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RESEARCH ARTICLE - CIVIL ENGINEERING

A Theoretical Method for Fracture Resistance of Shear-Strengthened RC Beams with FRP

Shahriar Shahbazpanahi · Abang Abdullah Abang Ali · Farah Nora Aznieta · Alaleh Kamgar · Nima Farzadnia

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Abstract So far, the conventional, theoretical, and numerical analyses in fracture mechanics have been applied to study concrete flexural beams, which are strengthened using fiber-reinforced polymer (FRP) composite sheets. However, there is still little knowledge about the shear capacity of a side face FRP-strengthened cracked beam. A theoretical analysis is herein presented to obtain the fracture resistance in a four-point reinforced concrete beam, with two inclined initial notch on the supports, which is strengthened with side face FRP strips. The fracture process zone (FPZ) at the head of the crack is used as the base of a fictitious crack to obtain shear stress distribution in the cross section of the beam. Based on equilibrium equation in the beam notch cross section, the change of shear force against the FPZ length and the tensile forces due to FRP are obtained. Then, in double notch four-point beam, Mode II of the stress intensity factor due to the external load is determined. Finally, the relationship between the shear capacity and the FPZ length is used to express the fracture resistance as a function of the FPZ length. It is observed that the FPZ and the FRP sheets have positive effects on the fracture toughness and they play important roles in preventing the propagation of shear cracks.

Keywords Fictitious crack · FPZ · FRP · Shear strengthening

S. Shahbazpanahi (✉) · F. N. Aznieta · A. Kamgar
Department of Civil Engineering, University Putra Malaysia, Selangor, Malaysia
e-mail: sh.shahbazpanahi@gmail.com

A. A. A. Ali · N. Farzadnia
Housing Research Center, University Putra Malaysia, Selangor, Malaysia

الخلاصـة

تم حتى الآن تطبيق التحليل التقليدي، النظري والعددي في ميكانيكا الكسر لدراسة حزم الخرسانة المتعرجة التي يتم تعزيزها باستخدام صحائف مركب البوليمر المقوى بالألياف (FRP). ومع ذلك ، ما يزال هناك القليل من المعرفة حول قدرة القص من الوجه الجانبي لحزمة البوليمر المقوى بالألياف المعززة والمتصدعة. وسوف نقدم هنا التحليل النظري للحصول على مقاومة الكسر في حزمة خرسانية معاد تعزيزها ذات أربع نقاط، مع ثلمتين ابتدائيتين مائلتين على المدعمات التي تمت تقويتها بشرائط الوجه الجانبي لحزمة البوليمر المقوى بالألياف. وتم استخـدام منطقة عملية الكسر (FPZ) على رأس الشق كقاعدة لصدع وهمي للحصول على توزيع إجهاد القص في المقطع العرضي للحزمة. واستنادا إلى معادلة الاتزان في المقطع العرضي للحزمة المثلومة تم الحصول على تغيير قوة القص مقابل طول منطقة عملية الكسر وقوى الشد بسبب البوليمر المقوى بالألياف. وبعد ذلك وفي حزمة من أربع نقاط في الشق المزدوج، يتم تحديد الوضع الثاني من عامل شدة الإجهاد بسبب الحمل الخارجي. وأخيرا، يتم استخدام العلاقة بين قدرة القص وطول منطقة عملية الكسر للتعبير عن مقاومة الكسر بوصفها وظيفة من طول منطقة عملية الكسر. وقد لوحظ أن منطقة عملية الكسر و صحائف البوليمر المقوى بالألياف يكون لها آثار إيجابية في صلابة الكسر وتلعب دورا هاما في منع انتشار شقوق القص.

