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Quasitriangular Hopf Group Coalgebras and Braided Monoidal Categories

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Abstract Turaev introduced the notions of group coalgebras, Hopf group coalgebras and quasitriangular Hopf group coalgebras. Virelizier studied algebraic properties of Hopf π-coalgebras. In this paper, we give the definitions of a (crossed) left H-π-modules over a (crossed) Hopf π-coalgebra H, and show that the categories of (crossed) left H-π-modules are both monoidal categories. Finally, we show that a family $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of elements is a quasitriangular structure of a crossed Hopf π-coalgebra H if and only if the category of crossed left H-π-modules over H is a braided monoidal category with braiding defined by R.

Keywords Crossed left H-π-modules · Braided monoidal category

Mathematics Subject Classification (2010) 16W30

الملخص

قدم تيوراييف مفاهيم جبريات الزمرة المرافقة، وجبريات زمر هوبف المرافقة، وجبريات زمر هوبف شبه المثلثية المرافقة. درس فيرليزير الخصائص الجبرية لِـ π- جبريات هوبف المرافقة. في هذه الورقة البحثية، نعطي تعريفات لِـ H-π- حلقيات (متقاطعة) يسرى على π- جبرية هوبف مرافقة H، ونبين أن فئتي H-π- الحلقيات (المتقاطعة) اليسرى مونوئيدية. أخيراً، نبين أن أي عائلة $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ من العناصر تمتلك بناءً لِـ π- جبرية هوبف مرافقة متقاطعة شبه مثلثية H إذا وفقط إذا كانت فئة H-π- الحلقيات المتقاطعة اليسرى على H فئة مونوئيدية مضفرة مع تضفير معرف على R.

1 Introduction

The notion of a quasitriangular Hopf algebra was introduced by Drinfeld [1] when he studied the Yang–Baxter equation. Recently, Turaev [9] introduced Hopf π-coalgebra, which generalizes the notion of Hopf algebra. Hopf π-coalgebras are used by Virelizier [11] to construct Hennings-like (see [2,4]) and Kuperberg-like (see [5]) invariants of principal π-bundles over link complements and over three-manifolds. Virelizier also studied

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algebraic properties of Hopf group coalgebras and generalized the main properties of quasitriangular Hopf algebras to the setting of quasitriangular Hopf π-coalgebras in [10].

In this paper, we first give the definitions of left H-π-modules and crossed left H-π-modules over a Hopf π-coalgebra H. Then, we show that the category of left H-π-modules is a monoidal category and so is the category of crossed left H-π-modules. Finally, for a Hopf π-coalgebra H, we show that a family $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of elements is a quasitriangular structure of H if and only if the category of crossed left H-π-modules is a braided monoidal category with braiding defined by R.

2 Basic Definitions

Throughout the paper, we let π be a discrete group (with neutral element 1) and k be a fixed field. We set $k^* = k\backslash\{0\}$. All algebras and coalgebras, π-coalgebras and Hopf π-coalgebras are defined over k. The definitions and properties of algebra, coalgebra, Hopf algebra and category can be seen in [3,6,7]. We use the standard Sweedler notation for comultiplication. The tensor product $\otimes = \otimes_k$ is always assumed to be over k. If U and V are k-spaces, $\tau_{U,V} : U \otimes V \to V \otimes U$ will denote the twist map defined by $\tau_{U,V}(u \otimes v) = v \otimes u$. The following definitions and notations in this section can be found in [8–10].

Definition 2.1 A π-coalgebra (over k) is a family $C = \{C_\alpha\}_{\alpha\in\pi}$ of k-spaces endowed with a family $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ of k-linear maps (the comultiplication) and a k-linear map $\varepsilon : C_1 \to k$ (the counit) such that Δ is coassociative in the sense that for any $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (\mathrm{id}_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$$

$$(\mathrm{id}_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = \mathrm{id}_{C_\alpha} = (\varepsilon \otimes \mathrm{id}_{C_\alpha})\Delta_{1,\alpha}.$$

Note that $(C_1, \Delta_{1,1}, \varepsilon)$ is a coalgebra in the usual sense.

Sweedler's notation We extend the Sweedler notation for a comultiplication in the following way: for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, we write $\Delta_{\alpha,\beta}(c) = \sum c_{(1,\alpha)} \otimes c_{(2,\beta)} \in C_\alpha \otimes C_\beta$, or shortly, we write $\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}$. The coassociativity axiom gives that for any $\alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta\gamma}$,

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)},$$

this element of $C_\alpha \otimes C_\beta \otimes C_\gamma$ is written as $c_{(1,\alpha)} \otimes c_{(2,\beta)} \otimes c_{(3,\gamma)}$. By iterating the procedure, we define inductively $c_{(1,\alpha_1)} \otimes c_{(2,\alpha_2)} \otimes \cdots \otimes c_{(n,\alpha_n)}$ for any $c \in C_{\alpha_1\alpha_2\ldots\alpha_n}$.

Definition 2.2 A Hopf π-coalgebra is a π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)$ endowed with a family $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of k-linear maps (the antipode) such that each H_α is an algebra with multiplication m_α and neutral element $1_\alpha \in H_\alpha$, $\varepsilon : H_1 \to k$ and $\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ are algebra homomorphisms for all $\alpha, \beta \in \pi$, and such that for any $\alpha \in \pi$,

$$m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha = m_\alpha(\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}.$$

We remark that $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is a usual Hopf algebra.

Definition 2.3 A Hopf π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon, S)$ is said to be crossed provided it is endowed with a family $\varphi = \{\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ of k-linear maps (the crossing) such that the following conditions are satisfied

1. each $\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}$ is an algebra isomorphism,
2. each φ_β preserves the comultiplication, i.e., for all $\alpha, \beta, \gamma \in \pi$,

$$(\varphi_\beta \otimes \varphi_\beta)\Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}\varphi_\beta,$$

3. each φ_β preserves the counit, i.e., $\varepsilon\varphi_\beta = \varepsilon$,
4. φ is multiplicative in the sense that $\varphi_{\beta\beta'} = \varphi_\beta\varphi_{\beta'}$ for all $\beta, \beta' \in \pi$.

