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A Modified Mean Value of Performance Measure Approach for Reliability-Based Design Optimization

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Abstract The advanced mean value (AMV) is generally implemented to evaluate the probabilistic constraints of reliability-based design optimization (RBDO) problems based on performance measure approach (PMA). The PMA-based AMV is efficient method but yields unstable results for highly nonlinear probabilistic constraints. In this paper, a modified mean value (MMV) method is proposed to improve the efficiency and robustness of inverse reliability method to evaluate the reliable level in RBDO-based PMA. The modified PMA using MMV is adaptively evaluated using a modified search direction based on the two previous performance values. The modified search direction is determined using an adaptive step size, which is simply computed based on a power function and adaptive factor between 0.95 and 1. The robustness and efficiency of proposed MMV are compared with several reliability methods-based PMA including the AMV, hybrid mean value (HMV), enriched HMV (HMV^+) and modified chaos control (MCC) through four mathematical and structural RBDO problems with nonlinear probabilistic constraints. The results illustrated that the proposed MMV is as robust as the MCC and HMV^+ but is computationally more efficient. In addition, the MMV is more robust than the HMV and AMV for RBDO problems with highly probabilistic constraints.

Keywords Reliability-based design optimization · Performance measure approach · Modified mean value · Probabilistic constraints

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Nomenclature

AMV	Advanced mean value
A_g	Adaptive factor
$\boldsymbol{d}$	Design variables
$\boldsymbol{d}^L$	Lower bound of the design vector
$\boldsymbol{d}^U$	Lower bound of the design vector
DLA	Double loop approaches
EHMV, HMV^+	Enhanced hybrid mean value
f	Objective or cost function
FORM	First-order second-moment method
$f_X(x)$	Joint probability density function
k	Number of iterations
MCC	Modified chaos control
MPFP	Most probable failure point
MPTP	Minimum performance target point
$\boldsymbol{n}(u_k^{\mathrm{AMV}})$	Normalized steepest descent search direction
$\boldsymbol{n}(u_k^{\mathrm{MMV}})$	Normalized modified descent search direction
p	Number of performance functions
P_{f}	Acceptable failure probability
PMA	Performance measure approach
RBDO	Reliability-based design optimization
RIA	Reliability index approach
SLA	Single loop approaches
SORA	Sequential optimization and reliability assessment approach
$\boldsymbol{X}$	Random variables in X-space
$\boldsymbol{U}$	Independent standard normal random variable
$\boldsymbol{U}^*$	The most probable failure point
$\tilde{\boldsymbol{U}}_{k+1}^{\mathrm{MMV}}$	Modified mean value search direction
β	Reliability index

β_t^j	Target reliability index of the jth probabilistic constraint (g_j)
δ	Modified factor
Φ	Standard normal cumulative distribution function
$\boldsymbol{\mu}_{\boldsymbol{x}}^L$	Lower mean of the random design vector
$\boldsymbol{\mu}_{\boldsymbol{x}}^U$	Upper mean of the random design vector

1 Introduction

Recently, the reliability-based design optimization (RBDO) models were developed to consider the uncertainties of engineering systems. The uncertainties in dimensions, materials and loads of a structural system have been evaluated based on probabilistic constraints in RBDO models. Therefore, the RBDO such as double loop approach (DLA), single loop approach (SLA) and decoupled approach can provide an optimal design under uncertainties to achieve a good balance between the total cost and the confidence level [1–4]. The SLA, which is converted the double loop structure to single loop, is an efficient approach, but may produce inaccurate results for RBDO problems with nonlinear constraints [5]. The accuracy of SLA was improved using the reliable design method, within which any design satisfies the reliability requirements [6]. A hybrid SLA and DLA approaches, in which was used the sufficient descent condition to implement SLA, was proposed to improve the accuracy of SLA [5]. The SORA was proposed to convert the double loop structure into serial loop that the deterministic optimization and reliability analysis are performed, sequentially [2]. In SORA, the convex linearization was used to approximate the probabilistic constraints at the design point and was used to shift the probabilistic constraints to reliable region of RBDO problems [7,8]. The accuracy of SORA was improved using the dimension reduction method for reliability analysis [9].

Generally, the DLA is widely used in RBDO of a structural system due to accuracy. The DLA involved two loops at each cycle that the inner loop provides reliability analysis and deterministic optimization is obtained based on the outer loop [10,11]. The performance of reliability method to evaluate the probabilistic constraints is vital important in DLA that it may be led to an accurate optimum results, efficiently and robustly. Reliability information in inner loop can be obtained based on two probabilistic models such as reliability index approach (RIA) [4,5] and performance measure approach (PMA) [12,13] in RBDO-based DLA. In RIA, the first-order reliability method (FORM) [14,15] is applied to search the most probable failure point (MPFP) on the limit state surface using transformation of the probabilistic constraint to reliability index constraint that several reliability methods have been developed for MPFP search in Refs. [15–18]. In PMA, the probabilistic constraint is evaluated by searching the minimum performance target point (MPTP) on the target reliability surface [12,19]. The PMA has higher efficiency and robustness in comparison with the RIA in the double loop process [5,20]. Therefore, the most of investigations were focused on the PMA-based MPTP search in RBDO models that several iterative formula methods for MPTP search have been developed in Refs. [4,5,10–19].

