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Suboptimal Output Tracking and Regulation of a Class of Nonlinear Systems

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Abstract

This paper proposes an approximate optimal approach for output tracking and regulation of a class of nonlinear systems. This class includes systems which are decomposable into a linear part and a nonlinear part. It is shown that the Hamilton–Jacobi–Bellman equation associated with optimal control of such systems, under an assumption on form of its solution, can be approximated as an algebraic equation. Decomposition of this algebraic equation into a linear part and a nonlinear part and their subsequent solutions together form an approximate value function. This value function is found to be nearly optimal locally around the origin. Thus, a locally stabilizing controller with the proposed value function as feedback gain is proposed. This approach achieves output tracking and regulation by converting these into a stabilization problem through internal model principle resulting in a near-optimal tracking and regulation. This proposed algebraic equation can be used for estimation too, i.e., it plays the same role in nonlinear settings as that played by an algebraic Riccati equation in linear quadratic regulation and linear quadratic estimation. Finally, the proposed approach can compensate integral windup effects due to actuator saturation through modification of filtering characteristics of internal model. Simulation results for an undamped spring-mass system are given for efficacy of the approach.

Keywords Nonlinear tracking · Nonlinear regulation · Nonlinear estimation · Optimal control · Internal model principle

1 Introduction

Linear and nonlinear output regulation problems were solved, respectively, by Francis, Davison, and Wonham in [1] and Isidori and Byrnes in [2]. The subject of output regulation of dynamical systems had matured by mid-2000s marked by the publication of four monographs [2–5]. [2] focused on local output regulation of nonlinear systems, [3] provided results for semi-global output regulation, [4] attempted global regulation of uncertain nonlinear system, and finally, [5] dealt with regional output regulation through Jacobian analysis of nonlinear systems. Optimality has not been a goal in research on nonlinear output regulation except for a solitary attempt by Byrnes in [6]. Byrnes explored Riccati partial differential equation for nonlinear optimal control in order to seek a counterpart of algebraic Riccati equation (ARE) in nonlinear settings. The results presented in [6] were too abstract to be of practical use. Thus, optimal output regulation of nonlinear systems is an open problem. This paper provides an approach for output tracking/regulation of a class of nonlinear systems with some degree of optimality.

Traditionally, an optimal nonlinear control problem requires solving the Hamilton–Jacobi–Bellman equation (HJBE) that is a nonlinear partial differential equation (NPDE). Since there is no general theory for analytical solutions of NPDEs, various approximate solutions of HJBE have been reported in the literature. Four approximate-analytical approaches to optimal nonlinear control are inverse optimal control by Freeman and Kokotovic in [7], patchy solutions by Navasca and Krener in [8], dynamic value functions by Sassano and Astolfi in [9], and state-dependent-Riccati equations (SDRE) by Cloutier et al. in [10]. Approaches of Patchy Solutions and dynamic value functions solve the stabilization problem of nonlinear systems. On the other hand, output regulation of nonlinear systems is better investigated by the other two approaches of inverse optimal control and SDRE. For an excellent research on optimal tracking control, [11–13] can be referred.

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Inverse optimal control traces its origin to the discovery by Freeman and Kokotovic in [7,14] according to which solution of an HJBE associated with optimal control of a system is related to a control Lyapunov function of the system. Kristic and Tsitoris in [15] proposed inverse optimal control which uses control Lyapunov functions to stabilize nonlinear systems without bothering about cost functionals. However, determination of control Lyapunov function is not a trivial task for nonlinear systems. An optimal output regulation through an inverse optimal controller embedded in Lyapunov redesign framework is proposed by Memon in [16].

SDRE approach totally ignores the partial derivatives of an HJBE converting it into a functional equation. This approach has shown promising local results to optimal nonlinear control problems though it lacks rigorous justification for the elimination of partial derivatives. Nonetheless, easy implementation of SDRE approach for local and suboptimal regulation of practical nonlinear systems makes it a very popular choice. Two recent application of SDRE method for nonlinear tracking are given in [17,18]. A survey on the stability and optimality results of SDRE is given in [19].

The main contribution of this paper is an approach to control weakly nonlinear systems with regard to optimality and actuator saturation. Weakly nonlinear systems are those which can be decomposed into a linear part and a nonlinear part. Mathematically, such a system can be stated in matrix form as a linear system with constant parameters and a linear system with state-dependent nonlinear parameters. Now, the associated HJBE needed to be solved can be converted into an algebraic equation (AE) through two assumptions, which are (i) partial derivatives of the HJBE replaced with their first-order approximations and (ii) decomposition of state dynamics into two parts. This AE is found to be decomposable into a linear component and a nonlinear component. Linear component, being an ARE, is solved first for a constant solution matrix which is then used in solving the nonlinear component for a state-dependent solution matrix. These two solutions together form (i) an approximate value function, and (ii) state-dependent gain of a stabilizing state-feedback controller. The idea of a two-component control law is based upon an earlier work of ours [20]. In [20], we proposed a variable gain controller composed of an LQR controller whose gains were modifiable with respect to the size of parametric uncertainty.

There are two salient features of this paper. Firstly, it is shown that the tracking and regulation problems for weakly nonlinear systems can be solved by converting these into a stabilization problem through internal model principle. Augmentation of internal models of reference signals, generated by linear exosystems, into linear component of a decomposed system achieves tracking/regulation with acceptable steady-state accuracy. Motivated by the fact that linear quadratic regulation and linear quadratic estimation problems are dual of each other, this paper proposes a dual of the proposed AE for estimator design. It is to be noted that the estimator design is carried out for unaugmented system. Secondly, the proposed approach provides anti-windup compensation against actuator saturation. It is shown that the fast Fourier transform of closed-loop dynamics shows a signal with a significant gain at a frequency other than the frequency of reference signal. It is shown that this unwanted signal can be filtered out by relocation of zero(s) of the internal model.

The rest of the paper is organized as follows: Section 2 states the problem; Section 3 provides the main result; Section 4 discusses the approximation of HJBE; Section 5 discusses the observer-based tracking/regulation; Section 6 presents the anti-windup compensation; and finally, Section 7 concludes the paper.

2 Problem Statement

Consider a SISO nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$ represents the state vector, $u \in R$ represents the control input vector, $y \in R^v$ represents the measured output vector. It is assumed that origin $x = 0$ is an origin of the system. Moreover, $f(x) : R^n \rightarrow R^n$ is continuously differentiable in all its arguments. We consider a class of nonlinear systems in which the state dynamics are decomposable into a linear part and a nonlinear part, i.e., $\dot{x} = Ax + f_1(x) + Bu$. This decomposition is similar to 'Additive-State Decomposition' in [21]. Furthermore, the remaining nonlinear part can be factorized in an infinite number of ways, i.e., $f_1(x) = A(x)x$. This factorization is based on the idea of extended linearization used in [22] for a suboptimal nonlinear control technique referred to as SDRE control. Hence, the (1) is modified as

$$\dot{x} = Ax + A(x)x + Bu = f_2(x)x + Bu \tag{2}$$

where $f_2(x) = A + A(x)$. Note that the function $f_1(x)$ does not include unmodeled dynamics. Considering unmodeled dynamics too renders the proposed approach as more practical. This direction of research has been mentioned as the future work while concluding this paper. It is assumed that the system is driven by a scalar control input. The problem is to possibly devise a controller design procedure that deals with the following three items in the requirement list

1. *Approximation of HJBE* The intractability of solving an HJBE be reduced through an approximation.
2. *Observer-based output tracking and regulation* It is desired that the controlled system be able to track/regulate its actual or observed output at any of these reference signals, i.e., $\lim_{t\to\infty} error(t) = e(t) = r(t) - y(t) = 0$, where $y(t)$ is the tracked output. Reference signals such as step-input, ramp-input, and sinusoidal-input are to be regulated or tracked. All these signals are generated by a linear exosystem.
3. *Actuator Saturation* Control mechanism be able to deal with input constraints such as actuator saturation. The saturation limit, however, is not allowed to affect the asymptotic stability achieved through observer-based output tracking and regulation.

