1}^*(s) &= -(\lambda\beta + \theta_1)\pi_{1,1}^*(s) + \lambda\pi_{0,1}^*(s) \\ \pi_{1,1}^*(s) &= \frac{\lambda}{s + \lambda\beta + \theta_1}\pi_{0,1}^*(s) \end{aligned} \tag{20}$$

With the initial condition $\pi_{n,1}(0) = 0; n = 2, 3, 4, \dots$, Eq. (15) gives

$$\begin{aligned} s\pi_{n,1}^*(s) &= -(\lambda\beta + \theta_1)\pi_{n,1}^*(s) + \lambda\beta\pi_{n-1,1}^*(s); n=2, 3, 4, \dots \\ \pi_{n,1}^*(s) &= \frac{\lambda\beta}{s + \lambda\beta + \theta_1}\pi_{n-1,1}^*(s); n = 2, 3, 4, \dots \end{aligned}$$

which recursively yields

$$\pi_{n,1}^*(s) = \left(\frac{\lambda\beta}{s + \lambda\beta + \theta_1}\right)^{n-1} \pi_{1,1}^*(s); n = 2, 3, 4, \dots$$

Hence, using Eq. (20), we get

$$\pi_{n,1}^*(s) = \left(\frac{\lambda}{s + \lambda\beta + \theta_1}\right)^n \beta^{n-1}\pi_{0,1}^*(s); n = 1, 2, 3, \dots \tag{21}$$

We henceforth solve Eq. (21) by substituting the value of $\pi_{0,1}^*(s)$ from Eq. (19)

$$\begin{aligned} \pi_{n,1}^*(s) &= \frac{\lambda^n \beta^{n-1}}{(s + \lambda\beta + \theta_1)^{n+1}} \\ &+ \frac{\mu \lambda^n \beta^{n-1}}{(s + \lambda + \theta_1)(s + \lambda\beta + \theta_1)^{n+1}} \pi_{1,0}^*(s); \\ n &= 1, 2, 3, \dots \end{aligned} \tag{22}$$

Since $\pi_{0,2}(0) = 0$, Eq. (16) deduces as

$$\begin{aligned} (s + \lambda)\pi_{0,2}^*(s) &= \theta_1\pi_{0,1}^*(s) \\ \pi_{0,2}^*(s) &= \left(\frac{\theta_1}{s + \lambda}\right)\pi_{0,1}^*(s) \end{aligned} \tag{23}$$

Hence, from Eqs. (19) and (23) we get

$$\begin{aligned} \pi_{0,2}^*(s) &= \frac{\theta_1}{(s + \lambda)(s + \lambda + \theta_1)} \\ &+ \frac{\theta_1 \mu}{(s + \lambda)(s + \lambda + \theta_1)} \pi_{1,0}^*(s) \end{aligned} \tag{24}$$

Using the initial condition $\pi_{1,2}(0) = 0$, Eq. (17) reduces to:

$$\pi_{1,2}^*(s) = \left(\frac{\lambda}{s + \lambda\beta + \theta_2}\right)\pi_{0,2}^*(s) \tag{25}$$

Similarly, under the initial condition $\pi_{n,2}(0) = 0$, Eq. (18) reduces as:

$$\pi_{n,2}^*(s) = \left(\frac{\lambda\beta}{s + \lambda\beta + \theta_2}\right)\pi_{n-1,2}^*(s); n = 2, 3, 4, \dots \tag{26}$$

which recursively yields

$$\pi_{n,2}^*(s) = \left(\frac{\lambda}{s + \lambda\beta + \theta_2}\right)^n \beta^{n-1}\pi_{0,2}^*(s); n = 1, 2, 3, \dots \tag{27}$$

Using Eqs. (24) and (27), we have

$$\begin{aligned} \pi_{n,2}^*(s) &= \frac{\theta_1 \lambda^n \beta^{n-1}}{(s + \lambda)(s + \lambda + \theta_1)(s + \lambda\beta + \theta_2)^n} \\ &+ \frac{\theta_1 \mu \lambda^n \beta^{n-1}}{(s + \lambda)(s + \lambda + \theta_1)(s + \lambda\beta + \theta_2)^n} \\ &\times \pi_{1,0}^*(s); n = 0, 1, 2, \dots \end{aligned} \tag{28}$$