In general, the AMV scheme is utilized to search MPTP in PMA but could converge to unstable solutions as periodic and chaotic solutions for highly nonlinear probabilistic constraints [5,19,20]. The hybrid mean value method (HMV) was proposed to enhance the robustness and efficiency of MPTP search based on a conjugate line search for either concave or convex problems [10]. It showed the HMV yields unstable solutions for highly nonlinear convex performance functions [4,13,21]. The enriched hybrid mean value method (HMV$^+$) was proposed to improve the robustness of the HMV method using PMA [11,22]. The new point in HMV$^+$ method was evaluated based on interpolation of two successive previous points [22]. Recently, the modified chaos control was developed to improve the robustness of reliability method based on a modified line search by chaos feedback control for highly nonlinear concave performance functions [21]. The MCC method could be converged to find the MPTP by implementing more iteration due to select the small control factor to achieve stabilization. The hybrid modified chaos control [21], adaptive chaos control [4] and self-adaptive modified chaos control [19] were proposed to improve the efficiency of MCC using a step size less than 1. The relaxed mean value was proposed using the sufficient descent condition to improve the robustness and efficiency of RBDO-based PMA [13]. Thus, the computational demand and robust reliability algorithm are main issues to implement the PMA-based MPTP search methods.

In this paper, a modified mean value (MMV) is proposed to improve the robustness of AMV approach based on an adaptive search direction, which is computed based on a simple power relation using two previous performance values to determine a adaptive factor. A step size is introduced which can be reduced with a nonlinear rate by increasing the iteration number. The robustness and efficiency of MMV are evaluated through four RBDO problems with nonlinear constraints. The computational iterations and the converged results of the proposed MMV are compared with the AMV, HMV, HMV$^+$ and MCC in studied RDBO problems. The results demonstrated that the MMV has a good performance both robustness and efficiency. The MMV is as simple as the AMV but more robust and is robustly and efficiently converged in comparison with the HMV and MCC methods.

2 Reliability-Based Design Optimization Model

The RBDO problem is generally formulated as [1,2]:

$$\begin{aligned} &\text{find } \boldsymbol{d}, \boldsymbol{\mu}_x \quad \min f(\boldsymbol{d}) \\ &\text{S.t. } P_{\mathrm{f}}[g_j(\boldsymbol{d},\boldsymbol{X}) \le 0] \le \Phi\left(-\beta_t^j\right) \quad j = 1, 2, \ldots, p \\ &\boldsymbol{d}^L \le \boldsymbol{d} \le \boldsymbol{d}^U, \quad \boldsymbol{\mu}_x^L \le \boldsymbol{\mu}_x \le \boldsymbol{\mu}_x^U \end{aligned} \tag{1}$$

where f is the objective or cost function, g_j is the jth constraint function (i.e. performance function), β_t^j is the target reliability index for the jth probabilistic constraint, and p is the number of performance functions. Φ is the standard normal cumulative distribution function. Two types of variables include deterministic design variables $\boldsymbol{d}$, representing physical quantities with design lower bound $\boldsymbol{d}^L$ and upper bound $\boldsymbol{d}^U$; and random variables $\boldsymbol{X}$, representing uncertain quantities with lower bound $\boldsymbol{\mu}_x^L$ and upper bound $\boldsymbol{\mu}_x^U$. In RBDO, uncertainties can be considered based on the probabilistic model $g(\boldsymbol{d},\boldsymbol{X})$ in terms of random variables $\mathbf{X}$. Therefore, the acceptable/target failure probability (P_{f}) for the constraints of Eq. (1) can be computed by following multidimensional integration [23,24]:

$$P_{\mathrm{f}}\left[g(\boldsymbol{d},\boldsymbol{X}) \le 0\right] = \int\limits_{g(\boldsymbol{d},\boldsymbol{X})\le 0} \cdots \int f_{\boldsymbol{X}}(\boldsymbol{X})\mathrm{d}\boldsymbol{X} \approx \Phi(-\beta_t) \tag{2}$$

where $f_{\boldsymbol{X}}(\mathbf{x})$ is the joint probability density function of the basic random variables $\boldsymbol{X}$ and $g(\boldsymbol{d},\boldsymbol{X}) \le 0$ denotes the failure domain. The above relation can be rewritten alternatively by use of the cumulative distribution function (F_{g_j}) of the performance function (g_j) in PMA as

$$g_j(\boldsymbol{d},\boldsymbol{X}) = F_{g_j}^{-1}\left(\boldsymbol{d}, \Phi\left(-\beta_t^j\right)\right) \ge 0 \tag{3}$$

Eq. (3) is employed to evaluate the probabilistic constraints of Eq. (1) in the reliability loop of RBDO model (1). Thus, Eq. (1) is rewritten using the following RBDO model based on PMA

$$\begin{aligned} &\textit{find} \quad \boldsymbol{d}, \boldsymbol{\mu}_x \quad \min f(\boldsymbol{d}) \\ &\text{S.t.} \quad g_j(\boldsymbol{d},\boldsymbol{X}) \ge 0 \quad j = 1, 2, \ldots, p \\ &\boldsymbol{d}^L \le \boldsymbol{d} \le \boldsymbol{d}^U, \quad \boldsymbol{\mu}_x^L \le \boldsymbol{\mu}_x \le \boldsymbol{\mu}_x^U \end{aligned} \tag{4}$$

