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Further Results on Containment Control for Multi-Agent Systems with Variable Communication Delay

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Abstract

This paper considers the problem on containment control of general linear multi-agent systems (MASs) with communication time-varying delay. Based on directed interaction topology, some sufficient conditions on the existence of feedback controller gains are provided to ensure the desired control. Through choosing an augmented Lyapunov–Krasovskii (K–L) functional and using some novel integral inequalities to estimate the derivative of Lyapunov functional, the previously ignored information can be reconsidered and the application area of the derived results can be greatly extended. Moreover, a novel constructive method is proposed to compute out the controller gains based on LMI technique. Finally, a numerical example with some simulations is provided to illustrate the effectiveness of the obtained results.

Keywords Containment control · General linear multi-agent systems · Time-varying delay · LMI technique

1 Introduction

In recent years, the research on containment control of multi-agent systems has received increasing attention owing to its widely practical applications in many fields, such as animal group behaviors, obstacle avoidance of robots and formation of underwater unmaned vehicles. Prior to the discussions, the agents are always classified into leaders and followers. Then the purpose of this control is to ensure the states/ outputs of the followers converge to the convex hull formed by the states/outputs of the leaders, and a great deal of results have been reported [1–33].

Initial studies on containment control for MASs could be found in [1], and aiming at this issue, the MASs with nonlinear dynamics were studied [2–5]. The works [6–8] focused on the event-triggered containment control. In [9], the algebraic graph theory was used to establish some sufficient conditions to make the UAVs complete a pre-specified formation containment and in [10], the finite-time attitude containment control was also studied for multiple rigid bodies. Owing to more generality, containment control was also applied to discuss the tracking problem and the controller law was designed for multi-vehicles on Lie Group [11]. In [12], static and adaptive protocols were proposed to keep the distances among the followers with the safe ones. Under a directed random graph, the problem on leader–follower consensus for multi-agent systems was studied [13]. Through transforming containment problem into a asymptotic stability one, some sufficient conditions were provided for second-order MASs in [14]. Under directed interaction topologies, the containment control via dynamic output feedback was analyzed and Markov jumping parameters were involved [15]. In [16], the uncertain topologies were applied to study containment control and the derived results were based on LMIs. Particularly, [17] studied constant time delays in fixed-networks. In [18], the containment control of leader-following MASs with jointly connected topologies and time-varying delays were fully studied and sampled-data-based protocol was put forward [19]. Moreover, In [20,21], distributed control protocols were proposed by using the relative states among neighboring agents via LMI technique. In [22,23], the state and

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output feedback protocols were separately designed to solve the MASs with input saturation. Furthermore, in [24–26], the H∞ containment control for MASs with external disturbance was also discussed. Wang et al. [27] developed a follower-based observer to estimate the relative states of neighbors. In [28,29,31], the containment control of general linear MASs was studied. Particularly, variable topologies were involved in [28] and communication variable delays were studied in [29,31].

In this paper, we consider an improved approach to deal with the problem on containment control on general high-order multi-agent systems and a control protocol with time-varying delay is proposed. By choosing some augmented K–L functionals and using some Wirtinger-based inequalities, some less conservative results can be achieved. The derived results are presented in terms of LMIs. Particularly, a constructive method can help to design and compute out the controller gain by solving the derived LMIs.

The contributions of this work can be illustrated in three points. Firstly, since the results in [25] were presented in the forms of complicated inequalities, they could not be conveniently checked by resorting to the most developed algorithms and applied to real systems. Thus in this work, the derived results are presented in terms of LMIs and the controller gains can be easily achieved. Secondly, in [12,14,24,32,33], the dynamics of each agent were always expressed as first-order or second-order one, which makes the results not applicable to more general cases. Since the lower-order agent can be regarded as a special case of general higher-order ones, the problem considered in this work can be more meaningful. Particularly, in [23], the higher-order MASs were studied but communication delay was not involved. Thirdly, in this work, two augmented L–K functionals are constructed and some novel techniques are used to extend the application area. Even though in [14,29], the containment control on MASs have studied the effect of communication delay. As for the issue of delay dependence, the methods still need to be improved. On the one hand, most recently, many effective techniques have been put forward to tackle time delay [34–39]. Yet they have not been utilized to consider the containment control of MASs, especially as time delay is variable. On the other hand, as we know, since it is always assumed that the communication delay is variable and bounded, both the upper bound and lower one of its variation rate should be measured. However, most works only took its upper bound into consideration but ignored the lower one [34–40]. Thus it will be more meaningful to consider the information on the communication delay as much as possible.

The paper is organized as follows. In Sect. 2, some valuable concepts on graph theory are introduced and the model is described briefly. In Sect. 3, the containment control of MASs with communication time-varying delay is investigated and some LMI-based results are given to compute out the controller gains. Numerical simulations are given to prove the efficiency of our work in Sect. 4 and conclusions are given in Sect. 5.

Notations. The set $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices. Define Λ_M as $\Lambda_M = \mathrm{diag}\{\mathrm{M}, \mathrm{M}\}$, $\mathbf{sym}\{X\}$ means $\mathbf{sym}\{X\} = X + X^{\mathrm{T}}$. For the symmetric matrices X, Y, $X > Y$ (respectively, $X \geq Y$) means that $X - Y > 0$ ($X - Y \geq 0$) is a positive-definite (respectively, positive-semidefinite) matrix; I_m represents an identity matrix of the $m \times m$ dimensions, $0_{m\cdot n}$ represents the Zero matrix of $m \times n$ dimensions; $A \otimes B$ means the Kronecker product of the matrices A and B; and $*$ denotes the symmetric term in a symmetric matrix, i.e., $\begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ * & Z \end{bmatrix}$.

2 Model Descriptions and Preliminaries

In this work, a directed digraph $G = \{V, \varepsilon, A\}$ is used to represent the interaction topology of MASs, where $V = \{1, 2, \cdots, N\}$ is a vertex set, $\varepsilon \subset V \times V$ is a link set and $A = [a_{ij}] \in \mathbf{R}^{n\times n}$ is a nonnegative weighted adjacency matrix. A directed link from vertex j to vertex i is indicated by $e_{ij} = \{j, i\}$. If and only if $e_{ij} \in \varepsilon$, the elements of matrix A satisfies that $a_{ij} > 0$. If not, $a_{ij} = 0$. $N_i = \{j \in V \mid (j, i) \in \varepsilon\}$ can be used to denote neighbors' set of the node i. It is supposed that there are no self-loops $a_{ii} = 0$ for all $i \in V$. In addition, let $\deg(i) = \sum_{j=1}^{n} a_{ij}$ and $D = \mathrm{diag}\{\deg(1), \cdots, \deg(n)\}$, the Laplacian matrix of G is defined as $L = D - A$.

Definition 1 In MASs, an agent is called a leader if it has no neighbor and an agent is called a follower if it has at least one neighbor.

Consider a general high-order LTI multi-agent system, including M followers and $N - M$ leaders. The interaction strength is represented by a_{ij}. The dynamic of the multi-agent system is described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \tag{1}$$

where $x_i(t) \in \mathbf{R}^n$ is state vector, $A = [p_{ij}]_{n\times n}$, $B = [q_{ij}]_{n\times n}$, and $u_i(t) = \left[u_{i1}(t), \ldots, u_{in}(t)\right]^{\mathrm{T}}$.

Based on the mentioned above, there are M followers and $N - M$ leaders in system (1). The subscript set of the follower and leaders are, respectively, described as $F = \{1, 2 \ldots M\}$

and $E = \{M+1, M+2 \dots N\}$. Then the matrix L can be partitioned as

$$L = \begin{bmatrix} L_1 & L_2 \\ 0 & 0 \end{bmatrix}, \tag{2}$$

where $L_1 \in \mathbf{R}^{M\times M}$, $L_2 \in \mathbf{R}^{M\times(N-M)}$ are nonsingular ones. Here L_1 represents the interactions among the followers and L_2 denotes the interactions from the leaders to followers.

Definition 2 Multi-agent system (1) is said to achieve state containment if for any given bounded initial states and $i \in F$, there exist nonnegative constants $\omega_{ij} (j \in E)$ satisfying $\sum_{j=M+1}^{N} \omega_{ij} = 1$ such that

$$\lim_{t\to\infty} \left(x_i(t) - \sum_{j=M+1}^{N} \omega_{ij} x_j(t) \right) = 0.$$

Here are some lemmas that will be used to derive main results.

Lemma 1 ([10]) *All the eigen values of L_1 have positive real parts and each entry of $-L_1^{-1}L_2$ is nonnegative, and each row of $-L_1^{-1}L_2$ has a sum equal to one if there is at least one leader has a directed path to each follower.*

Lemma 2 ([37]) *For an any constant matrix $M > 0$, the following inequality holds for all continuously differentiable function $\varphi(\cdot)$ in $[a, b] \longrightarrow \mathbf{R}^n$:*

$$\begin{aligned} &-(b-a)\int_a^b \varphi^{\mathrm{T}}(s) M \varphi(s)\mathrm{d}s \\ &\quad \le -\left(\int_a^b \varphi(s)\mathrm{d}s\right)^{\mathrm{T}} M \left(\int_a^b \varphi(s)\mathrm{d}s\right) - 3\Theta^{\mathrm{T}} M \Theta, \end{aligned}$$

where $\Theta = \int_a^b \varphi(s)\mathrm{d}s - \frac{2}{b-a}\int_a^b\int_a^s \varphi(u)\mathrm{d}u\mathrm{d}s.$

Lemma 3 ([41]) *For one given scalar $\alpha \in (0, 1)$, four constant matrices $R_i > 0\,(i = 1, 2)$, W_1, W_2, and the appropriately dimensional vector ξ, then the function $\Theta(\alpha, R)$ can be defined as*

$$\Theta(\alpha, R) = \frac{1}{\alpha}\xi^{\mathrm{T}} W_1^{\mathrm{T}} R_1 W_1 W \xi + \frac{1}{1-\alpha}\xi^{\mathrm{T}} W_2^{\mathrm{T}} R_2 W_2 W \xi.$$

Thus if there exists one appropriate dimensional matrix X such that $\begin{bmatrix} R_1 & X \\ X^{\mathrm{T}} & R_2 \end{bmatrix} \ge 0$, the following inequality holds,

$$\Theta(\alpha, R) \ge \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_1 & X \\ X^{\mathrm{T}} & R_2 \end{bmatrix} \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}.$$