Next section presents the main result of this paper.

3 Main Result

The main result of this paper is that a weakly nonlinear system

$$\dot{x} = Ax + A(x)x + Bu = f_2(x)x + Bu \tag{3}$$

can be stabilized near optimally by a two-component state-feedback controller given as

$$u = -R^{-1}B^T(P + \rho(x))x \tag{4}$$

with feedback gains P and $\rho(x)$ being, respectively, solutions of an ARE

$$\phi_1 := A^T P + PA + Q - PBR^{-1}B^T P = 0 \tag{5}$$

and a nonlinear algebraic equation

$$\begin{aligned}
&\phi_2(A, A(x), P, \rho(x), B, R)\\
&\quad := PBR^{-1}B^T\frac{3}{2}\rho(x) + \frac{3}{2}\rho(x)BR^{-1}B^T P\\
&\qquad + \frac{3}{2}\rho(x))BR^{-1}B^T\frac{3}{2}\rho(x) - A^T(x)P + PA(x)\\
&\qquad - f_2(x)^T\rho(x) - 2\rho(x)f_2(x) = 0.
\end{aligned} \tag{6}$$

The proposed controller in (4) is similar in form to the controller proposed in [11,13]. Now, we proceed to Sect. 4 that discusses the conversion of HJBE into an algebraic equation.

4 Approximation of Hamilton–Jacobi–Bellman Equation into an Algebraic Equation

The aforementioned approximation of an HJBE into an AE is predicated upon an assumption on form of the value function which is the solution of HJBE. This value function is assumed to consist of two components as $V(x) = x^T Px + x^T\rho(x)x$. Determination of matrices P and $\rho(x)$ is preceded by an analysis of stabilizing capability of the proposed two-component controller $u = -R^{-1}B^T(P + \rho(x))x$. If this controller is found to be stabilizing at least locally around the origin, then we can proceed to find an approximate value function. This idea is based on converse theorem used in nonlinear control systems by which existence of a Lyapunov function is guaranteed upon existence of a stabilizing controller, refer to Khalil [23].

Proving the boundedness of state trajectories of open-loop weakly nonlinear systems is trivial. Banks in [24] has proven the boundedness in case of closed-loop weakly nonlinear systems controlled by a general two-component controller. The proposed approach modifies this proof in context of optimal control theory. Closed-loop system is given as

$$\dot{x} = (A - BR^{-1}B^T P)x + (A(x) - BR^{-1}B^T\rho(x))x \tag{7}$$

The proof is to be preceded by a few necessary assumption to make the problem well-posed.

Assumption 1 Pair $(f_2(x), B)$ is stabilizable in a ball ϵ_1 around the origin, i.e., $x \in \epsilon_1$.

Assumption 2 Continuous differentiability of $f(x)$ requires $A(x)$ to be continuous as well around a ball ϵ_2 around the origin, i.e., $\mathcal{B}_{\epsilon_2}(0) = \{x : \|x - 0\| < \epsilon_2\}$. Subsequently, the state-dependent gain $\rho(x)$ is required to be continuous around the origin.

Remark 1 Ball of stabilizability is smaller than or equal to the ball of continuity, i.e., $\epsilon_1 \subseteq \epsilon_2$

Assumption 3 The state-dependent part of the control signal is proportional to the state vector such that $\lim_{\|x\|\to 0} \|\rho(x)\| = 0$

Proof begins here. A stable linear part of the closed-loop systems dynamics in (7) means that all Eigenvalues λ of $A - BR^{-1}B^T P$ must be such that $Re(\lambda) < -\beta$ for $\beta > 0$. As for the nonlinear part in (7), lets equate it to $h(x) = g(x)x$ resulting in the following closed-loop dynamics representation

$$\dot{x} = (A - BR^{-1}B^T P)x + h(x). \quad (8)$$

By assumptions 2 and 3, we can conclude that $h(x)$ satisfies the following property locally around the origin

$$\lim_{\|x\|\to 0} \frac{\|h(x)\|}{\|x\|} = 0 \quad (9)$$

for $\|x(t)\| \leq \epsilon_3$, where $\epsilon_3 > 0$. This property leads to the bounding of $g(x)$, i.e., $\|g(x)\| \leq \zeta$, where $\zeta > 0$ leads to $\|h(x)\| \leq \zeta\|x\|$. By variations of parameters formula, solution of (8) can be given as

$$x(t) = e^{(A-BR^{-1}B^T P)t}x(0) + \int_0^t e^{(A-BR^{-1}B^T P)(t-s)}h(x(s))\mathrm{d}s. \quad (10)$$

Taking the norm on both sides results in

$$\|x(t)\| = \|e^{(A-BR^{-1}B^T P)t}\|\|x(0)\| + \zeta\int_0^t \|e^{(A-BR^{-1}B^T P)(t-s)}\|\|x(s)\|\mathrm{d}s. \quad (11)$$

The factor $\|e^{(A-BR^{-1}B^T P)t}\|$ is bounded by
$\|e^{(A-BR^{-1}B^T P)t}\| \leq Ge^{-\beta t}$ for $G > 0$. Thus, (11) becomes

$$\|x(t)\| = \|e^{(A-BR^{-1}B^T P)t}\|\|x(0)\| + \zeta\int_0^t \|e^{(A-BR^{-1}B^T P)(t-s)}\|\|x(s)\|\mathrm{d}s. \quad (12)$$

Now for a positive constant G the factor $\|e^{(A-BR^{-1}B^T P)t}\|$ is bounded as follows

$$\|e^{(A-BR^{-1}B^T P)t}\| \leq Ge^{-\beta t} \quad (13)$$

that modifies (12) as

$$\|x(t)\| \leq Ge^{-\beta t}\|x(0)\| + \zeta G\int_0^t e^{-\beta(t-s)}\|x(s)\|\mathrm{d}s. \quad (14)$$

Finally, multiplying both sides by $e^{\beta t}$ and utilizing Gronwall inequality, we get

$$\|x(t)\| \leq G\|x(0)\|e^{-(\beta-\zeta G)}. \quad (15)$$

Hence, if the factor $\beta - \zeta G > 0$, we can guarantee local asymptotic stability of origin, i.e., $x = 0$. Proof ends here.

After the aforementioned establishment of local asymptotic stability of the origin, we begin constructing the approximate value function. This process starts by equating a quadratic cost functional to a quadratic Lyapunov function, i.e., $V(x(t)) = x^T(t)(P + \rho(x(t)))x(t)$. Here, matrix P is assumed as a symmetric-positive-definite (s.p.d.) matrix and $\rho(x)$ as a symmetric-indefinite (s.i.) matrix. This equation is mathematically written as

$$\int_{t_0}^{T}(x^T(t)Qx(t) + u^T(t)Ru(t))\mathrm{d}t = V(x(t)) \quad (16)$$

with differential form stated as

$$x^T(t)Qx(t) + u^T(t)Ru(t) = \left.\frac{\partial V(x(t))}{\partial t}\right|_{t_0}^{T} + \left.\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\right|_{t_0}^{T} \quad (17)$$

The aforementioned equation is an HJBE whose time-invariant form makes the first term zero on the right-hand side of equality. Moreover, with $t_0 = 0$ and $T = \infty$, it becomes an infinite-horizon optimal control problem

$$\begin{aligned} x^T(t)Qx(t) + u^T(t)Ru(t) &= \left.\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\right|_{t_0}^{T} \\ &= \left.\frac{\partial V(x(t))}{\partial x}\dot{x}\right|_{0}^{\infty} \end{aligned} \quad (18)$$