1 Introduction

Several years after being built, concrete members may need strengthening due to extra load, cracks, environmental factors, and damages. Considering budget constraints, concrete structures should be strengthened rather than reconstructed. fiber-reinforced polymer (FRP) composite sheets are increasingly used in concrete as it offers high resistance against corrosion; they have low weight and high tensile strength; and they are easily applicable with low labor costs. FRP composite sheets have been applied mainly to improve the flexural capacity, to provide shear strengthening, and to prevent crack growth in the soffit or/and side face reinforced concrete (RC) [1,2]. Studying the effects of FRP composite sheets in shear concrete cracks plays an essential role in preventing the growth of the crack and to enhance load bearing. Numerous research studies have been conducted to study the shear capacity of RC beams which are shear-strengthened using FRP composite sheets [3–9]. Each study has led to different methods for analyzing shear strengthening of such beams.

Although the application of FRP composite sheets in concrete structures increases the ultimate load capacity and enhances the stiffness and the strength of these structures, it lowers the ultimate displacement and the ductility of the concrete. The ductile failure allows alerting the danger of failure. Due to the elastic behavior of the FRP composite sheets and due to the quasi-brittle nature of the concrete, there is no clear definition for ductility in concrete structures, where FRP composite sheets are used. However, some efforts have been made to measure the ductility of flexural members for hybrid structures [10–12].

One of the methods used to explain the behavior of hybrid structures is the *R*-curve technique. This technique deals with tensile crack in fracture mechanics [13]. The *R*-curve method is used to study the crack initiation and the stress at the tip of a macro-crack as well as the stability and the propagation of the crack. To create *R*-curve, two approaches are used. One approach is to use experimental methods, while the other one is analytical. In experimental approaches, the crack propagation is measured by increasing the load on pre-cracked structure, whereas in analytical methods, the mathematical relationship between the fracture resistance and the geometry of the structure is obtained. The *R*-curve depends on the property of the material yet independent of the geometry and size of the structure. Compared to ductile materials, quasi-brittle materials (such as concrete) have different *R*-curves due to the softening zone in front of the macro-cracks [14].

Hillerborg et al. [15] used a fictitious crack modeling in concrete based on fracture mechanics. He has shown that there is a region, called fracture process zone (FPZ), in front of the real crack tip, which leads to crack closure. This fairly large zone, which contains micro-cracks, bridging, and crack branches, is located ahead of the macro-cracks. Since a significant amount of energy is stored in this region, a crack can have stable growth before peak load. In addition, the existence of this zone justifies the strain softening behavior in the stress-displacement curve after peak load. Beyond the peak load, the interlocking crack surfaces in the FPZ contribute to a gradual decline in stress and prevent sudden failure. The FPZ size depends on the size of the member, and on the initial crack length, as well as on the loading and the properties of concrete under consideration [16,17]. The length of the FPZ (l_p) is of more interest as compared to its width [18].

Based on the theory of linear elastic fracture mechanics (LEFM), a fairly high coefficient is applied to the stress in the vicinity of the crack tip. This coefficient is called stress intensity factor. The LEFM converts stress to a unique form of distribution. The stress intensity factor depends on the material properties, on the size of the crack, on the load, and on the geometry of the structure. This factor presents a relationship between the material and the reaction of the structure.

There have been some investigations on how to obtain the fracture resistance of concrete beams, with an initial notch, that have flexural strengthening achieved using FRP composite sheets [19,20]. However, to the best of our knowledge, the fracture resistance of the shear crack beams which strengthened with FRP composite sheets has not been studied in the literature. As concrete is relatively weak in shear, obtaining the stress intensity factor and the *R*-curve as well as investigating the role of the FPZ and the effect of the FRP on shear crack are of particular importance.

In this study, a theoretical analysis is presented to obtain the fracture resistance in a four-point RC beam, with two inclined initial notch on supports, which is strengthened with side face FRP composite sheets. In linear fracture mechanics, the FPZ at the tip of the crack is considered as the base of a fictitious crack to obtain shear stress distribution in the crack cross section. Interaction between shear stress distribution and the normal stress in damage zone is used to estimate the cohesive stress of the fracture. Based on the equilibrium equation in the notch cross section, the change of the shear force versus the FPZ length and the tensile force due to the FRP composite sheets are determined. Then, Mode II of the stress intensity factor in a double notch four-point beam due to the external load is determined. The fracture resistance of the material is expressed in terms of the fracture toughness of the plain concrete, the shear stress due to the FPZ, the force of the FRP slips, and the FRP tensile forces. Finally, the relationship between the shear capacity and the FPZ length is used to express the fracture resistance as a function of the FPZ length. It is shown that the FPZ and the FRP sheets have positive effects on the fracture toughness and they play important roles in preventing the propagation of shear cracks.