Lemma 4 ([35]) *For an any constant matrix $M > 0$, the following inequality is true for all continuously differentiable function $\varphi(\cdot)$ in $[a, b] \to \mathbf{R}^n$:*

$$\begin{aligned} &-\frac{(b-a)^2}{2}\int_a^b\int_a^s \varphi^{\mathrm{T}}(u) M \varphi(u)\mathrm{d}u\mathrm{d}s \\ &\le -\left(\int_a^b\int_a^s \varphi(u)\mathrm{d}u\mathrm{d}s\right)^{\mathrm{T}} M \left(\int_a^b\int_a^s \varphi(u)\mathrm{d}u\mathrm{d}s\right) \\ &\quad -2\Theta^{\mathrm{T}} M \Theta, \end{aligned}$$

where

$$\Theta = \int_a^b\int_a^s \varphi(u)\mathrm{d}u\mathrm{d}s - \frac{3}{b-a}\int_a^b\int_a^s\int_a^u \varphi(v)\mathrm{d}v\mathrm{d}u\mathrm{d}s.$$

Lemma 5 ([41]) *As for the constant matrix A of appropriate dimension, the following statements are equivalent:*

(a) *There exists two symmetric and positive-definitive matrices P, Q satisfying $\begin{bmatrix} -P & A^{\mathrm{T}} \\ A & -Q^{-1} \end{bmatrix} < 0$;*
(b) *There exists two symmetric and positive-definitive matrices P, Q, and constant matrix Y such that*

$$\begin{bmatrix} -P & (YA)^{\mathrm{T}} \\ YA & \mathbf{sym}(-Y) + Q \end{bmatrix} < 0.$$

Lemma 6 ([30]) *Assuming that P is a nonsingular matrix and $P^{-1}L_1P = J$, where J is the Jordan canonical form of L_1. Denoting the eigen values of L_1 as λ_i $(i = 1, 2, \dots, M)$. Describing that the maximum and minimum of the real part of J as $\mathrm{Re}(\lambda_{\max})$ and $\mathrm{Re}(\lambda_{\min})$, respectively. Furthermore, we assume that the maximum of the imaginary part of J as $\mathrm{Im}(\lambda_{\max})$. Then let $\bar{\lambda}_{1,2} = \mathrm{Re}(\lambda_{\min}) \pm \mathrm{Im}(\lambda_{\max})i$ and $\bar{\lambda}_{3,4} = \mathrm{Re}(\lambda_{\max}) \pm \mathrm{Im}(\lambda_{\max})i$. Let $\Theta_0, \Theta_1, \Theta_2$ be real symmetric matrices independent of λ_i $(i \in F)$ and $\bar{\lambda}_i$ $(i = 1, 2, 3, 4)$. Then by [30], we can conclude that if for all $i \in \{1, 2, 3, 4\}$, $\Theta_0 + \mathrm{Re}(\bar{\lambda}_i)\Theta_1 + \mathrm{Im}(\bar{\lambda}_i)\Theta_2 < 0$, then for $i \in F$, $\Theta_0 + \mathrm{Re}(\lambda_i)\Theta_1 + \mathrm{Im}(\lambda_i)\Theta_2 < 0$ holds.*

Lemma 7 ([40]) *Suppose that Ω, Ξ_{ij}, Ξ_{mn} $(i, m = 1, 2, 3, 4;\ j, n = 1, 2)$ are the constant matrices of appropriate dimensions, $\alpha \in [0, 1]$, $\beta \in [0, 1]$, $\gamma \in [0, 1]$, and $\delta \in [0, 1]$, then*

$$\begin{aligned} &\Omega + [\alpha\Xi_{11} + (1-\alpha)\Xi_{12}] + [\beta\Xi_{21} + (1-\beta)\Xi_{22}] \\ &\quad + [\gamma\Xi_{31} + (1-\gamma)\Xi_{32}] + [\delta\Xi_{41} + (1-\delta)\Xi_{42}] < 0 \end{aligned}$$

holds, if and only if the following inequalities hold simultaneously,

$$\Omega + \Xi_{ij} + \Xi_{mn} < 0 \quad (i, m = 1, 2, 3, 4;\ j, n = 1, 2).$$

The containment control protocol for the system (1) is proposed, which can be described as

$$\begin{aligned} u_i(t) &= K_1 x_i(t) + K_2 \sum_{j\in N_i} a_{ij}\big[x_i(t-\tau(t)) \\ &\quad - x_j(t-\tau(t))\big],\ i \in F; \\ u_i(t) &= K_1 x_i(t),\quad i \in E. \end{aligned} \tag{3}$$

Assumption 1 The term $\tau(t)$ denotes the communication delay satisfying

$$0 \le \tau(t) \le \tau_m,\quad \mu_0 \le \dot{\tau}(t) \le \mu_m < +\infty,$$

where τ_m, μ_0, μ_m are known constants.

Assumption 2 For each follower, at least one leader has a directed path to it.

Let $x_F(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_M^{\mathrm{T}}(t)]^{\mathrm{T}}$, $x_E(t) = [x_{M+1}^{\mathrm{T}}(t), x_{M+2}^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. Under the control protocol in (3), we will get that the dynamics of the leaders and followers can be separately described as

$$\begin{aligned} \dot{x}_F(t) &= (L_1 \otimes BK_2)x_F(t-\tau(t)) + (L_2 \otimes BK_2) \\ &\quad \times x_E(t-\tau(t)) + (I_M \otimes (A+BK_1))x_F(t), \\ \dot{x}_E(t) &= (I_{N-M} \otimes (A+BK_1))x_E(t). \end{aligned} \tag{4}$$

3 Containment Analysis and Controller Design

In what follows, in order for the simplification, some denotation will be presented as

$$\bar{\tau}(t) = \tau_m - \tau(t),\ \bar{\mu}_m = \mu_m - \mu_0; \tag{5}$$

$$\sigma_i(t) = \frac{1}{\bar{\tau}(t)} \int_{t-\tau_m}^{t-\tau(t)} z_i(s)\mathrm{d}s; \tag{6}$$

$$\rho_i(t) = \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} z_i(s)\mathrm{d}s; \tag{7}$$

$$\upsilon_i(t) = \frac{2}{\tau^2(t)} \int_{t-\tau(t)}^{t} \int_{t-\tau(t)}^{s} z_i(u)\mathrm{d}u\mathrm{d}s; \tag{8}$$

$$\omega_i(t) = \frac{2}{\bar{\tau}^2(t)} \int_{t-\tau_m}^{t-\tau(t)} \int_{t-\tau_m}^{s} z_i(u)\mathrm{d}u\mathrm{d}s; \tag{9}$$

$$\psi_i(t) = \frac{2}{\bar{\tau}^2(t)} \int_{t-\tau_m}^{t-\tau(t)} \int_{s}^{t-\tau(t)} z_i(u)\mathrm{d}u\mathrm{d}s; \tag{10}$$

$$\mu_i(t) = \frac{2}{\tau^2(t)} \int_{t-\tau(t)}^{t} \int_{s}^{t} z_i(u)\mathrm{d}u\mathrm{d}s; \tag{11}$$

$$\begin{aligned} \eta_i^{\mathrm{T}}(t) = \Big[& z_i^{\mathrm{T}}(t)\ \ z_i^{\mathrm{T}}(t-\tau(t))\ \ z_i^{\mathrm{T}}(t-\tau_m)\ \ \sigma_i^{\mathrm{T}}(t) \\ & \rho_i^{\mathrm{T}}(t)\ \ \upsilon_i^{\mathrm{T}}(t)\ \ \omega_i^{\mathrm{T}}(t)\ \ \psi_i^{\mathrm{T}}(t)\ \ \delta_i^{\mathrm{T}}(t)\ \ \dot{z}_i{}^{\mathrm{T}}(t)\Big]; \end{aligned} \tag{12}$$

$$\begin{aligned} M_1 = &- e_5^{\mathrm{T}}(4R_2)e_5 + \mathbf{sym}\Big(e_5^{\mathrm{T}}(3R_2)e_6\Big) - e_6^{\mathrm{T}}(3R_2)e_6 \\ &- e_2^{\mathrm{T}}\left(\frac{R_3}{\tau_m}\right)e_2 + \mathbf{sym}\left(e_2^{\mathrm{T}}\frac{R_3}{\tau_m}e_3\right) \\ &- e_3^{\mathrm{T}}\left(\frac{R_3}{\tau_m}\right)e_3; \end{aligned}$$

$$M_2 = - e_4^{\mathrm{T}}(4R_2)e_4 + \mathbf{sym}\Big(e_4^{\mathrm{T}}(3R_2)e_7\Big) - e_7(3R_2)e_7 \tag{13}$$

$$\begin{aligned} &- e_1^{\mathrm{T}}\Big(\frac{R_4}{\tau_m}\Big)e_1 - e_2^{\mathrm{T}}\Big(\frac{R_4}{\tau_m}\Big)e_2 \\ &+ \mathbf{sym}\Big(e_1^{\mathrm{T}}\frac{R_4}{\tau_m}e_2\Big); \end{aligned} \tag{14}$$

$$\bar{R} = \begin{bmatrix} \frac{R_1}{\tau_m} & 0 \\ 0 & \frac{3R_1}{\tau_m} \end{bmatrix},\ M = -\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{R} & X \\ X & \bar{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \tag{15}$$

where e_i are described as

$$e_i^{\mathrm{T}} = \begin{bmatrix} 0_{n\times(i-1)n} & I_n & 0_{n\times(10-i)n} \end{bmatrix}^{\mathrm{T}}\ (1 \le i \le 10), \tag{16}$$

and

$$E_1 = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_5 \end{bmatrix},\ E_2 = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_4 \end{bmatrix}. \tag{17}$$

Denoting the eigen values of L_1 as λ_i $(i = 1, 2, \ldots, M)$. $\bar{\lambda}_i$ $(i = 1, 2, 3, 4)$ are defined as $\bar{\lambda}_{1,2} = \mathrm{Re}(\lambda_{\min}) \pm \mathrm{Im}(\lambda_{\max})i$ and $\bar{\lambda}_{3,4} = \mathrm{Re}(\lambda_{\max}) \pm \mathrm{Im}(\lambda_{\max})i$, and for any λ, the expression Ψ_λ is represented as

$$\Psi_\lambda = \begin{bmatrix} \mathrm{Re}(\lambda)I & -\mathrm{Im}(\lambda)I \\ \mathrm{Im}(\lambda)I & \mathrm{Re}(\lambda)I \end{bmatrix}.$$

In what is next, we will give some sufficient conditions on containment control for the system (1).