Furthermore, with fixed terminal state, i.e., $x(\infty) = 0$, (18) modifies to

$$x_0^T Qx_0 + u_0^T Ru_0 = -\dot{x}_0^T(P + \rho(x_0))x_0 - x_0^T(P + \rho(x_0))\dot{x}_0 - x_0^T(P + \dot{\rho}(x_0))x_0 \quad (19)$$

where $x_0 = x(0)$ and $u_0 = u(0)$ are the initial values of states and control signal, respectively. We deal with the first two terms on the right-hand side of Eq. (19) separately from the third term on this side. Straightforward algebraic manipulation of the first two terms results in

$$\begin{aligned} &\dot{x}_0^T(P + \rho(x_0))x_0 + x_0^T(P + \rho(x_0))\dot{x}_0 \\ &= x_0^T\left\{f_2(x_0)^T(P + \rho(x_0)) + (P + \rho(x_0))f_2(x_0)\right\}x_0 \\ &+ 2B^T(P + \rho(x_0))x_0u_0 \end{aligned} \quad (20)$$

After dealing with the first two terms, we focus on the third term which is important with respect to optimality requirement in problem statement. The third term, $x_0^T(P + \dot{\rho}(x_0))x_0$, is modified into $x_0^T\dot{\rho}(x_0)x_0$ as time derivative of constant matrix P is a zero-matrix. By Chain rule, $x_0^T\dot{\rho}(x_0)x_0$ can be changed into $x_0^T\{\nabla\rho(x_0).\dot{x}_0\}x_0$, where ∇ represents gradient of matrix $\rho(x_0)$. We use a second-order multivariable Taylor series to approximate $\nabla\rho(x_0)$ resulting in $\nabla\rho(x) = \frac{\rho(x_0)-\rho(0)}{x_0} + \frac{1}{2}Hx_0$, where H represents the Hessian matrix. We ignore the term $\frac{1}{2}Hx_0$ simplifying remaining analysis at a price of rendering the proposed approach suboptimal. Hence, this approximation results in

$$\nabla\rho(x_0) = \frac{\rho(x_0)-\rho(0)}{x_0} + O(x_0) \approx \frac{\rho(x_0)-\rho(0)}{x_0} \tag{21}$$

There are two important aspects of the aforementioned approximation. Firstly, the difference between an absolutely optimal design and the proposed approach varies only linearly with farther initial values of system states from origin. Secondly, matrix $\rho(x)$ is a zero-matrix at the origin, i.e., $\rho(0) = 0$ which further simplifies Eq. (21). This means that our proposed suboptimal approach is almost optimal near the origin. Third term in Eq. (19) thus becomes

$$x_0^T \dot{\rho}(x_0)x_0 = x_0^T \rho(x_0) f_2(x_0)x_0 + x_0^T \rho(x_0)Bu_0 \tag{22}$$

Inserting Eqs. (20) and (22) into Eq. (19) yields a multivariable function. Minimization of this function by exploiting its geometry results in form of the minimizer, i.e., the optimal control law. Note that this function is required to be minimized at all times from $t = 0$ to $t = \infty$, thus dropping the initial condition dependence yields the following multivariable function

$$F(u,x) = \min_u \Big[u^T Ru + B^T(2P + 3\rho(x))xu + x^T\Big(f_2(x)^T(P+\rho(x)) + (P + 2\rho(x))f_2(x) + Q\Big)x\Big] \tag{23}$$

The goal is to find a u that will minimize the above objective function. This objective function is quadratic in both control signal u and state vector x. This is a case of multivariable quadratic equation which can be solved by considering it as a single-variable quadratic equation. Such a consideration is also mentioned in [25]. A single-variable quadratic equation is solved through the following quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{24}$$

Locus of solution points of the above equation form a *parabola*. The position of a parabola on the coordinate axes plane depends upon a factor known as a discriminant, $D = b^2 - 4ac$. When $D = 0$ and $a > 0$, the minimum of the parabola occurs on $x - axis$ at vertex. Figure (1) shows the position of a parabola with respect to difference cases of D. It is possible to consider the argument of the objective function as a single-variable quadratic equation. Using the quadratic formula in (24) to obtain a solution for u yields

$$u = \frac{-B^T(2P + 3\rho(x))x \pm \sqrt{\phi}}{2R} \tag{25}$$

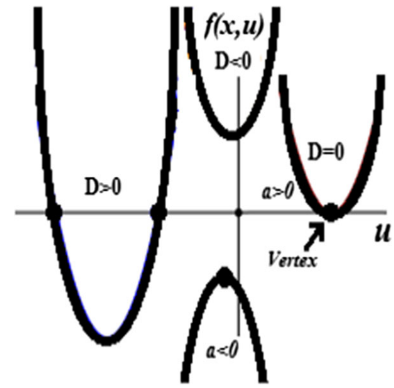

Fig. 1 Position of a parabola with respect to discriminant D

where

$$\phi = \Big(B^T(2P + 3\rho(x))x\Big)^2 - 4Rx^T\Big(f_2(x)^T(P+\rho(x)) + (P+2\rho(x))f_2(x) + Q\Big)x \tag{26}$$

Since we know that both the matrices P and $\rho(x)$ are symmetric, this expression for ϕ becomes

$$\phi = 4Rx^T\Big(\Big(P + \frac{3}{2}\rho(x)\Big)BR^{-1}B^T\Big(P+\frac{3}{2}\rho(x)\Big) - f_2(x)^T(P+\rho(x)) - (P+2\rho(x))f_2(x) - Q\Big)x \tag{27}$$

Geometric analysis of the foregoing expression allows stating conditions for feasibility of minimization of objective function in (23). These conditions are

1. *Existence of Minimum* In order to ensure that a minimum of the aforementioned objective function exists, the coefficient R of term u^2 must be positive-definite, i.e., $R > 0$.
2. *Asymptotic Stability* Setting $\phi = 0$ yields (i) form of the optimal control law u appearing as
$$u = -R^{-1}B^TPx - \frac{3}{2}R^{-1}B^T\rho(x)x, \tag{28}$$
and (ii) minimum of the objective function lies at the axis $f(x,u) = 0$ leading to the asymptotic stability of the system. Since $u = f(x)$, minimum of $F(x,u)$ occurs only when $x = 0$.
3. *Optimality* Setting $\phi = 0$ is also required in the proposed suboptimal approach as would be required in an optimal case without the aforementioned approximation. We know that the system evolves with state(s) starting from an initial condition, i.e., $x(0) \neq 0$ meaning

that the factor $\left((P + \frac{3}{2}\rho(x))BR^{-1}B^T(P + \frac{3}{2}\rho(x)) - f_2(x)^T(P + \rho(x)) - (P + 2\rho(x))f_2(x) - Q\right) = 0$ at all times from $t = 0$ to $t = \infty$.

It is now shown how the proposed approach simplifies to traditional equations for optimal control of linear systems. For a linear system $\dot{x} = Ax + Bu$ with a quadratic cost functional and Lyapunov function $V = x^T Px$-objective function, multivariable quadratic equation, ϕ, and resulting optimal control are given, respectively, as

$$\begin{aligned}&F(u, x) = \\ &\min_u \left\{u^T Ru + 2B^T Pxu + x^T(A^T P + PA + Q)x\right\} \\ &u^T Ru + 2B^T Pxu + x^T(A^T P + PA + Q)x = 0 \end{aligned} \tag{29}$$

$$\phi := A^T P + PA + Q - PBR^{-1}B^T P = 0 \ \ (ARE) \tag{30}$$

$$u = -R^{-1}B^T Px \ \ (LQR) \tag{31}$$

The next subsection discusses the solution of the equation $\phi = 0$. Solving $\phi = 0$ will consist of finding both the matrices P and $\rho(x)$.