Definition 2.4 A quasitriangular Hopf π-coalgebra is a crossed Hopf π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon, S, \varphi)$ endowed with a family $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of invertible elements (the R-matrix) such that the following conditions are satisfied

1. $R_{\alpha,\beta} \cdot \Delta_{\alpha,\beta}(x) = \tau_{\beta,\alpha}(\varphi_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha})\Delta_{\alpha\beta\alpha^{-1},\alpha}(x) \cdot R_{\alpha,\beta}$,
2. $(\mathrm{id}_{H_\alpha} \otimes \Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma}) = (R_{\alpha,\gamma})_{1\beta 3} \cdot (R_{\alpha,\beta})_{12\gamma}$,
3. $(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma}) = [(\mathrm{id}_{H_\alpha} \otimes \varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta 3} \cdot (R_{\beta,\gamma})_{\alpha 23}$,
4. $(\varphi_\beta \otimes \varphi_\beta)(R_{\alpha,\gamma}) = R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}$,

where $\alpha, \beta, \gamma \in \pi$, $x \in H_{\alpha\beta}$, $\tau_{\beta,\alpha}$ denotes the twist map $H_\beta \otimes H_\alpha \to H_\alpha \otimes H_\beta$. For k-spaces P, Q and $r = \Sigma p_j \otimes q_j \in P \otimes Q$, we set $r_{12\gamma} = r \otimes 1_\gamma \in P \otimes Q \otimes H_\gamma$, $r_{\alpha 23} = 1_\alpha \otimes r \in H_\alpha \otimes P \otimes Q$, and $r_{1\beta 3} = \Sigma p_j \otimes 1_\beta \otimes q_j \in P \otimes H_\beta \otimes Q$.

Note that $R_{1,1}$ is a (classical) R-matrix for the Hopf algebra H_1.

3 The Category of Left H-π-Modules

Definition 3.1 Let $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S)$ be a Hopf π-coalgebra. A left H-π-module over H is a family $M = \{M_\alpha\}_{\alpha\in\pi}$ of k-spaces such that M_α is a left H_α-module for any $\alpha \in \pi$. We denote the structure maps of left H_α-module M_α and left H-π-module M by $\Gamma_{M_\alpha} : H_\alpha \otimes M_\alpha \to M_\alpha$ and $\Gamma_M = \{\Gamma_{M_\alpha}\}_{\alpha\in\pi}$, respectively.

Definition 3.2 Let $M = \{M_\alpha\}_{\alpha\in\pi}$, $N = \{N_\alpha\}_{\alpha\in\pi}$ be left H-π-modules. A left H-π-module morphism is a family $f = \{f_\alpha : M_\alpha \to N_\alpha\}_{\alpha\in\pi}$ of k-linear maps such that f_α is an H_α-module morphism for any $\alpha \in \pi$.

Let $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S)$ be a Hopf π-coalgebra. We denote by ${}_H\mathcal{M}$ the category of all left H-π-modules, whose morphisms are left H-π-module morphisms.

Suppose that $M = \{M_\alpha\}_{\alpha\in\pi}$ and $N = \{N_\alpha\}_{\alpha\in\pi}$ are left H-π-modules. Then, $M_\beta \otimes N_\gamma$ is a left $H_\beta \otimes H_\gamma$-module for any $\beta, \gamma \in \pi$. Because $\Delta_{\beta,\gamma} : H_{\beta\gamma} \to H_\beta \otimes H_\gamma$ is an algebra morphism, $M_\beta \otimes N_\gamma$ is a left $H_{\beta\gamma}$-module with the action given by $h \cdot (m \otimes n) = \Delta_{\beta,\gamma}(h)(m \otimes n)$, $h \in H_{\beta\gamma}$, $m \in M_\beta$, $n \in N_\gamma$. So $(M \otimes N)_\alpha := \bigoplus_{\beta\gamma=\alpha} M_\beta \otimes N_\gamma$ is a left H_α-module. Thus, $M \otimes N = \{(M \otimes N)_\alpha\}_{\alpha\in\pi}$ is a left H-π-module, where the structure maps $\Gamma_{M\otimes N} = \{\Gamma_{(M\otimes N)_\alpha}\}_{\alpha\in\pi}$ are given by

$$\Gamma_{(M\otimes N)_\alpha} = \bigoplus_{\beta\gamma=\alpha} (\Gamma_{M_\beta} \otimes \Gamma_{N_\gamma})(\mathrm{id}_{H_\beta} \otimes \tau_{H_\gamma,M_\beta} \otimes \mathrm{id}_{N_\gamma})(\Delta_{\beta,\gamma} \otimes \mathrm{id}_{M_\beta} \otimes \mathrm{id}_{N_\gamma}).$$

Suppose that $P = \{P_\alpha\}_{\alpha\in\pi}$ is also a left H-π-module. Then, we have two left H-π-modules $(M \otimes N) \otimes P$ and $M \otimes (N \otimes P)$. By definition, for any $\alpha \in \pi$, we have

$$\begin{aligned}((M \otimes N) \otimes P)_\alpha &= \bigoplus_{\beta\gamma=\alpha} (M \otimes N)_\beta \otimes P_\gamma \\ &= \bigoplus_{\beta\gamma=\alpha} (\bigoplus_{\theta\sigma=\beta} (M_\theta \otimes N_\sigma) \otimes P_\gamma) \\ &= \bigoplus_{\theta\sigma\gamma=\alpha} (M_\theta \otimes N_\sigma) \otimes P_\gamma.\end{aligned}$$

and

$$\begin{aligned}(M \otimes (N \otimes P))_\alpha &= \bigoplus_{\theta\beta=\alpha} M_\theta \otimes (N \otimes P)_\beta \\ &= \bigoplus_{\theta\beta=\alpha} M_\theta \otimes (\bigoplus_{\sigma\gamma=\beta} (N_\sigma \otimes P_\gamma)) \\ &= \bigoplus_{\theta\sigma\gamma=\alpha} M_\theta \otimes (N_\sigma \otimes P_\gamma).\end{aligned}$$