Theorem 1 *For given scalars $\tau_m \ge 0$, μ_m, $\bar{\mu}_m$, and $\Psi_{\bar{\lambda}_i}$ $(i = 1, 2, 3, 4)$ in Lemma 6, the multi-agent system (1) under the time delay protocol (3) can achieve the desired containment control, if there exist constant matrices $P > 0$, $Q_i > 0$ $(i = 1, 2)$, $R_i > 0$ $(i = 1, 2, 3, 4)$, and X, N_1, N_2 with appropriate dimensions such that $\begin{bmatrix} \bar{R} & X \\ * & \bar{R} \end{bmatrix} \ge 0$ and the matrix inequalities in (18) hold*

$$\begin{aligned} &\mathbf{\Omega} + M + \tau_m M_j + \bar{\mu}_m e_2^{\mathrm{T}} Q_h e_2 \\ &+ \boldsymbol{sym}\Big(e_{10}^{\mathrm{T}} N_1^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B \Lambda_{K_2} e_2\Big) \\ &+ \boldsymbol{sym}\Big(e_2^{\mathrm{T}} N_2^{T} \Psi_{\bar{\lambda}_i} \Lambda_B \Lambda_{K_2} e_2\Big) < 0, \end{aligned} \tag{18}$$

where $j, h \in \{1, 2\}$ and $M, M_1, M_2, \bar{R}$ are expressed in (15), (13), (14), (15) respectively. Particularly, e_i $(i =$

$1, 2, \ldots, 10)$ *is defined in* (16) *and part elements of the matrix* $\boldsymbol{\Omega} = [\Omega_{ij}]_{10n\times 10n}$ *can be listed as*

$$\begin{aligned}
\Omega_{11} &= -\frac{3}{2}R_4 + \tau_m R_2 + Q_1 - \frac{1}{2}R_3,\\
\Omega_{12} &= \Lambda^{\mathrm{T}}_{A+BK_1}N_2,\ \Omega_{15} = \Omega_{24} = -R_3,\\
\Omega_{16} &= \Omega_{27} = \frac{3}{2}R_3,\ \Omega_{19} = \Omega_{28} = \frac{3}{2}R_4,\\
\Omega_{1,10} &= P + \Lambda^{\mathrm{T}}_{A+BK_1}N_1,\ \Omega_{2,10} = -N_2^{\mathrm{T}},\\
\Omega_{22} &= -\frac{3}{2}(R_3 + R_4) + (1-\mu_m)Q_2 + (\mu_0 - 1)Q_1,\\
\Omega_{25} &= \Omega_{34} = R_3,\quad \Omega_{33} = -Q_2 - R_3,\\
\Omega_{44} &= \Omega_{55} = -3R_4 - 3R_3,\\
\Omega_{47} &= \Omega_{56} = 3R_3,\ \Omega_{48} = \Omega_{59} = 3R_4,\\
\Omega_{66} &= \Omega_{77} = -\frac{9}{2}R_3,\ \Omega_{88} = \Omega_{99} = -\frac{9}{2}R_4,\\
\Omega_{10,10} &= -N_1^{\mathrm{T}} - N_1 + \tau_m R_1 + \frac{\tau_m^2}{4}(R_4 + R_3).
\end{aligned}$$

Proof Firstly, assume that

$$\phi_i(t) = \sum_{j\in N_i}\omega_{ij}(x_i(t) - x_j(t))\ (i \in F). \tag{19}$$

Then (19) can be written in a vector form as

$$\phi_F(t) = (L_1 \otimes I)x_F(t) + (L_2 \otimes I)x_E(t), \tag{20}$$

where $\phi_F(t) = [\phi_1^{\mathrm{T}}, \phi_2^{\mathrm{T}}, \ldots, \phi_M^{\mathrm{T}}]^{\mathrm{T}}$, and L_1, L_2 are defined in (2). Now by some alternation, we can obtain that

$$x_F(t) = (L_1^{-1} \otimes I)\phi_F(t) - (L_1^{-1}L_2 \otimes I)x_E(t). \tag{21}$$

It follows from Lemma 1 that if ϕ converges to zero, then the multi-agent system (1) can achieve the desired containment control.

Now taking the derivative of $\phi_F(t)$, we have

$$\dot\phi_F(t) = (L_1 \otimes I)\dot x_F(t) + (L_2 \otimes I)\dot x_E(t). \tag{22}$$

Substituting (4) and (21) into (22), one gets

$$\begin{aligned}
\dot\phi_F(t) = {}& I \otimes (A + BK_1)\phi_F(t)\\
&+ (L_1 \otimes BK_2)\phi(t - \tau(t)).
\end{aligned} \tag{23}$$

Describing the eigen values of L_1 as λ_i $(i = 1, 2, \ldots, M)$ and J as the Jordan canonical form of L_1. The matrix P satisfies $P^{-1}L_1P = J$. Then by using the diagonal transformation, we let $\iota_F(t) = (P^{-1} \otimes I)\phi_F(t)$. The multi-agent system (23) can be further alternated into

$$\dot\iota_F(t) = I \otimes (A + BK_1)\iota_F(t) + (J \otimes BK_2)\iota(t - \tau(t)). \tag{24}$$

Then from the term (24), we can transfer it into M subsystems and obtain

$$\dot\Phi_i(t) = (A + BK_1)\Phi_i(t) + \lambda_i BK_2\Phi_i(t - \tau(t)). \tag{25}$$

Consider the asymptotic stability of (25) and let $z_i(t) = [\mathrm{Re}(\mathrm{fl_i(t)})^{\mathrm{T}}, \mathrm{Im}(\mathrm{fl_i(t)})^{\mathrm{T}}]^{\mathrm{T}}$. For $i = 1, 2, \ldots, M$, through the decomposition of real and imaginary parts, the system (25) can be transferred as

$$\dot z_i(t) = \Lambda_{A+BK_1}z_i(t) + \Psi_{\lambda_i}\Lambda_B\Lambda_{K_2}z_i(t - \tau(t)). \tag{26}$$

Now based on the term (26), we can construct the Lyapunov–Krasovskii functional candidate

$$V(z_i(t)) = V_1(z_i(t)) + V_2(z_i(t)), \tag{27}$$

where

$$\begin{aligned}
V_1(z_i(t)) = {}& z_i^{\mathrm{T}}(t)Pz_i(t) + \int_{t-\tau(t)}^{t} z_i^{\mathrm{T}}(\theta)Q_1z_i(\theta)\mathrm{d}\theta\\
&+ \int_{t-\tau_m}^{t-\tau(t)} z_i^{\mathrm{T}}(\theta)Q_2z_i(\theta)\mathrm{d}\theta,\\
V_2(z_i(t)) = {}& \int_{-\tau_m}^{0}\int_{t+s}^{t} \dot z_i^{\mathrm{T}}(\theta)R_1\dot z_i(\theta)\mathrm{d}\theta\mathrm{d}s\\
&+ \int_{-\tau_m}^{0}\int_{t+s}^{t} z_i^{\mathrm{T}}(\theta)R_2z_i(\theta)\mathrm{d}\theta\mathrm{d}s\\
&+ \frac{1}{2}\int_{t-\tau_m}^{t}\int_{t-\tau_m}^{\varrho}\int_{\theta}^{t} \dot z_i^{\mathrm{T}}(s)R_3\dot z_i(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho\\
&+ \frac{1}{2}\int_{t-\tau_m}^{t}\int_{\varrho}^{t}\int_{\theta}^{t} \dot z_i^{\mathrm{T}}(s)R_4\dot z_i(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\varrho
\end{aligned}$$

with the matrices $P > 0$, $Q_i > 0$ $(i = 1, 2)$, $R_i > 0$ $(i = 1, 2, 3, 4)$ waiting to be determined.

Now the derivative of $V_j(z_i(t))$ $(j = 1, 2)$ along the system (26) can be directly computed out as

$$\begin{aligned}
\dot V_1(z_i(t)) = {}& \dot z_i^{\mathrm{T}}(t)Pz_i(t) + z_i^{\mathrm{T}}(t)P\dot z_i(t) + z_i^{\mathrm{T}}(t)Q_1z_i(t)\\
&+ (\dot\tau(t) - \mu_0)\left(z_i^{\mathrm{T}}(t - \tau(t)\right)Q_1z_i\,(t - \tau(t))\\
&+ (\mu_0 - 1)\left(z_i^{\mathrm{T}}(t - \tau(t)\right)Q_1z_i(t - \tau(t)))\\
&+ (1 - \mu_m)z_i^{\mathrm{T}}(t - \tau(t))Q_2z_i\,(t - \tau(t))\\
&+ (\mu_m - \dot\tau(t))z_i^{\mathrm{T}}\,(t - \tau(t))\,Q_2z_i(t - \tau(t))\\
&- z_i^{\mathrm{T}}(t - \tau_m)Q_2z_i(t - \tau_m),
\end{aligned} \tag{28}$$

$$\begin{aligned}
\dot V_2(z_i(t)) = {}& \dot z_i^{\mathrm{T}}(t)\Big(\tau_mR_1 + \frac{\tau_m^2}{4}R_3 + \frac{\tau_m^2}{4}R_4\Big)\dot z_i(t)\\
&+ z_i^{\mathrm{T}}(t)(\tau_mR_2)z_i(t)\\
&- \int_{t-\tau_m}^{t}\Big[\dot z_i^{\mathrm{T}}(\theta)R_1\dot z_i(\theta) + z_i^{\mathrm{T}}(\theta)R_2z_i(\theta)\Big]\mathrm{d}\theta
\end{aligned}$$

$$
-\frac{1}{2}\int_{t-\tau_m}^{t}\int_{t-\tau_m}^{\theta}\dot{z}_i^{\mathrm{T}}(\theta)R_3\dot{z}_i(s)\mathrm{d}s\mathrm{d}\theta
-\frac{1}{2}\int_{t-\tau_m}^{t}\int_{\theta}^{t}\dot{z}_i^{\mathrm{T}}(\theta)R_4\dot{z}_i(s)\mathrm{d}s\mathrm{d}\theta. \quad (29)
$$