After taking partial fraction and the inverse Laplace transform in Eqs. (22) and (28), we have

$$\begin{aligned} \pi_{n,1}(t) &= \frac{\lambda^n \beta^{n-1} e^{-(\lambda\beta + \theta_1)t}}{n!} \\ &+ \mu \lambda^n \beta^{n-1} \left\{ \frac{t^n e^{-(\lambda + \theta_1)t}}{n!} \times \frac{t^n e^{-(\lambda\beta + \theta_1)t}}{n!} \times \pi_{1,0}(t) \right\}; \\ n &= 1, 2, 3, \dots \\ \pi_{n,2}(t) &= \theta_1 \lambda^n \beta^{n-1} \left\{ e^{-\lambda t} \times e^{-(\lambda + \theta_1)t} \times \frac{t^{(n-1)} e^{-(\lambda\beta + \theta_2)t}}{(n-1)!} \right\} \\ &+ \theta_1 \mu \lambda^n \beta^{n-1} \\ &\left\{ e^{-\lambda t} \times e^{-(\lambda + \theta_1)t} \times \frac{t^{(n-1)} e^{-(\lambda\beta + \theta_2)t}}{(n-1)!} \times \pi_{1,0}(t) \right\}; \\ n &= 0, 1, 2, \dots \end{aligned}$$

Define the probability-generating function (PGF) as

$$\mathcal{P}(z, t) = \sum_{n=1}^{\infty} \pi_{n,0}(t) z^n$$

then

$$\frac{\partial \mathcal{P}(z, t)}{\partial t} = \sum_{n=1}^{\infty} \pi'_{n,0}(t) z^n$$

Using Eqs. (1) and (2), after some algebra we have

$$\begin{aligned} \frac{\partial \mathcal{P}(z, t)}{\partial t} - (\xi(1-z)) \frac{\partial \mathcal{P}(z, t)}{\partial z} &= \left((1-z^{-1})(\xi-\mu) + \lambda\beta(z-1) \right) \mathcal{P}(z, t) \\ &+ \sum_{n=1}^{\infty} \theta_1 \pi_{n,1}(t) z^n + \sum_{n=1}^{\infty} \theta_2 \pi_{n,2}(t) z^n - \mu \pi_{1,0}(t) \end{aligned} \tag{29}$$

On solving Eq. (29), we obtain

$$\begin{aligned} \mathcal{P}(z, t) &= \exp\left[((z^{-1}-1)(\mu-\xi) + \lambda\beta(z-1))t \right] \\ &+ \int_0^t \exp\left([(z^{-1}-1)(\mu-\xi) + \lambda\beta(z-1)](t-u) \right) \\ &\times \left[\sum_{n=1}^{\infty} \theta_1 \pi_{n,1}(t) z^n + \sum_{n=1}^{\infty} \theta_2 \pi_{n,2}(t) z^n - \mu \pi_{1,0}(t) \right] du \end{aligned} \tag{30}$$

It is well known that if

$$\Psi = 2\sqrt{\beta\lambda(\mu-\xi)} \quad \& \quad \Omega = \sqrt{\frac{\beta\lambda}{\mu-\xi}} \tag{31}$$

then

$$\exp\left\{ \left(\lambda\beta z + \frac{\mu-\xi}{z} \right) t \right\} = \sum_{n=-\infty}^{\infty} (\Omega z)^n I_n(\Psi t) \tag{32}$$

