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RESEARCH ARTICLE - ELECTRICAL ENGINEERING

Visual Recurrence Analysis of Chaotic and Regular Motion of a Multiple Pendulum System

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Abstract In this article, nonlinear dynamics of pendulum systems is studied. The system of governing differential equations is derived using the Euler–Lagrangian approach. The recurrence plot method has been used for the nonlinear dynamics analysis. The natural frequency of a lumped pendulum system is smaller than the natural frequency corresponding to the distributed system. It is also observed that the bottom pendulum is the most chaotic than the middle and the top pendulums. It is also shown that a triple pendulum system with distributed mass is more chaotic than corresponding to the lumped system.

Keywords Chaos · Recurrence plot · Pendulum system · Euler–Lagarangian

1 Introduction

Multiple pendulum system such as double pendulum (DP) or triple pendulum (TP) system is used for gait analysis, bionics of human body and robotic manipulators [1,2]. Awrejcewicz et.al have studied TP with lumped system for both numerically and experimentally. They have reported the evidence of hyper chaos in such a system [3,4].

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Recurrence plots (RP) [5,6] help in analyzing m-dimensional phase space trajectory by 2-D recurrences representations. Such re-occurrence of a state (i'th time) at j'th time is recognized within 2-D square matrix with ones and zeros dots, along time axes. RP is mathematically represented as [5]

$$R_{i,j} = \Theta(\varepsilon_i - ||x_i - x_j||), x_i \in \Re^m, i, j = 1, \ldots, N$$

where N = no. of states x_i, ε_i is a threshold distance, $||.||$ Euclidian norm and $\Theta(.)$ the Heaviside function.

Chaotic behavior of a DP has been widely studied experimentally as well as numerically [7]. A DP system shows regular motion at low initial energy, chaotic motion for intermediate energy and again regular motion at higher initial energy [8–11]. Nevertheless there is not much work in the literature on nonlinear behavior of the TP systems free from any external force.

Examples of chaotic systems include Lorenz system, Duffing force oscillator system etc [2]. More interestingly, multiple pendula such as DP and TP systems are the simplest mechanical models which show chaos in the absence of external force. The general requirement of a chaotic system is that the governing differential equation must be nonlinear and its phase form should be at least order of three. But all nonlinear systems do not show chaos [12]. As mentioned that a distinguished feature of a chaotic system depends on its initial condition. In other words, even a slight change in the initial condition of the differential equation, it results in large change in its solution.

2 Modeling of Pendulum

In the literature, mathematical derivations of the single and double link are reported widely for both lumped and distributed masses [1,13,14]. In lumped systems, mass is not considered to be distributed on the entire length of the pendulum, i.e., mass is lumped in to a certain location whereas in a distributed system, mass is considered to be distributed continuously throughout the length of the pendulum. The linearized governing differential equations of the triple pendulum with lumped mass (TPLM) is also reported in the literature [15]. The triple pendulum with distributed mass (TPDM) is shown in Fig 1.

The m_1, m_2, m_3 and l_1, l_2, l_3 are the mass and length of the top, middle and bottom pendulum, respectively. Further, θ_1, θ_2 and θ_3 represent angular displacement of the top, middle and the bottom pendulum, respectively.

The governing equation of motion of triple-link pendulum system is given by

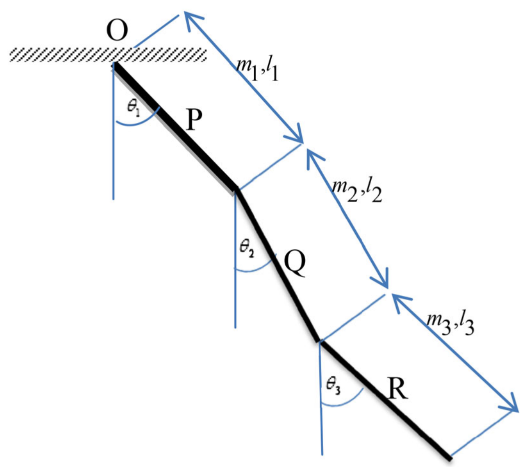


Fig. 1 Triple pendulum with distributed mass (TPDM)

$$\begin{aligned}
c_{11}\ddot{\theta}_1 + c_{12}\ddot{\theta}_2 + c_{13}\ddot{\theta}_3 + d_{11} &= 0 \\
c_{21}\ddot{\theta}_1 + c_{22}\ddot{\theta}_2 + c_{23}\ddot{\theta}_3 + d_{22} &= 0 \\
c_{31}\ddot{\theta}_1 + c_{32}\ddot{\theta}_2 + c_{33}\ddot{\theta}_3 + d_{33} &= 0
\end{aligned} \tag{2}$$