2 Mathematical Formulation

2.1 Equilibrium Equation in Cross Section

Let us consider a four-point loading RC beam. Shear cracks appear along the shear span in an intermediate position, between the loading point and the support, depending on the longitudinal reinforcement amount. Initially, a shear crack follows a vertical trajectory and then turns toward the loading point [21]. But in the present study, the shear crack is forced to conduct using an initial notch in supports. Since shear force is large and flexural moment is small in the supports, this moment can be ignored to study shear cracks in Mode II.

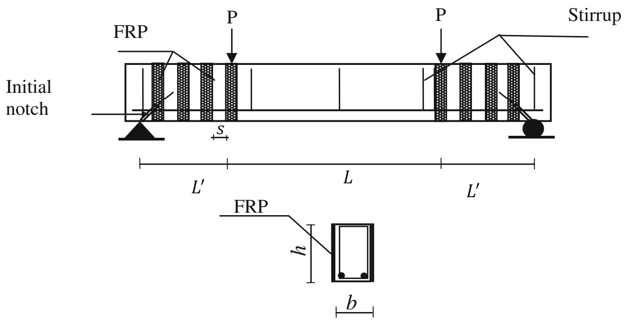

Fig. 1 Four-point loading FRP shear-strengthened beam

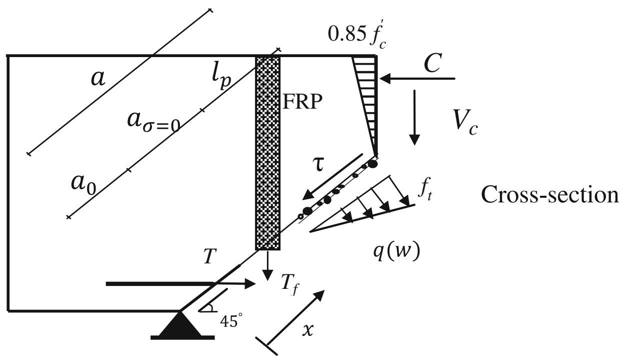

Fig. 2 The shear crack cross section and the corresponding stress distribution

The shear crack angle has an important effect on the shear capacity and fracture resistant. Although actual shear cracks angle is seldom reported, an angle of 45° has been used to design RC beams [22]. Thus, the assumptions that the shear crack is 45° and that it starts from the supports are not far from reality.

Figure 1 shows an FRP-strengthened beam with two inclined initial notches. These notches have shear crack angles of 45° and are located on the supports. The four-point loading RC beam has a rectangular cross section with depth h and width b. The lengths of the initial notches are a_0. An FRP sheet has a thickness t. The distance of the steel stirrups is selected very large to make the beam weak in shear. Also, for shear failure to happen, the locations of loads are selected close to the supports [6].

Figure 2 illustrates the cross section of one of the shear cracks and the corresponding stress distribution. In this figure, the parameters l_p, $a_{\sigma=0}$, and a are the FPZ length, the length of the stress-free region and the effective crack length, respectively. As shown in this figure, if load is increased, the crack grows due to extended FPZ.

Based on the equilibrium equation, the shear strength V_n of the beam is given by:

$$V_n = V_c + V_{FPZ} + n(T_f - T_s) \tag{1}$$

where V_c is the shear force sustained by the concrete, V_{FPZ} is the force applied to FPZ as a result of stress distribution, T_f is the tensile force due to the extension of FRP, T_s is shear force because of FRP sheet slip, and n is the number of side face FRP sheets.