Using Eq. (32), we have

$$\begin{aligned} \mathcal{P}(z, t) &= \exp\left(\lambda\beta z + \frac{\mu-\xi}{z} \right) t \times \exp\left(-(\mu-\xi) + \lambda\beta \right) t \\ &+ \int_0^t \exp\left(\lambda\beta z + \frac{\mu-\xi}{z} \right) (t-u) \\ &\times \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \sum_{n=1}^{\infty} \theta_1 \pi_{n,1}(t) z^n du \\ &+ \int_0^t \exp\left(\lambda\beta z + \frac{\mu-\xi}{z} \right) (t-u) \\ &\times \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \sum_{n=1}^{\infty} \theta_2 \pi_{n,2}(t) z^n du \\ &- \int_0^t \exp\left(\lambda\beta z + \frac{\mu-\xi}{z} \right) (t-u) \end{aligned}$$

$$\times \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \mu \pi_{1,0}(t) du \tag{33}$$

On equating the coefficient of n th power of z of Eq. (33) on the both sides for $n = 0, 1, 2, \dots$, we have

$$\begin{aligned} \pi_{n,0}(t) &= \Omega^n I_n(\Phi t) \exp\left(-(\mu-\xi) + \lambda\beta \right) t \\ &+ \theta_1 \int_0^t \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \\ &\left[\sum_{k=0}^n \Omega^k I_k(\cdot) \pi_{n-k,1}(t) + \sum_{k=0}^n \Omega^{-k} I_k(\cdot) \pi_{n+k,1}(u) \right] du \\ &+ \theta_2 \int_0^t \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \\ &\left[\sum_{k=0}^n \Omega^k I_k(\cdot) \pi_{n-k,2}(t) + \sum_{k=0}^n \Omega^{-k} I_k(\cdot) \pi_{n+k,2}(u) \right] du \\ &- \mu \int_0^t \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \Omega^n I_n(\Phi t) \pi_{1,0}(t) du \end{aligned} \tag{34}$$

where $I_n = I_n(\alpha(t-u))$. Equation (34) holds for negative integer $n = -1, -2, -3, \dots$ with the LHS substituted as zero. Using $I_{-n}(\cdot) = I_n(\cdot)$ for $n = 1, 2, 3, \dots$

$$\begin{aligned} 0 &= \Omega^{-n} I_{-n}(\Phi t) \exp\left(-(\mu-\xi) + \lambda\beta \right) t \\ &+ \theta_1 \int_0^t \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \\ &\left[\sum_{k=0}^{\infty} \Omega^{-(n+k)} I_{n+k}(\cdot) \pi_{n+k,1}(t) \right] du \\ &+ \theta_2 \int_0^t \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \\ &\left[\sum_{k=0}^{\infty} \Omega^{-(n+k)} I_{n+k}(\cdot) \pi_{n+k,2}(t) \right] du \\ &- \mu \int_0^t \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \Omega^{-n} I_{-n}(\Phi t) \pi_{1,0}(t) du \end{aligned} \tag{35}$$

By Eqs. (34) and (35), for $n = 1, 2, 3, \dots$, we have state probabilities when service provider is in busy state at instant t as:

$$\begin{aligned} \pi_{n,0}(t) &= \exp\left(-(\mu-\xi) + \lambda\beta \right) t \\ &\left[\Omega^n I_n(\Phi t) - \Omega^{-n} I_{-n}(\Phi t) \right] \\ &+ \theta_1 \int_0^t \exp\left(-(\mu-\xi) + \lambda\beta \right) (t-u) \\ &\times \left[\sum_{k=0}^n \Omega^k I_k(\cdot) \pi_{n-k,1}(t) + \sum_{k=0}^n \Omega^{-k} I_k(\cdot) \pi_{n+k,1}(u) \right] \end{aligned}$$