The derivation of coefficient of the various term used in Eq. (2) $c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, c_{33}, d_{11}, d_{22}, d_{33}$ is given in the literature [16].

3 Results and Discussion

Time series analysis and RP method has been used for the analysis of nonlinear dynamics.

3.1 Time Series Analysis

From the linear analysis, it can be concluded the natural frequency of the bottom pendulum is largest to any other pendulum of a pendula system [3]. Matlab ODE 45 solver has been used for obtaining the dynamics of triple-link pendulum. The nonlinear governing differential equations in Eq. (2) are simulated for a very small initial angular displacement of magnitude 10^{-6} degree to each pendulum in the TPDM from the rest. The initial condition of the TP in each simulation is represented as $[\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \theta_3, \dot{\theta}_3]$. This initial condition $[10^{-8}, 0, 10^{-8}, 0, 10^{-8}, 0]$ lies in the linear range. The time series plots in Fig. 2 show the increasing amplitudes of oscillations with time; thus, the TP system is unstable.

In general, a multiple pendulum system does not show chaotic behavior for small displacement. However, as initial displacement of the pendulum increases, there is a minimum value of displacement of the pendulum at which it begins to show chaos. The results in Fig. 3 show that paired time

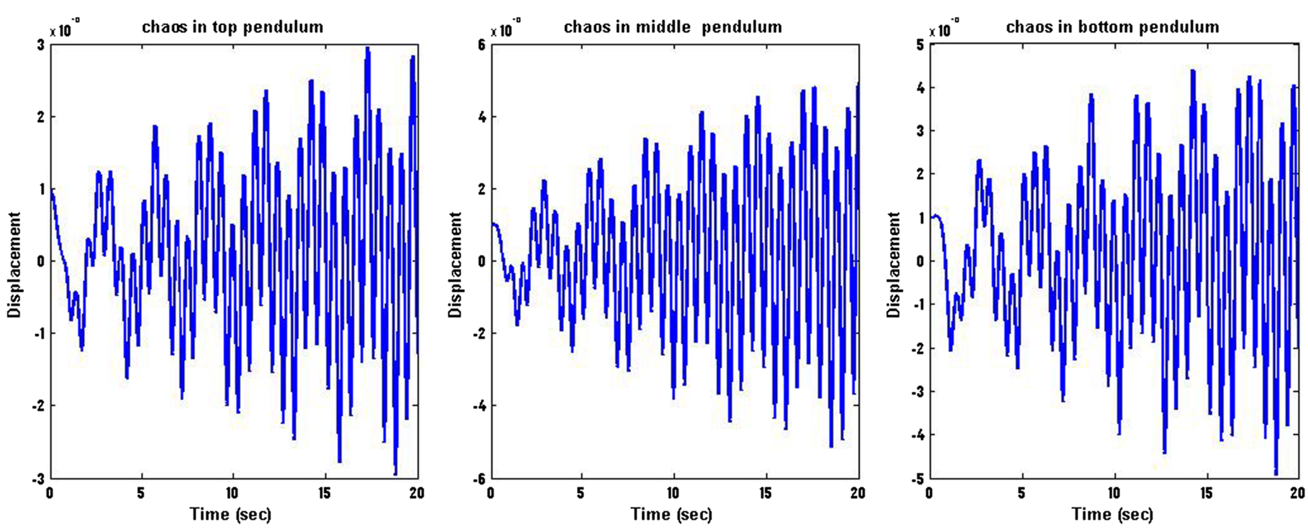


Fig. 2 Time series plot for top, middle and bottom pendulum of the TPDM with equal mass $m_1 = m_2 = m_3 = 1.0$ kg and equal length $l_1 = l_2 = l_3 = 1.0$ m of each pendulum for initial angular displacement 10^{-8} of each pendulum and angular velocity is zero, i.e., $[10^{-8}, 0, 10^{-8}, 0, 10^{-8}, 0]$