A simple formula for shear strength contributions of concrete members under flexure and shear load [23] is:

$$V_c = 0.1\sqrt{f_c'}bh \tag{2}$$

In Eq. (2), f_c' is the compressive strength, Mpa, while h and b are the depth and the thickness of the beam, respectively. Equation (2) is the expression of concrete shear strength with reinforcement.

Normal force due to the cohesive fracture zone is expressed as [20]:

$$F = \int_{a-l_p}^{a} q(w_x)b\,dx \tag{3}$$

where $q(w_x)$ is the distribution of the stress in the FPZ and w_x is one-half of crack opening where the origin of x lies (Fig. 2). The lower limit of normal force is corresponding to the cases of zero length l_p and the upper limit is corresponding to the cases of uniform distribution f_t [13] along the process zone. Equation (3) implies that normal stress depends on the opening of crack and that the damage zone is still capable to sustain the load. A simple linear stress distribution and linear crack opening function are used in this study as [17]:

$$q(w_x) = f_t\left(1 - \frac{(a-x)}{l_p}\right) \tag{4}$$

and

$$w_x = \frac{w_c(a-x)}{l_p} \tag{5}$$

where w_c is the critical crack opening displacement in the stress-crack opening displacement (COD) curve.

On the other hand the interaction between the shear and normal stress distributions in the damage zone was derived by Jefferson [24] as:

$$\tau = r(q(w_x) - f_t)^{2/3} \tag{6}$$

where f_t is the tensile strength of the concrete and the r factor is given by:

$$r = 0.78 f_c'^{1/3}. \tag{7}$$

Equation (6) was obtained by computing the strength envelope of the sequential Mohr's circle of stress. Using the assumption that the tensile strength of the concrete is approximately one-tenth of compressive strength, the r factor is calculated.

The tensile force T_f can be written as:

$$T_f = A_F E_F \varepsilon_{F_c} \tag{8}$$

where A_F and E_F are the area and the longitudinal elastic modulus of the FRP sheet, respectively, and ε_{F_c} is the effective strain for FRP sheet [25].

To calculate shear force due to slip of the FRP sheet, constitutive model presented by Nakaba et al. [26] is used in elastic zone. Thus, T_s can be written as:

$$T_s = l_F b_F \tau_{max} \tag{9}$$

where l_f and b_f are the length and width of FRP, respectively, τ_{max} is the maximum shear slip stress between the FRP and the concrete. Shear capacity, V_n, is equal to the external load, P, of the beam as:

$$V_n = P = 0.1\sqrt{f_c'}bh + 0.707\left[\int_{a-l_p}^{a} q(w_x)b\,dx + rb\int_{a-l_p}^{a}(q(w_x) - f_t)^{2/3}dx\right] + nA_F E_F \varepsilon_{F_c} - nl_F b_F \tau_{max} \tag{10}$$

Using Eq. (10), the shear capacity can be determined when l_p is known. The shear capacity of the beam is obtained based on the shear strength of the concrete, the shear and normal stresses in the FPZ, and tensile forces due to the FRP effect. In the next section, Eq. (10) is used to obtain the fracture resistance of the crack. The expression of the shear capacity given by Eq. (10) does not consider the effects of the steel longitudinal and stirrup reinforcing bars.

2.2 Fracture Resistance of Crack with FRP

Since shear force prevails the flexure moment in the shear span, a sliding mode (Mode II) is developed. In the present study, Mode II is used to formulate the fracture resistance of the crack. A crack will grow if the stress intensity factor, due to the external load $K_{p_{II}}$, reaches the fracture resistance of the material, $K_{R_{II}}$, i.e.,

$$K_{p_{II}} = K_{R_{II}} \tag{11}$$

The stress intensity factor of Mode II for the double notch four-point beam, due to the external load P, is given by [27]:

$$K_{p_{II}} = \frac{P}{bh^{1/2}} f(a/h) \tag{12}$$

where $f(a/h)$ is geometric shape function, respectively. Note that $f(a/h)$ can be written as:

$$f(a/h) = -3.40(a/h)^4 + 15.78(a/h)^3 - 16.04(a/h)^2 + 9.70(a/h) - 0.85 \tag{13}$$