Then, as for the integral terms in (29), it follows from Lemmas 2–3 and denotations (6)–(7) that

$$
\begin{aligned}
&-\int_{t-\tau_m}^{t}\dot{z}_i^{\mathrm{T}}(\theta)R_1\dot{z}_i(\theta)\mathrm{d}\theta\\
&\le -\frac{1}{\tau(t)}\left[z_i(t)-z_i(t-\tau(t))\right]^{\mathrm{T}}R_1\left[z_i(t)-z_i(t-\tau(t))\right]\\
&\quad -\frac{1}{\tau(t)}\left[z_i(t)+z_i(t-\tau(t))-2\rho_i(t)\right]^{\mathrm{T}}(3R_1)\\
&\quad \times\left[z_i(t)+z_i(t-\tau(t))-2\rho_i(t)\right]\\
&\quad -\frac{1}{\bar{\tau}(t)}\left[z_i(t-\tau(t))-z_i(t-\tau_m)\right]^{\mathrm{T}}R_1\\
&\quad \times\left[z_i(t-\tau(t))-z_i(t-\tau_m)\right]\\
&\quad -\frac{1}{\bar{\tau}(t)}\left[z_i(t-\tau(t))+z_i(t-\tau_m)-2\sigma_i(t)\right]^{\mathrm{T}}(3R_1)\\
&\qquad \times\left[z_i(t-\tau(t))+z_i(t-\tau_m)-2\sigma_i(t)\right]\\
&= -\frac{\tau_m}{\tau(t)}\eta_i^{\mathrm{T}}(t)\big(E_1^{\mathrm{T}}\bar{R}E_1\big)\eta_i(t)-\frac{\tau_m}{\bar{\tau}(t)}\eta_i^{\mathrm{T}}(t)\big(E_2^{\mathrm{T}}\bar{R}E_2\big)\eta_i(t)\\
&\le -\eta_i^{\mathrm{T}}(t)\begin{bmatrix}E_1\\E_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\bar{R} & X\\ X & \bar{R}\end{bmatrix}\begin{bmatrix}E_1\\E_2\end{bmatrix}\eta_i(t), \qquad (30)
\end{aligned}
$$

where $\bar{R}$ is defined in (15), and E_1, E_2 are expressed in (17). Now by utilizing Lemma 2 and denotations in (6)–(11), we can check that $-\int_{t-\tau_m}^{t} z_i^{\mathrm{T}}(\theta)R_2 z_i(\theta)\mathrm{d}\theta$ in (29) satisfies

$$
\begin{aligned}
&-\int_{t-\tau_m}^{t} z_i^{\mathrm{T}}(\theta)R_2 z_i(\theta)\mathrm{d}\theta\\
&= -\left[\int_{t-\tau(t)}^{t}+\int_{t-\tau_m}^{t-\tau(t)}\right]z_i^{\mathrm{T}}(\theta)R_2 z_i(\theta)\mathrm{d}\theta\\
&\quad \le -\tau(t)\rho_i^{\mathrm{T}}(t)R_2\rho_i(t)\\
&\quad -3\tau(t)\left[\rho_i(t)-\upsilon_i(t)\right]^{\mathrm{T}}R_2\left[\rho_i(t)-\upsilon_i(t)\right]\\
&\quad -\bar{\tau}(t)\sigma_i^{\mathrm{T}}(t)R_2\sigma_i(t)\\
&\quad -3\bar{\tau}(t)\left[\sigma_i(t)-\omega_i(t)\right]^{\mathrm{T}}R_2\left[\sigma_i(t)-\omega_i(t)\right]. \qquad (31)
\end{aligned}
$$

Furthermore, as for the double integral terms

$-\frac{1}{2}\int_{t-\tau_m}^{t}\int_{t-\tau_m}^{\theta}\dot{z}_i^{\mathrm{T}}(s)R_3\dot{z}_i^{\mathrm{T}}(s)\mathrm{d}s\mathrm{d}\theta$ and $-\frac{1}{2}\int_{t-\tau_m}^{t}\int_{\theta}^{t}\dot{z}_i^{\mathrm{T}}(s)R_4\dot{z}_i(s)\mathrm{d}s\mathrm{d}\theta$ in (29), it follows from Lemma 4 that

$$
\begin{aligned}
&-\frac{1}{2}\int_{t-\tau_m}^{t}\int_{t-\tau_m}^{\theta}\dot{z}_i^{\mathrm{T}}(s)R_3\dot{z}_i^{\mathrm{T}}(s)\mathrm{d}s\mathrm{d}\theta\\
&= -\frac{1}{2}\left[\int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{\theta}+\int_{t-\tau_m}^{t-\tau(t)}\int_{t-\tau_m}^{\theta}\right.\\
&\quad -\frac{1}{2}\int_{t-\tau(t)}^{t}\left.\int_{t-\tau_m}^{t-\tau(t)}\right]\dot{z}_i^{\mathrm{T}}(s)R_3\dot{z}_i(s)\mathrm{d}s\mathrm{d}\theta\\
&\le -\left[\rho_i(t)-z_i(t-\tau(t))\right]^{\mathrm{T}}R_3\left[\rho_i(t)-z_i(t-\tau(t))\right]\\
&\quad -\left[\frac{1}{2}z_i(t)+\rho_i(t)-\frac{3}{2}\upsilon_i(t)\right]^{\mathrm{T}}(2R_3)\\
&\quad \times\left[\frac{1}{2}z_i(t)+\rho_i(t)-\frac{3}{2}\upsilon_i(t)\right]\\
&\quad -\left[\sigma_i(t)-z_i(t-\tau_m)\right]^{\mathrm{T}}R_3\left[\sigma_i(t)-z_i(t-\tau_m)\right]\\
&\quad -\left[\frac{1}{2}z_i(t-\tau(t))+\sigma_i(t)-\frac{3}{2}\omega_i(t)\right]^{\mathrm{T}}(2R_3)\\
&\quad \times\left[\frac{1}{2}z_i(t-\tau(t))+\sigma_i(t)-\frac{3}{2}\omega_i(t)\right]\\
&\quad -\frac{\tau(t)}{\tau_m}\left[z_i(t-\tau(t))-z_i(t-\tau_m)\right]^{\mathrm{T}}R_3\\
&\quad \times\left[z_i(t-\tau(t))-z_i(t-\tau_m)\right], \qquad (32)\\
&-\frac{1}{2}\int_{t-\tau_m}^{t}\int_{\theta}^{t}\dot{z}_i^{\mathrm{T}}(s)R_4\dot{z}_i(s)\mathrm{d}s\mathrm{d}\theta\\
&= -\frac{1}{2}\left[\int_{t-\tau(t)}^{t}\int_{\theta}^{t}+\int_{t-\tau_m}^{t-\tau(t)}\int_{\theta}^{t-\tau(t)}\right.\\
&\quad -\frac{1}{2}\int_{t-\tau_m}^{t-\tau(t)}\left.\int_{t-\tau(t)}^{t}\right]\dot{z}_i^{\mathrm{T}}(s)R_4\dot{z}_i(s)\mathrm{d}s\mathrm{d}\theta\\
&\le -\left[z_i(t)-\rho_i(t)\right]^{\mathrm{T}}R_4\left[z_i(t)-\rho_i(t)\right]\\
&\quad -\left[\frac{1}{2}z_i(t)+\rho_i(t)-\frac{3}{2}\delta_i(t)\right]^{\mathrm{T}}(2R_4)\\
&\quad \times\left[\frac{1}{2}z_i(t)+\rho_i(t)-\frac{3}{2}\delta_i(t)\right]\\
&\quad -\left[z_i(t-\tau(t))-\sigma_i(t)\right]^{\mathrm{T}}R_4\left[z_i(t-\tau(t))-\sigma_i(t)\right]\\
&\quad -\left[\frac{1}{2}z_i(t-\tau(t))+\sigma_i(t)-\frac{3}{2}\psi_i(t)\right]^{\mathrm{T}}(2R_4)\\
&\quad \times\left[\frac{1}{2}z_i(t-\tau(t))+\sigma_i(t)-\frac{3}{2}\psi_i(t)\right]\\
&\quad -\frac{\bar{\tau}(t)}{\tau_m}\left[z_i(t)-z_i(t-\tau(t))\right]^{\mathrm{T}}R_4\\
&\quad \times\left[z_i(t)-z_i(t-\tau(t))\right]. \qquad (33)
\end{aligned}
$$

For any $n\times n$ matrices N_i $(i=1,2)$, it follows from the closed-loop system (26) that

$$
\begin{aligned}
0 &= 2\Big[\dot{z}_i^{\mathrm{T}}(t)N_1^{\mathrm{T}}+z_i^{\mathrm{T}}(t-\tau(t))N_2^{\mathrm{T}}\Big]\Big[-\dot{z}_i(t)\\
&\quad +\Lambda_{A+BK_1}z_i(t)+\Psi_{\lambda_i}\Lambda_B\Lambda_{K_2}z_i(t-\tau(t))\Big]. \qquad (34)
\end{aligned}
$$

Now by combining the right terms from (28) to (34), then $\dot{V}(z_i(t))$ can be estimated to satisfy

$$
\begin{aligned}
\dot{V}(z_i(t)) \le \eta_i^{\mathrm{T}}(t)\Big[&\mathbf{\Omega}+M+\tau(t)M_1+\bar{\tau}(t)M_2+\\
&\Big(\dot{\tau}(t)-\mu_0)e_2^{\mathrm{T}}Q_1e_2+(\mu_m-\dot{\tau}(t)\Big)e_2^{\mathrm{T}}Q_2e_2\\
&+\mathbf{sym}\Big(e_{10}^{\mathrm{T}}N_1^{\mathrm{T}}\Psi_{\lambda_i}\Lambda_B\Lambda_{K_2}e_2\Big)
\end{aligned}
$$

$$+\,\mathbf{sym}\Big(e_2^{\mathrm{T}} N_2^{\mathrm{T}} \Psi_{\lambda_i} \Lambda_B \Lambda_{K_2} e_2\Big)\Big]\eta_i(t)$$
$$\doteq \eta_i^{\mathrm{T}}(t)\boldsymbol{\Delta}(t)\eta_i(t), \tag{35}$$

where the terms $\boldsymbol{\Omega}$, M, M_1, M_2 are denoted in (18), (15), (13), (14), respectively.