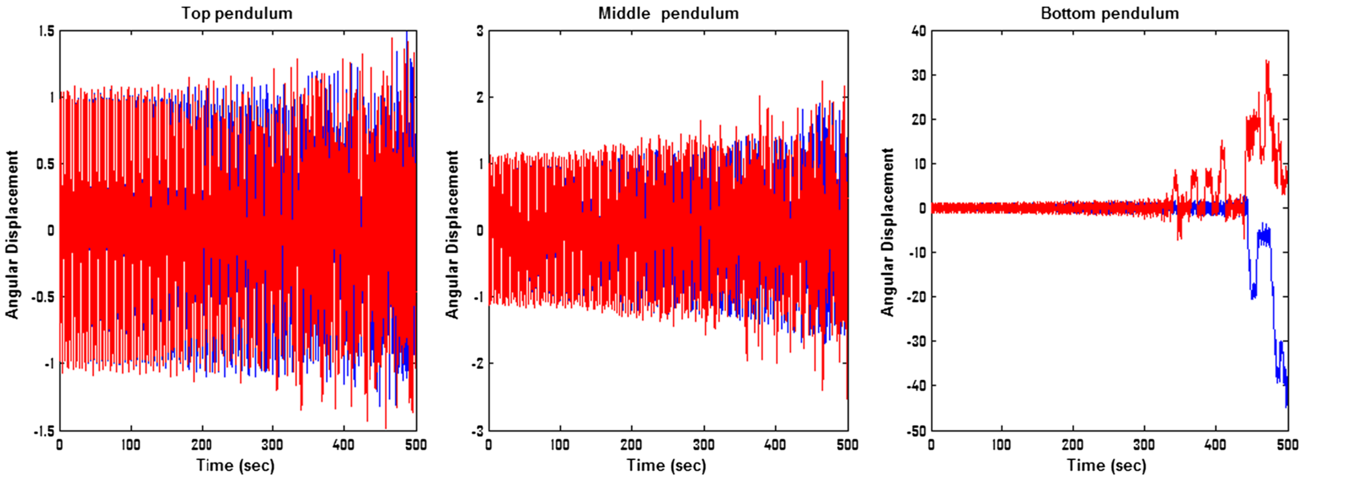


Fig. 3 Time series plots for the top, middle and the bottom pendulum of the simple TPDM system with equal mass $m_1 = m_2 = m_3 = 1.0$ kg and equal length $l_1 = l_2 = l_3 = 1.0$ m of each pendulum and the minimum angular displacement (θ_{ic}) of each pendulum is same and equal to 56° (*blue*) and ($56° + 10^{-8}$) (*red*)

