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Some Generalized Complex Intuitionistic Fuzzy Aggregation Operators and Their Application to Multicriteria Decision-Making Process

Harish Garg[1] · Dimple Rani[1]

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Abstract
The objective of this manuscript is to present some generalized weighted averaging aggregation operators for aggregating the different complex intuitionistic fuzzy sets using t-norm operations. In the existing studies of fuzzy and its extension, the uncertainties present in the data are handled with the help of degrees of membership which are the subset of real numbers, which may loose some useful information and hence consequently affect on the decision results. As a modification to these, complex intuitionistic fuzzy set handles the uncertainties with the degrees whose ranges are extended from real subset to the complex subset with the unit disk and hence handle the two-dimensional information in a single set. Thus, motivated by this, we developed some new averaging aggregation operators, namely complex intuitionistic fuzzy (CIF) weighted averaging, CIF ordered weighted averaging and CIF hybrid averaging in conjunction with their desirable properties. Then, we utilized these operators to propose a multicriteria decision-making approach and illustrated a numerical example to demonstrate the working of the proposed approach. Finally, the proposed results are compared with existing approaches results.

Keywords Complex fuzzy set · Complex intuitionistic fuzzy set · Averaging operators · Multicriteria decision-making · t-norm operations

1 Introduction

Multicriteria decision-making (MCDM) is concerned with structuring and solving decision and planning problems in which all the criteria are considered simultaneously. This area of decision-making has attracted the interest of many researchers, and they have worked in this field by utilizing various approaches [1–3]. Traditionally, MCDM problems often require decision makers to provide evaluation information about the criteria and the alternatives with a fuzzy set (FS) [4], intuitionistic fuzzy (IF) set (IFS) [5], interval-valued intuitionistic fuzzy set (IVIFS) [6] and other extended sets [7–9]. Under these existing sets, various researchers have proposed different types of methods in processing the information values using different operators [10–17], information measures [18–20], score and accuracy functions [21,22] under these environments. Among them, an aggregation operator is an important part of the decision-making which usually takes the form of mathematical function to aggregate all the input individual data into a single one. By taking the advantage of this, Yager [23] proposed the ordered weighted aggregation (OWA) operator by giving weights to all the inputs according to their ranking positions. Xu and Yager [24] presented geometric aggregation operator, while Xu [2] presented weighted averaging operator for aggregating the different intuitionistic fuzzy numbers (IFNs). Xu and Yager [25] presented intuitionistic fuzzy Bonferroni mean aggregation operators. Wang and Liu [26] presented some aggregation operations using Einstein norm operations. He et al. [27] presented some interactive averaging aggregation operators to solve the MCDM problems. Garg [28,29] presented some improved interactive aggregation operators for different IFNs. Wei and Wang [30] developed ordered weighted geometric aggregation operators for interval-valued IFNs (IVIFNs). Kaur and Garg [31] presented some aggregation operators for cubic IF set. Ye [32] presented some hybrid averaging and geometric aggregation operators under the IFS environment to solve the

✉ Harish Garg
harishg58iitr@gmail.com
http://sites.google.com/site/harishg58iitr/

Dimple Rani
dimplegoyal938@gmail.com

[1] School of Mathematics, Thapar Institute of Engineering and Technology (Deemed University), Patiala, Punjab 147004, India

MCDM problems. Recently, Garg and Singh [33] presented a novel triangular interval type-2 IF set and their aggregation operators.

As for the above existing studies, it has been analyzed that they have investigated the MCDM problems under the FS, IFS or its generalizations, which are only able to deal with the uncertainty and vagueness that exists in preferences given by the decision makers. None of these models are able to represent the partial ignorance of the data and its fluctuations at a given phase of time. However, in complex data sets, uncertainty and vagueness in the data occur concurrently with changes to the phase (periodicity) of the data. In order to handle these, Ramot et al. [34] proposed the concept of complex fuzzy set (CFS) in which the range of membership function is extended from the subset of real number to the unit disk and the membership function is represented in the form of $\zeta e^{i w_\zeta}$ where $\zeta \in [0, 1]$ and w_ζ is real valued. CFS is widely used in many of the data sets which include the large amounts of data that are generated from medical research, as well as government databases for biometric and facial recognition, audio and images all of which may contain large amounts of incomplete, uncertain and vague information. Ramot et al. [34,35] discussed several properties of CFSs such as complement, union and the intersection with sufficient amount of illustrative examples. As the CFS model does not tell anything about the disagreeness of any element in any set so, Alkouri and Salleh [36] extended the concept of CFS to complex intuitionistic fuzzy set (CIFS) by adding the degree of non-membership and defined their basic operations such as union, intersection, complement, etc. Alkouri and Salleh [37], further, introduced the concepts of complex intuitionistic fuzzy relation, composition, projections and hence proposed a distance measure between the two CIFSs. Kumar and Bajaj [38] proposed some distance and entropy measures for complex intuitionistic fuzzy soft sets. Rani and Garg [39] presented some series of distance measures under CIFS environment and their application in the decision-making process. Rani and Garg [40] presented power aggregation operators for CIFS and their application in decision-making problems.

CIFS is a generalization of the existing studies such as complex fuzzy sets [34], intuitionistic fuzzy sets [5], fuzzy set [4] by considering much more information related to an object during the process and to handle the two-dimensional information in a single set. In CIFS theory, membership and non-membership degrees are complex valued and are represented in polar coordinates. The amplitude term corresponding to the membership (non-membership) degree gives the extent of belongings (not-belongings) of an object in a CIFS and the phase term associated with membership (non-membership) degree gives the additional information, generally related with periodicity. The phase terms are novel parameters of the membership and non-membership degrees, and these are the parameters which distinguish the traditional IFS and CIFS theory. IFS theory deals with only one dimension at a time, which results in information loss in some instances. However, in day-to-day life, we come across complex natural phenomena where it becomes essential to add the second dimension to the expression of membership and non-membership grades. By introducing this second dimension, the complete information can be projected in one set, and hence, loss of information can be avoided. To illustrate the significance of the phase term, consider an example of a certain company where he decides to install new data processing and analysis software. For this, the company consults an expert who gives the information regarding (i) different alternatives of software (ii) corresponding software version. The company wants to select the most optimal alternative(s) of software with its latest version simultaneously. Here, the problem is two-dimensional, namely to select the optimal alternative of software and its latest version. This problem cannot be modeled accurately using traditional IFS theory. So, the best way to represent all of the information provided by the expert is by using CIFS theory. The amplitude terms in CIFS may be employed to give a company's decision regarding alternative of software, and the phase terms may be used to represent company's decision regarding software version.

Therefore, keeping the advantages of this set and taking the importance of aggregation operators, this paper presents the theory of the weighted averaging aggregation operators among the CIFS. As per our knowledge, in the aforementioned studies, the operators cannot be utilized to handle the CIFS information. Thus, in order to achieve it, we first define some operational laws between the pairs of the CIFSs and studied their properties that involve both uncertainty and periodicity semantics. Then, based on these, we propose some generalized t-norm-based aggregation operators named as complex intuitionistic fuzzy (CIF) weighted averaging, CIF ordered weighted geometric, CIF hybrid averaging to aggregate the different complex intuitionistic fuzzy numbers (CIFNs). The various properties of these operators are investigated in details. Furthermore, we propose an MCDM approach based on the proposed operators for CIFSs. The feasibility, as well as superiority of the approach, has been demonstrated through an illustrative example.

For this, the remaining text is organized as follows. In Sect. 2, we discuss some existing work on CFS, CIFS, t-norms and t-conorms. In Sect. 3, some basic generalized operational laws of CIFNs and their basic properties are presented. Further, some weighted averaging aggregation operators, namely CIFWA, CIFOWA and CIFHA are presented in conjunction with their desirable properties. In Sect. 4, we present a multicriteria decision-making approach based on the proposed operators under CIFSs environment, where each element of the set is characterized by complex intuitionistic fuzzy numbers. An illustrative example is presented to discuss the functionality of the proposed approach and their results are compared with some of the existing approaches results. Finally, a conclusion is given in Sect. 5.

2 Preliminaries

In this section, we review some basic concepts of CIFSs, t-norms and t-conorms over the universal set U.

Definition 1 [34] A CFS A defined on U is a set of ordered pairs which is defined as

$$A = \{(x, \mu_A(x)) : x \in U\} \tag{1}$$

where $\mu_A : U \to \{a : a \in C, |a| \leq 1\}$ is a complex-valued membership function. Corresponding to each element $x \in U$ the value of $\mu_A(x)$ is expressed as: $\mu_A(x) = \zeta_A(x)e^{iw_{\zeta_A}(x)}$ where $i = \sqrt{-1}$, $\zeta_A(x) \in [0, 1]$ and $0 \leq w_{\zeta_A}(x) \leq 2\pi$.

Definition 2 [36] A CIFS A defined on U is given by

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in U\} \tag{2}$$

where $\mu_A : U \to \{a : a \in C, |a| \leq 1\}$ and $\gamma_A : U \to \{a : a \in C, |a| \leq 1\}$ are complex-valued membership and non-membership functions, respectively, of the element x and are given by $\mu_A(x) = \zeta_A(x)e^{iw_{\zeta_A}(x)}$ and $\gamma_A(x) = \vartheta_A(x)e^{iw_{\vartheta_A}(x)}$, where $\zeta_A(x), \vartheta_A(x) \in [0, 1]$ such that $0 \leq \zeta_A(x) + \vartheta_A(x) \leq 1$ and $0 \leq w_{\zeta_A}(x), w_{\vartheta_A}(x) \leq 2\pi$ such that $0 \leq w_{\zeta_A}(x) + w_{\vartheta_A}(x) \leq 2\pi$.

For convenience, we denote this pair as $\delta = \langle \zeta e^{iw_\zeta}, \vartheta e^{iw_\vartheta} \rangle$ and named it as complex intuitionistic fuzzy number (CIFN), where $\zeta, \vartheta \in [0, 1]$ such that $\zeta + \vartheta \leq 1$ and $w_\zeta, w_\vartheta, [0, 2\pi]$ with $0 \leq w_\zeta + w_\vartheta \leq 2\pi$.

Definition 3 [36] Consider two CIFSs $A = \{(x, \zeta_A(x) e^{iw_{\zeta_A}(x)}, \vartheta_A(x)e^{iw_{\vartheta_A}(x)}) : x \in U\}$ and $B = \{(x, \zeta_B(x) e^{iw_{\zeta_B}(x)}, \vartheta_B(x)e^{iw_{\vartheta_B}(x)}) : x \in U\}$ defined on U. Then, we have

(i) $A \subseteq B$ if $\zeta_A(x) \leq \zeta_B(x)$, $\vartheta_A(x) \geq \vartheta_B(x)$ and $w_{\zeta_A}(x) \leq w_{\zeta_B}(x)$, $w_{\vartheta_A}(x) \geq w_{\vartheta_B}(x)$
(ii) $A = B \Leftrightarrow \zeta_A(x) = \zeta_B(x)$, $\vartheta_A(x) = \vartheta_B(x)$ and $w_{\zeta_A}(x) = w_{\zeta_B}(x)$, $w_{\vartheta_A}(x) = w_{\vartheta_B}(x)$
(iii) $A^c = \{(x, \vartheta_A(x)e^{iw_{\vartheta_A}(x)}, \zeta_A(x)e^{iw_{\zeta_A}(x)}) : x \in U\}$

Definition 4 [41] A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is called t-norm if it satisfies the boundary condition, monotonicity, commutativity and associativity. On the other hand, a function K defined by $K(a, b) = 1 - T(1 - a, 1 - b)$ $\forall a, b \in [0, 1]$ is called t-conorm.

Definition 5 [41] An Archimedean t-norm (t-conorm) function, T (or K), is a continuous t-norm (t-conorm) satisfying the condition $T(a, a) < a$(or $K(a, a) > a$) for $a \in (0, 1)$. A strict Archimedean t-norm (t-conorm) is strictly increasing t-norm (t-conorm).

Strict Archimedean t-norm T and t-conorm K can be expressed using continuous functions $t, s : [0, 1] \to [0, \infty)$, respectively, as $T(a, b) = t^{-1}\big(t(a) + t(b)\big)$ and $K(a, b) = s^{-1}\big(s(a) + s(b)\big)$ where t is decreasing function with $t(1) = 0$; s is an increasing function with $s(0) = 0$ and $s(a) = t(1 - a)$.

3 Operational Laws and Averaging Operators of CIFNs

In this section, some elementary operational laws of the CIFNs and some series of the averaging operators are defined.

3.1 Score and Accuracy Functions

Definition 6 For CIFN $\delta = \langle \zeta e^{iw_\zeta}, \vartheta e^{iw_\vartheta} \rangle$, the score function of δ is defined as

$$S(\delta) = \big(\zeta - \vartheta\big) + \frac{1}{2\pi}\big(w_\zeta - w_\vartheta\big), \tag{3}$$

and an accuracy function H of δ is stated as

$$H(\delta) = \big(\zeta + \vartheta\big) + \frac{1}{2\pi}\big(w_\zeta + w_\vartheta\big). \tag{4}$$

It is clear that $S(\delta) \in [-2, 2]$ and $H(\delta) \in [0, 2]$.

Based on these two functions, an order relation between two CIFNs δ and β is stated as

(a) if $S(\delta) > S(\beta)$ then $\delta > \beta$.
(b) if $S(\delta) = S(\beta)$
  (i) if $H(\delta) > H(\beta)$ then $\delta > \beta$
  (ii) if $H(\delta) = H(\beta)$ then δ and β represent the same information, denoted by $\delta = \beta$.

To study the properties of score function and accuracy function, we propose the following results.

Theorem 1 (Monotonicity of score function) *Let $\delta = \langle \zeta e^{iw_\zeta}, \vartheta e^{iw_\vartheta} \rangle$ be a CIFN. Then, the score function $S(\delta) = \big(\zeta - \vartheta\big) + \frac{1}{2\pi}\big(w_\zeta - w_\vartheta\big)$ is a monotonic increasing function with ζ, w_ζ, and a monotone decreasing function with ϑ, w_ϑ.*

Proof Omitted. □

Theorem 2 (Symmetry of score function) *Let $\delta_j = \big\langle \zeta_j e^{iw_{\zeta_j}}, \vartheta_j e^{iw_{\vartheta_j}} \big\rangle$, $j = 1, 2$ be two CIFNs, $\delta_j^c = \big\langle \vartheta_j e^{iw_{\vartheta_j}}, \zeta_j e^{iw_{\zeta_j}} \big\rangle$,*

$j=1,2$ be their associated inverse (complement) function, respectively, then we have the following conclusion $S(\delta_1)\le S(\delta_2)\Leftrightarrow S(\delta_1^c)\ge S(\delta_2^c)$.