Then together with Lemma 7, if four following cases can be true simultaneously, i.e.,
Case 1: $\dot{\tau}(t) = \mu_0$, $\tau(t) = 0$, $\bar{\tau}(t) = \tau_m$, it has

$$\begin{aligned}&\boldsymbol{\Omega} + M + \tau_m M_2 + (\mu_m - \mu_0)\, e_2^{\mathrm{T}} Q_2 e_2\\ &+\mathbf{sym}\Big(e_{10}^{\mathrm{T}} N_1^{\mathrm{T}} \Psi_{\lambda_i} \Lambda_B \Lambda_{K_2} e_2\Big)\\ &+\mathbf{sym}\Big(e_2^{\mathrm{T}} N_2^{\mathrm{T}} \Psi_{\lambda_i} \Lambda_B \Lambda_{K_2} e_2\Big) < 0;\end{aligned} \tag{36}$$

Case 2: $\dot{\tau}(t) = \mu_m$, $\tau(t) = 0$, $\bar{\tau}(t) = \tau_m$, it has

$$\begin{aligned}&\boldsymbol{\Omega} + M + \tau_m M_2 + (\mu_m - \mu_0) e_2^{\mathrm{T}} Q_1 e_2\\ &+\mathbf{sym}\Big(e_{10}^{\mathrm{T}} N_1^{\mathrm{T}} \Psi_{\lambda_i} \Lambda_B \Lambda_{K_2} e_2\Big)\\ &+\mathbf{sym}\Big(e_2^{\mathrm{T}} N_2^{\mathrm{T}} \Psi_{\lambda_i} \Lambda_B \Lambda_{K_2} e_2\Big) < 0;\end{aligned} \tag{37}$$

Case 3: $\dot{\tau}(t) = \mu_0$, $\tau(t) = \tau_m$, $\bar{\tau}(t) = 0$, it has

$$\begin{aligned}&\boldsymbol{\Omega} + M + \tau_m M_1 + (\mu_m - \mu_0) e_2^{\mathrm{T}} Q_2 e_2\\ &+\mathbf{sym}\Big(e_{10}^{\mathrm{T}} N_1^{\mathrm{T}} \Psi_{\lambda_i} \Lambda_B \Lambda_{K_2} e_2\Big)\\ &+\mathbf{sym}\Big(e_2^{\mathrm{T}} N_2^{\mathrm{T}} \Psi_{\lambda_i} \Lambda_B \Lambda_{K_2} e_2\Big) < 0;\end{aligned} \tag{38}$$

Case 4: $\dot{\tau}(t) = \mu_m$, $\tau(t) = \tau_m$, $\bar{\tau}(t) = 0$, it has

$$\begin{aligned}&\boldsymbol{\Omega} + M + \tau_m M_1 + (\mu_m - \mu_0) e_2^{\mathrm{T}} Q_1 e_2\\ &+\mathbf{sym}\Big(e_{10}^{\mathrm{T}} N_1^{\mathrm{T}} \Psi_{\lambda_i} \Lambda_B \Lambda_{K_2} e_2\Big)\\ &+\mathbf{sym}\Big(e_2^{\mathrm{T}} N_2^{\mathrm{T}} \Psi_{\lambda_i} \Lambda_B \Lambda_{K_2} e_2\Big) < 0,\end{aligned} \tag{39}$$

the terms in (36)–(39) can guarantee the term in (18) to be true, which indicates that there must exist a scalar $\varsigma > 0$ such that (35) satisfies

$$\dot{V}(z_i(t)) \le -\varsigma \parallel \eta_i(t) \parallel \le -\varsigma \parallel z_i(t) \parallel^2 < 0. \tag{40}$$

Then on the basis of Lemma 6, the multi-agent system (1) can achieve the containment by resorting to the time delay protocol (3) with $\Psi_{\lambda_i} = \Psi_{\bar{\lambda}_i}$ $(i = 1, 2, 3, 4)$. Therefore the proof is completed. □

Remark 1 During proving Theorem 1, the Lyapunov–Krasovskii functional terms in $V(z_i(t))$ have effectively utilized the information on the communication delay and several multiple integral Lyapunov functional terms have been constructed. Particularly, some novel Wirtinger-based integral inequalities have been utilized to consider those previously ignored information and the application area can be greatly extended.

Remark 2 Our work focuses on the general linear multi-agent systems and the communication delay is time-varying, while most existent results concentrate on first-order MASs or second-order ones, such as [14,16,32,33]. Particularly, Theorem 1 considers both the lower bound and upper one on variation rate of time delay, which has not been studied presently.

It is worth noting that, the derived inequalities in Theorem 1 are nonlinear and the gain K_2 cannot be tested by resorting to the most recently developed algorithms. In what follows, we will use Lemma 5 to obtain the controller gain via LMI approach.

Theorem 2 *For any given scalars $\tau_m \ge 0$, μ_m, $\bar{\mu}_m$, $\varepsilon > 0$, and $\Psi_{\bar{\lambda}_i}$ $(i = 1, 2, 3, 4)$, the system (1) under the time-delayed protocol (3) can reach the containment control with the gain matrix $K_2 = Z^{-1}L$, if there exist constant matrices $P > 0$, $Q_i > 0$ $(i = 1, 2)$, $J > 0$, $R_i > 0$ $(i = 1, 2, 3, 4)$, H_i $(i = 1, 2, 3, 4)$ X, N_1, Z, L with appropriate dimensions such that $\begin{bmatrix}\bar{R} & X\\ * & \bar{R}\end{bmatrix} \ge 0$ and the LMIs in (41) hold*

$$\begin{bmatrix}\bar{\mathbf{H}}_j & \Gamma_1^{\mathrm{T}} Z^{\mathrm{T}}\Big(\Psi_{\bar{\lambda}_i}\Lambda_B\Big)^{\mathrm{T}}\Psi_{\bar{\lambda}_i}\Lambda_B & 0\\ * & \boldsymbol{sym}\Big(-\Big(\Psi_{\bar{\lambda}_i}\Lambda_B\Big)^{\mathrm{T}}\Psi_{\bar{\lambda}_i}\Lambda_B Z\Big) & \Pi_1\\ * & * & -J\end{bmatrix} < 0 \tag{41}$$

where $\bar{\mathbf{H}}_j = \mathbf{H}_j + \boldsymbol{sym}\Big(\Pi_0^{\mathrm{T}}\Gamma_0\Big) + J$ $(j = 1, 2, 3, 4)$ with

$$\begin{aligned}\mathbf{H}_1 &= \boldsymbol{\Omega} + M + \tau_m M_2 + \bar{\mu}_m e_2^{\mathrm{T}} Q_2 e_2,\\ \mathbf{H}_2 &= \boldsymbol{\Omega} + M + \tau_m M_2 + \bar{\mu}_m e_2^{\mathrm{T}} Q_1 e_2,\\ \mathbf{H}_3 &= \boldsymbol{\Omega} + M + \tau_m M_1 + \bar{\mu}_m e_2^{\mathrm{T}} Q_2 e_2,\\ \mathbf{H}_4 &= \boldsymbol{\Omega} + M + \tau_m M_1 + \bar{\mu}_m e_2^{\mathrm{T}} Q_1 e_2,\\ \Pi_1^{\mathrm{T}} &= \begin{bmatrix}0\\ \varepsilon\Big(N_1^{\mathrm{T}}\Psi_{\bar{\lambda}_i}\Lambda_B - \Psi_{\bar{\lambda}_i}\Lambda_B Z\Big)\\ 0\\ \Big(N_1^{\mathrm{T}}\Psi_{\bar{\lambda}_i}\Lambda_B - \Psi_{\bar{\lambda}_i}\Lambda_B Z\Big)\end{bmatrix},\\ \Gamma_1 &= \big[0\ Z^{-1}L\ 0\big],\\ \Pi_0^{\mathrm{T}} &= \begin{bmatrix}0\\ \varepsilon\Psi_{\bar{\lambda}_i}\Lambda_B\\ 0\\ \bar{\Lambda}_B\end{bmatrix},\ \ \Gamma_0 = \big[0\ L\ 0\big].\end{aligned}$$

Proof Firstly, we denote

$$\mathbf{H}(t) = \boldsymbol{\Omega} + M + \tau(t)M_1 + \bar{\tau}(t)M_2 + (\dot{\tau}(t) - \mu_0)$$

$$\times e_2^{\mathrm{T}} Q_1 e_2 + (\mu_m - \dot{\tau}(t)) e_2^{\mathrm{T}} Q_2 e_2. \tag{42}$$

Then based on the derived results in Theorem 1 and using the terms $K_2 = Z^{-1}L$, $N_2 = \varepsilon N_1$ ($\varepsilon > 0$) to replace the relevant ones in $\mathbf{\Delta}(t)$ of (35), we can deduce that

$$\begin{aligned}
&\mathbf{H}(t) + \mathbf{sym}\left(\begin{bmatrix} 0 \\ N_2^{\mathrm{T}} \\ 0 \\ N_1^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 & \Psi_{\bar{\lambda}_i} \Lambda_B \Lambda_K & 0 \end{bmatrix}\right) \\
&= \mathbf{H}(t) + \mathbf{sym}\left(\begin{bmatrix} 0 \\ \varepsilon N_1^{\mathrm{T}} \\ 0 \\ N_1^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 & \Psi_{\bar{\lambda}_i} \Lambda_B \Lambda_K & 0 \end{bmatrix}\right) \\
&= \mathbf{H}(t) \\
&\quad + \mathbf{sym}\left(\begin{bmatrix} 0 \\ \varepsilon\left(N_1^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B - \Psi_{\bar{\lambda}_i} \Lambda_B Z\right) \\ 0 \\ \left(N_1^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B - \Psi_{\bar{\lambda}_i} \Lambda_B Z\right) \end{bmatrix} \begin{bmatrix} 0 & Z^{-1}L & 0 \end{bmatrix}\right) \\
&\quad + \mathbf{sym}\left(\begin{bmatrix} 0 \\ \varepsilon \Psi_{\bar{\lambda}_i} \Lambda_B Z \\ 0 \\ \Psi_{\bar{\lambda}_i} \Lambda_B Z \end{bmatrix} \begin{bmatrix} 0 & Z^{-1}L & 0 \end{bmatrix}\right).
\end{aligned}$$