The material fracture resistance is given by Wu and Davies [19]:

$$K_{R_{II}} = K_{c_{II}} + K_{t_{II}} + K_{f_{II}} + K_{s_{II}} \tag{14}$$

where $K_{c_{II}}$ is the fracture resistance of the plain concrete, $K_{t_{II}}$ is the fracture resistance due of the shear stress of the FPZ, $K_{f_{II}}$ is the fracture resistance due to the force of FRP sheet, and $K_{s_{II}}$ is the fracture resistance due to the force of FRP sliping for Mode II. The shear fracture resistance due to FPZ is given by Tada et al. [28]:

$$K_{TII} = \tau\sqrt{\pi a}G(a/h) \tag{15}$$

where

$$G(a/h) = \frac{1.122 - 0.561(a/h) + 0.085(a/h)^2 + 0.18(a/h)^3}{\sqrt{1 - a/h}} \tag{16}$$

and is a dimensionless factor which is obtained from handbooks of the stress intensity factors. Fracture resistance caused by the FRP sheet force sheet is expressed as [28]:

$$K_{f_{II}} = \frac{T_f \sin 45}{\sqrt{\pi a}} k(a/h) \tag{17}$$

where

$$k(a/h) = \frac{1.3 - 0.65(a/h) + 0.37(a/h)^2 + 0.28(a/h)^3}{\sqrt{1 - a/h}} \tag{18}$$

Fracture resistance due to FRP slip is obtained using Eq. (15) when T_f is replaced with T_s.

When the material properties of the plain concrete, force of FRP sheet, force of FRP slip, and the geometry of the beam are given, the R-curves can be estimated [13].

Fracture toughness is the fracture resistance at a special crack length [13]. Thus, the critical crack length, a_c, is obtained numerically by solving the following equation:

$$\left.\frac{\partial K_{p_{II}}}{\partial a}\right|_{a=a_c} = \left.\frac{\partial K_{R_{II}}}{\partial a}\right|_{a=a_c} \tag{19}$$

Substituting Eqs. (12) and (14) into (11) and (19), respectively, and then solving the equation using trial and error method, the critical crack length can be obtained for a certain l_p. The fracture toughness can be determined using Eq. (11)or (19) and applying known critical crack length.

3 Numerical Results and Discussions

A simply supported FRP shear-strengthened beam under four-point load is considered to study the fracture resistance that was tested by Mosallam and Banerjee [6]. The total length, depth, width, and compressive strength are 3,460, 250, 150 mm and 27.54 Mpa, respectively. The distance

Table 1 Results for shear capacity obtained by the present study and the methods in Mosallam and Banerjee [6]

Experimental (kN)	ACI 440 model (kN)	Colotti et al. model (kN)	Matthys and Triantafillou model (kN)	Present study (kN)
55.46	68.16	55.18	63.98	61.08

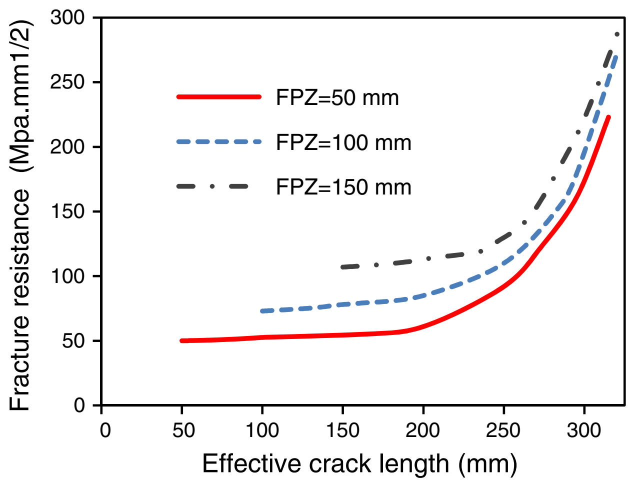

Fig. 3 Fracture resistance versus effective crack length (l_p = 100 mm)