Proof By the definition of score function for CIFNs $\delta_j\,(j=1,2)$ we obtain

$$S(\delta_1)=(\zeta_1-\vartheta_1)+\frac{1}{2\pi}(w_{\zeta_1}-w_{\vartheta_1}) \text{ and}$$
$$S(\delta_2)=(\zeta_2-\vartheta_2)+\frac{1}{2\pi}(w_{\zeta_2}-w_{\vartheta_2})$$

Since $S(\delta_1)\le S(\delta_2)$, then

$$\begin{aligned}&\Leftrightarrow (\zeta_1-\vartheta_1)+\frac{1}{2\pi}(w_{\zeta_1}-w_{\vartheta_1})\\ &\le (\zeta_2-\vartheta_2)+\frac{1}{2\pi}(w_{\zeta_2}-w_{\vartheta_2})\\ &\Leftrightarrow (-\zeta_1+\vartheta_1)+\frac{1}{2\pi}(-w_{\zeta_1}+w_{\vartheta_1})\\ &\ge (-\zeta_2+\vartheta_2)+\frac{1}{2\pi}(-w_{\zeta_2}+w_{\vartheta_2})\\ &\Leftrightarrow S(\delta_1^c)\ge S(\delta_2^c)\end{aligned}$$

□

Theorem 3 (Monotonicity of accuracy function) *Let $\delta=\left\langle \zeta e^{iw_\zeta},\vartheta e^{iw_\vartheta}\right\rangle$ is CIFN, the accuracy function $S(\delta)=(\zeta+\vartheta)+\frac{1}{2\pi}(w_\zeta+w_\vartheta)$ is a monotonic increasing function with ζ, w_ζ, ϑ and w_ϑ.*

Proof Omitted. □

Theorem 4 (Symmetry of accuracy function) *Let $\delta=\left\langle \zeta e^{iw_\zeta},\vartheta e^{iw_\vartheta}\right\rangle$ be CIFN and $\delta^c=\left\langle \vartheta e^{iw_\vartheta},\zeta e^{iw_\zeta}\right\rangle$ be their associated complement, then we have $H(\delta)=H(\delta^c)$.*

Proof Omitted. □

3.2 Operational Laws of CIFNs

Next, we define the basic operational laws of CIFNs based on the Archimedean t-norm operations as follows.

Definition 7 Let $\delta_j=\left\langle \zeta_j e^{iw_{\zeta_j}},\vartheta_j e^{iw_{\vartheta_j}}\right\rangle$, $j=1,2$ be any two CIFNs and let $\lambda>0$ be any real number. Then, we have

(i) $$\delta_1\oplus\delta_2=\left\langle \left(s^{-1}\left(s(\zeta_1)+s(\zeta_2)\right)\right)e^{i2\pi\left(s^{-1}\left(s\left(\frac{w_{\zeta_1}}{2\pi}\right)+s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)\right)e^{i2\pi\left(t^{-1}\left(t\left(\frac{w_{\vartheta_1}}{2\pi}\right)+t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)}\right\rangle$$

(ii) $$\delta_1\otimes\delta_2=\left\langle \left(t^{-1}\left(t(\zeta_1)+t(\zeta_2)\right)\right)e^{i2\pi\left(t^{-1}\left(t\left(\frac{w_{\zeta_1}}{2\pi}\right)+t\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)}, \left(s^{-1}\left(s(\vartheta_1)+s(\vartheta_2)\right)\right)e^{i2\pi\left(s^{-1}\left(s\left(\frac{w_{\vartheta_1}}{2\pi}\right)+s\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)}\right\rangle$$

(iii) $$\lambda\delta_1=\left\langle \left(s^{-1}\left(\lambda s(\zeta_1)\right)\right)e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\lambda t(\vartheta_1)\right)\right)e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)}\right\rangle$$

(iv) $$(\delta_1)^\lambda=\left\langle \left(t^{-1}\left(\lambda t(\zeta_1)\right)\right)e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)}, \left(s^{-1}\left(\lambda s(\vartheta_1)\right)\right)e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)}\right\rangle$$

Theorem 5 *If δ_1 and δ_2 be any two CIFNs and $\lambda>0$ be any real number, then $\delta_1\oplus\delta_2$, $\lambda\delta_1$, $\delta_1\otimes\delta_2$ and δ_1^λ are also CIFNs.*

Proof Let $\delta_1=\left\langle \zeta_1 e^{iw_{\zeta_1}},\vartheta_1 e^{iw_{\vartheta_1}}\right\rangle$ and $\delta_2=\left\langle \zeta_2 e^{iw_{\zeta_2}},\vartheta_2 e^{iw_{\vartheta_2}}\right\rangle$ are two CIFNs. So, by definition of CIFN, we have $\zeta_1,\zeta_2\in[0,1]$, $\vartheta_1,\vartheta_2\in[0,1]$, $w_{\zeta_1},w_{\zeta_2}\in[0,2\pi]$, $w_{\vartheta_1},w_{\vartheta_2}\in[0,2\pi]$, $\zeta_1+\vartheta_1\le 1$, $\zeta_2+\vartheta_2\le 1$, $w_{\zeta_1}+w_{\vartheta_1}\le 2\pi$ and $w_{\zeta_2}+w_{\vartheta_2}\le 2\pi$. Now, by using Definition 7, we obtain $\delta_1\oplus\delta_2=\left\langle \zeta_3 e^{iw_{\zeta_3}},\vartheta_3 e^{iw_{\vartheta_3}}\right\rangle$ where $\zeta_3=s^{-1}\left(s(\zeta_1)+s(\zeta_2)\right)$, $\vartheta_3=t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)$, $w_{\zeta_3}=2\pi\left(s^{-1}\left(s\left(\frac{w_{\zeta_1}}{2\pi}\right)+s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)$ and $w_{\vartheta_3}=2\pi\left(t^{-1}\left(t\left(\frac{w_{\vartheta_1}}{2\pi}\right)+t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)$. In order to show $\delta_1\oplus\delta_2$ is CIFN, it is enough to show that $\zeta_3,\vartheta_3\in[0,1]$, $w_{\zeta_3},w_{\vartheta_3}\in[0,2\pi]$, $\zeta_3+\vartheta_3\le 1$ and $w_{\zeta_3}+w_{\vartheta_3}\le 2\pi$.

Since $t,s:[0,1]\to[0,\infty)$ are the continuous function with $t(1)=0$ and $s(a)=t(1-a)$, so it is clearly seen that $\zeta_3,\vartheta_3\in[0,1]$, $w_{\zeta_3},w_{\vartheta_3}\in[0,2\pi]$. Further, using the conditions $\zeta_j+\vartheta_j\le 1$ for $j=1,2$, and s is an increasing function, we have

$$\begin{aligned}\zeta_3+\vartheta_3&=s^{-1}\left(s(\zeta_1)+s(\zeta_2)\right)+t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)\\ &\le s^{-1}\left(s(1-\vartheta_1)+s(1-\vartheta_2)\right)+t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)\\ &=1-t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)+t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)\\ &=1\end{aligned}$$

Thus, $\zeta_3+\vartheta_3\le 1$. Also, $\zeta_3+\vartheta_3\ge 0$ as $\zeta_3,\vartheta_3\ge 0$. Hence, $0\le\zeta_3+\vartheta_3\le 1$. Similarly, $0\le w_{\zeta_3}+w_{\vartheta_3}\le 2\pi$. Therefore, $\delta_1\oplus\delta_2$ is a CIFN. Similarly, we can prove that $\delta_1\otimes\delta_2$, δ_1^λ, $\lambda\delta_1$ are also CIFNs. □

Theorem 6 *Let δ_1, δ_2 be two CIFNs and let $\lambda,\lambda_1,\lambda_2>0$ be three real numbers. Then, we have*

(i) $\delta_1\oplus\delta_2=\delta_2\oplus\delta_1$;
(ii) $\delta_1\otimes\delta_2=\delta_2\otimes\delta_1$;
(iii) $\lambda(\delta_1\oplus\delta_2)=\lambda\delta_1\oplus\lambda\delta_2$;
(iv) $(\delta_1\otimes\delta_2)^\lambda=\delta_1^\lambda\otimes\delta_2^\lambda$;
(v) $\lambda_1\delta_1\oplus\lambda_2\delta_1=(\lambda_1+\lambda_2)\delta_1$;
(vi) $\delta_1^{\lambda_1}\otimes\delta_1^{\lambda_2}=\delta_1^{\lambda_1+\lambda_2}$.

Proof Here, we have proved the parts (i), (iii) and (v), while others can be deduced similarly for two CIFNs $\delta_1 = \langle \zeta_1 e^{iw_{\zeta_1}}, \vartheta_1 e^{iw_{\vartheta_1}} \rangle$ and $\delta_2 = \langle \zeta_2 e^{iw_{\zeta_2}}, \vartheta_2 e^{iw_{\vartheta_2}} \rangle$.

(i) By Definition 7, we have

$$\begin{aligned}
&\delta_1 \oplus \delta_2 \\
&= \Bigg\langle \left(s^{-1}\left(s(\zeta_1)+s(\zeta_2)\right)\right) e^{i2\pi\left(s^{-1}\left(s\left(\frac{w_{\zeta_1}}{2\pi}\right)+s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)\right) e^{i2\pi\left(t^{-1}\left(t\left(\frac{w_{\vartheta_1}}{2\pi}\right)+t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)} \Bigg\rangle \\
&= \Bigg\langle \left(s^{-1}\left(s(\zeta_2)+s(\zeta_1)\right)\right) e^{i2\pi\left(s^{-1}\left(s\left(\frac{w_{\zeta_2}}{2\pi}\right)+s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(t(\vartheta_2)+t(\vartheta_1)\right)\right) e^{i2\pi\left(t^{-1}\left(t\left(\frac{w_{\vartheta_2}}{2\pi}\right)+t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)} \Bigg\rangle \\
&= \delta_2 \oplus \delta_1
\end{aligned}$$

(iii) Since δ_1, δ_2 are CIFNs and $\lambda > 0$ is a real number. So, by Definition 7, we have

$$\begin{aligned}
&\lambda(\delta_1 \oplus \delta_2) \\
&= \lambda \Bigg\langle \left(s^{-1}\left(s(\zeta_1)+s(\zeta_2)\right)\right) e^{i2\pi\left(s^{-1}\left(s\left(\frac{w_{\zeta_1}}{2\pi}\right)+s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)\right) e^{i2\pi\left(t^{-1}\left(t\left(\frac{w_{\vartheta_1}}{2\pi}\right)+t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)} \Bigg\rangle \\
&= \Bigg\langle \left(s^{-1}\left(\lambda s\left(s^{-1}\left(s(\zeta_1)+s(\zeta_2)\right)\right)\right)\right) \\
&\quad e^{i2\pi\left(s^{-1}\left(\lambda s\left(s^{-1}\left(s\left(\frac{w_{\zeta_1}}{2\pi}\right)+s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(\lambda t\left(t^{-1}\left(t(\vartheta_1)+t(\vartheta_2)\right)\right)\right)\right) \\
&\quad e^{i2\pi\left(t^{-1}\left(\lambda t\left(t^{-1}\left(t\left(\frac{w_{\vartheta_1}}{2\pi}\right)+t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)\right)\right)} \Bigg\rangle \\
&= \Bigg\langle \left(s^{-1}\left(\lambda s(\zeta_1)+\lambda s(\zeta_2)\right)\right) \\
&\quad e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_{\zeta_1}}{2\pi}\right)+\lambda s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(\lambda t(\vartheta_1)+\lambda t(\vartheta_2)\right)\right) \\
&\quad e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_{\vartheta_1}}{2\pi}\right)+\lambda t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)} \Bigg\rangle \\
&= \Bigg\langle \left(s^{-1}\left(s\left(s^{-1}\left(\lambda s(\zeta_1)\right)\right)+s\left(s^{-1}\left(\lambda s(\zeta_2)\right)\right)\right)\right) \times \\
&\quad \times e^{i2\pi\left(s^{-1}\left(s\left(s^{-1}\left(\lambda s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)+s\left(s^{-1}\left(\lambda s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(t\left(t^{-1}\left(\lambda t(\vartheta_1)\right)\right)+t\left(t^{-1}\left(\lambda t(\vartheta_2)\right)\right)\right)\right) \times \\
&\quad \times e^{i2\pi\left(t^{-1}\left(t\left(t^{-1}\left(\lambda t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)+t\left(t^{-1}\left(\lambda t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)\right)\right)} \Bigg\rangle \\
&= \Bigg\langle \left(s^{-1}\left(\lambda s(\zeta_1)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(\lambda t(\vartheta_1)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)} \Bigg\rangle \\
&\oplus \Bigg\langle \left(s^{-1}\left(\lambda s(\zeta_2)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)}, \\
&\quad (t^{-}1(\lambda t(\vartheta_2)))e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)} \Bigg\rangle \\
&= \lambda\delta_1 \oplus \lambda\delta_2
\end{aligned}$$

Hence, $\lambda(\delta_1 \oplus \delta_2) = \lambda\delta_1 \oplus \lambda\delta_2$.

(v) Since δ_1 is CIFN and $\lambda_1, \lambda_2 > 0$ are real numbers.

$$\begin{aligned}
&\lambda_1\delta_1 \oplus \lambda_2\delta_1 \\
&= \Bigg\langle \left(s^{-1}\left(\lambda_1 s(\zeta_1)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda_1 s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(\lambda_1 t(\vartheta_1)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda_1 t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)} \Bigg\rangle \\
&\oplus \Bigg\langle \left(s^{-1}\left(\lambda_2 s(\zeta_1)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda_2 s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(\lambda_2 t(\vartheta_1)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda_2 t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)} \Bigg\rangle \\
&= \Bigg\langle \left(s^{-1}\left(s\left(s^{-1}\left(\lambda_1 s(\zeta_1)\right)\right)+s\left(s^{-1}\left(\lambda_2 s(\zeta_1)\right)\right)\right)\right) \\
&\quad \times e^{i2\pi\left(s^{-1}\left(s\left(s^{-1}\left(\lambda_1 s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)+s\left(s^{-1}\left(\lambda_2 s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)\right)\right)}, \\
&\quad \left(t^{-1}\left(t\left(t^{-1}\left(\lambda_1 t(\vartheta_1)\right)\right)+t\left(t^{-1}\left(\lambda_2 t(\vartheta_1)\right)\right)\right)\right) \\
&\quad \times e^{i2\pi\left(t^{-1}\left(t\left(t^{-1}\left(\lambda_1 t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)+t\left(t^{-1}\left(\lambda_2 t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)\right)\right)} \Bigg\rangle \\
&= \Bigg\langle \left(s^{-1}\left((\lambda_1+\lambda_2) s(\zeta_1)\right)\right) e^{i2\pi\left(s^{-1}\left((\lambda_1+\lambda_2) s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)}, \\
&\quad \left(t^{-1}\left((\lambda_1+\lambda_2) t(\vartheta_1)\right)\right) e^{i2\pi\left(t^{-1}\left((\lambda_1+\lambda_2) t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)} \Bigg\rangle \\
&= (\lambda_1+\lambda_2)\delta_1
\end{aligned}$$

Hence, $\lambda_1\delta_1 \oplus \lambda_2\delta_1 = (\lambda_1+\lambda_2)\delta_1$.