Let

$$\Pi_1^{\mathrm{T}} = \begin{bmatrix} 0 \\ \varepsilon\left(N_1^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B - \Psi_{\bar{\lambda}_i} \Lambda_B Z\right) \\ 0 \\ \left(N_1^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B - \Psi_{\bar{\lambda}_i} \Lambda_B Z\right) \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & Z^{-1}L & 0 \end{bmatrix},$$

$$\Pi_0^{\mathrm{T}} = \begin{bmatrix} 0 \\ \varepsilon \Psi_{\bar{\lambda}_i} \Lambda_B \\ 0 \\ \bar{\Lambda}_B \end{bmatrix}, \quad \Gamma_0 = \begin{bmatrix} 0 & L & 0 \end{bmatrix}.$$

Since the term $\mathbf{\Delta}(t)$ in (35) is equivalent to

$$\mathbf{H}(t) + \mathbf{sym}\left(\Pi_0^{\mathrm{T}} \Gamma_0\right) + \mathbf{sym}\left(\Pi_1^{\mathrm{T}} \Gamma_1\right) < 0,$$

there exists any constant matrix $J > 0$ of appropriate dimension such that

$$\mathbf{H}(t) + \mathbf{sym}\left(\Pi_0^{\mathrm{T}} \Gamma_0\right) + \Gamma_1^{\mathrm{T}} \Pi_1 J^{-1} \Pi_1^{\mathrm{T}} \Gamma_1 + J < 0.$$

In what is next, we denote $\bar{\mathbf{H}}(t) = \mathbf{H}(t) + \mathbf{sym}\left(\Pi_0^{\mathrm{T}} \Gamma_0\right) + J$ and use the Schur-complement to derive

$$\begin{bmatrix} \bar{\mathbf{H}}(t) & \Gamma_1^{\mathrm{T}} \\ * & -\left(\Pi_1 J^{-1} \Pi_1^{\mathrm{T}}\right)^{-1} \end{bmatrix} < 0. \tag{43}$$

Then it follows from Lemma 5 and (43) that

$$\begin{bmatrix} \bar{\mathbf{H}}(t) & \Gamma_1^{\mathrm{T}} Z^{\mathrm{T}} \left(\Psi_{\bar{\lambda}_i} \Lambda_B\right)^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B \\ * & \mathbf{sym}\left(-\left(\Psi_{\bar{\lambda}_i} \Lambda_B\right)^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B Z\right) + \left(\Pi_1 J^{-1} \Pi_1^{\mathrm{T}}\right) \end{bmatrix} < 0.$$

Therefore, by resorting to definition on Schur-complement, we can deduce that

$$\begin{bmatrix} \bar{\mathbf{H}}(t) & \Gamma_1^{\mathrm{T}} Z^{\mathrm{T}} \left(\Psi_{\bar{\lambda}_i} \Lambda_B\right)^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B & 0 \\ * & \mathbf{sym}\left(-\left(\Psi_{\bar{\lambda}_i} \Lambda_B\right)^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B Z\right) & \Pi_1 \\ * & * & -J \end{bmatrix} < 0.$$

Thus the term in (41) can guarantee the following inequality to be true

$$\mathbf{\Delta}(t) = \mathbf{H}(t) + \mathbf{sym}\left(\Pi_0^{\mathrm{T}} \Gamma_0\right) + \mathbf{sym}\left(\Pi_1^{\mathrm{T}} \Gamma_1\right) < 0,$$

which means that the system (1) can achieve the containment control with controller gain $K_2 = Z^{-1}L$.

By using Lemma 7 and (42), similar to Theorem 1, we can get following cases:
Case 1: $\dot{\tau}(t) = \mu_0$, $\tau(t) = 0$, $\bar{\tau}(t) = \tau_m$, one has

$$\mathbf{H}(t) = \mathbf{H}_1 = \mathbf{\Omega} + M + \tau_m M_2 + (\mu_m - \mu_0) e_2^{\mathrm{T}} Q_2 e_2;$$

Case 2: $\dot{\tau}(t) = \mu_m$, $\tau(t) = 0$, $\bar{\tau}(t) = \tau_m$, one has

$$\mathbf{H}(t) = \mathbf{H}_2 = \mathbf{\Omega} + M + \tau_m M_2 + (\mu_m - \mu_0) e_2^{\mathrm{T}} Q_1 e_2;$$

Case 3: $\dot{\tau}(t) = \mu_0$, $\tau(t) = \tau_m$, $\bar{\tau}(t) = 0$, one has

$$\mathbf{H}(t) = \mathbf{H}_3 = \mathbf{\Omega} + M + \tau_m M_1 + (\mu_m - \mu_0) e_2^{\mathrm{T}} Q_2 e_2;$$

Case 4: $\dot{\tau}(t) = \mu_m$, $\tau(t) = \tau_m$, $\bar{\tau}(t) = 0$, one has

$$\mathbf{H}(t) = \mathbf{H}_4 = \mathbf{\Omega} + M + \tau_m M_1 + (\mu_m - \mu_0) e_2^{\mathrm{T}} Q_1 e_2.$$

It completes the proof.

Since many existing works have not considered the information on the lower bound of delay's variation rate, thus based on the proof of Theorem 1 we also can derive the following corollary.

Corollary 1 *For any given scalars $\tau_m \geq 0$, μ_m, $\varepsilon > 0$, and $\Psi_{\bar{\lambda}_i}$ $(i = 1, 2, 3, 4)$, the system (1) under the time-delayed protocol (3) can achieve the containment control with the gain matrix $K_2 = Z^{-1}L$, if there exist constant matrices $P > 0$, $J > 0$, $Q_i > 0$ $(i = 1, 2)$, $R_i > 0$ $(i = 1, 2, 3, 4)$,*

H_i $(i = 1, 2)$, X, N_1, Z, L with appropriate dimensions such that $\begin{bmatrix} \bar{R} & X \\ * & \bar{R} \end{bmatrix} \geq 0$ *and the LMI in (44) hold*

$$\begin{bmatrix} \tilde{\mathbf{H}}_j & \Gamma_1^{\mathrm{T}} Z^{\mathrm{T}} \left(\Psi_{\bar{\lambda}_i} \Lambda_B\right)^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B & 0 \\ * & \boldsymbol{sym}\left(-\left(\Psi_{\bar{\lambda}_i} \Lambda_B\right)^{\mathrm{T}} \Psi_{\bar{\lambda}_i} \Lambda_B Z\right) & \Pi_1 \\ * & * & -J \end{bmatrix} < 0, \tag{44}$$

where $\tilde{\mathbf{H}}_j = \bar{\mathbf{H}}_j + \boldsymbol{sym}\left(\Pi_0^{\mathrm{T}} \Gamma_0\right) + J$ *with*

$$\bar{\mathbf{H}}_1 = \bar{\boldsymbol{\Omega}} + M + \tau_m M_1,$$
$$\bar{\mathbf{H}}_2 = \bar{\boldsymbol{\Omega}} + M + \tau_m M_2$$

with the term $\bar{\boldsymbol{\Omega}} = [\bar{\boldsymbol{\Omega}}_{ij}]_{10\times 10}$. *Particularly, most elements of* $\bar{\boldsymbol{\Omega}}$ *are identical to the relevant ones of* $\boldsymbol{\Omega}$ *in Theorem 2 except for*

$$\bar{\Omega}_{11} = -\frac{3}{2} R_4 + \tau_m R_2 + Q_1 + Q_3 - \frac{1}{2} R_2,$$
$$\bar{\Omega}_{22} = -\frac{3}{2} R_4 - \frac{3}{2} R_3 + (\mu_m - 1) Q_1, \ \bar{\Omega}_{33} = -R_3 - Q_2.$$

Proof Since the lower bound of time delay variation rate cannot be available, then we can choose the following Lyapunov–Krasovskii functional

$$\begin{aligned} V(z_i(t)) &= z_i(t)^{\mathrm{T}} P z_i(t) + \int_{t-\tau(t)}^{t} {z_i}^{\mathrm{T}}(\theta) Q_1 z_i(\theta) dt \\ &\quad + \int_{t-\tau_m}^{t} z_i^{\mathrm{T}}(\theta) Q_2 x(\theta) dt \\ &\quad + \int_{-\tau_m}^{0} \int_{t+s}^{t} \dot{z_i}^{\mathrm{T}}(\theta) R_1 \dot{z_i}(\theta) \mathrm{d}\theta \mathrm{d}s \\ &\quad + \int_{-\tau_m}^{0} \int_{t+s}^{t} z_i^{T}(\theta) R_2 z_i(\theta) \mathrm{d}\theta \mathrm{d}s \\ &\quad + \frac{1}{2} \int_{t-\tau_m}^{t} \int_{t-\tau_m}^{\varrho} \int_{\theta}^{t} \dot{z_i}^{\mathrm{T}}(s) R_3 \dot{z_i}(s) \mathrm{d}s \mathrm{d}\theta \mathrm{d}\varrho \\ &\quad + \frac{1}{2} \int_{t-\tau_m}^{t} \int_{\varrho}^{t} \int_{\theta}^{t} \dot{z_i}^{\mathrm{T}}(s) R_4 \dot{z_i}(s) \mathrm{d}s \mathrm{d}\theta \mathrm{d}\varrho. \end{aligned}$$

Then based on Theorems 1 and 2, this corollary can be easily achieved and the proof is omitted.

Remark 3 Together with Theorems 1–2 and Corollary 1, it is shown that regardless of the numbers of the followers and leaders, the gain matrix K_2 can be determined by $\bar{\lambda}_i$ $(i = 1, 2, 3, 4)$. Furthermore, because some novel integral inequalities have been used and both the upper and lower bound of $\tau(t)$ and its variation rate have been fully studied, we can get much less conservative results. By solving the derived LMIs, the maximum allowable upper bound of $\tau(t)$ can be greatly extended.

Remark 4 It should be mentioned that the conditions for designing K_2 in the protocol (3) given in (18) are non-convex, which cannot be solved directed via the LMI Toolbox. However, based on Lemma 5 and the Schur-complement, a new feasible method for the controller gain K_2 is presented in Theorem 2 and Corollary 1 by resorting to a novel constructive technique to separate the coupling among control gains.

4 Numerical Example

In this section, a numerical example will be provided to illustrate the efficiency of the derived results.

Example 1 Consider the third-order MASs which has an interaction topology shown in Fig. 1 and the weight is set as 0–1.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{45}$$

Specifying the eigen values of $(A + BK_1)$ at $0.8i$, $-0.8i$, -2 with $i^2 = -1$, we can choose the controller gain K_1 as

$$K_1 = [-8.84, -12.74, -16].$$

If one of the real parts of $(A + BK_1)$ is equal to 0, the leaders' trajectories will achieve the oscillation.