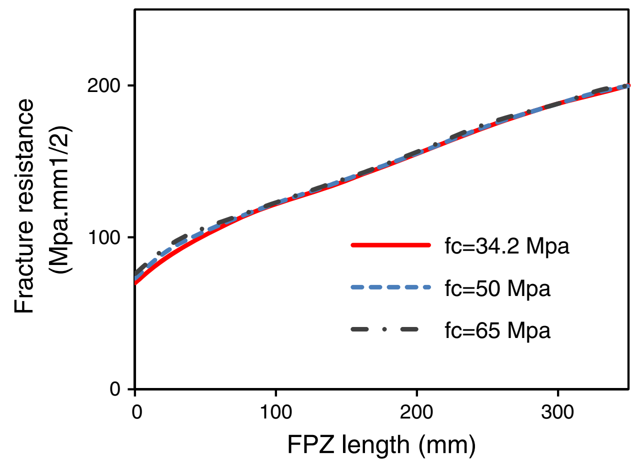

Fig. 4 Fracture resistance versus FPZ length with different compressive strength

between two point load positions, L, is 1,700 mm which to causes the shear failure in (Fig. 1) while $L^{'}$ is 370 mm. The space of steel transverse was selected 610 mm so that it has no effect on the shear capacity. The steel bars and stirrups have 169 and 45 mm^2 cross-sectional areas, respectively. Modulus of elasticity, thickness, width and length of FRP are 151.7 GPa, 1.19, 50, 150 mm, respectively. The spacing of FRP in Fig. 1 is 51 mm. Maximum shear slip stress between FRP and concrete, $\tau_{\max}$, is 6.65 Mpa, while ε_{fc}, the effective strain for FRP sheet, is 0.004 [25]. Plain concrete fracture toughness, $K_{C_{II}}$, [29] is 6.65 Mpa mm$^{1/2}$.

Table 1 shows the ultimate shear capacity of RC beam using the proposed method as well as the model and the experimental data of Mosallam and Banerjee [6]. Ultimate shear capacity is calculated by considering fully developed FPZ while effective crack reaches top of the beam. Thus, when a is 350 mm, the FPZ length is around 0.77a, based on the approach of Zhang and Wu [30] that can calculate ultimate shear capacity. It can be seen that the result of the proposed method is close to other results.

Figure 3 shows the relationship between the fracture resistance curves and effective crack length for three different FPZ lengths when l_p is 100 mm. The resistance rises with increasing the effective crack length, although it increases slowly at the beginning. The fracture resistance increases very fast when the effective crack length reaches almost 250 mm and then internal additional load is appeared to prevent the growth of the shear crack. This additional load is provided by FRP which leads ductility to a lower in this zone. It can be seen that after crack propagated at a certain point, depending on the FPZ length, the fracture resistance is climbed. This means when effective crack length turns to the beam depth, fracture resistance must be infinity. Another important observation is that at the same value for effective crack length, the FPZ plays a significant role in increasing the fracture resistance of the shear crack. It is seen that FPZ has a major role in preventing the propagation of shear crack, as it does into preventing the propagation of flexural crack [13]. Also, the effect of FPZ length is reduced in high effective crack length due to the creation of stress-free region at the tip of the notch [31].

Figure 4 shows the fracture resistance as the FPZ length varies with the different compressive strength of the concrete. It is seen that in the shear crack, the fracture resistance does not change with compressive strength. This may be due to concrete being weak in shear. Also, it is noticed that fracture resistance increases as FPZ length grows. It is shown that in shear crack, the FPZ directly influences the R-curve.