□

Theorem 7 *Let δ_1, δ_2, δ be three CIFNs and $\lambda > 0$ be any real number. Then, we have*

(i) $(\delta^c)^\lambda = (\lambda\delta)^c$;
(ii) $\lambda(\delta^c) = (\delta^\lambda)^c$;
(iii) $(\delta_1 \oplus \delta_2)^c = \delta_1^c \otimes \delta_2^c$;
(iv) $(\delta_1 \otimes \delta_2)^c = \delta_1^c \oplus \delta_2^c$.

Proof Let $\delta_1 = \langle \zeta_1 e^{i w_{\zeta_1}}, \vartheta_1 e^{i w_{\vartheta_1}} \rangle$, $\delta_2 = \langle \zeta_2 e^{i w_{\zeta_2}}, \vartheta_2 e^{i w_{\vartheta_2}} \rangle$ and $\delta = \langle \zeta e^{i w_\zeta}, \vartheta e^{i w_\vartheta} \rangle$ be three CIFNs. Then, $\delta^c = \langle \vartheta e^{i w_\vartheta}, \zeta e^{i w_\zeta} \rangle$. Therefore, we have

(i) $$(\delta^c)^\lambda = \left\langle \left(t^{-1}\left(\lambda t(\vartheta)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_\vartheta}{2\pi}\right)\right)\right)}, \left(s^{-1}\left(\lambda s(\zeta)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_\zeta}{2\pi}\right)\right)\right)} \right\rangle = (\lambda\delta)^c$$

(ii) $$\lambda(\delta^c) = \left\langle \left(s^{-1}\left(\lambda s(\vartheta)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_\vartheta}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\lambda t(\zeta)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_\zeta}{2\pi}\right)\right)\right)} \right\rangle = (\delta^\lambda)^c$$

(iii) $$(\delta_1 \oplus \delta_2)^c = \left\langle \left(t^{-1}\left(t(\vartheta_1) + t(\vartheta_2)\right)\right) e^{i2\pi\left(t^{-1}\left(t\left(\frac{w_{\vartheta_1}}{2\pi}\right)+t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)}, \left(s^{-1}\left(s(\zeta_1) + s(\zeta_2)\right)\right) e^{i2\pi\left(s^{-1}\left(s\left(\frac{w_{\zeta_1}}{2\pi}\right)+s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)} \right\rangle = \delta_1^c \otimes \delta_2^c$$

(iv) $$(\delta_1 \otimes \delta_2)^c = \left\langle \left(s^{-1}\left(s(\vartheta_1) + s(\vartheta_2)\right)\right) e^{i2\pi\left(s^{-1}\left(s\left(\frac{w_{\vartheta_1}}{2\pi}\right)+s\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(t(\zeta_1) + t(\zeta_2)\right)\right) e^{i2\pi\left(t^{-1}\left(t\left(\frac{w_{\zeta_1}}{2\pi}\right)+t\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)} \right\rangle = \delta_1^c \oplus \delta_2^c$$

□

Remark 1 Consider here some special cases of CIFN $\delta = \langle \zeta e^{i w_\zeta}, \vartheta e^{i w_\vartheta} \rangle$ and any positive real number λ.

(i) If $\delta = \langle 1e^{i2\pi}, 0e^{i0} \rangle$ then
$$\lambda\delta = \left\langle \left(s^{-1}\left(\lambda s(1)\right)\right) e^{i2\pi\left(s^{-1}(\lambda s(1))\right)}, \left(t^{-1}\left(\lambda t(0)\right)\right) e^{i2\pi\left(t^{-1}(\lambda t(0))\right)} \right\rangle = \langle 1e^{i2\pi}, 0e^{i0} \rangle$$

(ii) If $\delta = \langle 0e^{i0}, 1e^{i2\pi} \rangle$ then
$$\lambda\delta = \left\langle \left(s^{-1}\left(\lambda s(0)\right)\right) e^{i2\pi\left(s^{-1}(\lambda s(0))\right)}, \left(t^{-1}\left(\lambda t(1)\right)\right) e^{i2\pi\left(t^{-1}(\lambda t(1))\right)} \right\rangle = \langle 0e^{i0}, 1e^{i2\pi} \rangle$$

(iii) If $\delta = \langle 0e^{i0}, 0e^{i0} \rangle$ then
$$\lambda\delta = \left\langle \left(s^{-1}\left(\lambda s(0)\right)\right) e^{i2\pi\left(s^{-1}(\lambda s(0))\right)}, \left(t^{-1}\left(\lambda t(0)\right)\right) e^{i2\pi\left(t^{-1}(\lambda t(0))\right)} \right\rangle = \langle 0e^{i0}, 0e^{i0} \rangle$$

(iv) If $\delta = \langle \zeta e^{i w_\zeta}, \vartheta e^{i w_\vartheta} \rangle$ and $\lambda \to 0$ then
$$\lambda\delta = \left\langle \left(s^{-1}\left(\lambda s(\zeta)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_\zeta}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\lambda t(\vartheta)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_\vartheta}{2\pi}\right)\right)\right)} \right\rangle \to \langle 0e^{i0}, 1e^{i2\pi} \rangle$$

(v) If $\delta = \langle \zeta e^{i w_\zeta}, \vartheta e^{i w_\vartheta} \rangle$ and $\lambda \to \infty$ then
$$\lambda\delta = \left\langle \left(s^{-1}\left(\lambda s(\zeta)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_\zeta}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\lambda t(\vartheta)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_\vartheta}{2\pi}\right)\right)\right)} \right\rangle \to \langle 1e^{i2\pi}, 0e^{i0} \rangle$$

(vi) If $\delta = \langle \zeta e^{i w_\zeta}, \vartheta e^{i w_\vartheta} \rangle$ and $\lambda = 1$ then
$$\lambda\delta = \left\langle \left(s^{-1}\left(\lambda s(\zeta)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_\zeta}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\lambda t(\vartheta)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_\vartheta}{2\pi}\right)\right)\right)} \right\rangle = \langle \zeta e^{i w_\zeta}, \vartheta e^{i w_\vartheta} \rangle$$

Next, based on the above-defined operational laws of CIFNs, we propose some new averaging aggregation operators named as CIFWA, CIFOWA and CIFHA under complex intuitionistic fuzzy environment.

3.3 Weighted Averaging Operators

In this section, weighted averaging aggregation operator for a collection of CIFNs are defined.

Definition 8 Let Ω is the collection of all CIFNs δ_j $(j = 1, 2, \ldots, n)$ with corresponding weights $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ such that $\xi_j > 0$ and $\sum_{j=1}^n \xi_j = 1$. If CIFWA: $\Omega^n \to \Omega$, is a mapping defined by

$$\text{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n) = \xi_1\delta_1 \oplus \xi_2\delta_2 \oplus \ldots \oplus \xi_n\delta_n \tag{5}$$

then, CIFWA is called complex intuitionistic fuzzy weighted averaging operator.

Theorem 8 *For any collection of CIFNs $\delta_j = \langle \zeta_j e^{i w_{\zeta_j}}, \vartheta_j e^{i w_{\vartheta_j}} \rangle$ $(j = 1, 2, \ldots, n)$, the combined value obtained by using CIFWA operator is still CIFN and is given as*

$$CIFWA(\delta_1, \delta_2, \ldots, \delta_n) = \left\langle \left(s^{-1}\left(\sum_{j=1}^n \xi_j s(\zeta_j)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^n \xi_j s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\sum_{j=1}^n \xi_j t(\vartheta_j)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^n \xi_j t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)} \right\rangle \tag{6}$$

Proof The fact that, the value obtained after applying CIFWA operator is still CIFN, follows from Theorem 5. Now, by making use of mathematical induction, we will show that Eq. (6) holds.

Since for each j, δ_j is a CIFN and for real numbers $\xi_j > 0$, we have $\xi_j\delta_j$ is also CIFN by using Theorem 5. Then, by using mathematical induction we have:

Step 1: For $n = 2$, we get $\delta_1 = \langle \zeta_1 e^{iw_{\zeta_1}}, \vartheta_1 e^{iw_{\vartheta_1}} \rangle$ and $\delta_2 = \langle \zeta_2 e^{iw_{\zeta_2}}, \vartheta_2 e^{iw_{\vartheta_2}} \rangle$. Thus, by the operation laws of CIFNs, we get

$$\xi_1\delta_1 = \left\langle \left(s^{-1}(\xi_1 s(\zeta_1))\right) e^{i2\pi\left(s^{-1}\left(\xi_1 s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)}, \left(t^{-1}(\xi_1 t(\vartheta_1))\right) e^{i2\pi\left(t^{-1}\left(\xi_1 t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)} \right\rangle$$

and

$$\xi_2\delta_2 = \left\langle \left(s^{-1}(\xi_2 s(\zeta_2))\right) e^{i2\pi\left(s^{-1}\left(\xi_2 s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)}, \left(t^{-1}(\xi_2 t(\vartheta_2))\right) e^{i2\pi\left(t^{-1}\left(\xi_2 t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)} \right\rangle$$

Hence, by addition law of CIFNs, we get

$$\begin{aligned}
&\text{CIFWA}(\delta_1, \delta_2) \\
&= \xi_1\delta_1 \oplus \xi_2\delta_2 \\
&= \Big\langle \left(s^{-1}\left(s\left(s^{-1}(\xi_1 s(\zeta_1))\right) + s\left(s^{-1}(\xi_2 s(\zeta_2))\right)\right)\right) \\
&\quad \times e^{i2\pi\left(s^{-1}\left(s\left(s^{-1}\left(\xi_1 s\left(\frac{w_{\zeta_1}}{2\pi}\right)\right)\right)+s\left(s^{-1}\left(\xi_2 s\left(\frac{w_{\zeta_2}}{2\pi}\right)\right)\right)\right)\right)}, \\
&\left(t^{-1}\left(t\left(t^{-1}(\xi_1 t(\vartheta_1))\right) + t\left(t^{-1}(\xi_2 t(\vartheta_2))\right)\right)\right) \\
&\quad \times e^{i2\pi\left(t^{-1}\left(t\left(t^{-1}\left(\xi_1 t\left(\frac{w_{\vartheta_1}}{2\pi}\right)\right)\right)+t\left(t^{-1}\left(\xi_2 t\left(\frac{w_{\vartheta_2}}{2\pi}\right)\right)\right)\right)\right)} \Big\rangle \\
&= \left\langle \left(s^{-1}\left(\sum_{j=1}^{2} \xi_j s(\zeta_j)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{2} \xi_j s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\sum_{j=1}^{2} \xi_j t(\vartheta_j)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{2} \xi_j t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)} \right\rangle
\end{aligned}$$

Thus, results holds for $n = 2$.

Step 2: If Eq. (6) holds for $n = m$, where m is any natural number, then

$$\text{CIFWA}(\delta_1, \delta_2, \ldots, \delta_m) = \left\langle \left(s^{-1}\left(\sum_{j=1}^{m} \xi_j s(\zeta_j)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{m} \xi_j s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\sum_{j=1}^{m} \xi_j t(\vartheta_j)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{m} \xi_j t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)} \right\rangle$$

then for $n = m + 1$, we have

$$\begin{aligned}
&\text{CIFWA}(\delta_1, \delta_2, \ldots, \delta_{m+1}) \\
&= \text{CIFWA}(\delta_1, \delta_2, \ldots, \delta_m) \oplus \xi_{m+1}\delta_{m+1} \\
&= \left\langle \left(s^{-1}\left(\sum_{j=1}^{m} \xi_j s(\zeta_j)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{m} \xi_j s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\sum_{j=1}^{m} \xi_j t(\vartheta_j)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{m} \xi_j t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)} \right\rangle \\
&\oplus \left\langle \left(s^{-1}\left(\xi_{m+1} s(\zeta_{m+1})\right)\right) e^{i2\pi\left(s^{-1}\left(\xi_{m+1} s\left(\frac{w_{\zeta_{m+1}}}{2\pi}\right)\right)\right)}, \left(t^{-1}(\xi_{m+1} t(\vartheta_{m+1}))\right) e^{i2\pi\left(t^{-1}\left(\xi_{m+1} t\left(\frac{w_{\vartheta_{m+1}}}{2\pi}\right)\right)\right)} \right\rangle \\
&= \left\langle \left(s^{-1}\left(\sum_{j=1}^{m+1} \xi_j s(\zeta_j)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{m+1} \xi_j s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\sum_{j=1}^{m+1} \xi_j t(\vartheta_j)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{m+1} \xi_j t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)} \right\rangle
\end{aligned}$$

Thus, the result is true for $n = m + 1$, and hence, Eq. (6) holds for all natural numbers n. □

Proposition 1 *For $w_{\zeta_j}, w_{\vartheta_j} = 0$ for all j, CIFWA operator reduces to IF weighted averaging (IFWA) operator in IFS environment.*

Proof Since $w_{\zeta_j}, w_{\vartheta_j} = 0$. Therefore, Eq. (6) reduces to:

$$\text{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n) = \left\langle s^{-1}\left(\sum_{j=1}^{n} \xi_j s(\zeta_j)\right), t^{-1}\left(\sum_{j=1}^{n} \xi_j t(\vartheta_j)\right) \right\rangle$$

which is the weighted averaging operator in IFS environment. Hence, the proposed CIFWA operator is an extension of existing IFWA operator. □

Proposition 2 *If for all j, $w_{\zeta_j}, w_{\vartheta_j} = 0$ and $\zeta_j + \vartheta_j = 1$ then, the CIFWA operator reduces to weighted operator in fuzzy environment.*

Proof Similar to the above Proposition. □

The working of the proposed CIFWA aggregation operator is explained with a numerical example as follows:

Example 1 Let $\delta_1 = \langle 0.6e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.1)} \rangle$, $\delta_2 = \langle 0.8e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.1)} \rangle$, $\delta_3 = \langle 0.5e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)} \rangle$, $\delta_4 = \langle 0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.2)} \rangle$ be four CIFNs and $\xi = (0.35, 0.3, 0.1, 0.25)^T$ be the corresponding weight vector of $\delta_j (j = 1, 2, 3, 4)$. Without loss of generality, we consider the additive generators $t(a) = -\log a$ if $0 < a \leq 1$ with $t(0) = \infty$ and $s(a) = -\log(1 - a)$ if $0 \leq a < 1$ with

$s(1) = \infty$ corresponding to t-norm and t-conorm, respectively. Then, Eq. (6) becomes

$$\text{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n) = \left\langle \left(1 - \prod_{j=1}^{n} (1-\zeta_j)^{\xi_j}\right) e^{i2\pi\left(1-\prod_{j=1}^{n}\left(1-\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}\right)}, \prod_{j=1}^{n}\left(\vartheta_j^{\xi_j}\right) e^{i2\pi \prod_{j=1}^{n}\left(\frac{w_{\vartheta_j}}{2\pi}\right)^{\xi_j}} \right\rangle \tag{7}$$

Now, based on the given information, we have

$$\prod_{j=1}^{4}(1-\zeta_j)^{\xi_j} = (1-0.6)^{0.35} \times (1-0.8)^{0.3} \times (1-0.5)^{0.1} \times (1-0.6)^{0.25} = 0.3322$$

$$\prod_{j=1}^{4}\vartheta_j^{\xi_j} = (0.2)^{0.35} \times (0.2)^{0.3} \times (0.3)^{0.1} \times (0.3)^{0.25} = 0.2305$$

$$\prod_{j=1}^{4}\left(1-\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j} = (1-0.8)^{0.35} \times (1-0.7)^{0.3} \times (1-0.6)^{0.1} \times (1-0.7)^{0.25} = 0.2679$$

$$\prod_{j=1}^{4}\left(\frac{w_{\vartheta_j}}{2\pi}\right)^{\xi_j} = (0.1)^{0.35} \times (0.1)^{0.3} \times (0.4)^{0.1} \times (0.2)^{0.25} = 0.1366$$

Thus, by using Eq. (7), we get

$$\text{CIFWA}(\delta_1, \delta_2, \delta_3, \delta_4) = \left\langle (1-0.3322)\, e^{i2\pi(1-0.2679)}, 0.2305 e^{i2\pi(0.1366)} \right\rangle = \left\langle 0.6678 e^{i2\pi(0.7321)}, 0.2305 e^{i2\pi(0.1366)} \right\rangle$$

Based on Theorem 8, it is observed that the CIFWA operator satisfies some properties which are stated as below.