4.1 Solving $\phi = 0$

Solving ϕ for a closed-form solution is challenging as $\phi = 0$ is apparently an equation with two variables, P and $\rho(x)$, and there cannot be a unique solution. Rearranging $\phi = 0$ reveals a possibility of solving it sequentially. Proceeding with the derivation, $\phi = 0$ can be rearranged with a left-hand side and a right-hand side as shown below

$$\begin{aligned}&f_2(x)^T(P + \rho(x)) + (P + 2\rho(x))f_2(x) + Q \\ &\quad = \left(P + \frac{3}{2}\rho(x)\right)BR^{-1}B^T\left(P + \frac{3}{2}\rho(x)\right)\end{aligned} \tag{32}$$

Rearranging both sides of the above equation results in

$$\begin{aligned}&A^T P + PA + Q + A^T(x)P + PA(x) \\ &\quad + f_2(x)^T\rho(x) + 2\rho(x)f_2(x) \\ &\quad = PBR^{-1}B^T P + PBR^{-1}B^T\frac{3}{2}\rho(x) \\ &\quad + \frac{3}{2}\rho(x)BR^{-1}B^T P + \frac{3}{2}\rho(x))BR^{-1}B^T\frac{3}{2}\rho(x)\end{aligned} \tag{33}$$

A further rearrangement results in

$$\begin{aligned}&A^T P + PA + Q - PBR^{-1}B^T P + A^T(x)P + PA(x) \\ &\quad + f_2(x)^T\rho(x) + 2\rho(x)f_2(x) \\ &\quad = PBR^{-1}B^T\frac{3}{2}\rho(x) + \frac{3}{2}\rho(x)BR^{-1}B^T P \\ &\quad + \frac{3}{2}\rho(x))BR^{-1}B^T\frac{3}{2}\rho(x)\end{aligned} \tag{34}$$

In the aforementioned matrix equation, there are two unknown matrices P and $\rho(x)$. Hence, solution of this matrix equation is not unique. Now, there can be infinitely many solutions of this matrix equation by determining one unknown matrix and finding the other with respect to the first one. Note that the above rearrangement clearly shows the appearance of an ARE which can be solved for P. Once P is found, $\rho(x)$ can also be found. The splitting of (34) into an ARE and a remaining nonlinear part is based upon the notion of splitting of PDEs. Splitting a PDE is routinely carried out in numerical solution method such as Galerkin methods. Nonetheless, splitting a PDE in analytical domain is possible too as reported in [26]. Hence, we divide ϕ into ϕ_1 and ϕ_2 and sequentially solve these where former preceding the latter. So,

$$\phi_1 := A^T P + PA + Q - PBR^{-1}B^T P = 0 \tag{35}$$

and

$$\begin{aligned}&\phi_2(A, A(x), P, \rho(x), B, R) := PBR^{-1}B^T\frac{3}{2}\rho(x) \\ &\quad + \frac{3}{2}\rho(x)BR^{-1}B^T P + \frac{3}{2}\rho(x))BR^{-1}B^T\frac{3}{2}\rho(x) \\ &\quad - A^T(x)P - PA(x) - f_2(x)^T\rho(x) - 2\rho(x)f_2(x) \\ &\quad = 0\end{aligned} \tag{36}$$

The possibility of solving $\phi = 0$ sequentially, as stated above, justifies the choice of dynamical systems—systems separable into linear and nonlinear subdynamics. Linear subdynamics are treated acausally, and nonlinear subdynamics are treated causally. Such a treatment is discussed in [27] too. Determination of solution of ϕ_2 for $\rho(x)$ is a formidable task. This difficulty can be reduced by assuming specific forms of input matrix B and $\rho(x)$.

Assumption 4 For second-order nonlinear systems, as in (3), control inputs are required to be scalar, i.e., $B = [0 \ \ 1]^T$ or $B = [1 \ \ 0]^T$.

Remark 2 Matrix $P + \rho(x)$ can be dominated by matrix P to render $P + \rho(x)$ positive-definite.

Typical second-order nonlinear systems include a pendulum system and a spring-mass system as given in [28]. Thus, a general second-order nonlinear system with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $A(x) = \begin{bmatrix} a_x & b_x \\ c_x & d_x \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ and $\rho(x) = \begin{bmatrix} 0 & \rho_2 \\ \rho_2 & \rho_3 \end{bmatrix}$; the resulting equations in terms of ρ_2 and ρ_3,
respectively, are

$$\frac{9\rho_2^2}{4} + 3p_2\rho_2 - 2p_1a_x - 3c\rho_2 - 2p_2c_x = 0 \qquad (37)$$

$$\frac{9\rho_3^2}{4} + 3p_3\rho_3 - 2d\rho_3 - 2p_2b_x - 3b\rho_2 - d\rho_3 - 2p_3d_x = 0 \qquad (38)$$

Assumptions 4 and 5 lead to a modified form of (28) with respect to the considered second-order nonlinear system as shown below

$$u = -\left(p_2 + \frac{3}{2}\rho_2\right)x_1 - \left(p_3 + \frac{3}{2}\rho_3\right)x_2 \qquad (39)$$

where ρ_2 and ρ_3 are solutions of (37) and (38), respectively.

The aforementioned proposition is applicable to higher-dimensional nonlinear systems too. For example, a three-dimensional system with a scalar inputs $B = [0\ 0\ 1]^T$, $B = [0\ \ 1\ \ 0]^T$, and $B = [1\ \ 0\ \ 0]^T$ will, respectively, have the forms $\rho(x) = \begin{bmatrix} 0 & 0 & \rho_3 \\ 0 & 0 & \rho_5 \\ \rho_3 & \rho_5 & \rho_6 \end{bmatrix}$, $\rho(x) = \begin{bmatrix} 0 & \rho_2 & 0 \\ \rho_2 & \rho_4 & \rho_5 \\ 0 & \rho_5 & 0 \end{bmatrix}$, and $\rho(x) = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 \\ \rho_2 & 0 & 0 \\ \rho_3 & 0 & 0 \end{bmatrix}$. We now show the optimality and stabilizing capability of the proposed controller as in (39) through a motivating example

4.2 Motivating Example

Lets consider stabilization problem for the following nonlinear undamped spring-mass system,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_1^3 + u, \qquad (40)$$

and assume the following quadratic cost functional

$$J = \int_0^\infty (q_1x_1^2 + q_2x_2^2 + Ru^2)dt \qquad (41)$$

and

$$P + \rho(x) = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 & \rho_2 \\ \rho_2 & \rho_3 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 + \rho_2 \\ p_2 + \rho_2 & p_3 + \rho_3 \end{bmatrix}. \qquad (42)$$

Decomposition of the system into a linear part and a nonlinear part results in

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_1^2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u. \qquad (43)$$

Now, solution of $\phi_1 = 0$ is

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}. \qquad (44)$$

Solving $\phi_2 = 0$, solutions ρ_2 and ρ_3 from equations (37) and (38), respectively, can be obtained as follows

$$\rho_2 = \frac{2}{3}\left(-1 + x_1^2 - p_2 + \sqrt{p_2^2 + 2p_2 + x_1^4 - 2x_1^2 + 1}\right) \qquad (45)$$

$$\rho_3 = \frac{2}{3}\left(-p_3 + \sqrt{p_3^2 + 3\rho_2}\right). \qquad (46)$$

Resulting stabilizing controller, through (39), is given as

$$u = -\left\{-1 + x_1^2 + \sqrt{p_2^2 + 2p_2 + x_1^4 - 2x_1^2 + 1}\right\}x_1 - \left\{\sqrt{p_3^2 + 2(-1 + x_1^2 - p_2 + \sqrt{p_2^2 + 2p_2 + x_1^4 - 2x_1^2 + 1})}\right\}x_2 \qquad (47)$$

$$u = x_1 - x_1^3 - \left\{\sqrt{p_2^2 + 2p_2 + x_1^4 - 2x_1^2 + 1}\right\}x_1 - \left\{\sqrt{p_3^2 + 2(-1 + x_1^2 - p_2 + \sqrt{p_2^2 + 2p_2 + x_1^4 - 2x_1^2 + 1})}\right\}x_2 \qquad (48)$$

Remark 3 The aforementioned control law in (48) has a feedback linearizing part in the form of $x_1 - x_1^3$ for the system in (40).

