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On ring extensions pinched at the integral closure

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Abstract

The notion of the *unique maximal overring* of an integral domain is introduced and the domains for which the integral closure is the unique maximal overring are characterized. We characterize ring extensions $J \subseteq S$, satisfying a certain technical condition, for which $|\mathrm{MSupp}_J(S/J)| = 1$. The applications of the preceding result include, an extension of Ben Nasr and Zeidi's result (Nasr and Zeidi 2017, Theorem 2.7) (*Ben Nasr and Zeidi, When is the integral closure comparable to all intermediate rings, Bulletin of Australian Mathematical Society, 95 (2017), 14–21*) for a ring extension $R \subseteq S$ to be pinched at the integral closure $\overline{R}$ of R in S and an extension of a result by Dobbs and Jarboui (Dobbs and Noomen 2022, Theorem 3.9) (*D.E. Dobbs and Noomen Jarboui, Prüfer-closed extensions and FCP λ-extensions of commutative rings, Palestine Journal of Mathematics, Vol. 11(3) (2022), 362–378.*). We generalize the main result of Gilbert ((6, Proposition 2.8, Chapter II), *Extensions of Commutative Rings With Linearly Ordered Intermediate Rings, Doctoral Dissertation, University of Tennessee-Knoxville*). This result plays a key role in Gilbert's alternative proof of Ferrand and J Olivier's characterization of minimal ring extensions. We are also able to partially characterize the local rings which are not Prüfer closed and have the unique minimal overring without assuming any finiteness hypothesis.

Keywords Comparable overring · Integral closure · Valuation domain · Pseudovaluation domain · I-domain · Prüfer hull · Prüfer-closed

Mathematics Subject Classification Primary 13B02 · 13A18; Secondary 13A35 · 13B25 · 13B35

1 Introduction

In an important paper titled "Intersections of quotient rings of an integral domain", Gilmer and Heinzer were led to the definition (See (Gilmer and Heinzer 1967, Page 132)) of the *unique minimal overring* of an integral domain. They proved that if

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(R, M) is a local non-integrally closed domain such that the integral closure $\overline{R}$ of R is Prüfer and there are no domains properly contained between R and $\overline{R}$, then $\overline{R}$ is the unique minimal overring of R. In general they asked the following question: Which domains possess the unique minimal overring? The set of overrings of an integral domain is a key invariant for understanding the underlying base domain. The notion of the unique minimal overring is closely related to the order structure of the set of overrings (Nasr and Zeidi 2017; Jarboui 2022; Ayache 2022), QQR-domains (Gilmer and Heinzer 1967), "normal pairs" (Jarboui 2022) and minimal morphisms [5], among other things. We believe that the question of Gilmer and Heinzer in full generality has still not been settled yet.

Motivated by the notion of the unique minimal overring of an integral domain, recently Nasr and Zeidi (2017) studied the comparability of the integral closure to all the intermediate overrings, under an additional finiteness hypothesis (i.e. under "Finite Chain Property" or FCP) on the set of overrings of a domain. They characterized domains possessing the unique minimal overring. Jarboui (2022) answered the question of Gilmer and Heinzer in a case when a certain pair of rings is a "normal pair".

Gilbert [6] introduced the notion of "λ-extension of rings" which are defined as follows: a ring extension $R \subseteq S$ is called *λ-extension of rings* if the set $[R, S]$, the set of all R-subalgebras of S, is totally ordered under inclusion. The prototypical example being the extension $V \subset K$, where V is a valuation domain with the quotient field K. These extensions are also known as *chained extensions*. The class of λ-extensions forms a proper subclass of the class of *Δ-extensions* (see Gilmer and Huckaba (1974) for the Definition) and properly contains the class of minimal extensions. The class of *minimal ring extensions* (see [5] for the Definition) is a proper subclass of the class of λ-extensions. It is interesting to note here that the notion of a fixed comparable overring introduced by Ayache (2022) generalizes the notion of λ-extension of domains $R \subseteq S$, if S coincides with the quotient field of R.

Let K be a field, if we set $D = K[[T]]$ and $D_n = K + T^n K[[T]]$, for each positive integer n, then (Gilmer and Huckaba 1974, Proposition 11) shows that $D_n \subset K((T))$ is a λ-extension if and only if $n \leq 3$. Thus for $n \geq 4$, it follows from (Gilmer and Huckaba 1974, Proposition 8) that $D_n \subset K((T))$ is an extension having a fixed ring $\overline{D_n} = K[[T]]$ comparable to all the intermediate rings, but the extension $D_n \subset K((T))$ is not a λ-extension. Thus the class of extensions having a fixed ring comparable to all intermediate rings is strictly larger than the class of λ-extensions.

In the light of the preceding example it thus becomes desirable to investigate the extent to which the work of Gilbert [6] can be generalized to the settings of ring extensions possessing a fixed comparable overring.