Let $\theta, \sigma, \gamma \in \pi$. Because Δ is coassociative, one knows that

$$\begin{aligned}a_{\theta,\sigma,\gamma} : (M_\theta \otimes N_\sigma) \otimes P_\gamma &\to M_\theta \otimes (N_\sigma \otimes P_\gamma) \\ (m \otimes n) \otimes p &\mapsto m \otimes (n \otimes p)\end{aligned}$$

is an isomorphism of $H_{\theta\sigma\gamma}$-module. Hence, for any $\alpha \in \pi$, $a_\alpha = \bigoplus_{\theta\sigma\gamma=\alpha} a_{\theta,\sigma,\gamma}$ is an isomorphism of left H_α-module from $((M \otimes N) \otimes P)_\alpha$ to $(M \otimes (N \otimes P))_\alpha$. And $a = \{a_\alpha\}_{\alpha\in\pi} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$ is a left H-π-module isomorphism. Obviously, it is a family of natural isomorphisms.

Because H_1 is a usual Hopf algebra, k is a left H_1-module with the action given by $h \cdot \lambda = \varepsilon(h)\lambda$, $h \in H_1$, $\lambda \in k$. Hence, $K = \{K_\alpha\}_{\alpha\in\pi}$ is a left H-π-module, where $K_1 = k$, $K_\alpha = 0$, $\forall\, 1 \neq \alpha \in \pi$. For any left H-π-module M, we have $(K \otimes M)_\alpha = K_1 \otimes M_\alpha = k \otimes M_\alpha$ and $(M \otimes K)_\alpha = M_\alpha \otimes K_1 = M_\alpha \otimes k$, $\alpha \in \pi$. So we have natural isomorphisms $l_M : K \otimes M \to M$ and $r_M : M \otimes K \to M$ defined by

$$(l_M)_\alpha : k \otimes M_\alpha \to M_\alpha,\ \lambda \otimes m \mapsto \lambda m,$$
$$(r_M)_\alpha : M_\alpha \otimes k \to M_\alpha,\ m \otimes \lambda \mapsto \lambda m.$$

That is, $\{l_M\}$ and $\{r_M\}$ are two families of natural isomorphisms of left H-π-modules.

We summarize the above discussion as follows.

Theorem 3.3 $({}_H\mathcal{M}, \otimes, K, a, l, r)$ *is a monoidal category, where K is the unit object.*

Example 3.4 The category π-$Vect_k$ of π-graded vector spaces becomes the category of left H-π-modules over the trivial Hopf π-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ defined by $H_\alpha = k$.

4 The Category of Crossed Left H-π-Modules

Definition 4.1 Let H be a crossed Hopf π-coalgebra. A left H-π-module M is called crossed if there exists a family $\varphi_M = \{\varphi_{M,\beta} : M_\alpha \to M_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ of k-linear maps such that the following conditions are satisfied

1. each $\varphi_{M,\beta} : M_\alpha \to M_{\beta\alpha\beta^{-1}}$ is a vector space isomorphism,
2. each $\varphi_{M,\beta}$ preserves the action, i.e., for all $\alpha, \beta \in \pi$, $h \in H_\alpha$, $m \in M_\alpha$, $\varphi_{M,\beta}(h \cdot m) = \varphi_\beta(h) \cdot \varphi_{M,\beta}(m)$.
3. each φ_M is multiplicative in the sense that $\varphi_{M,\beta}\varphi_{M,\beta'} = \varphi_{M,\beta\beta'}$ for all $\beta, \beta' \in \pi$.

Remark 4.2 The regular left H-π-module H is always crossed if we set $\varphi_H = \{\varphi_\beta\}_{\beta\in\pi}$.

Definition 4.3 Let $M = \{M_\alpha\}_{\alpha\in\pi}$, $N = \{N_\alpha\}_{\alpha\in\pi}$ be two crossed left H-π-modules. A crossed left H-π-module morphism is a left H-π-module morphism $f = \{f_\alpha\}_{\alpha\in\pi} : M \to N$ such that $\varphi_{N,\beta} f_\alpha = f_{\beta\alpha\beta^{-1}}\varphi_{M,\beta}$ for any $\alpha, \beta \in \pi$.

Let $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S, \varphi)$ be a crossed Hopf π-coalgebra. We denote by ${}_H\mathcal{M}_{\text{crossed}}$, the category of crossed left H-π-modules and crossed left H-π-module morphisms.

Let M, N be any crossed left H-π-modules. We have already proved that $M \otimes N$ is also a left H-π-module. Define $\varphi_{M\otimes N,z} : (M \otimes N)_\alpha \longrightarrow (M \otimes N)_{z\alpha z^{-1}}$ by $\varphi_{M\otimes N,z}|_{M_\beta\otimes N_\gamma} := \varphi_{M,z}|_{M_\beta} \otimes \varphi_{N,z}|_{N_\gamma}$, where $\alpha, \beta, \gamma, z \in \pi$ with $\beta\gamma = \alpha$. Since

$$(M \otimes N)_\alpha = \bigoplus_{\beta\gamma=\alpha} M_\beta \otimes N_\gamma$$

and

$$\begin{aligned}(M \otimes N)_{z\alpha z^{-1}} &= \bigoplus_{z\beta\gamma z^{-1}=z\alpha z^{-1}} M_{z\beta z^{-1}} \otimes N_{z\gamma z^{-1}}\\ &= \bigoplus_{\beta\gamma=\alpha} M_{z\beta z^{-1}} \otimes N_{z\gamma z^{-1}},\end{aligned}$$