In RBDO-based PMA, a probabilistic constraint is evaluated by searching the MPTP on the target reliability surface based on the following probabilistic optimization model [12]:

$$\begin{aligned} &\text{find } \boldsymbol{U}^* \\ &\min \; g_j(\boldsymbol{d},\boldsymbol{U}) \\ &\text{s.t. } \|\boldsymbol{U}\| = \beta_t^j \end{aligned} \tag{5}$$

in which $\boldsymbol{U}$ is the independent normal variable that it is computed by transforming the random variables from the original space (X-space) into the standard normal U-space as $\boldsymbol{U} = T(\boldsymbol{X})$, i.e. $u = \Phi^{-1}\{F_X(x^*)\}$ [15,19]. Generally, the main goal of the optimization model (5) is determined the MPTP ($\boldsymbol{U}^*$). The convergence of RBDO model is depended on the performance of reliability algorithm to MPTP search in PMA. It can use an iterative robust and efficient reliability analysis algorithm for evaluating the probabilistic constraints in Eq. (3).

3 A Modified Mean Value Method

An iterative formula is proposed to evaluate the probabilistic constraints in RBDO-based PMA in this section. The iterative formula of the proposed modified mean value (MMV) method is proposed as follows:

$$\begin{aligned} &\boldsymbol{U}_{k+1}^{\mathrm{MMV}} = \beta_t \boldsymbol{n}\left(u_k^{\mathrm{MMV}}\right) \\ &\boldsymbol{n}\left(u_k^{\mathrm{MMV}}\right) = \frac{\tilde{\boldsymbol{U}}_{k+1}^{\mathrm{MMV}}}{\left\|\tilde{\boldsymbol{U}}_{k+1}^{\mathrm{MMV}}\right\|} \end{aligned} \tag{6}$$

where $\boldsymbol{n}(u_k^{\mathrm{MMV}})$ stands for the normalized modified descent search direction. A modified search direction $\left(\tilde{\boldsymbol{U}}_{k+1}^{\mathrm{MMV}}\right)$ is established for MMV method by the following search direction:

$$\tilde{\boldsymbol{U}}_{k+1}^{\mathrm{MMV}} = \boldsymbol{U}_k^{\mathrm{MMV}} + \lambda_k\left[\boldsymbol{U}_{k+1}^{\mathrm{AMV}} - \boldsymbol{U}_k^{\mathrm{MMV}}\right] \tag{7}$$

in which $\tilde{\boldsymbol{U}}_{k+1}^{\mathrm{MMV}}$ is new modified mean value search direction, $\boldsymbol{U}_k^{\mathrm{MMV}}$ is point at kth iteration of MMV method, and $\boldsymbol{U}_{k+1}^{\mathrm{AMV}}$ is new point, which is computed based on AMV iterative formula as follows:

$$\begin{aligned} &\boldsymbol{U}_{k+1}^{\mathrm{AMV}} = \beta_t \boldsymbol{n}\left(u_k^{\mathrm{AMV}}\right) \\ &\boldsymbol{n}\left(u_k^{\mathrm{AMV}}\right) = -\frac{\nabla_u g(\boldsymbol{d}, \boldsymbol{U}_k^{\mathrm{MMV}})}{\|\nabla_u g(\boldsymbol{d}, \boldsymbol{U}_k^{\mathrm{MMV}})\|} \end{aligned} \tag{8}$$

where $\boldsymbol{n}\left(u_k^{\mathrm{AMV}}\right)$ stands for the normalized steepest descent search direction. λ_k in Eq. (7) is an adaptive step size, which is computed by the following relation:

$$\lambda_k = \frac{\delta^k}{A_g} \tag{9}$$

where δ is modified factor, which is considered as $0.95 \le \delta < 1$ and A_g is adaptive factor, which is determined as follows:

$$A_g = \max\left\{1, \frac{\left|g(\boldsymbol{d}, \boldsymbol{U}_k^{\mathrm{MMV}})\right|}{\left|g(\boldsymbol{d}, \boldsymbol{U}_{k-1}^{\mathrm{MMV}})\right|}\right\} \tag{10}$$

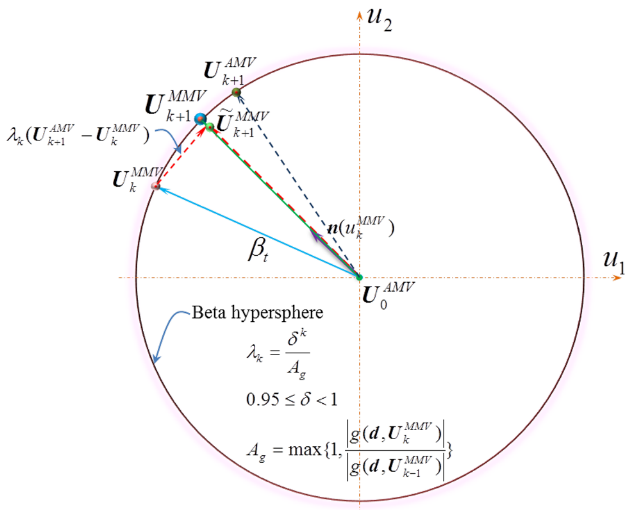

Fig. 1 The MPTP search using the proposed MMV

in which, $g\left(\boldsymbol{d}, \boldsymbol{U}_{k-1}^{\text{MMV}}\right)$ and $g\left(\boldsymbol{d}, \boldsymbol{U}_{k}^{\text{MMV}}\right)$ are the two previous performance values at kth and $k-1$th iterations, respectively.