In Sect. 2 we introduce the notion of the *unique maximal overring* (see Definition 2.5) of an integral domain which can be seen as a dual to the notion of the unique minimal overring mentioned above. We characterize the domains for which integral closure is the unique maximal overring. We also prove some additional related results in the same section. In the begining of Sect. 3 we characterize ring extensions $J \subseteq S$, satisfying a certain technical condition, for which $|\mathrm{MSupp}_J(S/J)| = 1$. The applications of the preceding result include an extension of M. Ben Nasr and N. Zeidi's result (Nasr and Zeidi 2017, Theorem 2.7) for a ring extension $R \subseteq S$ to be pinched at the integral closure $\overline{R}$ of R in S and an extension of a result by Dobbs and Jarboui (Dobbs and Noomen 2022, Theorem 3.9). We prove a necessary and a sufficient condition for a normal pair to be pinched at some intermediate ring. Additionally we also generalize one of the main results of Gilbert (6, Proposition 2.8, Chapter II), which says that if $R \subset T$ is a λ-extension, then the number of nonzero factors in the direct product decomposition of T is exactly two. Surprisingly the conclusion continues to hold even if the extension $R \subset T$ has some comparable overring (λ-extension hypothesis is not needed) (see Theorem 3.8). Next we prove that if a ring extension $R \subset S$, where (R, M) is a local ring satisfying $R \neq \widehat{R}_S$, has a unique minimal overring then Spec(R) has a very special property. A partial converse is also proved.

All the rings considered in this paper are commutative, and are assumed to contain the unity. If R is a ring, then Spec(R) (respectively; Max(R)) denotes the set of prime (respectively; maximal) ideals of R. A ring R is called local, if R has a unique maximal ideal. By an S-overring of a ring R, we mean a ring T such that $R \subseteq T \subseteq S$. If S happens to be the field of fractions K of an integral domain or in general the total quotient ring of R, we simply call T an overring of R. If $R \subseteq S$ is a ring extensions, then the set of all the R-subalgebras of S is denoted by $[R, S]$. The integral closure of R in S is denoted by $\overline{R}$ and the Prüfer hull of R in S is denoted by $\widehat{R}_S$. Sometimes the field of fractions of an integral domain R is denoted by qf(R). The conductor of R in S is denoted by $(R : S)$; $\mathrm{Supp}_R(S/R) = \{P \in \mathrm{Spec}(R) | R_P \neq S_P\}$; $\mathrm{MSupp}_R(S/R) = \mathrm{Supp}_R(S/R) \cap \mathrm{Max}(R)$; and $[R, S[= [R, S] \setminus \{S\}$. We denote the proper inclusion by the symbol $\subset$ i.e. $R \subset S$ means $R \subseteq S$ and $R \neq S$. Any unexplained notion or terminology can be found either in Gilmer (1992) or in Knebusch and Zhang (2002).

The extension $R \subseteq S$ is said to satisfy FIP (for the "finitely many intermediate algebras property") if $[R, S]$ is finite. An extension $R \subseteq S$ is said to have an FCP (or is called FCP extension) if the partially ordered set $([R, S], \subseteq)$ is both Artinian and Noetherian, this being equivalent to each chain in $[R, S]$ is finite. It is clear that each extension that satisfies FIP must also satisfy FCP; the converse however is false; as can be seen from the example $\mathbb{F}_2(X^2, Y^2) \subset \mathbb{F}_2(X, Y)$. Dobbs et al. characterized FCP and FIP extensions (Dobbs et al. 2012). Important examples of FCP extensions are given by extensions of number field orders. An extension $R \subseteq S$ is callled *minimal* if $[R, S] = \{R, S\}$ [5]. If $R \subseteq S$ has FCP, then any maximal (necessarily finite) chain of R-subalgebras of S, $R = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$, results from juxtaposing n minimal extensions $R_i \subset R_{i+1}$, $0 \leq i \leq n-1$. The *crucial maximal ideal* of the minimal extension $R_i \subset R_{i+1}$ is the ideal $M_i \in \mathrm{Max}(R_i)$ such that $(R_i)_P = (R_{i+1})_P$ for each $P \in \mathrm{Spec}(R_i) \setminus \{M_i\}$, and $(R_i)_{M_i} \subset (R_{i+1})_{M_i}$ is a minimal ring extension. A pair of rings (R, S) is called a *normal pair* if T is integrally closed in S for all $T \in [R, S]$. By (Knebusch and Zhang 2002, Theorem 5.2, Page 47), S is a Prüfer extension of R if and only if (R, S) is a normal pair.

2 The Unique maximal overring

We begin by recalling few definitions and a result which will be frequently used in this section.

Definition 2.1 An extension $R \subset T$ is called an *i-extension* (or *injective extension*) if the canonical map $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is injective. We say a domain R is an *i-domain* if $R \subset T$ is an i-extension for each overring T of R.

Definition 2.2 An integral domain R, with the field of fractions K, is called *seminormal*, if whenever $x \in K$ satisfies $x^2 \in R$ and $x^3 \in R$, then $x \in R$.