Assuming that the initial states $x_{ij}(0)$ $(i = 6, 7, 8, 9;\ j = 1, 2, 3)$ of four leaders are

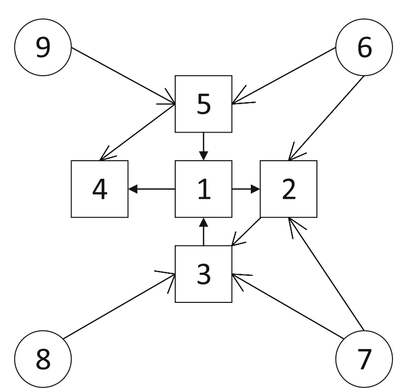

Fig. 1 Directed interaction topology G

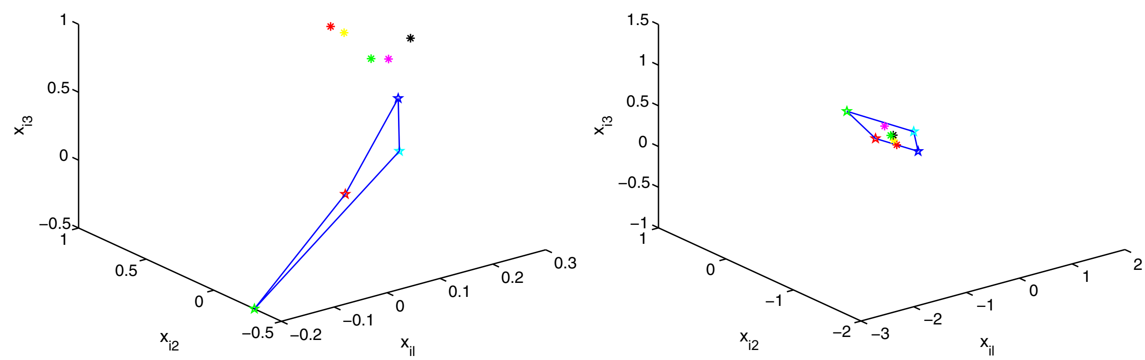

Fig. 2 Intial trajectory snapshots of leaders and followers

Table 1 Calculated MAUBs τ_m for various μ_m

μ_m	0.1	0.4	0.8	1	2
τ_m[29]	0.212	0.191	0.186	–	–
τ_m	0.267	0.264	0.259	0.256	0.247

$$x_{6j}(0)=\begin{bmatrix}0.1\\0,2\\-0.2\end{bmatrix},\ x_{7j}(0)=\begin{bmatrix}0.1\\-0,2\\0.3\end{bmatrix},$$
$$x_{8j}(0)=\begin{bmatrix}-0.2\\-0.3\\-0.5\end{bmatrix},\ x_{9j}(0)=\begin{bmatrix}0.2\\0,2\\0.4\end{bmatrix}.$$

The initial values above allow that the leaders can form the convex hull. Yet the initial states of followers $x_{ij}(0)$ $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3)$ are random values whose distributions are located on the interval (0,1). The initial states of leaders and followers are shown in Fig. 2. The pentagrams represent the leaders, and the asterisks represent the followers. The convex hull formed by leaders are also marked by the solid lines.

According to the feasibility of LMIs in Theorem 2, we can get Table 1 which shows the calculated maximum delay upper bounds (MAUBs) τ_m for various μ_m and $\mu_0 = -2$.

Comparing with [29], for different μ_m, the MAUBs in our work has increased a lot. Meanwhile, there is a limit $|\mu_m| < 1$ in [29], while our work still can be verified to be effective when $\mu_m > 1$. Furthermore, since the work [31] studies the case that $\tau(t)$ is a constant, which means $\mu_0 = \mu_m = 0$, the available MAUBs of τ is 0.17, while our result can reach to 0.267, which means that our theorem can be of larger application area.

Now in order to better illustrate the efficiency of our results, we will give some simulation results in two following cases.

Case 1: As for $\mu_m = 0.4$, we can choose $\tau(t) = 0.264\sin^2(1.51t)$, then $\tau_m = 0.264$ and $\mu_m = 0.4$, $\mu_0 = -0.4$. By using Theorem 2 and LMI Toolbox, K_2 can be computed out as

$$K_2 = \big[-0.3105 - 0.4013 - 0.5192\big].$$

The tendency of the leaders and followers is shown in Fig. 3, which demonstrates that the relationship of x_{ij} $(i = 1, 2, \ldots, 9;\ j = 1, 2, 3)$ at $t = 5.6$s, $t = 18.8$s and $t = 34$s, respectively.

Case 2: As for $\mu_m = 2$, let $\tau(t) = 0.247\sin^2(8.09t)$. By resorting to Theorem 2 and LMI Toolbox, the controller fain K_2 can be achieved as

$$K_2 = \big[-0.3017 - 0.3986 - 0.5011\big].$$

The trajectory snapshots of leaders and followers are shown in Fig. 4. Figure 4 verifies that the relationship of x_{ij} $(i = 1, 2, \ldots, 9;\ j = 1, 2, 3)$ and the one of $(x_{i1} - x_{i2}), (x_{i1} - x_{i3}), (x_{i2} - x_{i3})$ $(i = 1, 2, \ldots, 9)$ respectively at $t = 31$s.

Based on Figs. 3 and 4, we can check that MAS (45) can achieve the desired containment control with the protocol (3) when $\tau_m = 0.264$, $\mu_m = 0.4$ at $t = 18.8, 34$s and $\tau_m = 0.247$, $\mu_m = 2$ at $t = 31$s, which proves the efficiency of our work.

5 Conclusions

This paper has investigated the problem on containment control of MASs with the help of time delay protocol. In the protocol, we design K_1 to specify the motion mode of leaders and K_2 to enable the followers to form the containment control. By constructing two improved Lyapunov–Krasovskii functionals and employing some novel integral inequalities, the application area can be greatly extended. Meanwhile, in

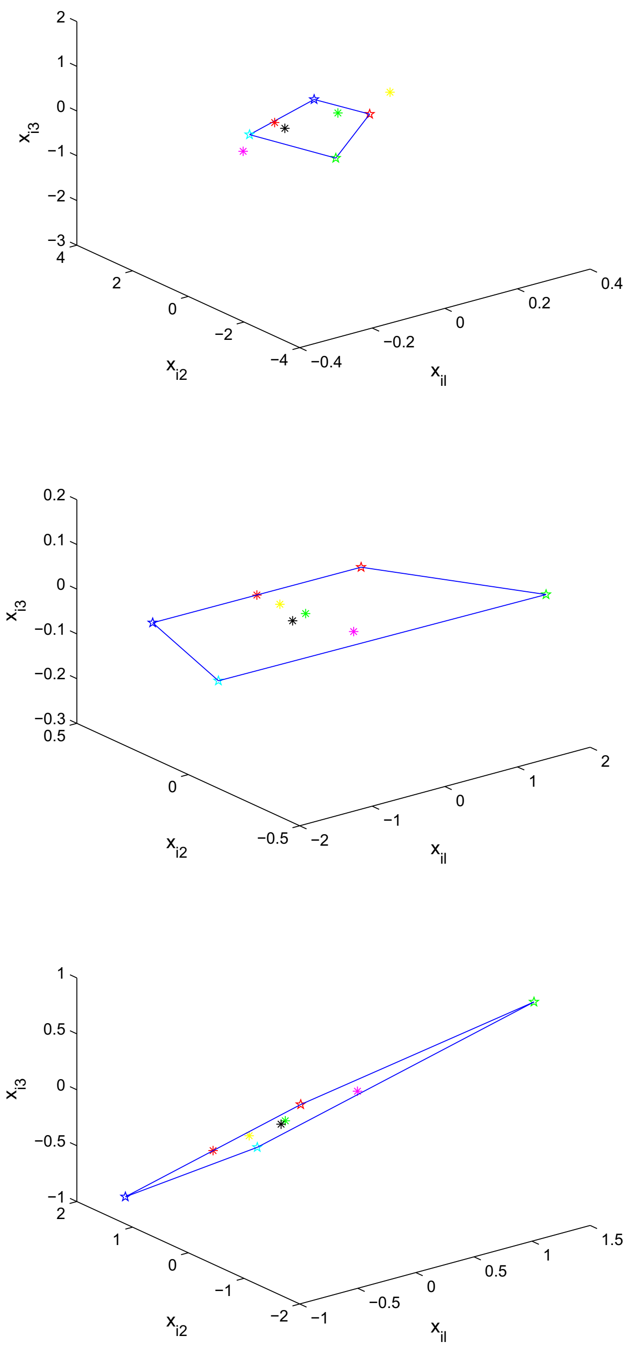

Fig. 3 The trajectory snapshots of $x_{ij}(t)$ at $t = 5.6$s, $t = 18.8$s, and $t = 34$s with $\mu_m = 0.4$, $\tau_m = 0.264$