It is now in order to evaluate the proposed approximate value function. This value function is given as

$$V(x_1, x_2) = [x_1\ \ x_2]\begin{bmatrix} p_1 & -1+x_1^2+\sqrt{p_2^2+2p_2+x_1^4-2x_1^2+1} \\ -1+x_1^2+\sqrt{p_2^2+2p_2+x_1^4-2x_1^2+1} & \sqrt{p_3^2+2(-1+x_1^2-p_2+\sqrt{p_2^2+2p_2+x_1^4-2x_1^2+1})} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (49)$$

Figure (2) shows comparison of value functions obtained through the proposed approximation and through linearization of system in (40) at origin. States are in ranges $-1 \leq x_1 \leq 1$ and $-1 \leq x_2 \leq 1$ with state penalizations $q1 = 1$ and $q2 = 1$. It is interesting to find the error between these two value functions which is defined as

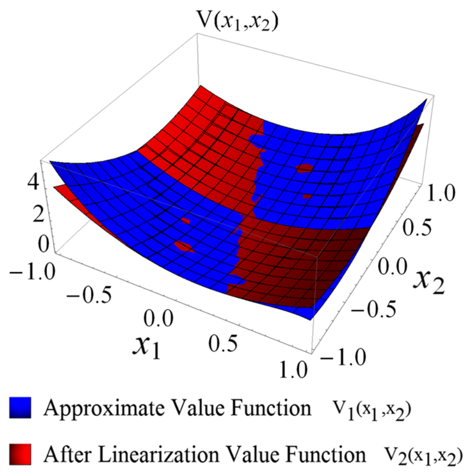

Fig. 2 Comparison of value functions obtained through the proposed approximation and after linearization of system in (40) in ranges $-1 \leq x_1 \leq 1$ and $-1 \leq x_2 \leq 1$ with state penalizations $q1 = 1$ and $q2 = 1$

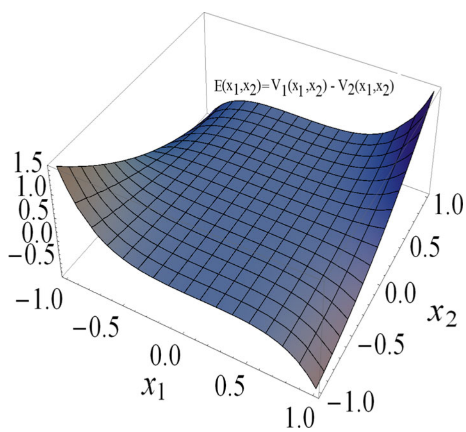

Fig. 3 Error between value functions obtained through the proposed approximation and after linearization of system in (40). Error is zero around the origin and increases when state trajectories are away

$E(x_1, x_2) = V(x_1, x_2)|_{approximate} - V(x_1, x_2)|_{linearized}$. Figure (3) shows the error $E(x_1, x_2)$ between value functions obtained through the proposed approximation and after linearization of system in (40).

5 Tracking/Regulation and Estimation

Both tracking/regulation and estimation can be achieved through the proposed approach explained in detail in the previous section. It is to be noted that the AE to be solved for tracking/regulation will have a higher dimension than that which is to be solved for estimation. This is due to the incorporation of internal model of the reference signal to achieve tracking/regulation. We proceed with tracking/regulation and later discuss estimation.

5.1 Tracker/Regulator Design

This section explains how tracking/regulation can be achieved within the proposed suboptimal stabilization approach. Since the reference signals such as step-input, ramp-input, and sinusoidal-input are generated by linear exosystems, their respective internal models can be straightforwardly augmented into the linear part of a decomposed system. These reference signals can all be generated by the following simple dynamical exosystem,

$$r^d + a_1 r^{(d-1)} + \cdots + a_d r = 0, \text{ and} \tag{50}$$

$$\dot{w} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ -a_d & \cdots & -a_2 & -a_1 \end{bmatrix} w = Mw \tag{51}$$

$$r = [1\ 0\ \cdots]w \tag{52}$$

where $M \in R^d$. Augmentation of the above mentioned internal model can be done using the following state-space model

$$\dot{\chi} = A^*\chi + A^*(\chi)\chi + B * u \tag{53}$$

$$y = C\chi \tag{54}$$

where $\chi \in R^{n+d}$ and $u \in R^{m+d}$ and partitioned matrices

$$A^* = \left[\begin{array}{c|c} M & \begin{array}{c} 0_{1\times n} \\ -H \end{array} \\ \hline 0_{n\times n} & A \end{array}\right], A^*(\chi) = \begin{bmatrix} 0 & 0 \\ 0 & A(x) \end{bmatrix}, B^* = \begin{bmatrix} 0 \\ B \end{bmatrix},$$

and $C^* = \begin{bmatrix} C & 0 \end{bmatrix}$

The proposition is now applied on the nonlinear undamped spring-mass system to enable one of its outputs track a sinusoidal reference signal. Let's introduce the error dynamics as

$$\dot{\chi}_1 = \chi_2$$
$$\dot{\chi}_2 = -\chi_1 + \chi_1^3 + u$$
$$\lim_{t\to\infty} e(t) = r(t) - \chi_1(t) = 0 \tag{55}$$

The task is to make the state track, i.e., $\chi_1(t) \to r(t) = 0.5sin(t)$, where $\omega = 1rad/s$ and $f = \frac{\omega}{2\pi} = .159$ Hz. The quadratic cost functional is given as

$$J = \int_0^\infty (q_1\chi_1^2 + q_2\chi_2^2 + q_{i1}\chi_{i1}^2 + q_{i2}\chi_{i2}^2 + Ru^2)dt, \tag{56}$$

where q_{i1} and q_{i2} are state penalizations for internal model and q_1 and q_2 are state penalizations for system. Decomposition and augmentation of internal model lead to the following system dynamics

$$\begin{bmatrix} \dot{\chi}_{i1} \\ \dot{\chi}_{i2} \\ \dot{\chi}_1 \\ \dot{\chi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \chi_{i1} \\ \chi_{i2} \\ \chi_1 \\ \chi_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \chi_1^2 & 0 \end{bmatrix} \begin{bmatrix} \chi_{i1} \\ \chi_{i2} \\ \chi_1 \\ \chi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} r \quad (57)$$

Solving $\phi_1 = 0$ for the nonlinear system in the above form would result in

$$P = \begin{bmatrix} p_1 & p_2 & p_3 & p_7 \\ p_2 & p_4 & p_5 & p_8 \\ p_3 & p_5 & p_6 & p_9 \\ p_7 & p_8 & p_9 & p_{10} \end{bmatrix} \quad (58)$$

To solve $\phi_2 = 0$, we choose

$$P + \rho(x) = \begin{bmatrix} p_1 & p_2 & p_3 & p_7 \\ p_2 & p_4 & p_5 & p_8 \\ p_3 & p_5 & p_6 & p_9 \\ p_7 & p_8 & p_9 & p_{10} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \rho_7 \\ 0 & 0 & 0 & \rho_8 \\ 0 & 0 & 0 & \rho_9 \\ \rho_7 & \rho_8 & \rho_9 & \rho_{10} \end{bmatrix} \quad (59)$$

Solving $\phi_2 = 0$ results in $\rho_7 = 0$, $\rho_8 = 0$,

$$\rho_9 = \frac{2}{3}\Big(-1 - p_9 + \chi_1^2 + \sqrt{p_9^2 + 2p_9 + \chi_1^4 - 2\chi_1^2 + 1}\Big),$$

$$\text{and} \quad \rho_{10} = \frac{2}{3}\Big(-p_{10} + \sqrt{p_{10}^2 + 3\rho_9}\Big). \quad (60)$$

Table 1 Three cases of state-feedback with corresponding state penalizations and gains

Cases	State penalizations q_{i1}, q_{i2}, q_1, q_2	Controller gains p_7, p_8, p_9, p_{10}
Case I	10,10,10,10	1.5, − 4.2, 2.6, 2.4
Case II	1,1,50,50	0.8, − 1.1, 3.5, 5.8
Case III	1,1,100,100	0.9, − 1.0, 5.7, 8.7