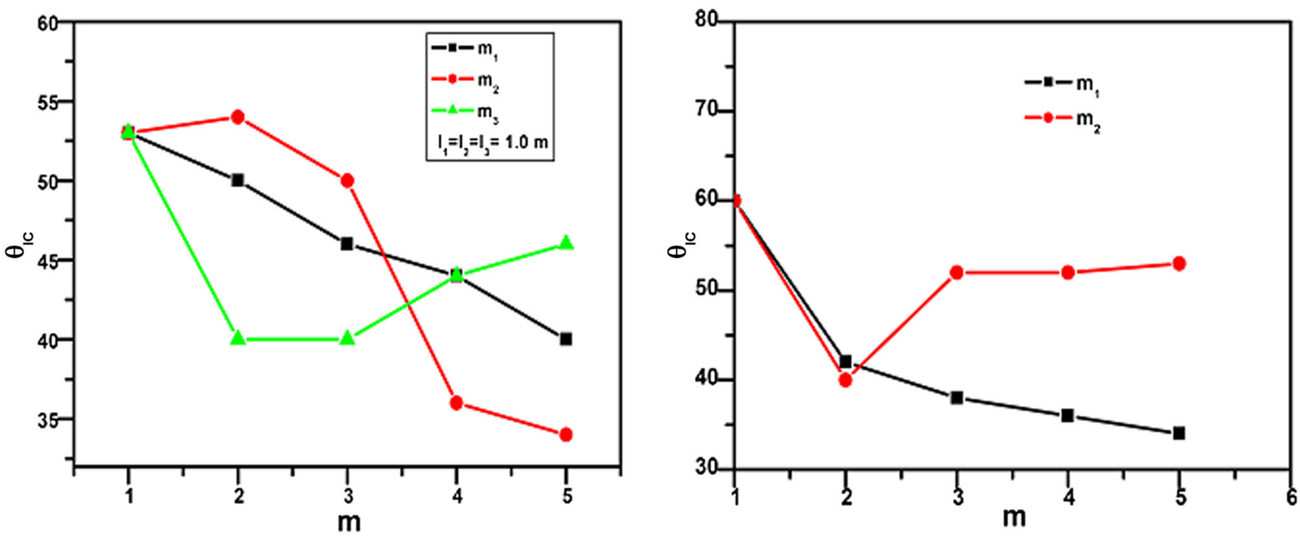


Fig. 4 Minimum initial angular displacement (θ_{ic}) required to initiate chaos in double pendulum (DP) and triple pendulum (TP) system for $l_1 = l_2 = 1.0$ m from rest

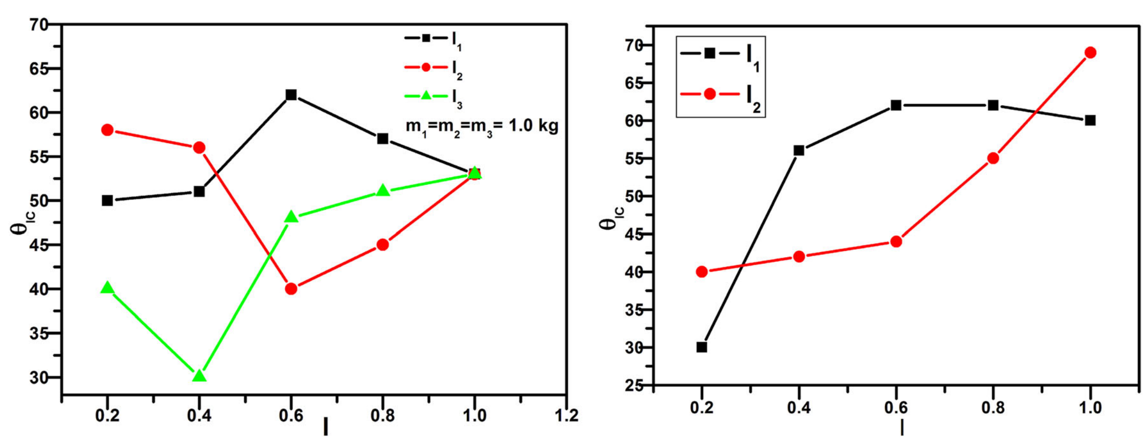


Fig. 5 Minimal angle for chaos θ_{ic} versus length of each pendulum in TPDM and DPDM for fixed mass of each pendulum