The proposed modified search direction in Eq. (7) for a cycle of the proposed MMV method is plotted in Fig. 1. As seen, the steepest descent search direction is modified using an adaptive step size in the MMV method. The proposed MMV is as simple as the AMV formula (see Eqs. 6, 8), but the major difference of the MMV and AMV is the modified search direction in Eq. (7) based on adaptive step size in Eq. (9). It can be funded from inverse FORM formula in Eq. (6) (e.g. proposed MMV) and Eq. (8) (e.g. AMV) that the MMV is as simple as the AMV, MCC and HMV methods. However, the steepest descent search direction is used in AMV with step size equal to 1, MCC in which a smaller (i.e. $\lambda = 0.1$) step size is selected at each iteration and MMV with an adaptive step size between 1 and 0. However, the HMV and HMV+ are formulated using a conjugate search direction. The adaptive step size may produce stable results for highly nonlinear performance function, adaptively. The MMV method with the adaptive step size is simpler than the HMV$^+$, because a cubic interpolation to approximate the performance function between two successive points and optimization to determine its parameters could be applied in the HMV$^+$.

A larger adaptive step size is determined at the beginning iterations (e.g. for $\delta = 0.99$, $\lambda_3 \approx 0.97$ and $\lambda_5 \approx 0.95$). Thus, the proposed method is converged similar to the AMV approach. Consequently, this approach is as efficient as the AMV approach for convex performance functions. The adaptive step size in Eq. (8) computes a small value at the final iterations, and also, the adaptive factor in Eq. (10) may be determined more than 1 for highly nonlinear performance functions. Consequently, the decreasing rate of adaptive step size is increased thus, if $k \to \infty$, then $\lambda_k \approx 0$. This means that the new modified search direction is located on the previous point, i.e. $\tilde{\boldsymbol{U}}_{k+1}^{\text{MMV}} \approx \boldsymbol{U}_{k}^{\text{MMV}}$; thus, it obtained a fixed point using iterative MMV formula in Eq. (6). Therefore, the proposed MMV inverse reliability method is robustly converged to stable MPTP for highly concave problems. Based on the above relations, the iterative procedure of proposed MMV is described by following steps:

Step 1 Define performance function $g(\boldsymbol{d},\boldsymbol{X})$ and β_t. Given parameters $0.95 \le \delta < 1$, statistical random variables μ and σ. Set $k = 0$, and ε (stopping criterion)
Step 2 Normalize random variables $\boldsymbol{U}_k = T(\boldsymbol{X}_k)$
Step 3 Compute $\nabla_u g(\boldsymbol{d}, \boldsymbol{U}_k^{\text{MMV}})$ and new point based on AMV ($\boldsymbol{U}_{k+1}^{\text{AMV}}$) using Eq. (8) Determine adaptive factor and step size using Eqs. (10) and (9), respectively.
Step 4 Compute the modified line search on the basis of Eq. (7)
Determine the new point using modified line search based on Eq. (6)
Step 5 If $\left\|\boldsymbol{U}_{k+1}^{\text{MMV}} - \boldsymbol{U}_{k}^{\text{MMV}}\right\| / \left\|\boldsymbol{U}_{k}^{\text{MMV}}\right\| < \varepsilon$, then print $g\left(\boldsymbol{d}, \boldsymbol{U}_{k+1}^{\text{MMV}}\right)$, else set $k = k + 1$ then go to step 2.

4 Illustrative Examples

The proposed MMV is coded in a computer program with MATLAB 7.10 to determine the optimum results of RBDO-based PMA problems that this program can consider the probabilistic constraints of RBDO examples with normal and non-normal variables. The converged results of MMV are compared with AMV [4], HMV [10], HMV$^+$ [22] and MCC methods using four mathematical and structural examples. The objective and number of evaluating probabilistic constraints are used to illustrate the efficiency and robustness of proposed method. The results from the MCC algorithm are obtained by the parameters of $\boldsymbol{c} = \boldsymbol{I}$ and $\lambda = 0.1$ [21]. The modified factor is set as $\delta = 0.975$for proposed MMV method, and also the stopping criterion (ε) is set as 10^{-6} for all examples in the reliability loop of PMA.