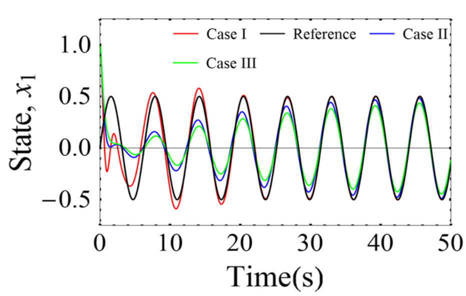

Fig. 4 Responses of state x_1 for cases I, II, and III of state-feedback. Initial values of tracked state is $x_1(0) = 1$

Thus, the optimal control law for tracking is given as

$$\begin{aligned} u = &- p_7\chi_{i1} - p_8\chi_{i2} + \chi_1 - \chi_1^3 \\ &- \left\{\sqrt{p_9^2 + 2p_9 + \chi_1^4 - 2\chi_1^2 + 1}\right\}\chi_1 \\ &- \left\{\sqrt{p_{10}^2 + 2(-1 - p_9 + \chi_1^2 + \sqrt{p_9^2 + 2p_9 + \chi_1^4 - 2\chi_1^2 + 1})}\right\}\chi_2 \end{aligned} \quad (61)$$

We describe three cases of controller gains in state-feedback, in Table 1, which will be used to evaluate corresponding transient and steady-states responses. Figure (4) shows state response of x_1 for Case I, Case II, and Case III of controller gains.

5.2 Estimator Design

Linear quadratic regulation and linear quadratic estimation problems are dual of each other, i.e., the same ARE with slight modifications can be used to deal with both the problems. In Sect. 4, the AE in (32) was solved to obtain a stabilizing controller. A dual of AE in (32) is now solved for state(s) estimation of considered class of systems. The state-space representation of the observer, with respect to the unaugmented system in (1), can be given as $\dot{\hat{x}} = (A(\hat{x}) + A(t))^T\hat{x} + C^T\tilde{x}$, where $\hat{x}$ and $\tilde{x}$ are estimated state vector and error vector $(x - \hat{x})$, respectively. An assumed quadratic cost functional is $J_o = \int_{t_0}^{T}(\hat{x}^T Q_o \hat{x} + \tilde{x}^T R_o \tilde{x})dt$ where $Q_o = \begin{bmatrix} q_{o1} & 0 \\ 0 & q_{o2} \end{bmatrix}$. Thus, the ARE to estimate states can be given as

$$A\Sigma + \Sigma A^T + Q_o - \Sigma C^T R_o^{-1} C \Sigma = 0 \quad (62)$$

where if the pair $\{A, C\}$ is observable then this ARE has a s.p.d. solution Σ. Matrices Q_0 and R_0 can be used as tuning parameters for the quality of state(s) estimation.

For nonlinear systems whose dynamics can be decomposed into an observable linear subdynamics, observer can be designed exactly the same way that the regulator was designed. Therefore, for the decomposable system dynamics $(A + A(\hat{x}))\hat{x}$, the solution matrix Σ is also modifiable to be $\Sigma + \sigma(\hat{x})$ with $\sigma(\hat{x}$ assumed as an s.i. matrix. The resulting observer is designed as follows

$$\dot{\hat{x}} = A\hat{x} + A(\hat{x})\hat{x} + Bu + L(y - \hat{y}), \quad (63)$$

where $\hat{x} \in R^n$. Similar observers are reported in [29] and [30]. Closed-loop system dynamics along with the observer is shown in Fig. (5). Now $L = (\Sigma + \sigma(\hat{x}))C^T R^{-1}$. Matrices Σ and $\sigma(\hat{x})$ can also be obtained sequentially as was done in stabilization and tracking. The algebraic observer equation, being dual of (36), is given as

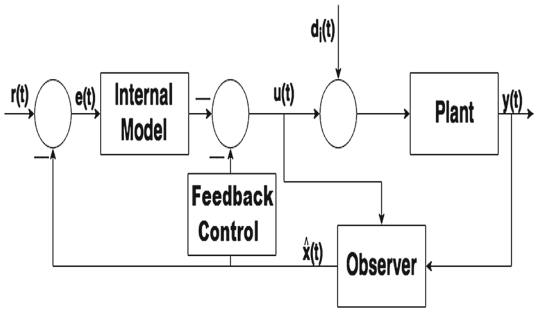

Fig. 5 Observer-based closed-loop system

$$f_2(\hat{x})(\Sigma + \sigma(\hat{x})) + (\Sigma + 2\sigma(\hat{x}))f_2(\hat{x})^T + Q_0 - \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right) C^T R_0^{-1} C \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right) = 0 \qquad (64)$$

As an example, we design an observer for the system given in (55). It is to be noted that the observer is to be designed for the unaugmented system. Thus, the stabilizing solution of (64) for (55) is given as

$$\Sigma_0 = \Sigma + \sigma(\hat{x}) = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_3 \end{bmatrix} + \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \qquad (65)$$

where

$$\sigma_1 = \frac{2}{3}\left(- s_1 + \sqrt{s_1^2 + 3\sigma_2}\right) \qquad (66)$$

and

$$\sigma_2 = \frac{2}{3}\left(- 1 + \hat{x}_1^2 - s_2 + \sqrt{s_2^2 + 2s_2 + x_1^2 - 2x_1 + 1}\right) \qquad (67)$$

Subsequently, resulting gain L can be given as

$$L = (\Sigma + \sigma(\hat{x}))C^T R^{-1} \qquad (68)$$

Finally, the two cases of observers are shown in Table 2 which will be used in unconstrained- and constrained tracking. As for the unconstrained tracking, two cases of observer gains are used with the Case II of controller gains. Figure (6) shows responses of states x_1 and $\hat{x}_1$ for Case II and Case I of controller and observer gains, respectively. It can be seen that the estimated state not only converges relatively late but also worsens the transient response of x_1. Figure (7) shows responses of states x_1 and $\hat{x}_1$ for Case II and Case II of controller and observer gains, respectively. It is obvious that convergence time is reduced as well as an improvement in transient response is seen. Finally, Figure (8) shows the phase portrait of convergence of estimated state trajectories to actual state trajectories.

Table 2 Two cases of output-feedback with corresponding state penalizations and gains

Cases	State penalizations q_{01}, q_{02}	Observer gains s_1, s_2
Case I	10,10	2.3, − 2.1
Case II	100,100	9.6, − 3.5

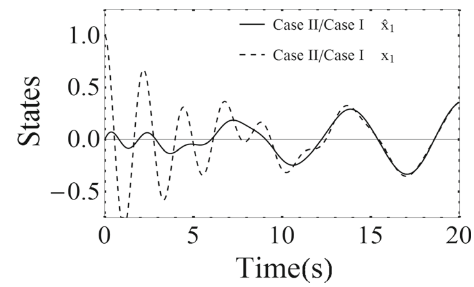

Fig. 6 Responses of state x_1 and estimated state $\hat{x}$ for Case II(Controller)/Case I(Observer) gains

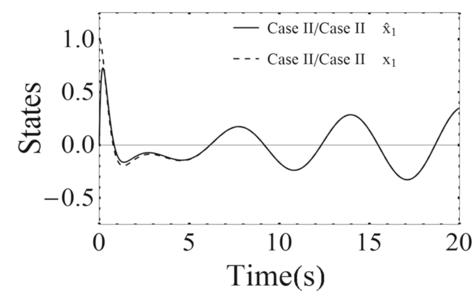

Fig. 7 Responses of state x_1 and estimated state $\hat{x}$ for Case II(Controller)/Case II(observer) gains

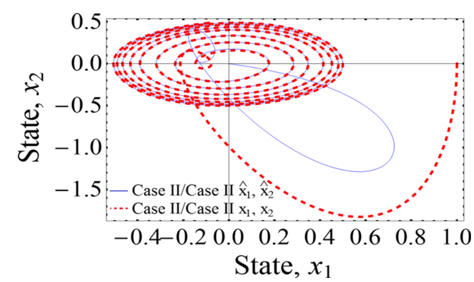

Fig. 8 Convergence of states $\hat{x}_1$ to x_1 and $\hat{x}_2$ to x_2 for Case II(controller)/Case II(observer) gains

6 Integral Windup Compensation

This section discusses compensation of integral windup effect in output tracking of nonlinear systems at sinusoidal reference signals. Integral windup due to actuator saturation results in a deteriorated transient response. Severe actuator saturation disturbs the steady-state response too in the form of reduced steady-state accuracy. The proposed approach can compensate for a deteriorated transient response provided the actuator saturation is not severe enough to affect the steady-state response.