$\varphi_{M\otimes N,z}$ is a well defined k-linear isomorphism from $(M \otimes N)_\alpha$ to $(M \otimes N)_{z\alpha z^{-1}}$ for any $\alpha, z \in \pi$. Moreover, for any $h \in H_\alpha$, $m \in M_\beta$ and $n \in N_\gamma$, we have

$$\begin{aligned}\varphi_{M\otimes N,z}(h \cdot (m \otimes n)) &= (\varphi_{M,z} \otimes \varphi_{N,z})(h_{(1,\beta)} \cdot m \otimes h_{(2,\gamma)} \cdot n)\\ &= \varphi_{M,z}(h_{(1,\beta)} \cdot m) \otimes \varphi_{N,z}(h_{(2,\gamma)} \cdot n)\\ &= \varphi_z(h_{(1,\beta)}) \cdot \varphi_{M,z}(m) \otimes \varphi_z(h_{(2,\gamma)}) \cdot \varphi_{N,z}(n).\end{aligned}$$

and

$$\begin{aligned}\varphi_z(h)\cdot\varphi_{M\otimes N,z}(m\otimes n) &= \varphi_z(h)\cdot(\varphi_{M,z}(m)\otimes\varphi_{N,z}(n))\\ &= \Delta_{z\beta z^{-1},z\gamma z^{-1}}(\varphi_z(h))\cdot(\varphi_{M,z}(m)\otimes\varphi_{N,z}(n)).\end{aligned}$$

Since φ_z preserves the comultiplication, we have

$$\Delta_{z\beta z^{-1},z\gamma z^{-1}}(\varphi_z(h)) = (\varphi_z\otimes\varphi_z)\Delta_{\beta,\gamma}(h) = \sum\varphi_z(h_{(1,\beta)})\otimes\varphi_z(h_{(2,\gamma)}).$$

Hence $\varphi_{M\otimes N,z}(h\cdot(m\otimes n)) = \varphi_z(h)\cdot\varphi_{M\otimes N,z}(m\otimes n)$. It is easy to see that $\varphi_{M\otimes N,z}\varphi_{M\otimes N,z'} = \varphi_{M\otimes N,zz'}$, i.e., $\varphi_{M\otimes N}$ is multiplicative. Thus, $M\otimes N$ is a crossed left H-π-module.

Now let M, N and P be crossed left H-π-module. Then, one can easily check that $\varphi_{M\otimes(N\otimes P),z}a_\alpha = a_{z\alpha z^{-1}}\varphi_{(M\otimes N)\otimes P,z}$ for any $\alpha, z\in\pi$, and hence $a : (M\otimes N)\otimes P\to M\otimes(N\otimes P)$ is a crossed left H-π-module morphism.

Note that $K_\alpha = K_{\beta\alpha\beta^{-1}} = k$ if $\alpha = 1$, and $K_\alpha = K_{\beta\alpha\beta^{-1}} = 0$ if $\alpha\neq 1$. Since $\varphi_1 = \mathrm{id} : H_\alpha\to H_\alpha$ is the identity map, the unit object $K = \{K_\alpha\}_{\alpha\in\pi}$ is a crossed left H-π-module if we set $\varphi_{K,\beta} = \mathrm{id} : K_\alpha\to K_{\beta\alpha\beta^{-1}}$. Then, one can easily check that the left and right unit constraints $l = \{l_M\}$ and $r = \{r_M\}$ are crossed left H-π-module morphisms. Thus, we have the following theorem.

Theorem 4.4 *$({}_H\mathcal{M}_{\mathrm{crossed}}, \otimes, K, a, l, r)$ is a monoidal category, where K is the unit object.*

Remark 4.5 There exists a natural monoidal functor F from $({}_H\mathcal{M}_{\mathrm{crossed}}, \otimes, K, a, l, r)$ to $({}_H\mathcal{M}, \otimes, K, a, l, r)$ defined by forgetting the crossing.

Example 4.6 Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be the trivial Hopf π-coalgebra defined by $H_\alpha = k$ as in Example 3.4. Then, H is a crossed Hopf π-coalgebra with the unique crossing $\varphi = \{\varphi_\beta = \mathrm{id} : H_\alpha\to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$. A left H-π-module is a π-graded vector space $V = \{V_\alpha\}_{\alpha\in\pi}$. Now assume that V is a crossed left H-π-module with a crossing $\varphi_V = \{\varphi_{V,\beta} : V_\alpha\to V_{\beta\alpha\beta^{-1}} | \alpha,\beta\in\pi\}$. Then, $\dim V_\alpha = \dim V_{\beta\alpha\beta^{-1}}$ for all $\alpha,\beta\in\pi$. Let $\mathcal{K}(\pi)$ denote the set of conjugate classes of π and assume $m_C := \dim V_\alpha < \infty$ for any $\alpha\in C\in\mathcal{K}(\pi)$. For any $\alpha\in\pi$, let us fix a basis of V_α. Then, a k-linear isomorphism $\varphi_{V,\beta} : V_\alpha\to V_{\beta\alpha\beta^{-1}}$ is determined by an invertible matrix $X_{\alpha,\beta}$ in $GL_{m_C}(k)$, where $\alpha\in C\in\mathcal{K}(\pi)$ and $\beta\in\pi$. Thus, a crossing of V is determined by a map

$$\varphi : \pi\times\pi\to\bigcup_{C\in\mathcal{K}(\pi)} GL_{m_C}(k)$$

with $\varphi(\alpha,\beta)\in GL_{m_C}(k)$ when $\alpha\in C\in\mathcal{K}(\pi)$, satisfying $\varphi(\beta\alpha\beta^{-1},\gamma)\varphi(\alpha,\beta) = \varphi(\alpha,\gamma\beta)$ for all $\alpha,\beta,\gamma\in\pi$.

5 The Braided Monoidal Category

Throughout the following, assume that $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S, \varphi)$ is a crossed Hopf π-coalgebra, and let $R = \{R_{\beta,\gamma}\in H_\beta\otimes H_\gamma\}_{\beta,\gamma\in\pi}$ be a family of elements. Let M and N be any crossed left H-π-modules. For any $\beta,\gamma\in\pi$, define

$$c_{M_\beta,N_\gamma} : M_\beta\otimes N_\gamma\to N_{\beta\gamma\beta^{-1}}\otimes M_\beta$$

by

$$c_{M_\beta,N_\gamma}(m\otimes n) = (\varphi_{N,\beta}\otimes\mathrm{id}_{M_\beta})\tau_{\beta,\gamma}(R_{\beta,\gamma}\cdot(m\otimes n)) \tag{1}$$

where $m\in M_\beta$ and $n\in N_\gamma$. For any $\alpha\in\pi$, define

$$(c_{M,N})_\alpha : (M\otimes N)_\alpha = \bigoplus_{\beta\gamma=\alpha} M_\beta\otimes N_\gamma\to(N\otimes M)_\alpha = \bigoplus_{\beta\gamma=\alpha} N_{\beta\gamma\beta^{-1}}\otimes M_\beta.$$

by $(c_{M,N})_\alpha = \bigoplus_{\beta\gamma=\alpha} c_{M_\beta,N_\gamma}$. Then it is obvious that $(c_{M,N})_\alpha$ is a k-linear isomorphism for any $\alpha\in\pi$ if and only if so is c_{M_β,N_γ} for any $\beta,\gamma\in\pi$.