Example 1 A highly nonlinear mathematical

A highly nonlinear mathematical for the RBDO model is used as [5,14]

$$\textit{Find}\quad \boldsymbol{d} = [d_1, d_2]^{\text{T}}$$

$$\min\quad f(\boldsymbol{d}) = -\frac{(d_1 + d_2 - 10)^2}{30} - \frac{(d_1 - d_2 + 10)^2}{120}$$

$$\text{S.t.}\quad P_{\text{f}}[g_j(\boldsymbol{X}) > 0] \le \Phi\left(-\beta_t^j\right),\quad j = 1, 2, 3$$

$$\textit{where}\quad g_1 = 1 - \frac{x_1^2 x_2}{20}$$

Table 1 RBDO results for highly nonlinear mathematical example

Method	Design variables $[d_1^*, d_2^*]$		Objective	Iterations	F-evaluations $g_1\backslash g_2\backslash g_3$
AMV	Not converged				
HMV	Not converged				
MCC	4.558115	1.964495	−1.724735	10	7260 (3399\1167\2694)
HMV$^+$	4.558115	1.964495	−1.724735	10	2013 (285\1548\180)
Proposed MMV	4.558115	1.964495	−1.724735	10	**1593** (372\981\240)

The bold numbers are the minimum iterations to achieve the stabilization

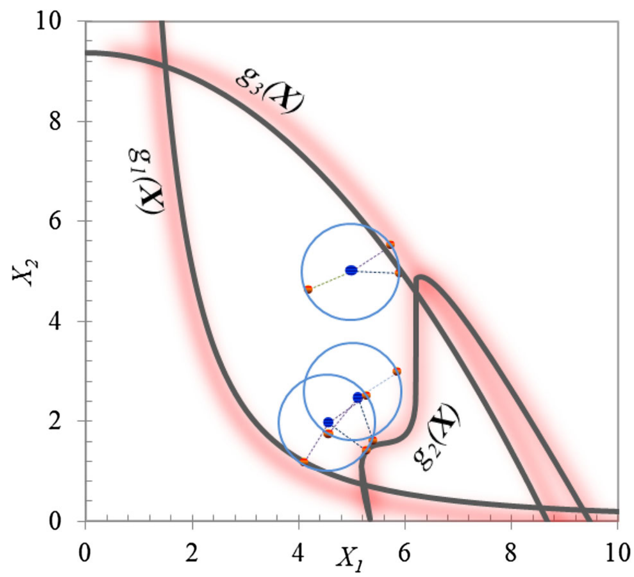

Fig. 2 The iterative histories of MPTP search using MMV method (iterations 1, 7, 10)

$$g_2 = -1 + (Y-6)^2 + (Y-6)^3 - 0.6(Y-6)^4 + Z$$
$$Y = 0.9063x_1 + 0.4226x_2, \quad Z = 0.4226x_1 - 0.9063x_2$$
$$g_3 = 1 - \frac{80}{x_1^2 + 8x_2 + 5}$$
$$0 \le d_i \le 10, x_i \sim N(d_i, 0.3^2) \quad \text{for } i = 1, 2$$
$$\boldsymbol{d}^0 = [5, 5], \quad \beta_t^1 = \beta_t^2 = \beta_t^3 = 3.0 \tag{11}$$

This example includes two normally distributed independent random variables x_1, x_2 and three probabilistic constraints g_1, g_2, g_3. The RBDO results of different PMA-based MPTP methods are listed in Table 1. As is evident from Table 1, the PMA-based AMV and HMV yield unstable solutions, but the MCC, HMV$^+$ and proposed MMV are accurately converged to optimum objective as −1.729546, which is equal to the results from Kestegar and Hao [5] and Meng et al. [21]. The MPTP of the proposed method is illustrated for 1, 7, and final iterations in Fig. 2. It can also be found from the results of Table 1 and Fig. 2 that the MMV is converged to stable results that the first and second probabilistic constraints are the active for this example. The MMV and HMV$^+$ are more slightly efficient than the MCC, and the MMV method is robustly converged about four times faster than the PMA-based MCC. The first and third constraints are converged computationally more efficient based on the proposed MMV in comparison with the MCC, while the HMV$^+$ evaluates the first and third constraints more efficient than the proposed MMV. The HMV$^+$ is robust than the AMV and HMV methods, but HMV$^+$ is inefficiently converged to stable results compared to MCC and MMV method for highly nonlinear constraints (see second constraint in Table 1). Therefore, it can be concluded that the proposed MMV method is more robust than the AMV and HMV methods and more efficient than the MCC for convex and HMV$^+$ for concave probabilistic constraints.

The effects of the stopping criterion (ε, i.e. 10^{-3}, 10^{-4}, 10^{-5} and 10^{-6}) are tabulated for the MCC, HMV$^+$ and proposed MMV method in Table 2. These reliability methods are converged to stable results. The HMV$^+$ is more efficient than the MCC because the HMV$^+$ is computationally efficient approach for convex probabilistic constraints 1 and 3 compared to the MCC. The efficiency of the HMV+ is more sensitive to smaller stepping criterion (i.e. $\varepsilon < 10^{-6}$), while the proposed MMV is insensitive to stopping criterion in

Table 2 RBDO results (F-evaluations $g_1\backslash g_2\backslash g_3$) using different stopping criterions (ε) for highly nonlinear mathematical example