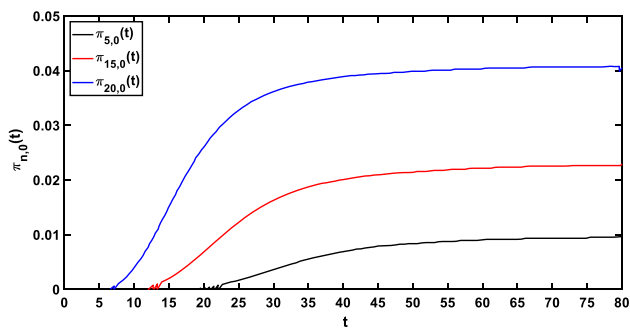


Fig. 1 The variation of the state probability $\pi_{n,0}(t)$ wrt t

$$\begin{aligned}
 & - \sum_{k=0}^{\infty} \Omega^{n-k} I_{n+k}(\cdot) \pi_{n+k,1}(t) \Big] du \\
 & + \theta_2 \int_0^t \exp\left(-(\mu - \xi) + \lambda\beta\right)(t - u) \\
 & \times \left[\sum_{k=0}^n \Omega^k I_k(\cdot) \pi_{n-k,2}(t) + \sum_{k=0}^n \Omega^{-k} I_k(\cdot) \pi_{n+k,2}(u) \right. \\
 & \left. - \sum_{k=0}^{\infty} \Omega^{n-k} I_{n+k}(\cdot) \pi_{n+k,2}(t) \right] du \tag{36}
 \end{aligned}$$

Hence, state probabilities at instant t for $n = 0, 1, 2, \dots$ when service provider is on type-1 vacation as

$$\begin{aligned}
 \pi_{n,1}(t) = & \frac{\lambda^n \beta^{n-1} e^{-(\lambda\beta+\theta_1)t}}{n!} \\
 & + \mu \lambda^n \beta^{n-1} \\
 & \left\{ \frac{t^n e^{-(\lambda+\theta_1)t}}{n!} \times \frac{t^n e^{-(\lambda\beta+\theta_1)t}}{n!} \times \pi_{1,0}(t) \right\}
 \end{aligned}$$

and is on type-2 vacation as

$$\begin{aligned}
 \pi_{n,2}(t) = & \theta_1 \lambda^n \beta^{n-1} \left\{ e^{-\lambda t} \times e^{-(\lambda+\theta_1)t} \times \frac{t^{(n-1)} e^{-(\lambda\beta+\theta_2)t}}{(n-1)!} \right\} \\
 & + \theta_1 \mu \lambda^n \beta^{n-1} \left\{ e^{-\lambda t} \times e^{-(\lambda+\theta_1)t} \times \frac{t^{(n-1)} e^{-(\lambda\beta+\theta_2)t}}{(n-1)!} \times \pi_{1,0}(t) \right\},
 \end{aligned}$$

respectively.

For the default value of the involved parameters $\lambda = 0.3$; $\mu = 0.5$, $\xi = 0.1$, $\beta = 0.6$, $\theta_1 = 0.3$ and $\theta_2 = 0.4$, we plot the variation of state-probabilities $\pi_{n,0}$, $\pi_{n,1}$, and $\pi_{n,2}$ in Figs. 1, 2, and 3, respectively, wherein the deviation is displayed for $n = 5, 15$, and 20 . Figures 1, 2, and 3 illustrate that the state probabilities become stable, which prompt the system to tend to steady state after a long time. Initially, there is much fluctuation in state probabilities, which shows the customers are getting service immediately.