Lemma 5.1 *With the above notations, we have*

1. $(c_{M,N})_\alpha$ *is a* k*-linear isomorphism for any crossed left* H*-*π*-modules* M *and* N*, and* $\alpha \in \pi$ *if and only if* R *is a family of invertible elements.*
2. $c_{M,N} : M \otimes N \to N \otimes M$ *is a left* H*-*π*-module morphism for any crossed left* H*-*π*-module* M *and* N *if and only if*

$$R_{\beta,\gamma} \cdot \Delta_{\beta,\gamma}(h) = \tau_{\gamma,\beta}(\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h) \cdot R_{\beta,\gamma}$$

for all $\beta, \gamma \in \pi$ *and* $h \in H_{\beta\gamma}$.

Proof 1. Assume that $R = \{R_{\beta,\gamma} \in H_\beta \otimes H_\gamma\}_{\beta,\gamma\in\pi}$ is a family of invertible elements. Then obviously, c_{M_β,N_γ} is a k-linear isomorphism for any $\beta, \gamma \in \pi$ since so is $\varphi_{N,\beta}$. Conversely, let $M = N = H$. Because φ_β is an isomorphism, from the hypothesis one knows that the map $H_\beta \otimes H_\gamma \to H_\beta \otimes H_\gamma$, $x \otimes y \mapsto R_{\beta,\gamma}(x \otimes y)$ is a k-linear isomorphism for all $\beta, \gamma \in \pi$. It follows that $R_{\beta,\gamma}$ is an invertible element in the algebra $H_\beta \otimes H_\gamma$.
2. From [10] we know that $\varphi_\beta\varphi_{\beta^{-1}} = \varphi_1 = \mathrm{id}$. Let M and N be crossed left H-π-modules. Then it is clear that $(c_{M,N})_\alpha$ is an H_α-module morphism for any $\alpha \in \pi$ if and only if c_{M_β,N_γ} is an $H_{\beta\gamma}$-module morphism for any $\beta, \gamma \in \pi$. Let $m \in M_\beta$, $n \in N_\gamma$ and $h \in H_{\beta\gamma}$ with $\beta, \gamma \in \pi$. Then, we have

$$\begin{aligned} c_{M_\beta,N_\gamma}(h \cdot (m \otimes n)) &= c_{M_\beta,N_\gamma}(\Delta_{\beta,\gamma}(h)(m \otimes n)) \\ &= (\varphi_{N,\beta} \otimes id_{M_\beta})\tau_{\beta,\gamma}(R_{\beta,\gamma}(\Delta_{\beta,\gamma}(h)(m \otimes n))) \\ &= (\varphi_{N,\beta} \otimes id_{M_\beta})\tau_{\beta,\gamma}((R_{\beta,\gamma}\Delta_{\beta,\gamma}(h))(m \otimes n)) \end{aligned}$$

and

$$\begin{aligned} &h \cdot c_{M_\beta,N_\gamma}(m \otimes n) \\ &\quad = \Delta_{\beta\gamma\beta^{-1},\beta}(h) \cdot c_{M_\beta,N_\gamma}(m \otimes n) \\ &\quad = ((\varphi_\beta \otimes \mathrm{id}_{H_\beta})(\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h)) \cdot (\varphi_{N,\beta} \otimes \mathrm{id}_{M_\beta})(\tau_{\beta,\gamma}(R_{\beta,\gamma}(m \otimes n))) \\ &\quad = (\varphi_\beta \otimes \mathrm{id}_{H_\beta})((\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h)) \cdot (\varphi_{N,\beta} \otimes \mathrm{id}_{M_\beta})(\tau_{\beta,\gamma}(R_{\beta,\gamma}(m \otimes n))) \\ &\quad = (\varphi_{N,\beta} \otimes \mathrm{id}_{M_\beta})(((\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h)) \cdot \tau_{\beta,\gamma}(R_{\beta,\gamma}(m \otimes n))) \\ &\quad = (\varphi_{N,\beta} \otimes \mathrm{id}_{M_\beta})(\tau_{\beta,\gamma}(\tau_{\gamma,\beta}((\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h))) \cdot \tau_{\beta,\gamma}(R_{\beta,\gamma} \cdot (m \otimes n))) \\ &\quad = (\varphi_{N,\beta} \otimes \mathrm{id}_{M_\beta})\tau_{\beta,\gamma}(\tau_{\gamma,\beta}((\varphi_{\beta^{-1}} \otimes \mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h)) \cdot R_{\beta,\gamma} \cdot (m \otimes n)) \end{aligned}$$

Because $\varphi_{N,\beta}$ is an isomorphism, $c_{M_\beta,N_\gamma}(h\cdot(m\otimes n)) = h\cdot c_{M_\beta,N_\gamma}(m\otimes n)$ if and only if $(R_{\beta,\gamma}\Delta_{\beta,\gamma}(h))(m\otimes n) = \tau_{\gamma,\beta}((\varphi_{\beta^{-1}}\otimes\mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h))\cdot R_{\beta,\gamma}\cdot(m\otimes n)$. Thus, if $R_{\beta,\gamma}\cdot\Delta_{\beta,\gamma}(h) = \tau_{\gamma,\beta}(\varphi_{\beta^{-1}}\otimes\mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h)\cdot R_{\beta,\gamma}$ for all $h \in H_{\beta\gamma}$, then c_{M_β,N_γ} is an isomorphism of left $H_{\beta\gamma}$-modules. Conversely, let $M = N = H$, $m = 1_\beta \in H_\beta$ and $n = 1_\gamma \in H_\gamma$. Then, from $(R_{\beta,\gamma}\Delta_{\beta,\gamma}(h))(m\otimes n) = \tau_{\gamma,\beta}((\varphi_{\beta^{-1}}\otimes\mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h))\cdot R_{\beta,\gamma}\cdot(m\otimes n)$, one gets $R_{\beta,\gamma}\cdot\Delta_{\beta,\gamma}(h) = \tau_{\gamma,\beta}((\varphi_{\beta^{-1}}\otimes\mathrm{id}_{H_\beta})\Delta_{\beta\gamma\beta^{-1},\beta}(h))\cdot R_{\beta,\gamma}$. □