Method	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
MCC	2016 (915\462\639)	3762 (1743\699\1320)	5511 (2574\927\2110)	7260 (3399\1167\2694)
HMV$^+$	**720** (165\435\120)	1023 (207\666\150)	1230 (243\837\150)	2013 (285\1548\180)
MMV	782 (204\446\132)	**969** (240\573\156)	**1177** (263\722\192)	**1593** (372\981\240)

The bold numbers are the minimum iterations to achieve the stabilization

Table 3 Statistical properties of random variables for steel column

Variables	Description	Mean	SD	Distributions
F_s (MPa)	Yield stress	400	35	Lognormal
F_0 (mm)	Initial deflection	30	10	Normal
E (MPa)	Young's modulus	21,000	4200	Weibull
P_1 (N)	Dead load	500,000	50,000	Normal
P_2 (N)	Live load	600,000	90,000	Gumbel
P_3 (N)	Live load	600,000	200,000	Gumbel
L (mm)	Column length	3000	300	Normal
b_f (mm)	Flange breadth	Design variable	30	Lognormal
t_f (mm)	Flange thickness	Design variable	2	Lognormal
h (mm)	Height of profile	Design variable	50	Lognormal

Table 4 RBDO results for steel column example

Method	Design variables $[b_f^*, t_f^*, h^*]$	Objective	*Iterations*	F-evaluations
AMV	[200, 10.3368, 486.7069]	4500.8886	22	8273
HMV	[200, 10.3368, 486.7069]	4500.8886	23	10,426
MCC	[200, 10.3368, 486.7069]	4500.8886	25	8789
HMV$^+$	[200, 10.3354, 486.7564]	4500.8659	26	9507
Proposed MMV	[200, 10.3368, 486.7069]	4500.8886	19	**4535**

The bold numbers are the minimum iterations to achieve the stabilization

comparison with the MCC and HMV+. The MCC and HMV+ show a similar efficiency for highly nonlinear performance function in second constraint, but the proposed method is converged, more efficiently.

Example 2 A steel T-column

A steel column is considered with the random section dimensions. The objective function is defined based on the mean values of dimensions with a probabilistic constant as follows:

$$\begin{aligned}
&\textit{Find} \quad \boldsymbol{d} = [b_f, t_f, h]^T \\
&\min \quad f(\boldsymbol{d}) = b_f t_f + 5h \\
&\text{S.t.} \quad P_f[g(\boldsymbol{X}) > 0] \le \Phi(-\beta_t), \\
&\textit{where} \\
&\quad g = F_s - F\left(\frac{1}{A_s} + \frac{F_0}{m_s} \times \frac{e_b}{e_b - F}\right) \\
&200 \le b_f \le 400,\ 10 \le t_f \le 30,\ 100 \le h \le 500 \\
&\boldsymbol{d}^0 = [300, 20, 300],\ \beta_t = 3.0 \qquad (12)
\end{aligned}$$

The parameters of the T-shaped steel profile of column are defined as $F = P_1 + P_2 + P_3$, $A_s = 2b_f t_f$, $m_s = b_f t_f h$, and $e_b = \frac{\pi^2 E}{2L^2} b_f t_f h^2$ [25]. This problems involve three non-normal random design variables and seven independent random variables as $\{F_s, P_1, P_2, P_3, F_0, E, L\}$, whose statistical characteristics are given in Table 3. This example involves a nonlinear constraint with normal and non-normal random variables.

The results of RBDO-based PMA for different reliability methods, i.e. AMV, HMV, MCC, HMV$^+$ and MMV are tabulated in Table 4 for steel column. It can see all reliability methods, which are implemented to evaluate the probabilistic constraint, are converged to a same optimum as 4500.8886 but the numbers of evaluating the performance function to achieve stable results are obtained different iterations for these reliability methods. The proposed MMV is slightly more efficient than other existing PMA-based MPTP search methods for steel column RBDO model (12). The AMV and MCC are more efficient than HMV and HMV$^+$, but the MMV is converged about twice faster than the AMV and MCC. The proposed method is accurately converged to stable results with less iteration than the modified versions of iterative inverse FORM formula to evaluate the constraint of this example.

Example 3 A rectangular reinforced concrete beam

A rectangular reinforced concrete beam, which is plotted in Fig. 3, is considered to determine the optimum dimensions and reinforced bars by the following RBDO model:

$$\begin{aligned}
&\textit{Find} \quad \boldsymbol{d} = [b, d, A_s]^T \\
&\min \quad f(\boldsymbol{d}) = 800bd + 2000A_s \\
&\text{S.t.} \quad P_f[g_j(\boldsymbol{X}) > 0] \le \Phi\big(-\beta_t^j\big), \quad j = 1, 2, 3 \\
&\textit{where} \quad g_1 = BA_s f_y \left(d - 0.59\frac{A_s.f_y}{b.f_c}\right) - M_D - M_L
\end{aligned}$$