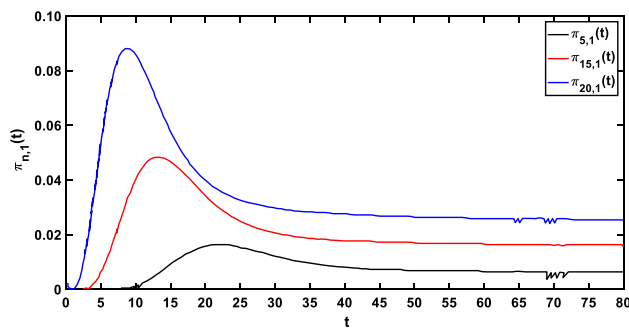


Fig. 2 The variation of the state probability $\pi_{n,1}(t)$ wrt t

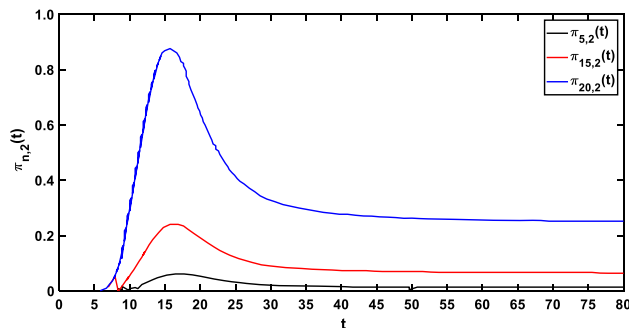


Fig. 3 The variation of the state probability $\pi_{n,2}(t)$ wrt t

5 Performance Measures

The acceptance of any queueing model is best evaluated in terms of its system characteristics. Evaluating queueing system performance indices is the most essential and promising method for improving any system. Systematic observation of the state genuinely aids decision-makers in enhancing the performance and efficiency of the queueing system.

5.1 Expectation of $N(t)$

Estimating the number of customers in the system $N(t)$ at arbitrary instant t is the primary goal of any queueing modeling. Here, it is expressed as:

$$\begin{aligned}
 m(t) = & E(N(t)) \\
 = & \sum_{n=1}^{\infty} n (\pi_{n,0}(t) + \pi_{n,1}(t) + \pi_{n,2}(t))
 \end{aligned}$$

On differentiating both sides wrt t , we have

$$m'(t) = \sum_{n=1}^{\infty} n (\pi'_{n,0}(t) + \pi'_{n,1}(t) + \pi'_{n,2}(t)) \tag{37}$$

On substituting the value from Eqs. (1)–(8) in Eq. (37) and using some mathematical manipulation, we get

$$\begin{aligned}
 m'(t) = & -(\lambda\beta + \mu)\pi_{1,0}(t) + (\mu + \xi)\pi_{2,0}(t) \\
 & - \lambda\beta(\pi_{1,1}(t) + \pi_{1,2}(t)) + \lambda(\pi_{0,1}(t) + \pi_{0,2}(t)) \\
 & + \sum_{n=2}^{\infty}(\mu - \lambda\beta - \xi)n\pi_{n,0}(t) \\
 & + \xi \sum_{n=2}^{\infty} n^2\pi_{n,0}(t) \\
 & + \lambda\beta \sum_{n=2}^{\infty} n\pi_{n-1,0}(t) - \lambda\beta \sum_{n=2}^{\infty} n\pi_{n,2}(t) \\
 & + \mu \sum_{n=2}^{\infty} n\pi_{n+1,0}(t) + \xi \sum_{n=2}^{\infty} n^2\xi\pi_{n+1,0}(t) \\
 & - \lambda\beta \sum_{n=2}^{\infty} n\pi_{n,1}(t) \\
 & + \lambda\beta \sum_{n=2}^{\infty} n\pi_{n-1,1}(t) + \lambda\beta \sum_{n=0}^{\infty} n\pi_{n-1,2}(t) \quad (38)
 \end{aligned}$$

$$\begin{aligned}
 m(t) = & -(\lambda\beta + \mu) \int_0^t \pi_{1,0}(y)dy + (\mu + \xi) \int_0^t \pi_{2,0}(y)dy \\
 & - \lambda\beta \int_0^t (\pi_{1,1}(y) + \pi_{1,2}(y)) dy \\
 & + \int_0^t \lambda (\pi_{0,1}(y) + \pi_{0,2}(y)) dy \\
 & + \sum_{n=2}^{\infty} \int_0^t (\mu - \lambda\beta - \xi)n\pi_{n,0}(y)dy \\
 & + \xi \sum_{n=2}^{\infty} \int_0^t n^2\pi_{n,0}(y)dy \\
 & + \lambda\beta \sum_{n=2}^{\infty} \int_0^t n\pi_{n-1,0}(y)dy - \lambda\beta \sum_{n=2}^{\infty} \int_0^t n\pi_{n,2}(y)dy \\
 & + \mu \sum_{n=2}^{\infty} \int_0^t n\pi_{n+1,0}(y)dy + \xi \sum_{n=2}^{\infty} \int_0^t n^2\xi\pi_{n+1,0}(y)dy \\
 & - \lambda\beta \sum_{n=2}^{\infty} \int_0^t n\pi_{n,1}(y)dy + \lambda\beta \sum_{n=2}^{\infty} \int_0^t n\pi_{n-1,1}(y)dy \\
 & + \lambda\beta \sum_{n=0}^{\infty} \int_0^t n\pi_{n-1,2}(y)dy \quad (39)
 \end{aligned}$$