Conventional methods of anti-windup compensation are conditional integration, back calculations, and observer-based anti-windup. Conditional integration is easy to be applied on controllers; however, it comes with problem of chattering. Back calculations are applicable to only those controllers which are driven by PID signal. Observer-based compensation is applicable to those controllers for which observers are easily designed. Interested reader is referred to [31] for a more elaborate explanation.

One way of looking at a deteriorated transient response is to consider a disturbance signal entering from plant input and having magnitude and frequency comparable to those of the sinusoidal reference signal. This disturbance signal gets propagated in the closed loop and appears at the input of the internal model too. Fourier transform of the signal appearing at the input verifies this consideration as shown in simulation results. The effect of an input disturbance is usually mitigated through incorporating an extra pole, having the same frequency as that of disturbance, in the transfer function of internal model. This new pole appears as a zero in numerator of transfer function of closed-loop system, thereby allowing for possible cancelation of pole generated by input disturbance. However, addition of pole(s) in internal model means addition of integrator(s) which only makes the problem worse as windup occurs due to the presence of integrators in feedback loop.

We propose frequency domain anti-windup compensation. The proposed compensation eliminates the undesired frequency by modifying the filtering characteristics of internal model. Knowing that increasing the number of poles in internal model will only make the problem worse, we alter the filtering characteristic of internal model by relocating its zero(s). A practical means of this relocation is through the choice of state penalization q_{i1}, q_{i2}, q_1, and q_2. Since windup deteriorates the transient response of the tracked state x_1, this state has to be penalized more than others . Actual system states, x_1 and x_2, are penalized through q_1 and q_2, respectively, in the cost functional. As a result, location of poles and zeros corresponding to the three cases of controller gains is given in following Table 3.

Table 3 State penalization with corresponding locations of pole and zero

Case	q_{i1}, q_{i2}, q_1, q_2	Zero (Hz)	Pole (Hz)
I	10,10,10,10	0.45	$-0.159i$,$0.159i$
II	1,1,50,50	0.22	$-0.159i$,$0.159i$
III	1,1,100,100	0.18	$-0.159i$,$0.159i$

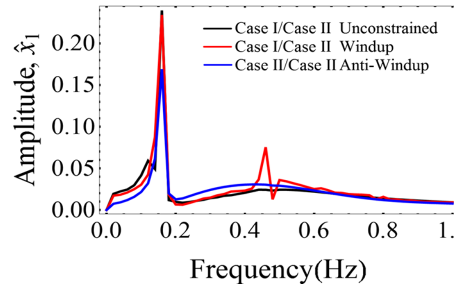

Fig. 9 FFT of $\hat{x}_1$ for unconstrained, windup and anti-windup tracking

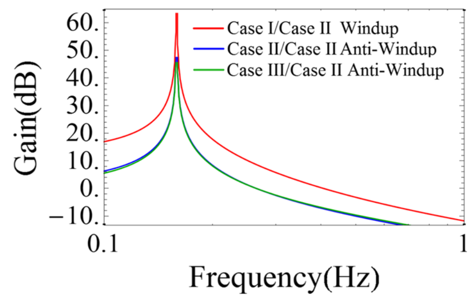

Fig. 10 Filter characteristics of internal model for a windup and two anti-windup trackings

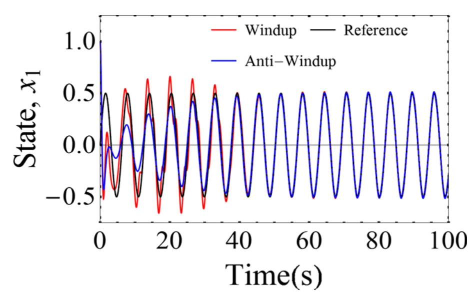

Fig. 11 Response of tracked state x_1 showing overshoots in windup and no-overshoots in anti-windup

The actuator saturation limit is set as ± 3.5 above and below which the undesired frequency does not have enough magnitude to hinder either the transient response or the steady-state response. At exactly this saturation limit, only the transient response gets worsened in form of overshoots. A saturation limit set within the range ± 3.5 and closer to 0 results in worsening of both transient and steady-state response beyond any compensation. Coming to the simulation results, Figure (9) shows the Fourier transform of the observed state $\hat{x}_1$ for unconstrained, windup and anti-windup trackings. In case of windup, an additional harmonic occurs at 0.47 Hz which is eliminated in anti-windup compensation. Figure (10) shows the Bode plot of the gain of filtering characteristics of internal model. It is obvious that Cases II & III of controller gains provide better attenuation of the disturbance than Case I. Figure (11) shows the overshoots experienced by the tracked state x_1 due to

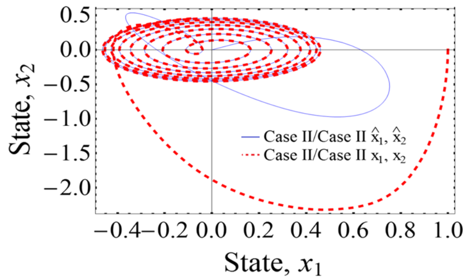

Fig. 12 Convergence of states $\hat{x}_1$ to x_1 and $\hat{x}_2$ to x_2 for Case II(controller)/Case II(observer) gains

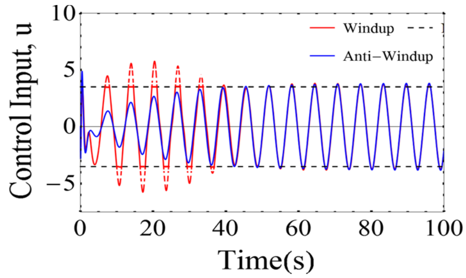

Fig. 13 Control signal u experiencing saturation in windup tracking and no-saturation in anti-windup tracking. Saturation limit is ± 3.5

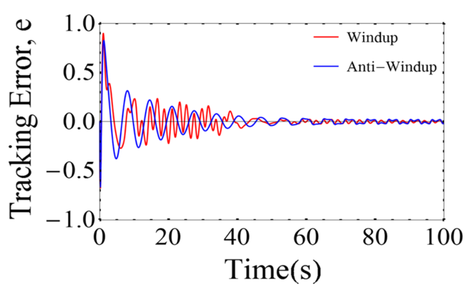

Fig. 14 Transient and steady-state responses of error $e = r - x_1$ for windup tracking and anti-windup tracking

windup and the suppression of these overshoots through anti-windup compensation. Figure (12) shows the phase portrait of convergence of estimated state trajectories to actual state trajectories with anti-windup compensation. Figure (13) shows the control signal experiencing saturation at the limit ± 3.5. Anti-windup compensation eliminates this clipping of control signal. In the end, Figure (14) shows the exponential stability achieved through anti-windup compensation compared with only the asymptotic stability with windup.

7 Stability Analysis of Closed-Loop System

The stability of the closed-loop system can be established by two different methods [32]. In the classical approach, a control system is designed and then it is checked for stability. In an opposite approach, conditions of stability are described first and then a controller is designed within limits of these conditions. We have used the latter method in our work as shown in Sects. 4 and 5 so the proposed regulator and estimator are already stable. But for the sake of completeness, we use the classical approach to determine the stability of the closed-loop system, independently with state-feedback and output-feedback.