Property 1 Let δ_0 be CIFN and if $\delta_j = \delta_0$ for all $j = 1, 2, \ldots, n$, then, we have

$$\text{CIFWA}(\delta_1, \delta_2 \ldots \delta_n) = \delta_0$$

This property is called Idempotency.

Proof Let $\delta_0 = \left\langle \zeta_0 e^{i w_{\zeta_0}}, \vartheta_0 e^{i w_{\vartheta_0}} \right\rangle$ and $\delta_j = \left\langle \zeta_j e^{i w_{\zeta_j}}, \vartheta_j e^{i w_{\vartheta_j}} \right\rangle (j = 1, 2, \ldots, n)$ be CIFNs such that $\delta_j = \delta_0$ which implies that $\zeta_j = \zeta_0$, $\vartheta_j = \vartheta_0$, $w_{\zeta_j} = w_{\zeta_0}$ and $w_{\vartheta_j} = w_{\vartheta_0}$ for all j. Also, we have $\xi_j > 0$ such that $\sum_{j=1}^{n} \xi_j = 1$. Then, by using Theorem 8, we get

$$\begin{aligned}
&\text{CIFWA}(\delta_1, \delta_2 \ldots \delta_n) \\
&= \left\langle \left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_0)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(\frac{w_{\zeta_0}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_0)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(\frac{w_{\vartheta_0}}{2\pi}\right)\right)\right)} \right\rangle \\
&= \left\langle \left(s^{-1}\left(s(\zeta_0)\right)\right) e^{i2\pi\left(s^{-1}\left(s\left(\frac{w_{\zeta_0}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(t(\vartheta_0)\right)\right) e^{i2\pi\left(t^{-1}\left(t\left(\frac{w_{\vartheta_0}}{2\pi}\right)\right)\right)} \right\rangle \\
&= \left\langle \zeta_0 e^{i w_{\zeta_0}}, \vartheta_0 e^{i w_{\vartheta_0}} \right\rangle \\
&= \delta_0
\end{aligned}$$

Hence, $\text{CIFWA}(\delta_1, \delta_2 \ldots \delta_n) = \delta_0$. □

Property 2 Consider two collections of CIFN $\delta_j = \left\langle \zeta_{\delta_j} e^{i w_{\zeta_{\delta_j}}}, \vartheta_{\delta_j} e^{i w_{\vartheta_{\delta_j}}} \right\rangle$ and $\beta_j = \left\langle \zeta_{\beta_j} e^{i w_{\zeta_{\beta_j}}}, \vartheta_{\beta_j} e^{i w_{\vartheta_{\beta_j}}} \right\rangle$ satisfying $\zeta_{\delta_j} \le \zeta_{\beta_j}$, $\vartheta_{\delta_j} \ge \vartheta_{\beta_j}$, $w_{\zeta_{\delta_j}} \le w_{\zeta_{\beta_j}}$ and $w_{\vartheta_{\delta_j}} \ge w_{\vartheta_{\beta_j}}$ for all j. Then, we have

$$\text{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n) \le \text{CIFWA}(\beta_1, \beta_2, \ldots, \beta_n).$$

This property is called monotonicity.

Proof For two CIFNs $\delta_j = \left\langle \zeta_{\delta_j} e^{i w_{\zeta_{\delta_j}}}, \vartheta_{\delta_j} e^{i w_{\vartheta_{\delta_j}}} \right\rangle$ and $\beta_j = \left\langle \zeta_{\beta_j} e^{i w_{\zeta_{\beta_j}}}, \vartheta_{\beta_j} e^{i w_{\vartheta_{\beta_j}}} \right\rangle$ such that $\zeta_{\delta_j} \le \zeta_{\beta_j}$, $\vartheta_{\delta_j} \ge \vartheta_{\beta_j}$, $w_{\zeta_{\delta_j}} \le w_{\zeta_{\beta_j}}$, $w_{\vartheta_{\delta_j}} \ge w_{\vartheta_{\beta_j}}$ for all j which implies that $\delta_j \le \beta_j$. Further, t and s are decreasing and increasing functions, respectively, then we have

$$s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_{\delta_j})\right) \le s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_{\beta_j})\right)$$

$$t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_{\delta_j})\right) \ge t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_{\beta_j})\right)$$

$$s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(\frac{w_{\zeta_{\delta_j}}}{2\pi}\right)\right) \le s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(\frac{w_{\zeta_{\beta_j}}}{2\pi}\right)\right) \quad \text{and}$$

$$t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(\frac{w_{\vartheta_{\delta_j}}}{2\pi}\right)\right) \ge t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(\frac{w_{\vartheta_{\beta_j}}}{2\pi}\right)\right)$$

Now, by using the score function as defined in Eq. (3), we get

$$S\Big(\mathrm{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n)\Big)$$
$$= s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_{\delta_j})\right) - t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_{\delta_j})\right)$$
$$+ \frac{1}{2\pi}\left(2\pi s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(\frac{w_{\zeta_{\delta_j}}}{2\pi}\right)\right)\right.$$
$$\left. - 2\pi t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(\frac{w_{\vartheta_{\delta_j}}}{2\pi}\right)\right)\right)$$
$$\le s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_{\beta_j})\right) - t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_{\beta_j})\right)$$
$$+ \frac{1}{2\pi}\left(2\pi s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(\frac{w_{\zeta_{\beta_j}}}{2\pi}\right)\right)\right.$$
$$\left. - 2\pi t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(\frac{w_{\vartheta_{\beta_j}}}{2\pi}\right)\right)\right)$$
$$= S\Big(\mathrm{CIFWA}(\beta_1, \beta_2, \ldots, \beta_n)\Big)$$

Hence, based on the order relation defined in Definition 6, we can obtain that $\mathrm{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n) \le \mathrm{CIFWA}(\beta_1, \beta_2, \ldots, \beta_n)$. □

Property 3 For a collection of CIFNs $\delta_j = \left\langle \zeta_j e^{i w_{\zeta_j}}, \vartheta_j e^{i w_{\vartheta_j}} \right\rangle$ $(j = 1, 2, \ldots n)$, let $\delta^- = \langle \zeta^- e^{i w_\zeta^-}, \vartheta^+ e^{i w_\vartheta^+} \rangle$ and $\delta^+ = \langle \zeta^+ e^{i w_\zeta^+}, \vartheta^- e^{i w_\vartheta^-} \rangle$ where $\zeta^- = \min_j\{\zeta_j\}$, $w_\zeta^- = \min_j\{w_{\zeta_j}\}$, $\zeta^+ = \max_j\{\zeta_j\}$, $w_\zeta^+ = \max_j\{w_{\zeta_j}\}$, $\vartheta^- = \min_j\{\vartheta_j\}$, $w_\vartheta^- = \min_j\{w_{\vartheta_j}\}$, $\vartheta^+ = \max_j\{\vartheta_j\}$, $w_\vartheta^+ = \max_j\{w_{\vartheta_j}\}$. Then, we have

$$\delta^- \le \mathrm{CIFWA}(\delta_1, \delta_2 \ldots \delta_n) \le \delta^+$$

This property is called boundedness.

Proof Let $\mathrm{CIFWA}(\delta_1, \delta_2 \ldots \delta_n) = \left\langle \zeta_S e^{i w_{\zeta_S}}, \vartheta_S e^{i w_{\vartheta_S}} \right\rangle$. For a CIFN δ_j, we have $\min_j\{\zeta_j\} \le \zeta_j \le \max_j\{\zeta_j\}$ and $\min_j\{\vartheta_j\} \le \vartheta_j \le \max_j\{\vartheta_j\}$. Since t, s are decreasing and increasing functions, respectively, therefore,

$$s^{-1}\left(\sum_{j=1}^{n}\xi_j s\Big(\min_j\{\zeta_j\}\Big)\right) \le s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_j)\right)$$
$$\le s^{-1}\left(\sum_{j=1}^{n}\xi_j s\Big(\max_j\{\zeta_j\}\Big)\right) \text{ and}$$

$$t^{-1}\left(\sum_{j=1}^{n}\xi_j t\Big(\min_j\{\vartheta_j\}\Big)\right)$$
$$\le t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_j)\right) \le t^{-1}\left(\sum_{j=1}^{n}\xi_j t\Big(\max_j\{\vartheta_j\}\Big)\right)$$

which implies that $\min_j\{\zeta_j\} \le \zeta_S \le \max_j\{\zeta_j\}$ and $\min_j\{\vartheta_j\} \le \vartheta_S \le \max_j\{\vartheta_j\}$, i.e., $\zeta^- \le \zeta_S \le \zeta^+$ and $\vartheta^- \le \vartheta_S \le \vartheta^+$. Similarly, we can obtain $w_\zeta^- \le w_{\zeta_S} \le w_\zeta^+$ and $w_\vartheta^- \le w_{\vartheta_S} \le w_\vartheta^+$. Hence, we get

$$S(\delta^-) = \zeta^- - \vartheta^+ + \frac{1}{2\pi}\left(w_\zeta^- - w_\vartheta^+\right)$$
$$\le \zeta_S - \vartheta_S + \frac{1}{2\pi}\left(w_{\zeta_S} - w_{\vartheta_S}\right)$$
$$= S(\mathrm{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n))$$

and

$$S(\delta^+) = \zeta^+ - \vartheta^- + \frac{1}{2\pi}\left(w_\zeta^+ - w_\vartheta^-\right)$$
$$\ge \zeta_S - \vartheta_S + \frac{1}{2\pi}\left(w_{\zeta_S} - w_{\vartheta_S}\right)$$
$$= S(\mathrm{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n))$$

Therefore, by using Definition 6, we get

$$\delta^- \le \mathrm{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n) \le \delta^+$$

□

Property 4 For a collection of CIFNs $\delta_j (j = 1, 2, \ldots, n)$ and CIFN β, we have

$$\mathrm{CIFWA}(\delta_1 \oplus \beta, \delta_2 \oplus \beta, \ldots, \delta_n \oplus \beta)$$
$$= \mathrm{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n) \oplus \beta$$

This property is called Shift Invariance.

Proof Let $\delta_j = \left\langle \zeta_j e^{i w_{\zeta_j}}, \vartheta_j e^{i w_{\vartheta_j}} \right\rangle$ and $\beta = \left\langle \zeta_\beta e^{i w_{\zeta_\beta}}, \vartheta_\beta e^{i w_{\vartheta_\beta}} \right\rangle$ be CIFNs. Then, by using addition law for CIFNs for all j, we get

$$\delta_j \oplus \beta = \left\langle \left(s^{-1}\left(s(\zeta_j) + s(\zeta_\beta)\right)\right) e^{i2\pi\left(s^{-1}\left(s\left(\frac{w_{\zeta_j}}{2\pi}\right) + s\left(\frac{w_{\zeta_\beta}}{2\pi}\right)\right)\right)}, \right.$$
$$\left. \left(t^{-1}\left(t(\vartheta_j) + t(\vartheta_\beta)\right)\right) e^{i2\pi\left(t^{-1}\left(t\left(\frac{w_{\vartheta_j}}{2\pi}\right) + t\left(\frac{w_{\vartheta_\beta}}{2\pi}\right)\right)\right)} \right\rangle$$

Now, using Eq. (6), we get

$$\begin{aligned}
&\mathrm{CIFWA}(\delta_1 \oplus \beta, \delta_2 \oplus \beta, \ldots, \delta_n \oplus \beta)\\
&=\left\langle \left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(s^{-1}\left(s(\zeta_j)+s(\zeta_\beta)\right)\right)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(s^{-1}\left(s\left(\frac{w_{\zeta_j}}{2\pi}\right)+s\left(\frac{w_{\zeta_\beta}}{2\pi}\right)\right)\right)\right)\right)},\right.\\
&\left.\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(t^{-1}\left(t(\vartheta_j)+t(\vartheta_\beta)\right)\right)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(t^{-1}\left(t\left(\frac{w_{\vartheta_j}}{2\pi}\right)+t\left(\frac{w_{\vartheta_\beta}}{2\pi}\right)\right)\right)\right)\right)}\right\rangle\\
&=\left\langle \left(s^{-1}\left(\sum_{j=1}^{n}\xi_j \left(s(\zeta_j)+s(\zeta_\beta)\right)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{n}\xi_j \left(s\left(\frac{w_{\zeta_j}}{2\pi}\right)+s\left(\frac{w_{\zeta_\beta}}{2\pi}\right)\right)\right)\right)},\right.\\
&\left.\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j \left(t(\vartheta_j)+t(\vartheta_\beta)\right)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j \left(t\left(\frac{w_{\vartheta_j}}{2\pi}\right)+t\left(\frac{w_{\vartheta_\beta}}{2\pi}\right)\right)\right)\right)}\right\rangle\\
&=\left\langle \left(s^{-1}\left(\left(\sum_{j=1}^{n}\xi_j s(\zeta_j)\right)+s(\zeta_\beta)\right)\right) e^{i2\pi\left(s^{-1}\left(\left(\sum_{j=1}^{n}\xi_j s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)+s\left(\frac{w_{\zeta_\beta}}{2\pi}\right)\right)\right)},\right.\\
&\left.\left(t^{-1}\left(\left(\sum_{j=1}^{n}\xi_j t(\vartheta_j)\right)+t(\vartheta_\beta)\right)\right) e^{i2\pi\left(t^{-1}\left(\left(\sum_{j=1}^{n}\xi_j t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)+t\left(\frac{w_{\vartheta_\beta}}{2\pi}\right)\right)\right)}\right\rangle\\
&=\left\langle \left(s^{-1}\left(s\left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_j)\right)\right)+s(\zeta_\beta)\right)\right) e^{i2\pi\left(s^{-1}\left(s\left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)+s\left(\frac{w_{\zeta_\beta}}{2\pi}\right)\right)\right)},\right.\\
&\left.\left(t^{-1}\left(t\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_j)\right)\right)+t(\vartheta_\beta)\right)\right) e^{i2\pi\left(t^{-1}\left(t\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)+t\left(\frac{w_{\vartheta_\beta}}{2\pi}\right)\right)\right)}\right\rangle\\
&=\mathrm{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n) \oplus \beta
\end{aligned}$$

which completes the proof of the theorem. □

Property 5 For a collection of CIFNs δ_j and any real number $\lambda > 0$, we have

$$\mathrm{CIFWA}(\lambda\delta_1, \lambda\delta_2, \ldots, \lambda\delta_n) = \lambda\mathrm{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n)$$

This property is called Homogeneity.