5.2 The variance of $N(t)$

The variance $V(t)$ of a number of customers in the system $N(t)$ at an arbitrary instant t is calculated as:

$$V(t) = E(N^2(t)) - (E(N(t)))^2 \quad (40)$$

where $E(N^2(t))$ represents the 2nd moment of $drv N(t)$ at instant t . Therefore,

$$E(N^2(t)) = \sum_{n=1}^{\infty} n^2 (\pi_{n,0}(t) + \pi_{n,1}(t) + \pi_{n,2}(t))$$

$$E(N(t)) = \sum_{n=1}^{\infty} n (\pi_{n,0}(t) + \pi_{n,1}(t) + \pi_{n,2}(t))$$

Differentiating both sides of Eq. (40) with respect to t yields:

$$V'(t) = E'(N^2(t)) - (E'(N(t)))^2 \quad (41)$$

On substituting the values of computed state probabilities, we get

$$\begin{aligned}
 V'(t) = & -(\mu + \lambda\beta)\pi_{1,0}(t) + (\mu + \xi)\pi_{2,0}(t) - \lambda\beta\pi_{1,1}(t) \\
 & + \lambda\pi_{0,1}(t) - \lambda\beta\pi_{1,2}(t) + \lambda\pi_{0,2}(t) \\
 & + (\mu - \lambda\beta) \sum_{n=2}^{\infty} n^2\pi_{n,0}(t) + \xi \sum_{n=2}^{\infty} n^2(n - 1)\pi_{n,0}(t) \\
 & + \theta_2 \sum_{n=2}^{\infty} n^2\pi_{n,2}(t) + \mu \sum_{n=2}^{\infty} n^2\pi_{n+1,0}(t) \\
 & + \xi \sum_{n=2}^{\infty} n^3\pi_{n+1,0}(t) - \lambda\beta \sum_{n=2}^{\infty} n^2\pi_{n,1}(t) \\
 & + \lambda\beta \sum_{n=2}^{\infty} n^2\pi_{n-1,1}(t) - (\lambda\beta + \theta_2) \sum_{n=2}^{\infty} n^2\pi_{n,2}(t) \\
 & + \lambda\beta \sum_{n=2}^{\infty} n^2\pi_{n-1,2}(t) - m'(t) \quad (42)
 \end{aligned}$$

Hence, we have

$$\begin{aligned}
 V(t) = & -(\mu + \lambda\beta) \int_0^t \pi_{1,0}(y)dy + (\mu + \xi) \int_0^t \pi_{2,0}(y)dy \\
 & - \lambda\beta \int_0^t \pi_{1,1}(y)dy + \lambda \int_0^t \pi_{0,1}(y)dy \\
 & - \lambda\beta \int_0^t \pi_{1,2}(y)dy + \lambda \int_0^t \pi_{0,2}(y)dy \\
 & - (\mu + \lambda\beta) \sum_{n=2}^{\infty} \int_0^t n^2\pi_{n,0}(y)dy \\
 & + \xi \sum_{n=2}^{\infty} \int_0^t n^2(n - 1)\pi_{n,0}(y)dy \\
 & + \theta_2 \sum_{n=2}^{\infty} \int_0^t n^2\pi_{n,2}(y)dy + \mu \sum_{n=2}^{\infty} \int_0^t n^2\pi_{n+1,0}(y)dy
 \end{aligned}$$