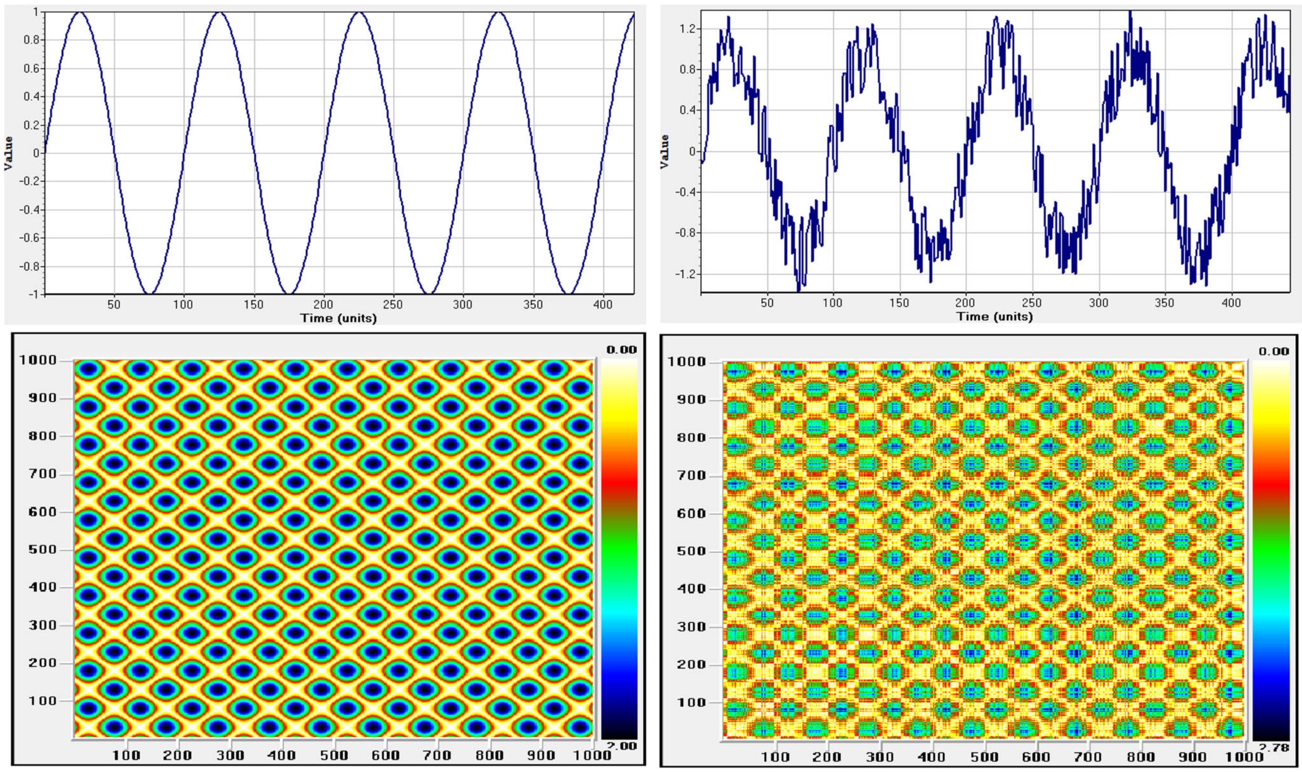


Fig. 6 Sample recurrence plot

Table 1 Recurrence plots and their meaning [6]

Observation	Interpretation
Homogeneity	Static process
Fading upper left and lower right corners	Dynamic; trend process
Disruptions	Dynamic; possible occurrence of transitions
Periodic/ quasi-periodic patterns	Cyclicties; quasi-periodic process reveals long diagonal lines with different distances to each other
Diagonal lines (parallel to the LOI)	The process could be chaotic if these diagonal lines occur beside single isolated points

series plots (blue and red colors) begin to differ at minimum angular displacement of 56°, i.e., [56°, 0, 56°, 0, 56°, 0] and [$56° + 10^{-8}$, 0, $56° + 10^{-8}$, 0, $56° + 10^{-8}$, 0]. This deviation is basically a signature of chaos in a dynamical system [2].

The minimum angular displacement (θ_{ic}) of the TPDM is also studied. For small angle, triple pendulum system shows periodic motion. When we increase the angle more than 50°, initially motion is periodic, then both the trajectories diverted from each other which leads to chaos [13]. The results establish that chaos in the TPDM occurs at smaller θ_{ic} than the TPLM. For example, θ_{ic} for the TPDM is found to be 56° and θ_{ic} for the TPLM is 79°. Similarly in the case of the DP systems, the DPDM begins to show chaos at $\theta_{ic} = 67°$ which is less than $\theta_{ic} = 78°$ for the DPLM.

The effect of pendulum mass on (θ_{ic}) is also investigated. The trend is quite erratic for instance, θ_{ic} decreases linearly owing to change in m_1 of the TPDM, θ_{ic} first increases and then decreases as m_2. Similarly, in the case of change in mass of the bottom pendulum m_3, θ_{ic} first decreases and then increases. Similar observation is also seen in the case of DPDM (Fig. 4).