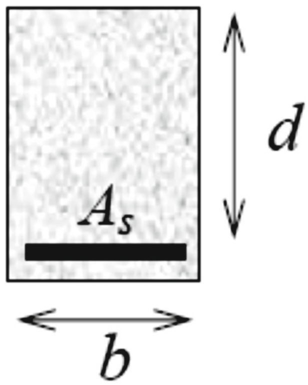

Fig. 3 A rectangular reinforced concrete beam

$$g_2 = A_s - \frac{1.4bd}{f_y} n$$
$$g_3 = 0.85\beta_1 \frac{600}{600 + f_y} . \frac{f_c}{f_y} bd - A_s$$
$$100 \le b \le 400, \quad 200 \le d \le 550, \quad 500 \le A_s \le 3000$$
$$\boldsymbol{d}^0 = [200, 400, 1500], \quad \beta_t^1 = \beta_t^2 = \beta_t^3 = 3.5 \qquad (13)$$

This problem involves three constraints that the first constraint is defined based on the moment capacity of beam that it is considered various uncertainties such as model, load and resistance. Two constraints g_2 and g_3 on parameters are given a maximum and minimum admissible area of reinforcement in relation to the total area of the concrete section. The statistical properties of random variables are described in Table 5 for reinforced concrete beam.

The RBDO results of different reliability methods of PMA are listed in Table 6 for reinforced concrete beam. A same optimum and converged design variables are obtained for all studied reliability analysis methods. As seen, the proposed MMV method has a top performance in comparison with the other methods. The MMV yields stable results about four and three times faster than the HMV and HMV$^+$ methods, respectively. The MCC is more computationally inefficient and needs more iterations to achieve the stabilization for constraints 1–3, but the proposed MMV is more efficient.

Example 4 A welded beam problem

A welded beam, which is extracted from [4], is shown in Fig. 4. The objective function is the welding cost, and the probabilistic constraints are related to physical quantities, such as shear stress, bending stress, buckling load and displacement that the RBDO model can be formulated as

$$\text{Find} \quad \boldsymbol{d} = [d_1, d_2, d_3, d_4]^{\mathrm{T}}$$
$$\min \quad f(\boldsymbol{d}) = c_1 d_1^2 d_2 + c_2 d_3 d_4 (z_2 + d_2)$$
$$\text{S.t.} \quad P_{\mathrm{f}}[g_j(\boldsymbol{X}, z) > 0] \le \Phi(-\beta_t^j), \quad j = 1, 2, \ldots, 5$$
$$\textit{where} \quad g_1(\boldsymbol{X}, z) = \frac{\tau(\boldsymbol{X}, z)}{z_6} - 1, \quad g_2(\boldsymbol{X}, z) = \frac{\sigma(\boldsymbol{X}, z)}{z_7} - 1,$$
$$g_3(\boldsymbol{X}, z) = \frac{x_1}{x_4} - 1, \; g_4(\boldsymbol{X}, z) = \frac{\delta(\boldsymbol{X}, z)}{z_5} - 1,$$
$$g_5(\boldsymbol{X}, z) = 1 - \frac{P_c(\boldsymbol{X}, z)}{z_1},$$

Table 5 Statistical properties of random variables for reinforced concrete beam

Variable	Description	Distribution	Mean	COV
f_{y}	Steel yield stress	Lognormal	400 (MPa)	0.1
f_{c}	Concrete comp. strength	Normal	30 (Mpa)	0.18
B	Model uncertainty	Normal	1.01	0.06
M_{D}	Dead bending moment	Normal	95.87 (kN-m)	0.10
M_{L}	Live bending moment	Gumbel	67.11 (kN-m)	0.25
β_1	–	Lognormal	0.85	0.05
A_s (mm^2)	Reinforced area	Normal	Design variable	0.1
b (mm)	Width	Normal	Design variable	0.12
d (mm)	Effective depth	Normal	Design variable	0.12

Table 6 RBDO results for reinforced concrete beam example

Method	Design variables $[b^*, d^*, A_s^*]$	Objective	Iterations	F-evaluations $g_1\backslash g_2\backslash g_3$
AMV	[361.1941, 550, 2236.004]	163,397,418.77	10	4415 (1835\980\1600)
HMV	[361.1941, 550, 2236.004]	163,397,418.77	14	14,978 (5846\3841\5291)
MCC	[361.1941, 550, 2236.004]	163,397,418.77	27	12,4382 (42,113\34,370\47,899)
HMV$^+$	[361.1985, 550, 2236.031]	163,399,388.03	12	10,015 (5313\751\3951)
Proposed MMV	[361.1941, 550, 2236.004]	163,397,418.77	8	**3616** (1617\700\1299)