Proof For a collection of CIFNs $\delta_j = \langle \zeta_j e^{iw_{\zeta_j}}, \vartheta_j e^{iw_{\vartheta_j}} \rangle (j = 1, 2, \ldots, n)$ and real number $\lambda > 0$, we have

$$\lambda\delta_j = \left\langle \left(s^{-1}\left(\lambda s(\zeta_j)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\lambda t(\vartheta_j)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)} \right\rangle$$

Now, using Eq. (6), we get

$$\begin{aligned}
&\mathrm{CIFWA}(\lambda\delta_1, \lambda\delta_2, \ldots, \lambda\delta_n)\\
&=\left\langle \left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(s^{-1}\left(\lambda s(\zeta_j)\right)\right)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(s^{-1}\left(\lambda s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)\right)\right)},\right.\\
&\left.\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(t^{-1}\left(\lambda t(\vartheta_j)\right)\right)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(t^{-1}\left(\lambda t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)\right)\right)}\right\rangle\\
&=\left\langle \left(s^{-1}\left(\sum_{j=1}^{n}\xi_j \lambda s(\zeta_j)\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{n}\xi_j \lambda s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)},\right.\\
&\left.\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j \lambda t(\vartheta_j)\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j \lambda t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)}\right\rangle\\
&=\left\langle \left(s^{-1}\left(\lambda s\left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s(\zeta_j)\right)\right)\right)\right) e^{i2\pi\left(s^{-1}\left(\lambda s\left(s^{-1}\left(\sum_{j=1}^{n}\xi_j s\left(\frac{w_{\zeta_j}}{2\pi}\right)\right)\right)\right)\right)},\right.\\
&\left.\left(t^{-1}\left(\lambda t\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t(\vartheta_j)\right)\right)\right)\right) e^{i2\pi\left(t^{-1}\left(\lambda t\left(t^{-1}\left(\sum_{j=1}^{n}\xi_j t\left(\frac{w_{\vartheta_j}}{2\pi}\right)\right)\right)\right)\right)}\right\rangle\\
&=\lambda\mathrm{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n)
\end{aligned}$$

which completes the proof. □

Furthermore, some of the special cases of the aggregation operators are obtained from the proposed CIFWA operator by assigning the different form of the generator t with $t(0)=\infty$ as follows:

(i) If $t(a)=-\log a,\ 0<a\leq 1$ then Eq. (6) becomes

$$\text{CIFWA}(\delta_1,\delta_2,\ldots,\delta_n)=\left\langle\left(1-\prod_{j=1}^{n}\left(1-\zeta_j\right)^{\xi_j}\right)e^{i2\pi\left(1-\prod_{j=1}^{n}\left(1-\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}\right)},\ \prod_{j=1}^{n}\left(\vartheta_j^{\xi_j}\right)e^{i2\pi\prod_{j=1}^{n}\left(\frac{w_{\vartheta_j}}{2\pi}\right)^{\xi_j}}\right\rangle$$

and is called as CIF Archimedean weighted averaging operator.

(ii) If $t(a)=\log\left(\frac{2-a}{a}\right)$, then Eq. (6) becomes

$$\text{CIFWA}(\delta_1,\delta_2,\ldots,\delta_n)=\left\langle\left(\frac{\prod_{j=1}^{n}\left(1+\zeta_j\right)^{\xi_j}-\prod_{j=1}^{n}\left(1-\zeta_j\right)^{\xi_j}}{\prod_{j=1}^{n}\left(1+\zeta_j\right)^{\xi_j}+\prod_{j=1}^{n}\left(1-\zeta_j\right)^{\xi_j}}\right)e^{i2\pi\left(\frac{\prod_{j=1}^{n}\left(1+\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}-\prod_{j=1}^{n}\left(1-\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}}{\prod_{j=1}^{n}\left(1+\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}+\prod_{j=1}^{n}\left(1-\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}}\right)},\right.$$
$$\left.\left(\frac{2\prod_{j=1}^{n}\vartheta_j^{\xi_j}}{\prod_{j=1}^{n}\left(2-\vartheta_j\right)^{\xi_j}+\prod_{j=1}^{n}\vartheta_j^{\xi_j}}\right)e^{i2\pi\left(\frac{2\prod_{j=1}^{n}\left(\frac{w_{\vartheta_j}}{2\pi}\right)^{\xi_j}}{\prod_{j=1}^{n}\left(2-\frac{w_{\vartheta_j}}{2\pi}\right)^{\xi_j}+\prod_{j=1}^{n}\left(\frac{w_{\vartheta_j}}{2\pi}\right)^{\xi_j}}\right)}\right\rangle$$

and is called as CIF Einstein weighted averaging (CIFEWA) operator.

(iii) If $t(a)=\log\left(\frac{\gamma+(1-\gamma)a}{a}\right)$, $\gamma\in(0,\infty)$ then Eq. (6) becomes

$$\text{CIFWA}(\delta_1,\delta_2,\ldots,\delta_n)=\left\langle\left(\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)\zeta_j\right)^{\xi_j}-\prod_{j=1}^{n}\left(1-\zeta_j\right)^{\xi_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\zeta_j\right)^{\xi_j}+\prod_{j=1}^{n}\left(1-\zeta_j\right)^{\xi_j}}\right)e^{i2\pi\left(\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}-\prod_{j=1}^{n}\left(1-\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}+\prod_{j=1}^{n}\left(1-\frac{w_{\zeta_j}}{2\pi}\right)^{\xi_j}}\right)},\right.$$
$$\left.\left(\frac{\gamma\prod_{j=1}^{n}\vartheta_j^{\xi_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(1-\vartheta_j)\right)^{\xi_j}+(\gamma-1)\prod_{j=1}^{n}\vartheta_j^{\xi_j}}\right)e^{i2\pi\left(\frac{\gamma\prod_{j=1}^{n}\left(\frac{w_{\vartheta_j}}{2\pi}\right)^{\xi_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-\frac{w_{\vartheta_j}}{2\pi}\right)\right)^{\xi_j}+(\gamma-1)\prod_{j=1}^{n}\left(\frac{w_{\vartheta_j}}{2\pi}\right)^{\xi_j}}\right)}\right\rangle$$

and is called as CIF Hamacher weighted averaging (CIFHWA) operator.

3.4 Ordered Weighted Averaging Operator

In this section, we propose a new operator named as complex intuitionistic fuzzy ordered weighted averaging (CIFOWA) operator.

Definition 9 Let Ω be a collection of CIFNs. We define a map CIFOWA : $\Omega^n\to\Omega$ by

$$\text{CIFOWA}(\delta_1,\delta_2,\ldots,\delta_n)=\xi_1\delta_{\sigma(1)}\oplus\xi_2\delta_{\sigma(2)}\oplus\ldots\oplus\xi_n\delta_{\sigma(n)}$$

for all $\delta_j\in\Omega$ where $\left(\sigma(1),\sigma(2),\ldots,\sigma(n)\right)$ is a permutation of $(1,2,\ldots,n)$ such that $\delta_{\sigma(j-1)}\geq\delta_{\sigma(j)}$ for $j=2,3,\ldots,n$ and $\xi=(\xi_1,\xi_2,\ldots,\xi_n)^T$ is the weight vector of δ_j with $\xi_j>0$ and $\sum_{j=1}^{n}\xi_j=1$. Then, CIFOWA is called complex intuitionistic fuzzy ordered weighted averaging operator.

Theorem 9 *The aggregated value by using CIFOWA operator for a collection of CIFNs* $\delta_j=\left\langle\zeta_je^{iw_{\zeta_j}},\vartheta_je^{iw_{\vartheta_j}}\right\rangle$ $(j=1,2,\ldots n)$ *is still a CIFN and is given by*

$$CIFOWA(\delta_1,\delta_2,\ldots,\delta_n)=\left\langle\left(s^{-1}\left(\sum_{j=1}^{n}\xi_js(\zeta_{\sigma(j)})\right)\right)e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{n}\xi_js\left(\frac{w_{\zeta_{\sigma(j)}}}{2\pi}\right)\right)\right)},\ \left(t^{-1}\left(\sum_{j=1}^{n}\xi_jt(\vartheta_{\sigma(j)})\right)\right)e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{n}\xi_jt\left(\frac{w_{\vartheta_{\sigma(j)}}}{2\pi}\right)\right)\right)}\right\rangle \tag{8}$$

In particular, if $w_{\zeta_j},w_{\vartheta_j}=0\ \forall\ j$ *then, Eq. (8) reduces to*

$$CIFOWA(\delta_1,\delta_2,\ldots,\delta_n)=\left\langle\left(s^{-1}\left(\sum_{j=1}^{n}\xi_js(\zeta_{\sigma(j)})\right)\right),\ \left(t^{-1}\left(\sum_{j=1}^{n}\xi_jt(\vartheta_{\sigma(j)})\right)\right)\right\rangle$$

which is an intuitionistic fuzzy OWA operator.

Proof Proof is similar to Theorem 8. □

Example 2 Let $\delta_1 = \langle 0.6e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.1)}\rangle$, $\delta_2 = \langle 0.8e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.1)}\rangle$, $\delta_3 = \langle 0.5e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)}\rangle$, $\delta_4 = \langle 0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.2)}\rangle$ be four CIFNs and $\xi = (0.35, 0.3, 0.1, 0.25)^T$ be the associated weight vector. Then, score function of each CIFN is calculated as $S(\delta_1) = 1.1$, $S(\delta_2) = 1.2$, $S(\delta_3) = 0.4$ and $S(\delta_4) = 0.8$. Since $S(\delta_2) > S(\delta_1) > S(\delta_4) > S(\delta_3)$ and hence by permutation, we have $\delta_{\sigma(1)} = \langle 0.8e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.1)}\rangle$, $\delta_{\sigma(2)} = \langle 0.6e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.1)}\rangle$, $\delta_{\sigma(3)} = \langle 0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.2)}\rangle$ and $\delta_{\sigma(4)} = \langle 0.5e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)}\rangle$. Without loss of generality, we consider the additive generator $t(a) = -\log a$ if $0 < a \le 1$ with $t(0) = \infty$ corresponding to t-norm. Then, Eq. (8) becomes

$$\text{CIFOWA}(\delta_1, \delta_2, \ldots, \delta_n) = \left\langle \left(1 - \prod_{j=1}^{n} (1 - \zeta_{\sigma(j)})^{\xi_j}\right) e^{i2\pi\left(1 - \prod_{j=1}^{n}\left(1 - \frac{w_{\zeta_{\sigma(j)}}}{2\pi}\right)^{\xi_j}\right)}, \prod_{j=1}^{n}\left(\vartheta_{\sigma(j)}^{\xi_j}\right) e^{i2\pi \prod_{j=1}^{n}\left(\frac{w_{\vartheta_{\sigma(j)}}}{2\pi}\right)^{\xi_j}} \right\rangle \tag{9}$$

Therefore,

$$\prod_{j=1}^{4} (1 - \zeta_{\sigma(j)})^{\xi_j} = (1 - 0.8)^{0.35} \times (1 - 0.6)^{0.3} \times (1 - 0.6)^{0.1} \times (1 - 0.5)^{0.25} = 0.3318$$

$$\prod_{j=1}^{4} \vartheta_{\sigma(j)}^{\xi_j} = (0.2)^{0.35} \times (0.2)^{0.3} \times (0.3)^{0.1} \times (0.3)^{0.25} = 0.2305$$

$$\prod_{j=1}^{4} \left(1 - \frac{w_{\zeta_{\sigma(j)}}}{2\pi}\right)^{\xi_j} = (1 - 0.7)^{0.35} \times (1 - 0.8)^{0.3} \times (1 - 0.7)^{0.1} \times (1.0.6)^{0.25} = 0.2854$$

$$\prod_{j=1}^{4} \left(\frac{w_{\vartheta_{\sigma(j)}}}{2\pi}\right)^{\xi_j} = (0.1)^{0.35} \times (0.1)^{0.3} \times (0.2)^{0.1} \times (0.4)^{0.25} = 0.1516$$

Thus, by using Eq. (9), we obtain

$$\text{CIFOWA}(\delta_1, \delta_2, \delta_3, \delta_4) = \left\langle (1 - 0.3318)\, e^{i2\pi(1-0.2854)}, 0.2305e^{i2\pi(0.1516)} \right\rangle = \left\langle 0.6682e^{i2\pi(0.7146)}, 0.2305e^{i2\pi(0.1516)} \right\rangle$$

Also, it is analyzed that CIFOWA operator satisfies the properties as CIFWA operator satisfies. Besides these properties, CIFOWA operator satisfies the additional property which are stated as below.

Property 6 Let δ_j $(j = 1, 2, \ldots, n)$ be the collection of CIFNs. Then,

(i) If $\xi = (1, 0, \ldots, 0)^T$ then, $\text{CIFOWA}(\delta_1, \delta_2, \ldots, \delta_n) = \max\{\delta_1, \delta_2, \ldots, \delta_n\}$
(ii) If $\xi = (0, 0, \ldots, 1)^T$ then, $\text{CIFOWA}(\delta_1, \delta_2, \ldots, \delta_n) = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$
(iii) If $\xi_j = 1$ and $\xi_t = 0$ for $t \neq j$ then, $\text{CIFOWA}(\delta_1, \delta_2, \ldots, \delta_n) = \delta_{\sigma(j)}$ where $\delta_{\sigma(j)}$ is the jth largest of δ_j.

3.5 Hybrid Averaging Operator

In this section, by taking the advantages of both weighted and ordered weighted averaging operators, we propose a hybrid averaging aggregation operator for a collection of CIFNs.

Definition 10 Let Ω be a collection of CIFNs. A map CIFHA : $\Omega^n \to \Omega$, having associated weight vector $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T$ with $\psi_j > 0$ and $\sum_{j=1}^{n} \psi_j = 1$, defined by

$$\text{CIFHA}(\delta_1, \delta_2, \ldots, \delta_n) = \psi_1\dot{\delta}_{\sigma(1)} \oplus \psi_2\dot{\delta}_{\sigma(2)} \oplus \ldots \oplus \psi_n\dot{\delta}_{\sigma(n)}$$

where $\dot{\delta}_j = n\xi_j\delta_j$, $j = 1, 2, \ldots, n$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ is the weight vector of δ_j with $\xi_j > 0$ and $\sum_{j=1}^{n} \xi_j = 1$, is called complex intuitionistic fuzzy hybrid averaging operator(CIFHA).