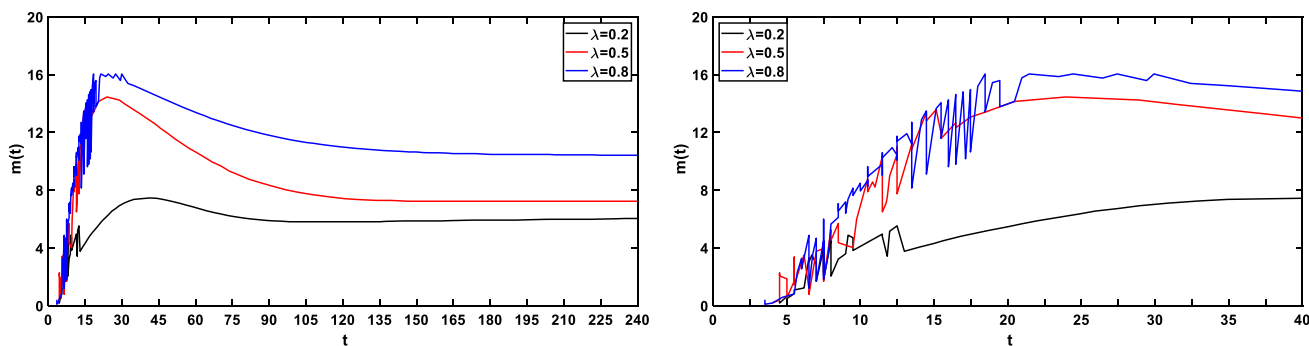


Fig. 4 The variation of the mean number of the customers in the system $m(t)$ wrt t

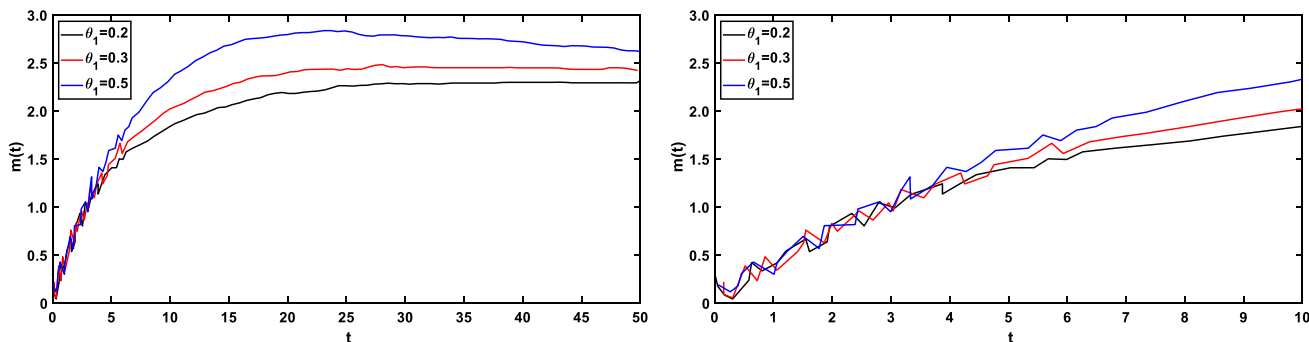


Fig. 5 The variation of the mean number of the customers in the system $m(t)$ wrt t

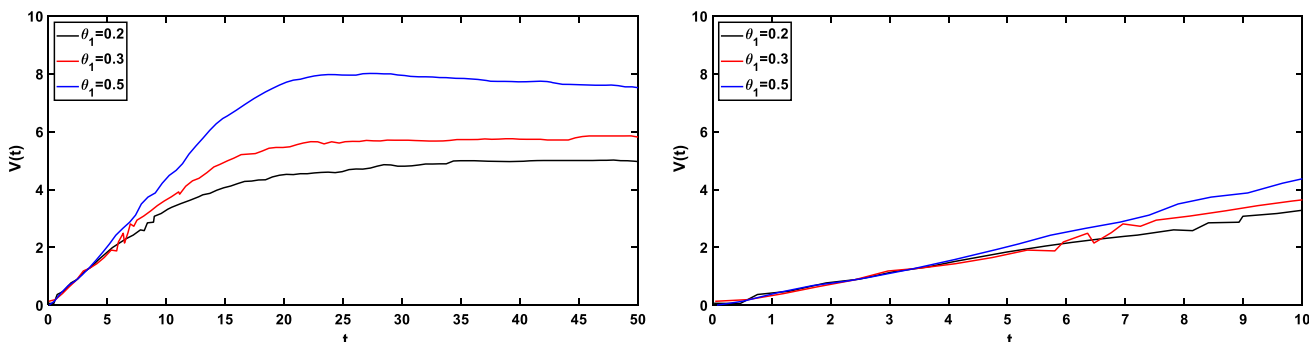


Fig. 6 The variation of the variance of the number of the customers in the system $V(t)$ wrt t