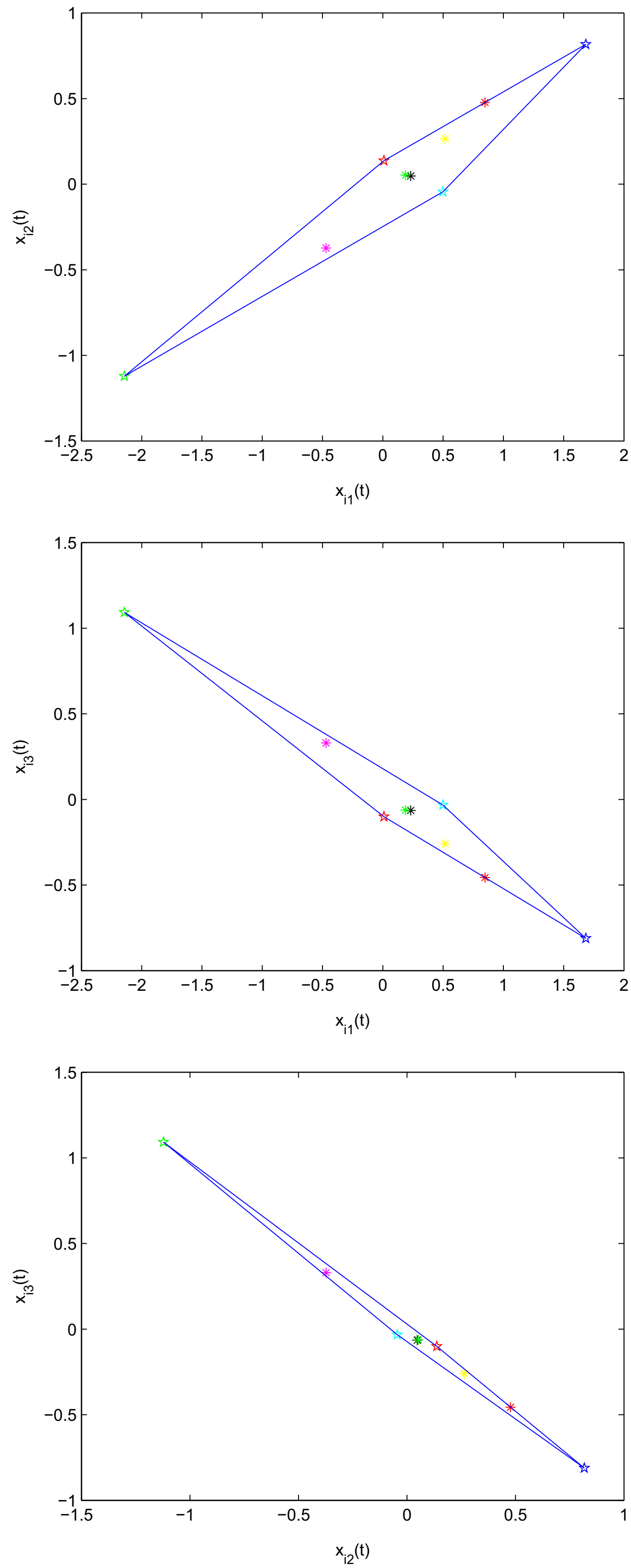

Fig. 4 The trajectory snapshots of $x_{ij}(t)$ from different views at $t = 31$s with $\mu_m = 2$ and $\tau_m = 0.247$

order to compute out the controller gain, a novel constructive method has been established in terms of LMI, which presents much less conservatism. Finally, some comparing results with simulations are given to illustrate the efficiency of our results.

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References

1. Ferrari-Trecate, G.; Egerstedt, M.; Buffa, A.; Ji, M.: A hybrid stop-go policy for leader-based containment control. In: Hybrid Systems Computation and Control, pp. 212–226 (2006)
2. Deng, C.; Yang, G.H.: Distributed adaptive fault-tolerant containment control for a class of multi-agent systems with non-identical matching non-linear functions. IET Control Theory Appl. **10**(3), 273–281 (2016)
3. Ma, L.; Min, H.; Wang, S.; et al.: Distributed containment control of networked nonlinear second-order systems with unknown parameters. IEEE/CAA J. Automat. Sin. **99**, 1–9 (2016)
4. Zhou, H.; Ji, Y.; Gao, S.; et al.: Distributed containment control for multiple-agent systems in nonlinear pure-feedback form via singular perturbation analysis. In: IEEE Conference on Control and Decision, pp. 2319–2324 (2016)
5. Hu, J.; Yu, J.; Cao, J.: Distributed containment control for nonlinear multi-agent systems with time-delayed protocol. Asian J. Control **18**(2), 747–756 (2016)
6. Huang, N.; Duan, Z.; Yu, C.: Distributed event-triggered containment control of multiple rigid bodies with combinational measurements. In: IEEE International Conference on Control, Automation, Robotics and Vision, pp. 1–6 (2017)
7. Sun, Y.; Li, Z.; Ma, D.: Event-based finite time containment control for multi-agent systems. In: The 31st Youth Academic Annual Conference of Chinese Association, pp. 129–134 (2017)
8. Zhang, W.; Tang, Y.; Liu, Y.: Event-triggering containment control for a class of multi-agent networks with fixed and switching topologies. IEEE Trans. Circuits Syst. **64**(3), 619–629 (2017)
9. Han, T.; Chi, M.; Guan, Z.; et al.: Distributed three-dimensional formation containment control of multiple unmanned aerial vehicle systems. Asian J. Control **19**(3), 157–163 (2017)
10. Meng, Y.Z.; Ren, W.; You, Z.: Distributed finite-time attitude containment control for multiple rigid bodies. Automatica **46**(12), 2092–2099 (2010)
11. Peng, X.; Geng, Z.; Tayefi, M.: Containment control based formation tracking for multi-vehicles on lie group. In: Chinese Control Conference, pp. 7751–7756 (2016)
12. Cao, H.Z.; Yan, X.; Zhou, F.: Research on containment control of second-order nonlinear multi-agent with collision avoidance mechanism. Acta Armamentarii **9**, 1646–1654 (2016)
13. Zhen, K.; She, J.M.; Dixon, W.E.: Leader–follower containment control over directed random graphs. Automatica **66**(C), 56–62 (2016)
14. Han, L.; Dong, X.W.; Li, Q.: Formation-containment control for second-order multi-agent systems with time-varying delays. Neurocomputing **218**, 439–447 (2016)
15. Ji, H.; Zhang, H.; Cui, B.: Containment analysis of Markov jump swarm systems with stationary distribution. IET Control Theory Appl. **11**(7), 901–907 (2017)
16. Wang, F.; Yang, H.; Liu, Z.: Second-order containment control of multi-agent systems in the presence of uncertain topologies with time-varying delays. J. Control Sci. Eng. https://doi.org/10.1155/2017/9489431
17. Li, B.; Chen, Z.Q.; Liu, Z.X.: Containment control of multi-agent systems with fixed time-delays in fixed directed networks. Neurocomputing **173**(P3), 2069–2075 (2016)
18. Wang, F.; Yang, H.; Liu, Z.: Containment control of leader-following multi-agent systems with jointly-connected topologies and time-varying delays. Neurocomputing **260**(80), 341–348 (2017)
19. Liu, Y.; Zhao, Y.; Shi, Z.: Specified-time containment control of multi-agent systems over directed topologies. IET Control Theory Appl. **11**(4), 576–585 (2017)
20. Li, B.; Chen, Q.Z.; Zhang, C.Y.: Containment control for directed networks multi-agent system with nonlinear dynamics and communication time-delays. Int. J. Control Autom. Syst. **15**(3), 1181–1188 (2017)
21. Liu, S.; Xie, L.; Zhang, H.: Containment control of discrete-time multi-agent systems based on delayed neighbours' information. In: The 9th Asian Control Conference, pp. 1–6 (2013)
22. Xie, X.; Yang, Z.; Fang, J.: Containment control of multi-agent systems with input saturation and intermittent communication. In: The 31st Youth Academic Annual Conference of Chinese Association, pp. 99–104 (2017)
23. Dong, X.W.; Shi, Z.; Lu, G.; et al.: Output containment control for high-order linear time-invariant swarm systems. Systems **8215**(2), 285–290 (2013)
24. Li, Z.G.; Huang, W.P.; Du, Y.J.: H∞ containment control for second-order multi-agent systems with time delay. In: Chinese Control Conference, pp. 7926–7931 (2016)
25. Liao, F.C.; Kong, M.; Liu, H.Y.: H∞ containment control for first-order multi-agent systems with multiple time-varying delays. Control Decis. **32**(4), 584–592 (2017)
26. Jia, Y.: Robust H∞ containment control for uncertain multi-agent systems with inherent nonlinear dynamics. Int. J. Syst. Sci. **47**(5), 1–11 (2016)
27. Wang, D.; Zhang, N.; Wang, J.; et al.: Cooperative containment control of multiagent systems based on follower observers with time delay. IEEE Trans. Syst. Man Cybern. Syst. **99**, 1–11 (2016)
28. Yang, Z.; Mu, X.; Liu, K.: Containment control of continuous-time multi-agent systems with general linear dynamics under time-varying communication topologies. Int. J. Control Autom. Syst. **15**(1), 442–449 (2017)
29. Dong, X.W.; Han, L.; Li, Q.; et al.: Containment analysis and design for general linear multi-agent systems with time-varying delays. Neurocomputing **173**(3), 2062–2068 (2016)
30. Xi, J.; Shi, Z.; Zhong, Y.: Consensus analysis and design for high-order linear swarm systems with time-varying delays. Phys. A Stat. Mech. Appl. **390**(2324), 4114–4123 (2011)
31. Dong, X.W.; Li, Q.D.; Zhang, R.; Zhong, Y.S.: Formation-containment control for high-order linear time-invariant multi-agent systems with time delays. J. Frankl. Inst. **352**(9), 3564–3584 (2015)
32. Wang, X.L.; et al.: Second-order consensus of multi-agent systems via periodically intermittent pinning control. Circuits Syst. Signal Process. **35**(7), 2413–2431 (2016)
33. Zhong, Z.; Sun, L.; Wang, J.; et al.: Consensus for first- and second-order discrete-time multi-agent systems with delays based on model predictive control schemes. Circuits Syst. Signal Process. **34**(1), 127–152 (2015)
34. Zeng, H.; He, Y.; Wu, M.; She, J.H.: New results on stability analysis for systems with discrete distributed delay. Automatica **63**, 189–192 (2015)
35. Liu, Y.; Ju, H.P.; Guo, B.Z.: Results on stability of linear systems with time varying delay. IET Control Theory Appl. **11**(1), 129–134 (2016)
36. Park, M.; Kwon, O.; Park, J.H.; Lee, S.: Stability of time-delay systems via Wirtinger-based double integral inequality. Automatica **55**, 204–208 (2015)

37. Park, P.; Lee, W.; Lee, S.Y.: Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems. J. Frankl. Inst. **352**(4), 1378–1396 (2016)
38. Hua, C.C.; Wu, S.S.; Yang, X.; Guan, X.P.: Stability analysis of time-delay systems via free-matrix-based double integral inequality. Int. J. Syst. Sci. **48**(2), 257–263 (2017)
39. Zhang, C.K.; He, Y.; Jiang, L.; Wu, M.; Zeng, H.B.: Stability analysis of systems with time-varying delay via relaxed integral inequalities. Syst. Control Lett. **92**, 52–61 (2016)
40. Li, T.; Wang, T.; Song, A.G.; Fei, S.M.: Delay-derivative-dependent stability for delayed neural networks with unbounded distributed delay. IEEE Trans. Neural Netw. **21**, 1365–1371 (2010)
41. Shen, M.Q.; Yan, S.; Zhang, G.M.: A new approach to event-triggered static output feedback control of networked control systems. ISA Trans. **65**(4), 468–474 (2016)