Theorem 10 *The combined value obtained after applying CIFHA operator for a collection of CIFNs* $\delta_j = \langle \zeta_j e^{iw_{\zeta_j}}, \vartheta_j e^{iw_{\vartheta_j}}\rangle$ $(j = 1, 2, \ldots n)$ *is still a CIFN and is given by*

$$\mathit{CIFHA}(\delta_1, \delta_2, \ldots, \delta_n) = \left\langle \left(s^{-1}\left(\sum_{j=1}^{n} \psi_j s(\dot{\zeta}_{\sigma(j)})\right)\right) e^{i2\pi\left(s^{-1}\left(\sum_{j=1}^{n} \psi_j s\left(\frac{\dot{w}_{\zeta_{\sigma(j)}}}{2\pi}\right)\right)\right)}, \left(t^{-1}\left(\sum_{j=1}^{n} \psi_j t(\dot{\vartheta}_{\sigma(j)})\right)\right) e^{i2\pi\left(t^{-1}\left(\sum_{j=1}^{n} \psi_j t\left(\frac{\dot{w}_{\vartheta_{\sigma(j)}}}{2\pi}\right)\right)\right)} \right\rangle \tag{10}$$

Proof Similar to Theorem 8. □

Example 3 Let $\delta_1 = \langle 0.6e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.1)}\rangle$, $\delta_2 = \langle 0.8e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.1)}\rangle$, $\delta_3 = \langle 0.5e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)}\rangle$, $\delta_4 = \langle 0.6e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.2)}\rangle$ be four CIFNs and $\psi = (0.4, 0.2, 0.1, 0.3)^T$ be the weight vector associated with CIFHA and $\xi = (0.35, 0.3, 0.1, 0.25)^T$ be the corresponding weight vector of δ_j for all $j = 1, 2, 3, 4$. Consider the additive generator $t(a) = -\log a$ if $0 < a \le 1$ with $t(0) = \infty$ corresponding to t-norm. Then, using Definition 7, we have $\lambda\delta_1 = \left\langle \left(1-(1-\zeta_1)^\lambda\right) e^{i2\pi\left(1-\left(1-\frac{w_{\zeta_1}}{2\pi}\right)^\lambda\right)}, \vartheta_1^\lambda e^{i2\pi\left(\frac{w_{\vartheta_1}}{2\pi}\right)^\lambda} \right\rangle$ and Eq. (10) becomes

$$\text{CIFHA}(\delta_1, \delta_2, \ldots, \delta_n) = \left\langle \left(1-\prod_{j=1}^n \left(1-\dot{\zeta}_{\sigma(j)}\right)^{\psi_j}\right) e^{i2\pi\left(1-\prod_{j=1}^n\left(1-\frac{\dot{w}_{\zeta_{\sigma(j)}}}{2\pi}\right)^{\psi_j}\right)}, \prod_{j=1}^n \left(\dot{\vartheta}_{\sigma(j)}^{\psi_j}\right) e^{i2\pi\prod_{j=1}^n\left(\frac{\dot{w}_{\vartheta_{\sigma(j)}}}{2\pi}\right)^{\psi_j}} \right\rangle \quad (11)$$

Now, by utilizing $\xi = (0.35, 0.3, 0.1, 0.25)^T$ and CIFNs $\delta_j (j = 1, 2, 3, 4)$, we have

$$\begin{aligned}\dot{\delta}_1 &= n\xi_1\delta_1 = 1.4\langle 0.6e^{i2\pi(0.8)}, 0.2e^{i2\pi(0.1)}\rangle \\ &= \left\langle \left(1-(1-0.6)^{1.4}\right) e^{i2\pi\left(1-(1-0.8)^{1.4}\right)}, (0.2)^{1.4} e^{i2\pi(0.1)^{1.4}} \right\rangle \\ &= \left\langle \left(1-0.4^{1.4}\right) e^{i2\pi\left(1-(0.2)^{1.4}\right)}, (0.2)^{1.4} e^{i2\pi(0.1)^{1.4}} \right\rangle \\ &= \left\langle 0.7227e^{i2\pi(0.8949)}, 0.1051e^{i2\pi(0.0398)} \right\rangle\end{aligned}$$

Similarly, $\dot{\delta}_2 = \langle 0.8550e^{i2\pi(0.7642)}, 0.1450e^{i2\pi(0.0631)}\rangle$, $\dot{\delta}_3 = \langle 0.2421e^{i2\pi(0.3069)}, 0.6178e^{i2\pi(0.6931)}\rangle$ and $\dot{\delta}_4 = \langle 0.6000e^{i2\pi(0.7000)}, 0.3000e^{i2\pi(0.2000)}\rangle$. Thus, score values of these numbers are $S(\dot{\delta}_1) = 1.4728$, $S(\dot{\delta}_2) = 1.4112$, $S(\dot{\delta}_3) = -0.7619$ and $S(\dot{\delta}_4) = 0.8000$ and hence $S(\dot{\delta}_1) > S(\dot{\delta}_2) > S(\dot{\delta}_4) > S(\dot{\delta}_3)$. Therefore, $\dot{\delta}_{\sigma(1)} = \langle 0.7227e^{i2\pi(0.8949)}, 0.1051e^{i2\pi(0.0398)}\rangle$, $\dot{\delta}_{\sigma(2)} = \langle 0.8550e^{i2\pi(0.7642)}, 0.1450e^{i2\pi(0.0631)}\rangle$, $\dot{\delta}_{\sigma(3)} = \langle 0.6000e^{i2\pi(0.7000)}, 0.3000e^{i2\pi(0.2000)}\rangle$ and $\dot{\delta}_{\sigma(4)} = \langle 0.2421e^{i2\pi(0.3069)}, 0.6178e^{i2\pi(0.6931)}\rangle$. Hence, based on these information and by using Eq. (11) we get $\text{CIFHA}(\delta_1, \delta_2, \delta_3, \delta_4) = \langle 0.6584e^{i2\pi(0.7584)}, 0.2118e^{i2\pi(0.1209)}\rangle$.

As similar to CIFWA and CIFOWA operators, the proposed CIFHA operator also satisfies the above stated properties.

Theorem 11 *If $\psi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ then, CIFHA operator becomes CIFWA operator.*

Proof Since, $\dot{\delta}_{\sigma(j)} = n\xi_j\delta_j$ and $\psi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ which implies that $\psi_j\dot{\delta}_{\sigma(j)} = \xi_j\delta_j$. Therefore,

$$\begin{aligned}\text{CIFHA}(\delta_1, \delta_2, \ldots, \delta_n) &= \psi_1\dot{\delta}_{\sigma(1)} \oplus \psi_2\dot{\delta}_{\sigma(2)} \oplus \ldots \oplus \psi_n\dot{\delta}_{\sigma(n)} \\ &= \xi_1\delta_1 \oplus \xi_2\delta_2 \oplus \ldots \oplus \xi_n\delta_n \\ &= \text{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n)\end{aligned}$$

Hence, $\text{CIFHA}(\delta_1, \delta_2, \ldots, \delta_n) = \text{CIFWA}(\delta_1, \delta_2, \ldots, \delta_n)$. □

Theorem 12 *If $\xi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ then, CIFHA operator becomes CIFOWA operator.*

Proof Similar to the proof of the above theorem. □

4 Multicriteria Decision-Making Approach Using Proposed Operators

In this section, a multicriteria decision-making (MCDM) approach is presented for designating the available choices in order to select the most suitable one, under CIFS environment, followed by an illustrative example.

4.1 Approach Based on Proposed Operators

Consider a decision-making problem which consists of 'm' alternatives $A_1, A_2, \ldots, A_m$ which are evaluated under the 'n' different criteria $C_1, C_2, \ldots, C_n$. For it, consider an expert who evaluates the each alternative under the CIFS environment and gives their rating values in terms of CIFNs $\delta_{pq} = \langle \zeta_{pq}e^{iw_{\zeta_{pq}}}, \vartheta_{pq}e^{iw_{\vartheta_{pq}}}\rangle$ where $p = 1, 2, \ldots, m$ and $q = 1, 2, \ldots n$, with $0 \le \zeta_{pq}, \vartheta_{pq} \le 1$ such that $0 \le \zeta_{pq} + \vartheta_{pq} \le 1$ and $0 \le w_{\zeta_{pq}}, w_{\vartheta_{pq}} \le 2\pi$ such that $0 \le w_{\zeta_{pq}} + w_{\vartheta_{pq}} \le 2\pi$. Further, assume that $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ be the weight of the different criteria such that $\xi_q > 0$ and $\sum_{q=1}^n \xi_q = 1$. Then, to determine the most desirable alternative(s), the proposed operators are utilized to develop a multicriteria decision-making method with complex intuitionistic fuzzy information, which involves the following steps:

Step 1: Collect the CIF decision matrix $M = \left(\delta_{pq}\right)_{m\times n}$ corresponding to the rating values of each alternative as

$$M = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{pmatrix} \delta_{11} & \delta_{12} & \ldots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \ldots & \delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{m1} & \delta_{m2} & \ldots & \delta_{mn} \end{pmatrix} \end{array}$$

Step 2: Aggregate the collective rating values $M = (\delta_{pq})$ of the alternative $A_p (p = 1, 2, \ldots, m)$ into the overall assessment value $\delta_p = \langle r_p e^{i w_{r_p}}, k_p e^{i w_{k_p}} \rangle$ based on the either CIFWA, CIFOWA or CIFHA operators as defined in Eqs. (6), (8) and (10), respectively. For instance, if we utilize CIFWA operator to aggregate each rating value of the alternative A_p, then we get the overall assessment value $\delta_p (p = 1, 2, \ldots, m)$ as

$$\delta_p = \text{CIFWA}(\delta_{p1}, \delta_{p2}, \ldots, \delta_{pn}) = \left\langle \left(s^{-1}\left(\sum_{q=1}^{n} \xi_q s(\zeta_{pq}) \right) \right) e^{i2\pi\left(s^{-1}\left(\sum_{q=1}^{n} \xi_q s\left(\frac{w_{\zeta_{pq}}}{2\pi}\right) \right) \right)}, \left(t^{-1}\left(\sum_{q=1}^{n} \xi_q t(\vartheta_{pq}) \right) \right) e^{i2\pi\left(t^{-1}\left(\sum_{q=1}^{n} \xi_q t\left(\frac{w_{\vartheta_{pq}}}{2\pi}\right) \right) \right)} \right\rangle \tag{12}$$

or by using CIFOWA operator as follows

$$\delta_p = \text{CIFOWA}(\delta_{p1}, \delta_{p2}, \ldots, \delta_{pn}) = \left\langle \left(s^{-1}\left(\sum_{q=1}^{n} \xi_q s(\zeta_{p\sigma(q)}) \right) \right) e^{i2\pi\left(s^{-1}\left(\sum_{q=1}^{n} \xi_q s\left(\frac{w_{\zeta_{p\sigma(q)}}}{2\pi}\right) \right) \right)}, \left(t^{-1}\left(\sum_{q=1}^{n} \xi_q t(\vartheta_{p\sigma(q)}) \right) \right) e^{i2\pi\left(t^{-1}\left(\sum_{q=1}^{n} \xi_q t\left(\frac{w_{\vartheta_{p\sigma(q)}}}{2\pi}\right) \right) \right)} \right\rangle \tag{13}$$

or by using CIFHA operator as follows

$$\delta_p = \text{CIFHA}(\delta_{p1}, \delta_{p2}, \ldots, \delta_{pn}) = \left\langle \left(s^{-1}\left(\sum_{q=1}^{n} \psi_q s(\dot{\zeta}_{p\sigma(q)}) \right) \right) e^{i2\pi\left(s^{-1}\left(\sum_{q=1}^{n} \psi_q s\left(\frac{w_{\dot{\zeta}_{p\sigma(q)}}}{2\pi}\right) \right) \right)}, \left(t^{-1}\left(\sum_{q=1}^{n} \psi_q t(\dot{\vartheta}_{p\sigma(q)}) \right) \right) e^{i2\pi\left(t^{-1}\left(\sum_{q=1}^{n} \psi_q t\left(\frac{w_{\dot{\vartheta}_{p\sigma(q)}}}{2\pi}\right) \right) \right)} \right\rangle \tag{14}$$

where σ is the permutation map of $\{1, 2, \ldots, n\}$ such that $\delta_{p\sigma(q-1)} \geq \delta_{p\sigma(q)}$ for $q = 2, 3, \ldots, n$ and t, s, respectively, be the decreasing and increasing t-norm function such that $s(a) = t(1-a)$ for $a \in [0, 1]$, ψ_q be the standardized weighted vector associated with CIFHA operator and $\dot{\delta}_{pq} = n\xi_q \delta_{pq}$.

Step 3: Compute the score values of the overall aggregated values $\delta_p = \langle r_p e^{i w_{r_p}}, k_p e^{i w_{k_p}} \rangle (p = 1, 2, \ldots, m)$ by using equation

$$S(\delta_p) = (r_p - k_p) + \frac{1}{2\pi}\left(w_{r_p} - w_{k_p}\right).$$

If there is no difference between two score values $S(\delta_{p_1})$ and $S(\delta_{p_2})$ for any two positive p_1, p_2, then we need to calculate the accuracy values of the alternatives as

$$H(\delta_p) = (r_p + k_p) + \frac{1}{2\pi}\left(w_{r_p} + w_{k_p}\right).$$

Step 4: Rank all the feasible alternatives $A_p (p = 1, 2, \ldots, m)$ according to Definition 6 and hence select the most desirable alternative(s).

4.2 Illustrative Example

In order to demonstrate the above-mentioned approach, we illustrate it with a numerical example which is stated as below:

Suppose an entrepreneur decides to purchase a new machine for his company from a machine maker X. The machine maker provides information on five models of machine $(A_1, A_2, A_3, A_4, A_5)$ with different production dates for each model. The entrepreneur decides to consider four criteria namely C_1: Reliability, C_2: Safety, C_3: Flexibility and C_4: Productivity for selecting machine. According to the changes in production date for the same model of machines, these criteria will also affect and change. The purpose of the entrepreneur is to select the best model of machine among the available alternatives. For it, the considered weight vectors corresponding to four preferences factors are $\xi = (0.4, 0.25, 0.15, 0.2)^T$, while the positional weight vectors of the factors are $\psi = (0.35, 0.3, 0.1, 0.25)^T$. To fulfill this purpose, he consults an expert who evaluates available models of machine and gives their preferences under CIF environment. In what follows, we utilize the multicriteria decision-making method proposed in above section to determine the most desirable alternative(s) under complex intuitionistic fuzzy environment.