Lemma 5.2 *The following two statements are equivalent*:

1. $\varphi_{N\otimes M,z}(c_{M,N})_\alpha = (c_{M,N})_{z\alpha z^{-1}}\varphi_{M\otimes N,z}$ *for any crossed left* H*-*π*-modules* M *and* N*, and* $\alpha, z \in \pi$.
2. $(\varphi_z \otimes \varphi_z)(R_{\beta,\gamma}) = R_{z\beta z^{-1},z\gamma z^{-1}}$ *for any* $\beta, \gamma, z \in \pi$.

Proof Let M and N be crossed left H-π-modules. For any $\beta, \gamma, z \in \pi$, $m \in M_\beta$ and $n \in N_\gamma$, we have

$$\begin{aligned} &\varphi_{N\otimes M,z}(c_{M,N})_{\beta\gamma}(m \otimes n) \\ &\quad = (\varphi_{N,z} \otimes \varphi_{M,z})(c_{M_\beta,N_\gamma})(m \otimes n) \\ &\quad = (\varphi_{N,z} \otimes \varphi_{M,z})(\varphi_{N,\beta} \otimes \mathrm{id}_{M_\beta})\tau_{\beta,\gamma}(R_{\beta,\gamma} \cdot (m \otimes n)) \\ &\quad = (\varphi_{N,z}\varphi_{N,\beta} \otimes \varphi_{M,z})(\tau_{\beta,\gamma}(R_{\beta,\gamma}) \cdot (n \otimes m)) \\ &\quad = (\varphi_{N,z\beta} \otimes \varphi_{M,z})(\tau_{\beta,\gamma}(R_{\beta,\gamma}) \cdot (n \otimes m)) \\ &\quad = (\varphi_{N,z\beta z^{-1}} \otimes \mathrm{id}_{M_{z\beta z^{-1}}})(\varphi_{N,z} \otimes \varphi_{M,z})(\tau_{\beta,\gamma}(R_{\beta,\gamma}) \cdot (n \otimes m)) \\ &\quad = (\varphi_{N,z\beta z^{-1}} \otimes \mathrm{id}_{M_{z\beta z^{-1}}})((\varphi_z \otimes \varphi_z)(\tau_{\beta,\gamma}(R_{\beta,\gamma})) \cdot (\varphi_{N,z}(n) \otimes \varphi_{M,z}(m))) \\ &\quad = (\varphi_{N,z\beta z^{-1}} \otimes \mathrm{id}_{M_{z\beta z^{-1}}})(\tau_{z\beta z^{-1},z\gamma z^{-1}}((\varphi_z \otimes \varphi_z)(R_{\beta,\gamma})) \cdot (\varphi_{N,z}(n) \otimes \varphi_{M,z}(m))) \end{aligned}$$

and

$$\begin{aligned}
&(c_{M,N})_{z\beta\gamma z^{-1}}\varphi_{M\otimes N,z}(m\otimes n)\\
&\quad = c_{M_{z\beta z^{-1}},N_{z\gamma z^{-1}}}(\varphi_{M,z}(m)\otimes\varphi_{N,z}(n))\\
&\quad = (\varphi_{N,z\beta z^{-1}}\otimes \mathrm{id}_{M_{z\beta z^{-1}}})\tau_{z\beta z^{-1},z\gamma z^{-1}}(R_{z\beta z^{-1},z\gamma z^{-1}}\cdot(\varphi_{M,z}(m)\otimes\varphi_{N,z}(n)))\\
&\quad = (\varphi_{N,z\beta z^{-1}}\otimes \mathrm{id}_{M_{z\beta z^{-1}}})(\tau_{z\beta z^{-1},z\gamma z^{-1}}(R_{z\beta z^{-1},z\gamma z^{-1}})\cdot(\varphi_{N,z}(n)\otimes\varphi_{M,z}(m))).
\end{aligned}$$

It follows that Part (2) implies Part (1). Now assume that Part (1) is satisfied. Let $M=N=H$, $m=1_\beta$ and $n=1_\gamma$. Then from $\varphi_{N\otimes M,z}(c_{M,N})_{\beta\gamma}(m\otimes n)=(c_{M,N})_{z\beta\gamma z^{-1}}\varphi_{M\otimes N,z}(m\otimes n)$, one gets

$$\begin{aligned}
&(\varphi_{z\beta z^{-1}}\otimes \mathrm{id}_{H_{z\beta z^{-1}}})(\tau_{z\beta z^{-1},z\gamma z^{-1}}((\varphi_z\otimes\varphi_z)(R_{\beta,\gamma})))\\
&\quad = (\varphi_{z\beta z^{-1}}\otimes \mathrm{id}_{H_{z\beta z^{-1}}})(\tau_{z\beta z^{-1},z\gamma z^{-1}}(R_{z\beta z^{-1},z\gamma z^{-1}})).
\end{aligned}$$

Since $\varphi_{z\beta z^{-1}}$ and $\tau_{z\beta z^{-1},z\gamma z^{-1}}$ are isomorphisms, it follows that $(\varphi_z\otimes\varphi_z)(R_{\beta,\gamma})=R_{z\beta z^{-1},z\gamma z^{-1}}$. □

Lemma 5.3 *The following two statements hold*:

1. $c_{M,N\otimes P}=(\mathrm{id}_N\otimes c_{M,P})(c_{M,N}\otimes \mathrm{id}_P)$ *for all crossed left* H-π*-modules* M, N *and* P, *if and only if for all* $\alpha,\beta,\gamma\in\pi$,
$$(\mathrm{id}_{H_\alpha}\otimes\Delta_{\beta,\gamma})(R_{\alpha,\beta\gamma})=(R_{\alpha,\gamma})_{1\beta3}\cdot(R_{\alpha,\beta})_{12\gamma}.$$
2. $c_{M\otimes N,P}=(c_{M,P}\otimes \mathrm{id}_N)(\mathrm{id}_M\otimes c_{N,P})$ *for all crossed left* H-π*-modules* M, N *and* P, *if and only if for all* $\alpha,\beta,\gamma\in\pi$,
$$(\Delta_{\alpha,\beta}\otimes \mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma})=[(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta3}\cdot(R_{\beta,\gamma})_{\alpha23}.$$

Proof We only prove Part (2). The proof of Part (1) is similar.