Figures 5 and 6 are presented to display the parameter study, such as effect of the number and the thickness of FRP strips upon the shear load capacity. Figure 5 shows how the shear capacity is affected by different number of FRP strips, n, when l_p is 100 mm. It can be seen that, initially, shear capacity decreases. This may be due to the crack growth. After this initial decrease, if the stress-free region appears about 0.7 a, the effect of FRP and its slip is more pronounced. This phenomenon occurs in flexural crack as well [13]. The condition, where load falls as the crack growth, is called stabilization [21]. Another observation is that loading capacity is increased, as expected, by increasing numbers of FRP strips. This observation verifies the rationality and efficiency of the method. Figure 6 illustrates the shear load-effective crack length with different thicknesses of FRP strips if l_p is

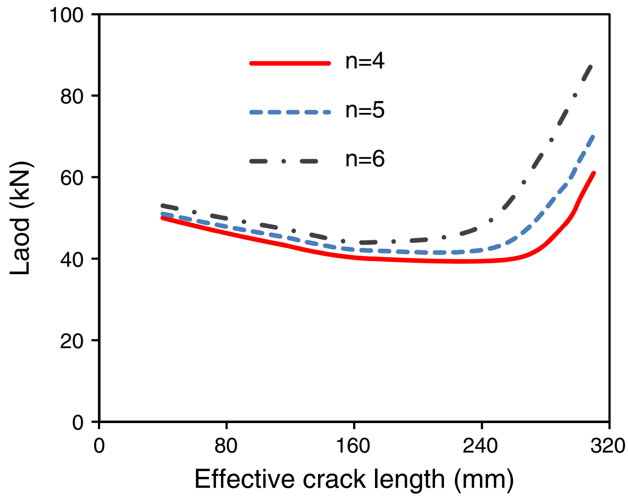

Fig. 5 Shear load versus the effective crack length for different number of FRP sheets (l_p = 100 mm)

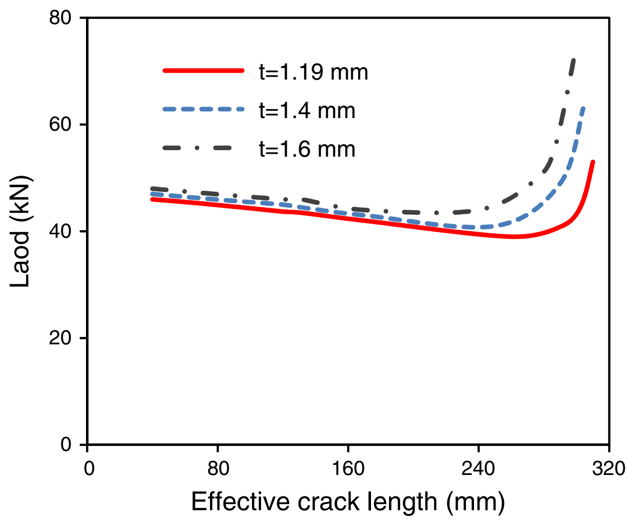

Fig. 6 Shear load versus effective crack length with different thicknesses of FRP sheets (l_p = 100 mm)

100 mm. When the thickness increases to 1.6 mm, the shear load slowly drops due to the crack propagation. After this drop, the shear load recovers due to the effect of FRP sheets as the influence of the thickness of the FRP sheets on the fracture resistance prevails.

4 Conclusions

In this study, a theoretical analysis is presented to obtain the fracture resistance in a four-point RC beam with two inclined initial notches on supports strengthened with FRP strips. The FPZ at the tip of the crack is considered as the base of a fictitious crack to obtain the shear stress distribution in the cross section. Interaction between the shear stress distribution and the normal stress in the damage zone is used to estimate the cohesive fracture. Based on the equilibrium equation in the notch cross section, shear force is expressed in terms of the FPZ length and the tensile force due to FRP. Then, Mode II of the stress intensity factor in double notch four-point beam due to the external load is determined. The fracture resistance of the materials is explained by plain concrete fracture resistance, the shear stress due to FPZ, and the FRP tensile force. Finally, the relationship between the shear capacity and the FPZ length is used to express the fracture resistance as a function of the FPZ length. It is seen that the FPZ and the FRP sheets have important effects on the fracture resistance and on the prevention of the propagation of shear cracks. It is seen that in shear cracks, the fracture resistance does not change with compressive strength. The proposed method considers the influence of geometric parameters and shows acceptable similarity to experimental data.

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