$$\begin{aligned}
 & + \xi \sum_{n=2}^{\infty} \int_0^t n^3 \pi_{n+1,0}(y) dy \\
 & - \lambda \beta \sum_{n=2}^{\infty} \int_0^t n^2 \pi_{n,1}(y) dy \\
 & + \lambda \beta \sum_{n=2}^{\infty} \int_0^t n^2 \pi_{n-1,1}(y) dy \\
 & - (\lambda \beta + \theta_2) \sum_{n=2}^{\infty} \int_0^t n^2 \pi_{n,2}(y) dy \\
 & + \lambda \beta \sum_{n=2}^{\infty} n^2 \int_0^t \pi_{n-1,2}(y) dy - m(t) \tag{43}
 \end{aligned}$$

6 Numerical Result

The numerical results for different experiments conducted on MAPLE software with a computing system of hardware configuration having processor Intel(R) Core(TM) i5-5200U CPU @ 2.20GHz and RAM 16.0 GB for various involved parameters are summarized in Figs. 4, 5, and 6. The depicted results show the effects of various system parameters on the system performance measures, namely expected customers count in the system ($m(t)$) and time-dependent variance ($V(t)$). Initially we set the default value of system parameters as $\lambda = 0.3$; $\mu = 0.5$, $\xi = 0.1$, $\beta = 0.6$, $\theta_1 = 0.3$ and $\theta_2 = 0.4$.

Figure 4 depicts the deviation in the mean number of customers in the system $wrt t$ for different values of λ as 0.2, 0.5, and 0.8. The apparent result is that $m(t)$ is increasing $wrt \lambda$. As the time t is large, the plot becomes uniform, revealing the system achieves stability after a long time, and the system tends to steady state. Initially, a lot of fluctuation of decreasing and increasing value is observed with customer accumulation before stability.

Figure 5 depicts the deviation of the expected number of customers in the system $m(t)$ wrt time t for varying the duration of type-1 vacation as $\theta_1 = 0.2, 0.3, \text{ and } 0.5$, which denotes the rate at which the server joins the system from type-1 vacation mode. Figure 5 indicates that $m(t)$ increases with time for all values of θ_1 with some fluctuation in the initial time. As the server's vacation time is longer, the system remains with no service provider in this period, and arriving customers either join or show balking behavior. This pattern can be easily inferred from Fig. 5 as the customers' count in the system rises for the lesser value of parameter θ_1 .

While Fig. 6 illustrates the graph of variance $V(t)$ with time t for varying the duration of type-1 vacation as $\theta_1 = 0.2, 0.3, \text{ and } 0.5$, Figure 6 reveals that $V(t)$ increases with time for all values of θ_1 . As the server's vacation time decreases, arriving customers' reluctance behavior decreases. This observation can be easily incidental from Fig. 6 as the customers' count variance in the system upsurges for the lesser parameter θ_1 .

7 Conclusion

In this article, we have analyzed the queueing-based congestion model incorporating strategic, differentiated-multiple vacations and customer impatience attributes like balking and reneging. The studied system has infinite differential-difference equations, which are solved with the help of Laplace transformation, Bessel modified function, and generating function techniques. The transient analysis of the system gives the explicit formula for transient-state probabilities of the proposed queueing-based congestion system. The investigation demonstrates the dynamic congestion behavior in the planning phase. These transient probabilities are helpful in the evaluation of the characteristic measure of the system. The numerical illustrations are performed, which also justify the theoretical results. The present model can be extended for service with general distribution and batch arrivals. The unreliability of the server can also be included in future work.

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Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest There is no conflict of interest among all authors regarding the contents and publication of any part of the article.

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