Step 1: The given expert evaluates each model of the machine taken as an alternative with respect to the four criteria under the CIFS environment, and their corresponding rating values are summarized in the decision matrix represented in Table 1. In this

Table 1 Input information in the form of the complex intuitionistic fuzzy decision matrix

	C_1	C_2	C_3	C_4
A_1	$\langle 0.7e^{i2\pi(0.9)}, 0.1e^{i2\pi(0.1)}\rangle$	$\langle 0.8e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.4)}\rangle$	$\langle 0.6e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.2)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.2)}\rangle$
A_2	$\langle 0.7e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.3)}\rangle$	$\langle 0.4e^{i2\pi(0.9)}, 0.2e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.3)}\rangle$	$\langle 0.4e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.1)}\rangle$
A_3	$\langle 0.3e^{i2\pi(0.4)}, 0.6e^{i2\pi(0.4)}\rangle$	$\langle 0.6e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)}\rangle$	$\langle 0.3e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.6)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.1)}\rangle$
A_4	$\langle 0.4e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.3)}\rangle$	$\langle 0.6e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.3)}\rangle$	$\langle 0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.4)}\rangle$
A_5	$\langle 0.9e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.2)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.2)}\rangle$	$\langle 0.8e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.1)}\rangle$

Table 2 Ordering position data

	C_1	C_2	C_3	C_4
A_1	$\langle 0.7e^{i2\pi(0.9)}, 0.1e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.2)}\rangle$	$\langle 0.8e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.4)}\rangle$	$\langle 0.6e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.2)}\rangle$
A_2	$\langle 0.4e^{i2\pi(0.9)}, 0.2e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.3)}\rangle$	$\langle 0.7e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.3)}\rangle$	$\langle 0.4e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.1)}\rangle$
A_3	$\langle 0.7e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.1)}\rangle$	$\langle 0.6e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)}\rangle$	$\langle 0.3e^{i2\pi(0.4)}, 0.6e^{i2\pi(0.4)}\rangle$	$\langle 0.3e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.6)}\rangle)$
A_4	$\langle 0.6e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.3)}\rangle$	$\langle 0.4e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.3)}\rangle$	$\langle 0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.4)}\rangle$
A_5	$\langle 0.8e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.1)}\rangle$	$\langle 0.9e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.2)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.2)}\rangle$

table, for instance, the rating value for a model of machine A_1 under "reliability" (C_1) criteria is given as $\langle 0.7e^{i2\pi(0.9)}, 0.1e^{i2\pi(0.1)}\rangle$ by an expert which describes that the expert agreed 70% with the suitability of the model A_1 at C_1 while disagree with 10%. On the other hand, the same expert satisfied 90% with the suitability of production date of the model at C_1 and not satisfied with the 10%. In a similar manner, the other data values can be interpreted.

Step 2: Without loss of generality, we take a generator $t(a) = -\log(a)$. Now, by utilizing CIFWA operator defined in Eq. (12) corresponding to the weight vectors $\xi = (0.4, 0.25, 0.15, 0.2)^T$, we get the aggregated values $\delta_p(p = 1, 2, 3, 4, 5)$ corresponding to each alternative $A_p(p = 1, 2, 3, 4, 5)$ as

$$\delta_1 = \left\langle 0.7170e^{i2\pi(0.7707)}, 0.1469e^{i2\pi(0.1803)}\right\rangle,$$

$$\delta_2 = \left\langle 0.5902e^{i2\pi(0.7291)}, 0.2551e^{i2\pi(0.1830)}\right\rangle,$$

$$\delta_3 = \left\langle 0.4863e^{i2\pi(0.5280)}, 0.3431e^{i2\pi(0.3222)}\right\rangle,$$

$$\delta_4 = \left\langle 0.5422e^{i2\pi(0.6230)}, 0.3121e^{i2\pi(0.2048)}\right\rangle$$

$$\delta_5 = \left\langle 0.8217e^{i2\pi(0.7112)}, 0.1320e^{i2\pi(0.1464)}\right\rangle.$$

Step 3: The score values of the alternative $A_p(p = 1, 2, 3, 4, 5)$ are obtained based on the overall assessment values $\delta_p(p = 1, 2, 3, 4, 5)$ as follows:

$$S(\delta_1) = 1.1605, \quad S(\delta_2) = 0.8812, \quad S(\delta_3) = 0.3491,$$
$$S(\delta_4) = 0.6484, \quad S(\delta_5) = 1.2545.$$

Step 4: Since $S(\delta_5) > S(\delta_1) > S(\delta_2) > S(\delta_4) > S(\delta_3)$ and hence based on it, the ranking of all the feasible alternatives $A_p(p = 1, 2, 3, 4, 5)$ is given as

$$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3,$$

where the symbol "$\succ$" means "preferred to." Thus, we conclude that the best alternative is A_5, i.e., A_5 is the most optimal model.

On the other hand, if we utilize CIFOWA operator instead of CIFWA operator to aggregate the given preferences, then the following steps of the proposed approach are executed to find the best alternative(s) as below

Step 1: The information is summarized in Table 1.

Step 2: Compute the score values of each CIFN and based on the ordering relation between them as defined in Definition 6, we get the permutation rating values of each alternative under different criteria. These values corresponding to each alternative are summarized in Table 2.
Now, take a generator $t(a) = -\log(a)$ and weight vector $\xi = (0.4, 0.25, 0.15, 0.2)^T$, we aggregate the preferences of the alternatives by using CIFOWA operator as defined in Eq. (13). The collective values corresponding to each alternative $A_p(p = 1, 2, 3, 4, 5)$ are obtained as

$$\delta_1 = \left\langle 0.7010e^{i2\pi(0.7789)}, 0.1639e^{i2\pi(0.1682)}\right\rangle,$$

$$\delta_2 = \left\langle 0.5453e^{i2\pi(0.7862)}, 0.2305e^{i2\pi(0.1552)}\right\rangle,$$

Table 3 Ranking of the alternatives based on the proposed operators

Additive generators	Proposed Operators	Score values of the alternatives					Ranking
		A_1	A_2	A_3	A_4	A_5	
$t(a) = -\log(a)$	CIFWA	1.1605	0.8812	0.3491	0.6484	1.2545	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
	CIFOWA	1.1478	0.9458	0.6687	0.6774	1.2719	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
	CIFHA	1.3249	1.0153	0.4290	0.7917	1.3421	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
$t(a) = \log\left(\frac{2-a}{a}\right)$	CIFEWA	1.1468	0.8650	0.3096	0.6193	1.2501	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
	CIFEOWA	1.1359	0.9306	0.6339	0.6539	1.2675	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
	CIFEHA	1.3352	1.0264	0.3598	0.8005	1.3611	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$

$$\delta_3 = \left\langle 0.5663e^{i2\pi(0.5891)}, 0.2376e^{i2\pi(0.2491)} \right\rangle,$$
$$\delta_4 = \left\langle 0.5567e^{i2\pi(0.5818)}, 0.2197e^{i2\pi(0.2415)} \right\rangle,$$
$$\delta_5 = \left\langle 0.8062e^{i2\pi(0.7298)}, 0.1275e^{i2\pi(0.1366)} \right\rangle.$$

Step 3: The score values of these aggregated numbers are computed by using Eq. (3) as

$$S(\delta_1) = 1.1478, \quad S(\delta_2) = 0.9458, \quad S(\delta_3) = 0.6687,$$
$$S(\delta_4) = 0.6774, \quad S(\delta_5) = 1.2719.$$

Step 4: The ranking order of these alternatives, based on optimal values of the score values given as

$$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3,$$

and hence, we obtain A_5 is the best alternative for the required machine.

From these computed results, we conclude that the best alternative by both the operator remains same, but the computational procedure is entirely different. In CIFWA operator, the weight vector is assigned directly to the CIFNs and then collect the aggregated values. On the other hand, in CIFOWA operator, firstly the importance of the numbers is arranged sequentially based on the importance of the numbers and then aggregate the different values in the collective one. Apart from the above analysis, if we utilize the different t-norm generators such as $t(a) = -\log(a)$ or $t(a) = \log\left(\frac{2-a}{a}\right)$ for $a \in (0, 1]$; $t(0) = \infty$ and by using CIFWA, CIFOWA and CIFHA operator to the considered data then we get the overall score values of the alternatives $A_p (p = 1, 2, 3, 4, 5)$. These values along with the final ranking order of the alternatives are summarized in Table 3. From this table, it is clearly seen that the ranking order is preserved by all the operator, and hence, it shows the stability of the result. Further, by taking the importance of the corresponding aggregation operator, the decision maker can choose the appropriate one and hence select the best alternative for the required task.

4.3 Validity Test

Since the different MCDM methods may give different evaluations (ranking) when applied to same decision-making problem, which leads to uncertain results. Therefore, there is a need to validate the results obtained from their corresponding approach to explain the reliability of the approach. For it, Wang and Triantaphyllou [8] presented the following three test criteria to validate the MCDM approach.

Test criterion 1: "An MCDM method is effective if on replacing a non-optimal alternative by another worse alternative without changing the relative importance of each decision criteria, the indication of the best alternative remains same."

Test criterion 2: "An effective MCDM method should follow transitive property."

Test criterion 3: "An MCDM method is effective if on decomposing the MCDM problem into smaller problems and by applying the same MCDM method to these subproblems for ranking the alternatives, the combined ranking of the alternatives remains same to the ranking of the original problem."

Now, we validate these criteria on our proposed MCDM approach as follows:

4.3.1 Validity Test by Applying Criterion 1

For testing the validity of proposed approach under the criterion 1, we replace the non-optimal alternative A_3 with the worse alternative A'_3 in original decision matrix of the expert with their rating values are summarized in Table 4.

Now by utilizing the CIFWA operator in Step 2 of the proposed approach corresponding to generator $t(a) = -\log(a)$ to this modified data, we get the collective values of each alternative are

$$\delta_1 = \left\langle 0.7170e^{i2\pi(0.7707)}, 0.1469e^{i2\pi(0.1803)} \right\rangle,$$
$$\delta_2 = \left\langle 0.5902e^{i2\pi(0.7291)}, 0.2551e^{i2\pi(0.1830)} \right\rangle,$$
$$\delta_3 = \left\langle 0.2617e^{i2\pi(0.3075)}, 0.4895e^{i2\pi(0.5370)} \right\rangle,$$

Table 4 Transformed input information

	C_1	C_2	C_3	C_4
A_1	$\langle 0.7e^{i2\pi(0.9)}, 0.1e^{i2\pi(0.1)}\rangle$	$\langle 0.8e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.4)}\rangle$	$\langle 0.6e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.2)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.2)}\rangle$
A_2	$\langle 0.7e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.3)}\rangle$	$\langle 0.4e^{i2\pi(0.9)}, 0.2e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.3)}\rangle$	$\langle 0.4e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.1)}\rangle$
A'_3	$\langle 0.1e^{i2\pi(0.3)}, 0.7e^{i2\pi(0.5)}\rangle$	$\langle 0.3e^{i2\pi(0.3)}, 0.5e^{i2\pi(0.6)}\rangle$	$\langle 0.2e^{i2\pi(0.2)}, 0.6e^{i2\pi(0.8)}\rangle$	$\langle 0.5e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.4)}\rangle$
A_4	$\langle 0.4e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.3)}\rangle$	$\langle 0.6e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.3)}\rangle$	$\langle 0.5e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.4)}\rangle$
A_5	$\langle 0.9e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.2)}\rangle$	$\langle 0.7e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.1)}\rangle$	$\langle 0.7e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.2)}\rangle$	$\langle 0.8e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.1)}\rangle$

$$\delta_4 = \left\langle 0.5422e^{i2\pi(0.6230)}, 0.3121e^{i2\pi(0.2048)}\right\rangle$$
$$\delta_5 = \left\langle 0.8217e^{i2\pi(0.7112)}, 0.1320e^{i2\pi(0.1464)}\right\rangle$$

The score values of the alternatives corresponding to these values are computed by using Eq. (3) and get

$$S(\delta_1) = 1.1605, \quad S(\delta_2) = 0.8812, \quad S(\delta_3) = -0.4573,$$
$$S(\delta_4) = 0.6484, \quad S(\delta_5) = 1.2545.$$

Therefore, the final ranking order of the alternatives is $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A'_3$ which indicates that A_5 is still the best alternative. Thus, the proposed MCDM method satisfies *test criterion 1*.

4.3.2 Validity Test by Using Criteria 2 and 3

Under these test, if we decompose original problem in four subparts which are: $\{A_1, A_2, A_3, A_4\}, \{A_2, A_3, A_4, A_5\}$, $\{A_3, A_4, A_5, A_1\}$ and $\{A_4, A_5, A_1, A_2\}$ and then applying the proposed approach individually to these subproblems, we get the final ranking order of the alternatives corresponding to each subproblem is $A_1 \succ A_2 \succ A_4 \succ A_3$, $A_5 \succ A_2 \succ A_4 \succ A_3$, $A_5 \succ A_1 \succ A_4 \succ A_3$ and $A_5 \succ A_1 \succ A_2 \succ A_4$, respectively. Thus, the overall combined ranking order of the alternative is $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ which is same as un-decomposed problem and show transitive property. Hence, the proposed MCDM approach is valid under the criteria 2 and 3.

4.4 Comparative Study

In order to validate the performance of the proposed MCDM approach with some of the existing approaches, an investigation has been done where we compare the results obtained by the proposed approach with existing approaches under CIFS as well as IFS environment.

4.4.1 Comparative Studies Under CIFS Environment

Firstly, an analysis has been conducted under the CIFS environment by applying the existing approaches [37,39] to the considered data. The results corresponding to these approaches from its ideal alternative $A^* = \langle r_q e^{i w_{r_q}}, k_q e^{i w_{k_q}}\rangle$; $q = 1, 2, \ldots, n$ where $r_q = \max_{1\le p\le m}\{r_{pq}\}$; $k_q = \min_{1\le p\le m}\{k_{pq}\}$ and $w_{r_p} = \max_{1\le p\le m}\{w_{r_{pq}}\}$; $w_{k_q} = \min_{1\le p\le m}\{w_{k_{pq}}\}$ are computed and summarized as below:

(i) By applying the approach of Alkouri and Salleh [37] using distance measures, denoted by d_1 to the considered data, we get the measurement values of each alternative from its ideal values are $d_1(A_1, A^*) = 0.1110$, $d_1(A_2, A^*) = 0.1758$, $d_1(A_3, A^*) = 0.3245$, $d_1(A_4, A^*) = 0.2503$ and $d_1(A_5, A^*) = 0.0700$. From these values, we observed that $d_1(A_5, A^*) < d_1(A_1, A^*) < d_1(A_2, A^*) < d_1(A_4, A^*) < d_1(A_3, A^*)$ and hence conclude that the best alternative is A_5.

(ii) By utilizing the distance measure (d_2) as proposed by Rani and Garg [39] to the considered problem, then the measurement values for each alternative are computed as $d_2(A_1, A^*) = 0.1593$, $d_2(A_2, A^*) = 0.2107$, $d_2(A_3, A^*) = 0.3655$, $d_2(A_4, A^*) = 0.2964$ and $d_2(A_5, A^*) = 0.0975$. Since measurement value of A_5 is minimum among all these, we conclude that the most optimal alternative is A_5 which again coincides with the proposed results.