The bold numbers are the minimum iterations to achieve the stabilization

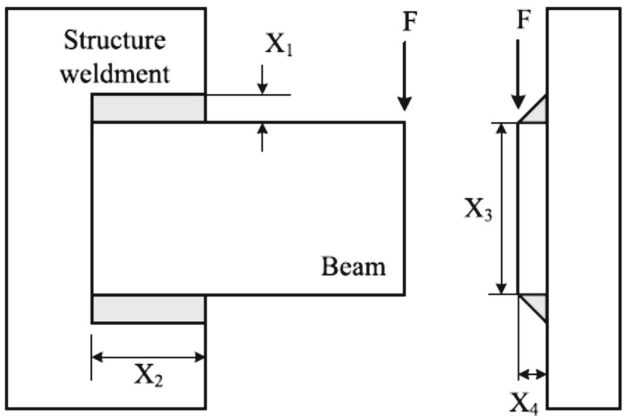

Fig. 4 A welded beam structure

$$x_i \sim N(d_i, 0.1693^2)\ for \quad i = 1, 2\ x_i \sim N(d_i, 0.0107^2)\ \ for \quad i = 3, 4$$

$$\beta_t^1 = \beta_t^2 = \cdots = \beta_t^5 = 3.0, \quad 3.175 \le d_1 \le 50.8, \ 0 \le d_2 \le 254,$$

$$0 \le d_3 \le 254, \quad 0 \le d_4 \le 50.8, \ \boldsymbol{d}^0 = [6.208, 157.82, 210.62, 6.208]^{\mathrm{T}}$$

$$\tau(\boldsymbol{X}, z) = \sqrt{t(\boldsymbol{X}, z)^2 + \frac{2t(\boldsymbol{X}, z)tt(\boldsymbol{X}, z)x_2}{2R(\boldsymbol{X})} + tt(\boldsymbol{X}, z)^2},$$

$$t(\boldsymbol{X}, z) = \frac{z_1}{\sqrt{2}x_1x_2}, tt(\boldsymbol{X}, z) = \frac{M(\boldsymbol{X}, z)R(\boldsymbol{X})}{J(\boldsymbol{X})},$$

$$\sigma(\boldsymbol{X}, z) = \frac{6z_1z_2}{x_3^2x_4},$$

$$\delta(\boldsymbol{X}, z) = \frac{4z_1z_2^3}{z_3x_3^3x_4}, M(\boldsymbol{X}, z) = z_1[z_2 + x_2/2], R(\boldsymbol{X}) = \frac{\sqrt{x_2^2 + (x_1 + x_3)^2}}{2},$$

$$J(\boldsymbol{X}) = \sqrt{2}x_1x_2\left\{x_2^2/12 + (x_1 + x_3)^2/4\right\},$$
$$P_c(\boldsymbol{X}, z) = \frac{4.013x_3x_4^3\sqrt{z_3z_4}}{6z_2^2}\left(1 - \frac{x_3}{4z_2}\sqrt{\frac{z_3}{z_4}}\right) \qquad (14)$$

This problem has four random variables and five probabilistic constraints. The fixed system parameters are listed in Table 7. All random variables are statistically independent and follow the normal distribution.

The optimal results of welded beam are listed in Table 8. The optimal results (i.e. objective value and design point of all the approaches) are almost equal to those in the obtained results of Li et al. [4]. The results from Table 8 demonstrated that all RBDO using PMA methods converge to the same optimum 2.59132.

The PMA-based AMV, HMV and HMV$^+$ are converged with a same iteration, but proposed PMA method based on MMV is converged faster than other existing PMA methods. The proposed MMV has a highly convergence rate in comparison with the MCC for this example. Thus, the proposed MMV can improve the efficiency and robustness of the DLA-based PMA.

5 Conclusions

Typically, the iterative schemes of minimum performance target point (MPTP) search could yield unstable solutions for evaluating the highly probabilistic constraints in reliability-based design optimization (RBDO) problems. In present paper, a simple iterative formula is developed based on a modified search direction to enhance the robustness and efficiency of reliability analyses-based MPTP search in performance measure approach (PMA), which is computed using two previous results of performance value. The proposed method is called modified mean value (MMV) that the

Table 7 System parameters for the welded beam

z_1	2.6688×10^4 (N)	z_4	8.274×10^4 (MPa)	z_7	2.0685×10^2(Mpa)
z_2	3.556×10^2 (mm)	z_5	6.35 (mm)	c_1	6.74135×10^{-5} (\$/mm^3)
z_3	2.0685×10^5 (Mpa)	z_6	9.377×10^1 (Mpa)	c_2	2.93585×10^{-6} (\$/mm^3)

Table 8 RBDO results for welded beam

Method	Design variables	Objective	Iteration	F-evaluations
AMV	(5.730022, 200.8981, 210.5977, 6.238936)	2.59132	15	1350
HMV	(5.730022, 200.8981, 210.5977, 6.238936)	2.59132	15	1350
MCC	(5.730022, 200.8981, 210.5977, 6.238936)	2.59132	15	12,170
HMV$^+$	(5.730022, 200.8981, 210.5977, 6.238936)	2.59132	15	1350
Proposed MMV	(5.730022, 200.8981, 210.5977, 6.238936)	2.59132	9	**1135**

The bold numbers are the minimum iterations to achieve the stabilization

efficiency and robustness of MMV are illustrated with four nonlinear mathematical and structural RBDO problems with nonlinear probabilistic constraints. The converged results of MMV are compared with several existing reliability methods such as AMV, HMV, HMV^+ and MCC. The MMV is as simple as PMA-based AMV but more robust and efficient. In addition, the proposed MMV has top convergence both efficiency and robustness in comparison with the AMV, HMV, HMV^+ and MCC methods.

The HMV^+ can be improved for highly nonlinear performance function with a simple interpolation and MMV can be combined with the AMV to improve its efficiency for convex problems in future.

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