Let M, N and P be crossed left H-π-modules. For any $m\in M_\alpha$, $n\in N_\beta$ and $p\in P_\gamma$ with $\alpha,\beta,\gamma\in\pi$, we have

$$\begin{aligned}
&((c_{M,P}\otimes \mathrm{id}_N)(\mathrm{id}_M\otimes c_{N,P}))_{\alpha\beta\gamma}(m\otimes n\otimes p)\\
&\quad = (c_{M_\alpha,P_{\beta\gamma\beta^{-1}}}\otimes \mathrm{id}_{N_\beta})(\mathrm{id}_{M_\alpha}\otimes c_{N_\beta,P_\gamma})(m\otimes n\otimes p)\\
&\quad = (c_{M_\alpha,P_{\beta\gamma\beta^{-1}}}\otimes \mathrm{id}_{N_\beta})((\mathrm{id}_{M_\alpha}\otimes\varphi_{P,\beta}\otimes \mathrm{id}_{N_\beta})(\mathrm{id}_{M_\alpha}\otimes\tau_{\beta,\gamma})((R_{\beta,\gamma})_{\alpha23}\cdot(m\otimes n\otimes p)))\\
&\quad = (\varphi_{P,\alpha}\otimes \mathrm{id}_{M_\alpha}\otimes \mathrm{id}_{N_\beta})(\tau_{\alpha,\beta\gamma\beta^{-1}}\otimes \mathrm{id}_{N_\beta})\\
&\qquad \times((R_{\alpha,\beta\gamma\beta^{-1}})_{12\beta}\cdot((\mathrm{id}_{M_\alpha}\otimes\varphi_{P,\beta}\otimes \mathrm{id}_{N_\beta})(\mathrm{id}_{M_\alpha}\otimes\tau_{\beta,\gamma})((R_{\beta,\gamma})_{\alpha23}\cdot(m\otimes n\otimes p))))\\
&\quad = (\varphi_{P,\alpha}\otimes \mathrm{id}_{M_\alpha}\otimes \mathrm{id}_{N_\beta})(\tau_{\alpha,\beta\gamma\beta^{-1}}\otimes \mathrm{id}_{N_\beta})((\mathrm{id}_{H_\alpha}\otimes\varphi_\beta\otimes \mathrm{id}_{H_\beta})([(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{12\beta})\cdot\\
&\qquad \times((\mathrm{id}_{M_\alpha}\otimes\varphi_{P,\beta}\otimes \mathrm{id}_{N_\beta})(\mathrm{id}_{M_\alpha}\otimes\tau_{\beta,\gamma})((R_{\beta,\gamma})_{\alpha23}\cdot(m\otimes n\otimes p))))\\
&\quad = (\varphi_{P,\alpha}\otimes \mathrm{id}_{M_\alpha}\otimes \mathrm{id}_{N_\beta})(\tau_{\alpha,\beta\gamma\beta^{-1}}\otimes \mathrm{id}_{N_\beta})(\mathrm{id}_{M_\alpha}\otimes\varphi_{P,\beta}\otimes \mathrm{id}_{N_\beta})\\
&\qquad \times([(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{12\beta}\cdot((\mathrm{id}_{M_\alpha}\otimes\tau_{\beta,\gamma})((R_{\beta,\gamma})_{\alpha23}\cdot(m\otimes n\otimes p))))\\
&\quad = (\varphi_{P,\alpha}\otimes \mathrm{id}_{M_\alpha}\otimes \mathrm{id}_{N_\beta})(\varphi_{P,\beta}\otimes \mathrm{id}_{M_\alpha}\otimes \mathrm{id}_{N_\beta})(\tau_{\alpha,\gamma}\otimes \mathrm{id}_{N_\beta})\\
&\qquad \times(\mathrm{id}_{M_\alpha}\otimes\tau_{\beta,\gamma})([(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta3}\cdot((R_{\beta,\gamma})_{\alpha23}\cdot(m\otimes n\otimes p)))\\
&\quad = (\varphi_{P,\alpha\beta}\otimes \mathrm{id}_{M_\alpha\otimes N_\beta})(\tau_{\alpha,\gamma}\otimes \mathrm{id}_{N_\beta})(\mathrm{id}_{M_\alpha}\otimes\tau_{\beta,\gamma})\\
&\qquad \times(([(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta3}(R_{\beta,\gamma})_{\alpha23})\cdot(m\otimes n\otimes p))
\end{aligned}$$

and

$$\begin{aligned}
&(c_{M\otimes N,P})_{\alpha\beta\gamma}(m\otimes n\otimes p)\\
&\quad = (\varphi_{P,\alpha\beta}\otimes \mathrm{id}_{M_\alpha\otimes N_\beta})\tau_{\alpha\beta,\gamma}(R_{\alpha\beta,\gamma}\cdot((m\otimes n)\otimes p))\\
&\quad = (\varphi_{P,\alpha\beta}\otimes \mathrm{id}_{M_\alpha\otimes N_\beta})(\tau_{\alpha,\gamma}\otimes \mathrm{id}_{N_\beta})(\mathrm{id}_{M_\alpha}\otimes\tau_{\beta,\gamma})((\Delta_{\alpha,\beta}\otimes \mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma})\cdot(m\otimes n\otimes p)).
\end{aligned}$$