4.4.2 Comparative Studies Under IFS Environment

In this section, we compare the performance of the proposed MCDM approach with some of the existing approaches [2,10–12,24–29,32,42] under an intuitionistic fuzzy set theory. For it, firstly the considered preferences of the expert are converted into the intuitionistic fuzzy numbers by taking the phase terms corresponding to each CIFN is zero. Then, based on this information, we applied the existing aggregation operators-based approaches to the considered data, and hence, their results are summarized in Table 5. From this table, it is concluded that the best alternative obtained from the final ranking of the alternative remains same, but the preferences of the other alternatives are different. This is mainly due to the changeable decision environment. For instance, in [2,10–12,24–29,32,42] approaches, weighted averaging and geometric aggregation operators were introduced by taking

Table 5 Comparative study with some existing approaches

Existing methods	Score values					Ranking
	A_1	A_2	A_3	A_4	A_5	
Xu and Yager [24]	− 0.1459	−0.2007	− 0.2343	−0.1767	− 0.0731	$A_5 \succ A_1 \succ A_4 \succ A_2 \succ A_3$
Xu [2]	− 0.1073	−0.1482	− 0.0732	−0.0931	− 0.0368	$A_5 \succ A_3 \succ A_4 \succ A_1 \succ A_2$
Wang and Liu [26]	0.5670	0.3276	0.1183	0.2181	0.6871	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
Garg [28]	0.6563	0.4787	0.0142	0.2849	0.7193	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
Xu and Yager [25]	− 0.3968	−0.5370	− 0.6319	−0.5754	− 0.3136	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
He et al. [27]	0.6484	0.4768	− 0.0085	0.2707	0.7172	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
Huang [10]	0.5658	0.3241	0.1064	0.2127	0.6860	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
Chen and Chang [11]	0.4339	0.1804	0.1000	0.0845	0.6435	$A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$
Goyal et al. [12]	0.7982	0.6623	0.3109	0.4510	0.8604	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
Ye [32]	0.5506	0.3084	0.0596	0.1876	0.6715	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
Zhou and Xu [42]	0.5868	0.3824	0.3288	0.3776	0.6979	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
Garg [29]	0.4316	0.1669	0.0809	0.0743	0.6392	$A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$

into account only one-dimensional grades of membership and non-membership.

It is noted from the study that the computational procedure of the proposed approach is entirely different from the existing approaches under CIFS as well as IFS environment, but the proposed result in this paper is more rational to reality in decision-making process due to the consideration of the two-dimensional information simultaneously into a single set. In addition, the advantages of the proposed operators are that it is based on the generalized t-norm operations, and hence by assigning a different function "t" and their equivalent "s," we can get the more generalized aggregation operations for different CIFNs.

Also, it is revealed that in IFS, the information contains a real-valued membership and non-membership degrees and only considered amplitude term which causes loss of information during the execution. On the other hand, a complex intuitionistic fuzzy set is a generalization of the existing studies such as complex fuzzy sets [34], intuitionistic fuzzy sets [5], fuzzy set [4] by considering much more information related to an object during the process and to handle the two-dimensional information in a single set.

4.5 Advantages of the Proposed Approach

From the existing studies and the proposed operators, we address the following merits of the proposed method to solve the decision-making problem under the CIFS environment.

(i) A complex intuitionistic fuzzy set is a generalization of the existing studies such as complex fuzzy sets [34], intuitionistic fuzzy sets [5], fuzzy set [4] by considering much more information related to an object during the process and to handle the two-dimensional information in a single set. For instance, CIFS contains information (both the membership and non-membership degrees are complex valued) with amplitude and phase terms than the CFS (contains only complex-valued membership degree), IFS (with a real-valued membership and non-membership degrees and only considered amplitude term), FS (with only crisp membership degrees with amplitude term only). Thus, the proposed aggregation operators under CIFSs environment are more generalizations than the existing operators.

(ii) It is revealed from the present study that the aggregation operators under IFSs, FSs [2,10–12,24–29,32,42] are the special cases of the proposed measures. Thus, the proposed operators can be equivalently utilized to solve the MCDM problem under these existing environment by setting phase term to be zero while the existing operators [2,10–12,24–29,32,42] are unable to solve the problems under the environment considered in the present paper.

(iii) The major advantages of the proposed decision-making approach are to consider the much more information to access the alternative to reduce the information loss. Further, by utilizing the various expressions to the t-norm and its equivalent s -norm will help the decision maker to select the best alternative(s) more accurately. In other words, we can say that the proposed generalized aggregation operators will give the various choices to the decision makers toward the decision-making process.

5 Conclusion

In this work, an attempt has been made to present different kinds of the weighted averaging aggregation operators based on the generalized t-norm operations in the decision-making process under the complex intuitionistic fuzzy set environment. Earlier, it has been observed that the various aggregation operators are defined under the IFSs environment where the range of their corresponding membership and non-membership degrees is the subset of the real numbers. But this condition has been relaxed in the present work by considering the ranges of the membership degrees are a subset of the complex numbers with the unit disk. Hence, IFSs can be considered as a special case of the CIFSs. Therefore, considering the importance of the phase angle, for representing the periodicity of the information, some weighted averaging aggregation operators, namely CIFWA, CIFOWA and CIFHA, are developed under the CIFS environment and studied their properties in detail. Further, depending on the standardize decision matrix and the proposed aggregation operators, a decision-making approach is presented to find the best alternative to the CIFSs environment. An illustrative example is taken for illustrating the developed approach, and their results are compared with some of the existing approaches under the CIFS and IFS environment to show the validity of it. A validity test has been conducted to show the stability of the proposed approach. From the studies, we conclude that the proposed approach is more generic and suitable for solving the stated problem, whereas the existing method under IFS environment fails to deal it. In the future, the result of this paper can be extended to some other uncertain and fuzzy environment [43–50].

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References

1. Garg, H.; Kumar, K.: An advanced study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making. Soft. Comput. **22**(15), 4959–4970 (2018). https://doi.org/10.1007/s00500-018-3202-1
2. Xu, Z.S.: Intuitionistic fuzzy aggregation operators. IEEE Trans. Fuzzy Syst. **15**, 1179–1187 (2007)
3. Arora, R.; Garg, H.: Prioritized averaging/geometric aggregation operators under the intuitionistic fuzzy soft set environment. Sci. Iran. **25**(1), 466–482 (2018)
4. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)
5. Atanassov, K.T.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**, 87–96 (1986)
6. Atanassov, K.; Gargov, G.: Interval-valued intuitionistic fuzzy sets. Fuzzy Sets Syst. **31**, 343–349 (1989)
7. Kumar, K.; Garg, H.: Connection number of set pair analysis based TOPSIS method on intuitionistic fuzzy sets and their application to decision making. Appl. Intell. (2017). https://doi.org/10.1007/s10489-017-1067-0
8. Wang, X.; Triantaphyllou, E.: Ranking irregularities when evaluating alternatives by using some electre methods. Omega Int. J. Manag. Sci. **36**, 45–63 (2008)
9. Garg, H.; Kumar, K.: Some aggregation operators for linguistic intuitionistic fuzzy set and its application to group decision-making process using the set pair analysis. Arab. J. Sci. Eng. **43**(6), 3213–3227 (2018)
10. Huang, J.Y.: Intuitionistic fuzzy Hamacher aggregation operator and their application to multiple attribute decision making. J. Intell. Fuzzy Syst. **27**, 505–513 (2014)
11. Chen, S.M.; Chang, C.H.: Fuzzy multiattribute decision making based on transformation techniques of intuitionistic fuzzy values and intuitionistic fuzzy geometric averaging operators. Inf. Sci. **352–353**, 133–149 (2016)
12. Goyal, M.; Yadav, D.; Tripathi, A.: Intuitionistic fuzzy genetic weighted averaging operator and its application for multiple attribute decision making in E-learning. Indian J. Sci. Technol. **9**(1), 1–15 (2016)
13. Garg, H.: Some robust improved geometric aggregation operators under interval-valued intuitionistic fuzzy environment for multi-criteria decision-making process. J. Ind. Manag. Optim. **14**(1), 283–308 (2018)
14. Chen, S.M.; Cheng, S.H.; Tsai, W.H.: Multiple attribute group decision making based on interval-valued intuitionistic fuzzy aggregation operators and transformation techniques of interval-valued intuitionistic fuzzy values. Inf. Sci. **367–368**(1), 418–442 (2016)
15. Kaur, G.; Garg, H.: Multi-attribute decision-making based on bonferroni mean operators under cubic intuitionistic fuzzy set environment. Entropy **20**(1), 65 (2018). https://doi.org/10.3390/e20010065
16. Wang, W.; Liu, X.: The multi-attribute decision making method based on interval-valued intuitionistic fuzzy Einstein hybrid weighted geometric operator. Comput. Math. Appl. **66**, 1845–1856 (2013)
17. Garg, H.: Some picture fuzzy aggregation operators and their applications to multicriteria decision-making. Arab. J. Sci. Eng. **42**(12), 5275–5290 (2017)
18. Arora, R.; Garg, H.: A robust correlation coefficient measure of dual hesitant fuzzy soft sets and their application in decision making. Eng. Appl. Artif. Intell. **72**, 80–92 (2018)
19. Singh, S.; Garg, H.: Distance measures between type-2 intuitionistic fuzzy sets and their application to multicriteria decision-making process. Appl. Intell. **46**(4), 788–799 (2017)
20. Kumar, K.; Garg, H.: TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment. Comput. Appl. Math. **37**(2), 1319–1329 (2018)
21. Garg, H.: A linear programming method based on an improved score function for interval-valued pythagorean fuzzy numbers and its application to decision-making. Int. J. Uncertain. Fuzziness Knowl. Based Syst. **26**(1), 67–80 (2018)
22. Garg, H.: A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems. Appl. Soft Comput. **38**, 988–999 (2016)
23. Yager, R.R.: On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Trans. Syst. Man Cybern. **18**(1), 183–190 (1988)
24. Xu, Z.S.; Yager, R.R.: Some geometric aggregation operators based on intuitionistic fuzzy sets. Int. J. Gen. Syst. **35**, 417–433 (2006)
25. Xu, Z.S.; Yager, R.R.: Intuitionistic fuzzy Bonferroni means. IEEE Trans. Syst. Man Cybern. **41**(2), 568–578 (2011)
26. Wang, W.; Liu, X.: Intuitionistic fuzzy information aggregation using Einstein operations. IEEE Trans. Fuzzy Syst. **20**(5), 923–938 (2012)

27. He, Y.; Chen, H.; Zhau, L.; Liu, J.; Tao, Z.: Intuitionistic fuzzy geometric interaction averaging operators and their application to multi-criteria decision making. Inf. Sci. **259**, 142–159 (2014)
28. Garg, H.: Generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making. Comput. Ind. Eng. **101**, 53–69 (2016)
29. Garg, H.: Novel intuitionistic fuzzy decision making method based on an improved operation laws and its application. Eng. Appl. Artif. Intell. **60**, 164–174 (2017)
30. Wei, G.; Wang, X.: Some geometric aggregation operators based on interval-valued intuitionistic fuzzy sets and their application to group decision making. In: Proceedings of the IEEE International Conference on Computational Intelligence and Security (2007), pp. 495–499
31. Kaur, G.; Garg, H.: Cubic intuitionistic fuzzy aggregation operators. Int. J. Uncertain. Quantif. (2018). https://doi.org/10.1615/Int.J.UncertaintyQuantification.2018020471
32. Ye, J.: Intuitionistic fuzzy hybrid arithmetic and geometric aggregation operators for the decision-making of mechanical design schemes. Appl. Intell. **47**, 743–751 (2017)
33. Garg, H.; Singh, S.: A novel triangular interval type-2 intuitionistic fuzzy sets and their aggregation operators. Iran. J. Fuzzy Syst. (2018). https://doi.org/10.22111/IJFS.2018.3559
34. Ramot, D.; Milo, R.; Fiedman, M.; Kandel, A.: Complex fuzzy sets. IEEE Trans. Fuzzy Syst. **10**(2), 171–186 (2002)
35. Ramot, D.; Friedman, M.; Langholz, G.; Kandel, A.: Complex fuzzy logic. IEEE Trans. Fuzzy Syst. **11**(4), 450–461 (2003)
36. Alkouri, A.; Salleh, A.: Complex intuitionistic fuzzy sets, vol. 1482. In: 2nd International Conference on Fundamental and Applied Sciences, 2012, pp. 464–470
37. Alkouri, A.U.M.; Salleh, A.R.: Complex Atanassov's intuitionistic fuzzy relation. Abstr. Appl. Anal. (2013). https://doi.org/10.1155/2013/287382
38. Kumar, T.; Bajaj, R.K.: On complex intuitionistic fuzzy soft sets with distance measures and entropies. J. Math. (2014). https://doi.org/10.1155/2014/972198
39. Rani, D.; Garg, H.: Distance measures between the complex intuitionistic fuzzy sets and its applications to the decision-making process. Int. J. Uncertain. Quantif. **7**(5), 423–439 (2017)
40. Rani, D.; Garg, H.: Complex intuitionistic fuzzy power aggregation operators and their applications in multi-criteria decision-making. Expert Syst. (2018). https://doi.org/10.1111/exsy.12325
41. Klir, G.J.; Yuan, B.: Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall of India Private Limited, New Delhi (2005)
42. Zhou, W.; Xu, Z.: Extreme intuitionistic fuzzy weighted aggregation operators and their applications in optimism and pessimism decision-making processes. J. Intell. Fuzzy Syst. **32**, 1129–1138 (2017)
43. Dey, A.; Pal, A.: Prim's algorithm for solving minimum spanning tree problem in fuzzy environment. Ann. Fuzzy Math. Inform. **12**(3), 419–430 (2016)
44. Garg, H.: New exponential operational laws and their aggregation operators for interval-valued Pythagorean fuzzy multicriteria decision-making. Int. J. Intell. Syst. **33**(3), 653–683 (2018)
45. Garg, H.: Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process. Int. J. Intell. Syst. **33**(6), 1234–1263 (2018)
46. Dey, A.; Pal, A.; Pal, T.: Interval type 2 fuzzy set in fuzzy shortest path problem. Mathematics **4**(4), 62 (2016). https://doi.org/10.3390/math4040062
47. Dey, A.; Pal, T.; Mondal, S.: Robust and minimum spanning tree in fuzzy environment. Int. J. Comput. Sci. Math. (2017). https://doi.org/10.1504/IJCSM.2017.10008767
48. Dey, A.; Pal, A.: Computing the shortest path with words. Int. J. Adv. Intell. Paradig. (2018). https://doi.org/10.1504/IJAIP.2018.10007917
49. Garg, H.; Arora, R.: Novel scaled prioritized intuitionistic fuzzy soft interaction averaging aggregation operators and their application to multi criteria decision making. Eng. Appl. Artif. Intell. **71C**, 100–112 (2018)
50. Garg, H.; Arora, R.: Dual hesitant fuzzy soft aggregation operators and their application in decision making. Cogn. Comput. (2018). https://doi.org/10.1007/s12559-018-9569-6