The effect of pendulum length on (θ_{ic}) is also explored in the present study. Like the effect of mass of pendulum on θ_{ic}, change in length of pendulum does not show any definite trend on θ_{ic}. This observation is also found in the case of DPDM as shown in Fig. 5.

3.2 Recurrence Plot Analysis

RP is a modern method of nonlinear data exploration. The recurrence of states is quite distinctive for nonlinear dynamical or chaotic systems [16]. Historically, recurrence of states has been deliberated in analyzing cosmic-ray intensity [17]. Since recurrence plot are visual representation of system properties, the structures in a typical RP (large-scale and small-scale patterns) possess crude information [18]. For, e.g., RPs for pure sinusoidal series and noise mixed sinu-

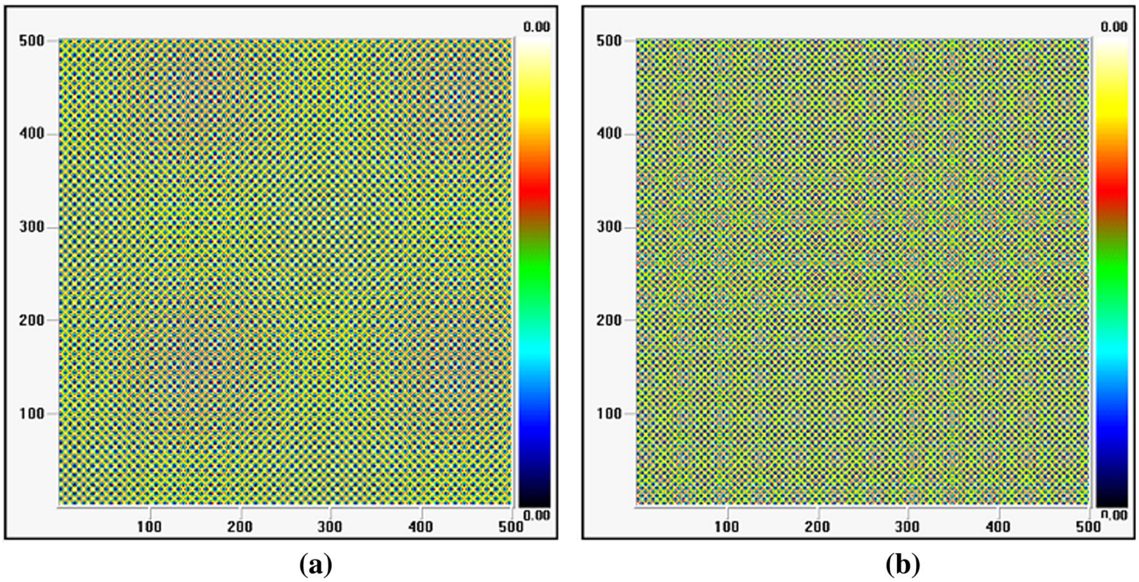


Fig. 7 Single-link. **a** Lumped analysis, **b** distributed analysis

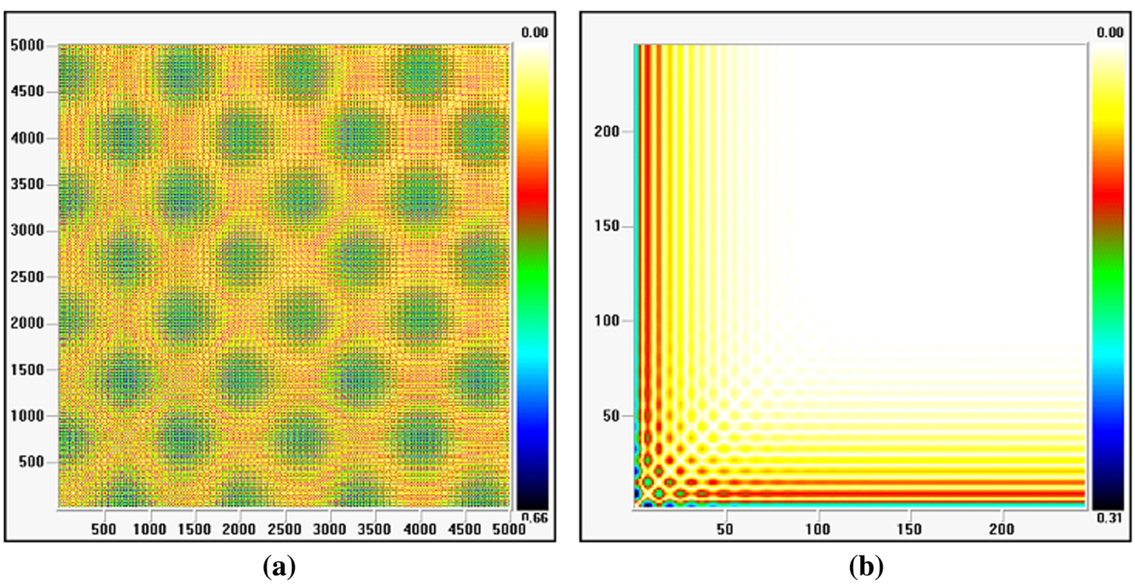


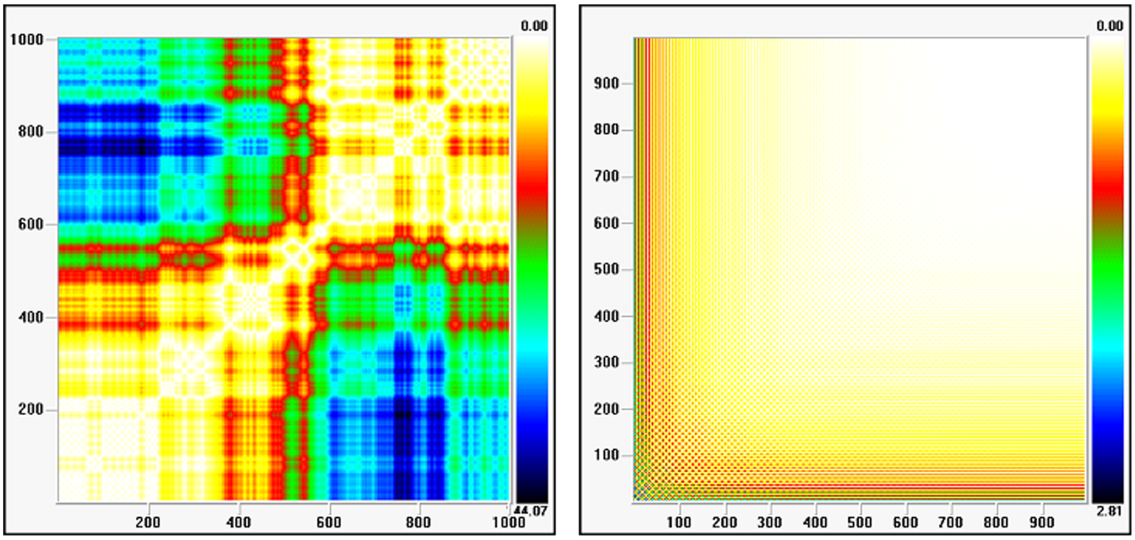
Fig. 8 Double-link. **a** Lumped analysis, **b** distributed analysis

soidal series are shown in Fig. 6. Assuming only noisy data being available for analysis, one can easily identify the hidden cyclicities present in the time series. Likewise, it becomes very easy to detect stationary, fluctuations, evolution, etc. for system states using RPs.

System dynamics for one- , two- and three-link triple pendulum, respectively, can be analyzed using RPs. The inferences from these plots significantly match with theoretical aspects.

With the help of Table 1 for single-link pendulum regular occurrences of the pattern shows the presence of periodicity. However, distributed analysis has slightly lesser periodicity in comparison to lumped analysis as shown in Fig. 7. RPs for lumped and distributed analysis of two-link pendulum are shown in Fig. 8. Presence of quasi-periodicity can be observed for distributed analysis along with a moderate periodicity for lumped analysis.

In the line of insight (LOI), the process could be chaotic if the diagonal lines occur beside single isolated points [6]. The same phenomenon can be found in triple-link recurrence plot for both lumped and distributed system as shown in Fig. 9. Another important feature from Table 1 is that if white fading occurs to the upper left and lower right corners then data will be non-stationary in nature, so when we move from lumped



(a) **(b)**

Fig. 9 Triple-link. **a** Lumped analysis, **b** distributed analysis

to distributed then system become more non-stationary in nature.