Thus, if $(\Delta_{\alpha,\beta}\otimes \mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma})=[(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta3}\cdot(R_{\beta,\gamma})_{\alpha23}$ for all $\alpha,\beta,\gamma\in\pi$, then $c_{M\otimes N,P}=(c_{M,P}\otimes \mathrm{id}_N)(\mathrm{id}_M\otimes c_{N,P})$ for all crossed left H-π-modules M, N and P. Conversely, let $M=N=P=H$, $m=1_\alpha$, $n=1_\beta$ and $p=1_\gamma$. Then from $(c_{M\otimes N,P})_{\alpha\beta\gamma}(m\otimes n\otimes p)=((c_{M,P}\otimes \mathrm{id}_N)(\mathrm{id}_M\otimes c_{N,P}))_{\alpha\beta\gamma}(m\otimes n\otimes p)$, one gets $(\Delta_{\alpha,\beta}\otimes \mathrm{id}_{H_\gamma})(R_{\alpha\beta,\gamma})=[(\mathrm{id}_{H_\alpha}\otimes\varphi_{\beta^{-1}})(R_{\alpha,\beta\gamma\beta^{-1}})]_{1\beta3}\cdot(R_{\beta,\gamma})_{\alpha23}$ since $\varphi_{P,\alpha\beta}$, $\tau_{\alpha,\gamma}$ and $\tau_{\beta,\gamma}$ are isomorphisms. □

Theorem 5.4 *Let $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S, \varphi)$ be a crossed Hopf π-coalgebra, and let $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ be a family of elements. Then, the monoidal category $({}_H\mathcal{M}_{\text{crossed}}, \otimes, K, a, l, r)$ of crossed left H-π-modules is a braided monoidal category with the braiding c if and only if $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S, \varphi, R)$ is a quasitriangular Hopf π-coalgebra, where c is defined by R as in (1).*

Proof If c is a braiding of the monoidal category $({}_H\mathcal{M}_{\text{crossed}}, \otimes, K, a, l, r)$, then it follows from Lemmas 5.1, 5.2 and 5.3 that R is a quasitriangular structure. Conversely, assume that R is a quasitriangular structure. Then by Lemmas 5.1, 5.2 and 5.3, it is enough to show that $c = \{c_{M,N}\}$ is natural.

Let $f : M \to M'$ and $g : N \to N'$ be two crossed morphisms of crossed left H-π-modules. For any $m \in M_\alpha$ and $n \in N_\beta$, where $\alpha, \beta \in \pi$, we have

$$\begin{aligned}
((g\otimes f)c_{M,N})_{\alpha\beta}(m\otimes n) &= (g_{\alpha\beta\alpha^{-1}}\otimes f_\alpha)c_{M_\alpha,N_\beta}(m\otimes n)\\
&= (g_{\alpha\beta\alpha^{-1}}\otimes f_\alpha)(\varphi_{N,\alpha}\otimes \mathrm{id}_{M_\alpha})\tau_{\alpha,\beta}(R_{\alpha,\beta}\cdot(m\otimes n))\\
&= (g_{\alpha\beta\alpha^{-1}}\otimes f_\alpha)(\varphi_{N,\alpha}\otimes \mathrm{id}_{M_\alpha})(\tau_{\alpha,\beta}(R_{\alpha,\beta})\cdot(n\otimes m))\\
&= (g_{\alpha\beta\alpha^{-1}}\otimes f_\alpha)(((\varphi_\alpha\otimes \mathrm{id}_{H_\alpha})\tau_{\alpha,\beta}(R_{\alpha,\beta}))\cdot(\varphi_{N,\alpha}(n)\otimes m))\\
&= ((\varphi_\alpha\otimes \mathrm{id}_{H_\alpha})\tau_{\alpha,\beta}(R_{\alpha,\beta}))\cdot((g_{\alpha\beta\alpha^{-1}}\otimes f_\alpha)(\varphi_{N,\alpha}(n)\otimes m))\\
&= ((\varphi_\alpha\otimes \mathrm{id}_{H_\alpha})\tau_{\alpha,\beta}(R_{\alpha,\beta}))\cdot(g_{\alpha\beta\alpha^{-1}}\varphi_{N,\alpha}(n)\otimes f_\alpha(m))\\
&= ((\varphi_\alpha\otimes \mathrm{id}_{H_\alpha})\tau_{\alpha,\beta}(R_{\alpha,\beta}))\cdot(\varphi_{N',\alpha}g_\beta(n)\otimes f_\alpha(m))
\end{aligned}$$

and

$$\begin{aligned}
(c_{M',N'}(f\otimes g))_{\alpha\beta}(m\otimes n) &= c_{M'_\alpha,N'_\beta}(f_\alpha\otimes g_\beta)(m\otimes n)\\
&= c_{M'_\alpha,N'_\beta}(f_\alpha(m)\otimes g_\beta(n))\\
&= (\varphi_{N',\alpha}\otimes \mathrm{id}_{M'_\alpha})\tau_{\alpha,\beta}(R_{\alpha,\beta}\cdot(f_\alpha(m)\otimes g_\beta(n)))\\
&= (\varphi_{N',\alpha}\otimes \mathrm{id}_{M'_\alpha})(\tau_{\alpha,\beta}(R_{\alpha,\beta})\cdot(g_\beta(n)\otimes f_\alpha(m)))\\
&= ((\varphi_\alpha\otimes \mathrm{id}_{H_\alpha})\tau_{\alpha,\beta}(R_{\alpha,\beta}))\cdot((\varphi_{N',\alpha}\otimes \mathrm{id}_{M'_\alpha})(g_\beta(n)\otimes f_\alpha(m)))\\
&= ((\varphi_\alpha\otimes \mathrm{id}_{H_\alpha})\tau_{\alpha,\beta}(R_{\alpha,\beta}))\cdot(\varphi_{N',\alpha}g_\beta(n)\otimes f_\alpha(m)).
\end{aligned}$$

Hence $(g\otimes f)c_{M,N} = c_{M',N'}(f\otimes g)$. This completes the proof. □

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