7.1 State-Feedback

The Lyapunov function is chosen as $V = \chi^T (P + \rho(\chi))\chi = \chi^T P_r \chi$ where $P > 0$ and $P(\chi) \geq 0$. Proceeding by taking the time derivative

$$\begin{aligned}\dot{V} &= \dot{\chi}^T P_r x + x^T P_r \dot{\chi} + \chi^T \dot{\rho(\chi)}\chi \\ &= \chi^T \{f_2^T(\chi) P_r + P_r f_2(\chi)\}\chi + 2B^T P_r \chi u \\ &\quad + \chi^T \rho(\chi) f_2(\chi)\chi + \chi^T \rho(\chi) B u \\ &= \chi^T \{f_2^T(\chi) P_r + (P + 2\rho(\chi)) f_2(\chi)\}\chi \\ &\quad + B^T (2P + 3\rho(\chi))\chi u \end{aligned} \tag{69}$$

Inserting $u = -R^{-1} B^T (P + \frac{3}{2}\rho(\chi))\chi$ and adding and subtracting $\chi^T Q \chi$ above give

$$\begin{aligned} &= \chi^T \{f_2^T(\chi) P_r + (P + 2\rho(\chi)) f_2(\chi) + Q\}\chi \\ &\quad - 2\chi^T \left\{\left(P + \frac{3}{2}\rho(\chi)\right) B R^{-1} B^T \left(P + \frac{3}{2}\rho(\chi)\right)\right\}\chi \\ &\quad - \chi^T Q \chi \\ &= \chi^T \{f_2^T(\chi) P_r + (P + 2\rho(\chi)) f_2(\chi) + Q\}\chi \\ &\quad - \chi^T \left\{\left(P + \frac{3}{2}\rho(\chi)\right) B R^{-1} B^T \left(P + \frac{3}{2}\rho(\chi)\right)\right\}\chi \\ &\quad - \chi^T \left\{Q + \left(P + \frac{3}{2}\rho(\chi)\right) B R^{-1} B^T \left(P + \frac{3}{2}\rho(\chi)\right)\right\}\chi \end{aligned} \tag{70}$$

The arguments inside the brackets of the first two terms in the foregoing expression form ϕ. Since $\phi = 0$ to obtain a minimum according to (36), thus

$$\dot{V} = -\chi^T \underbrace{\left\{Q + (P + \frac{3}{2}\rho(\chi)) B R^{-1} B^T \left(P + \frac{3}{2}\rho(\chi)\right)\right\}}_{>0}\chi \tag{71}$$

Since the time derivative of the Lyapunov function turns out to be negative-definite, the closed-loop system with state-feedback is asymptotically stable.

7.2 Output-Feedback

The Lypunov function for the closed-loop system with output-feedback is assumed to be as $V = \chi^T P_r \chi + \hat{x}^T \Sigma_0 \hat{x}^T$.

Stability of the closed-loop system with output-feedback will be established following the same line that was adopted for the state-feedback. We use the system in (55) without the loss of generality for determination of closed-loop stability with output-feedback. It is to be noted that the control law will now be dependent on estimated state(s), i.e., $u = -R^{-1}B^T(P + \frac{3}{2}\rho(\chi))x^\dagger$, where $x^\dagger = [\chi_{i1} \quad \chi_{i2} \quad \hat{x}_1 \quad \hat{x}_2]^T$. Since we know that

$$\begin{bmatrix} \chi_{i1} \\ \chi_{i2} \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} \chi_{i1} \\ \chi_{i2} \\ x_1 - \tilde{x}_1 \\ x_2 - \tilde{x}_2 \end{bmatrix}. \tag{72}$$

Hence, a bit of manipulation leads to

$$\begin{aligned} \dot{V} = &-\chi^T \left\{ Q + \left(P + \frac{3}{2}\rho(\chi)\right) BR^{-1}B^T \left(P + \frac{3}{2}\rho(\chi)\right)\right\}\chi \\ &- \hat{x}^T \left\{ Q_0 + \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right) C^T R_0^{-1} C \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right)\right\}\hat{x} \\ &+ \chi^T \left(P + \frac{3}{2}\rho(\chi)\right) BR^{-1}B^T \left(P + \frac{3}{2}\rho(\chi)\right) \begin{bmatrix} 0 \\ 0 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \end{aligned} \tag{73}$$

It is to be noted that the third term reduces the negativity in the above equation and may cause instability of the closed-loop system because, (i) $(P + \frac{3}{2}\rho(\chi))BR^{-1}B^T(P + \frac{3}{2}\rho(\chi))$ is positive by construction, (ii) state χ and error $\begin{bmatrix} 0 \\ 0 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$ vectors may become positive when actual states are more positive than the estimated states, i.e., $x > \hat{x}$. Such an unstable closed-loop system can be stabilized back by increasing either the gains P or Σ through tuning factors Q and R or Q_0 and R_0, respectively. Stabilization can also be achieved if the initial errors are taken small enough, i.e., $\tilde{x}_1 \approx 0$ and $\tilde{x}_2 \approx 0$ rendering the effect of third term negligible.

7.3 With Saturation-Type Input Constraint

Actuator saturation in the present context is mathematically stated as $u = sat(-R^{-1}B^T(P + \frac{3}{2}\rho(\chi))\chi)$. Assuming that the saturation limit is labeled as η, then $u = sat(-R^{-1}B^T(P + \frac{3}{2}\rho(\chi))\chi) \implies sat(R^{-1}B^T(P + \frac{3}{2}\rho(\chi))\chi) = -|\eta|$. Inserting this saturated control input into (73) results in

$$\begin{aligned} \dot{V} = &-\chi^T Q\chi \\ &- \chi^T \left(P + \frac{3}{2}\rho(\chi)\right) B\, sat\left[R^{-1}B^T\left(P + \frac{3}{2}\rho(\chi)\right)\chi\right] \\ &- \hat{x}^T \left\{ Q_0 + \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right) C^T R_0^{-1} C \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right)\right\}\hat{x} \\ = &-\chi^T Q\chi - \chi^T \left(P + \frac{3}{2}\rho(\chi)\right) B\,(-|\eta|) \\ &- \hat{x}^T \left\{ Q_0 + \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right) C^T R_0^{-1} C \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right)\right\}\hat{x} \\ = &-\chi^T Q\chi + \chi^T (P + \frac{3}{2}\rho(\chi))B\,|\eta| \\ &- \hat{x}^T \left\{ Q_0 + \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right) C^T R_0^{-1} C \left(\Sigma + \frac{3}{2}\sigma(\hat{x})\right)\right\}\hat{x}. \end{aligned} \tag{74}$$

The second term in (74) shown above can potentially render the time derivative of the Lyapunov function nonnegative which in turn shows the instability of closed-loop dynamics for larger initial values of states for a particular saturation limit, i.e., η. The instability in actuality can happen due to the positive nature of this second term as it is being added that reduces the negativity of the expression because, (i) $(P + \frac{3}{2}\rho(\chi))B$ and $|\eta|$ are both positive, and (ii) χ can have positive initial values of all its states.

8 Conclusion

This paper presents output tracking/regulation of weakly nonlinear systems addressing the optimality and actuator saturation too. Hamilton–Jacobi–Bellman equation has been approximated as an algebraic equation which is solved for both stabilization and estimation just as an algebraic Riccati equation can be solved for both cases in linear settings. It is found out that reference signals such as a sinusoidal one are appropriate for tracking because their linear internal models are easily augmented in the linear part of the overall system. This conversion of a tracking/regulation problem into a stabilization problem allows for near-optimal asymptotic stability locally around the origin. Performance of state-feedback is recoverable through output-feedback too. The proposed approach also provides a frequency domain anti-windup compensation through relocation of internal model zeros. This relocation does not warrant any additional hardware as it is carried out by altering the penalization of system states in the cost functional corresponding to regulator design. As for the future, work can be pursued in three directions. First direction is exploration of stability and optimality in the presence of external disturbances. Second direction is to repeat the results after including the effects of unmodeled dynamics. Third direction can be hardware implementation of the research work on a practical problem.

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