From the discussion it is clear that triple-link system is chaotic in nature which is validating the result obtained from time series analysis.

4 Conclusion

The present study establishes that natural frequency of a lumped pendula is smaller than corresponding to the distributed system. The nonlinear analysis based on time to chaos shows that tendency of chaos of a multiple pendula increases with DOF. It is also observed that bottom pendulum is the most chaotic than the middle and the top pendulum. Also distributed system is more chaotic than corresponding to the lumped system. Results obtained from RP method are similar to those obtained from classical method. In future, the same technique can be applied to analyze four-link pendulum which has application in gymnast robot and biped robot.

References

1. Zak, S.H.: Systems and Control. Oxford University Press, Oxford (2003)
2. Baker, G.; Gollub, J.: Chaotic Dynamics. Oxford Univeristy Press, Oxford (1990)
3. Awrejcewicz, J.; Kudra, G.; Wasilewski, G.: Experimental and numerical investigation of chaotic regions in the triple physical pendulum. Nonlinear Dyn. **50**, 755–766 (2007)
4. Awerjcewicz, J.; Supel, B.; Lamarque, C.; Wasilewski, G.; Kundra, G.; Olejnik, P.: Numerical and experimental study of regular and chaotic motion of physical pendulum. Int. J. Bifurcat. Chaos **18**, 2883 (2009)
5. www.recurrence-plot.tk
6. Marwan, N.; Romano, M.C.; Thiel, M.; Kurths, J.: Recurrence plots for the analysis of complex systems. Phys. Rep. **438**(5–6), 237–329 (2007)
7. Stachowiak, T.; Okada, T.: A numerical analysis of chaos in the double pendulum. Chaos Solitons Fractals **29**, 417–422 (2006)
8. Ohlhoff, A.; Richter, P.H.: Forces in the double pendulum. ZAMM **80**, 517–534 (2000)
9. Rafat, M.Z.; Wheatland, M.S.; Bedding, T.R.: Dynamics of a double pendulum with distributed mass. Am. J. Phys. **77**, 216–223 (2006)
10. Levin, R.B.; Tan, S.M.: Double pendulum: an experiment with chaos. Am. J. Phys. **61**, 1038–1044 (1993)
11. Shinbrot, T.; Grebogi, C.; Wisdom, J.; Yorke, J.A.: Chaos in a double pendulum. Am. J. Phys. **60**, 491–499 (1992)
12. Zhu, Q.; Ishitobi, M.: Experimental Study of Chaos in a driven triple pendulum. J. Sound Vib. **227**(1), 230–238 (1999)
13. Gupta, M. K.; Gupta, V.; Singh, A. K.: Linear and Non- Linear Dynamics of Double Pendula with Distributed Mass, Sixth International Conference on Theoretical, Applied, Computational and Experimental Mechanics (ICTACEM) at IIT Kharagpur 29–31 Dec.(2014)
14. Gupta, M. K.; Bansal, K.; Singh, A.K.: Mass and Length Dependent Chaotic Behavior of a Double Pendulum, Paper presented in third international conference on Advances in control and Dynamical System (ACODS) at IIT Kanpur from March 13–15 (2014)
15. Chen, K.F.; Zhang, C.; Huang, F.: Triple-pendulum model for studying the vibration of multi-degree-of-freedom systems. Lat. Am. J. Phys. Educ. **5**(1), 5 (2011)
16. Gupta, M.K.; Sinha, N.; Bansal, K.; Singh, A.K.: Natural frequencies of multiple pendulum systems under free condition. Arch. Appl. Mech. **85**, 1049 (2015)
17. Monk, A.T.; Compton, A.H.: Recurrence phenomena in cosmic-ray intensity. Rev Mod Phys **11**(3–4), 173 (1939)
18. Eckmann, J.P.; Kamphorst, S.O.; Ruelle, D.: Recurrence plots of dynamical systems. Europhys. Lett. **4**(9), 973–977 (1987)