Definition 2.3 An inegral domain R is called λ-*domain* if the set of all overrings of R is linearly ordered under inclusion.

One of the results that we frequently use, sometimes without any special mention, is due to I. Papick (Papick 1976, Corollary 2.15). We state it below.

Proposition 2.3.1 *Let R is an integral domain with the field of fractions K. Then R is a local i-domain if and only if $\overline{R}$ is a valuation ring.*

In [6, Example 1.18] Gilbert showed that seminormality is not enough to ensure that a local i-domain is a λ-domain. In Gilbert's example, the base ring, as well as all the overrings of the base ring, were seminormal. It is thus natural to ask: If (R, M) is a seminormal local i-domain, does there exist a fixed overring of R which is comparable to every other overring of R? Our starting point answers this question in affirmative. Since $(\overline{R}, N)$ is a valuation overring (by Proposition 2.3.1), so by (Anderson et al. 1982, Lemma 3.4 and Proposition 3.5) if dim $(R) \leq 1$, then R is a pseudo-valuation domain with an associated valuation ring $\overline{R}$ and $M = N$. Thus $\overline{R}$ is a comparable overring of R by (Ayache 2022, Corollary 21). If dim $(R) > 1$, then we have the following result.

Proposition 2.3.2 *If R is a local i-domain of dimension $n > 1$, then R has a comparable overring R_1 such that $\overline{R} \subset R_1$.*

Proof Since $(\overline{R}, N)$ is a valuation domain of dimension n, $\mathrm{Spec}(\overline{R})$ is linearly ordered, thus we get a chain of length n,

$$(0) \subset \cdots \subset P_i \subset \cdots \subset P_{n-1} \subset N.$$

Let P_1 be the nonzero minimal prime ideal containing the ideal (0). We claim that $R_1 = \overline{R}_{P_1}$ is the required comparable overring of R. Trivially R_1 is a proper valuation overring of $\overline{R}$ and its maximal ideal $P_1\overline{R}_{P_1}$ is a nonzero divided prime ideal of $\overline{R}$. By (Ayache 2022, Theorem 13 (ii)) it is enough to prove that R_P is a valuation domain for each prime ideal $P \subset P_1 \cap R$. Using the Incomparability and the Lying-over properties of the integral extension $R \subset \overline{R}$, one can easily check that the spectrum of R is linearly ordered and is exactly

$$(0) \subset P_1 \cap R \subset \cdots \subset P_{n-1} \cap R \subset N \cap R.$$

Now the localization of R at (0) is the quotient field of R, thus trivially a valuation ring. The claim now follows from (Ayache 2022, Theorem 13). □

Remark 2.4 In general, the integral closure $\overline{R}$ of R in an arbitrary ring S may fail to be comparable. For instance the example in (Anderson et al. 1982, Page 1444) provides the required counterexample. Let $2 \leq n \leq \infty$ and $S = \mathbb{C}((X)) + M$. Then S is an $(n-1)$-dimensional valuation domain of the form $D + M$. Set $R = \mathbb{R}[[X]] + M$, then it is easy to verify that R is an n-dimensional seminormal local i-domain and $\overline{R} = \mathbb{C}[[X]] + M$ is not comparable overring of R (e.g. the overring $\mathbb{R}((X)) + M$ of R is not comparable with $\mathbb{C}[[X]] + M$).

We now define the notion of the *unique maximal overring*.

Definition 2.5 We say a proper overring R_1 of R is *the unique maximal overring* of R if R_1 is distinct from $\mathrm{qf}(R)$ and for any overring R_2 of R we have $R_2 \subseteq R_1$.

Remark 2.6 One can think of the notion of the unique maximal overring as dual to the notion of the unique minimal overring of a domain introduced by Gilmer and Heinzer in Gilmer and Heinzer (1967).

Lemma 2.7 *Let R be an integral domain with quotient field K. If T is a maximal overring of R, then T is a rank one valuation domain with quotient field K.*

Proof As T is a maximal overring of R, therefore $T \subset K$ is a minimal ring extension. Thus T is a rank one valuation domain with quotient field K (14, Exercise 29, Page 43). □

Our main aim is to characterize the integral domains for which the integral closure is the unique maximal overring. We remark here that the examples of rings for which the unique maximal overring is not the integral closure, can easily be constructed using the classical $D + M$ construction as follows: Let V be a valuation ring of dimension one and containing a field k such that $V = k + M$, where M is the maximal ideal of V. Let D be a proper subring of k and set $D_1 = D + M$. It is easy to see that $D_1 \subset V \subset \mathrm{qf}(V) = \mathrm{qf}(D_1)$, and V is the unique maximal overring of D_1 (Bastida and Gilmer 1973, Theorem 3.1). Here V need not be the integral closure of D_1. For concreteness, one may consider $D_1 = \mathbb{Z} + X\mathbb{C}[[X]] \subset \mathbb{C}[[X]] = \mathbb{C} + X\mathbb{C}[[X]] = V$, where $\mathbb{C}[[X]]$ is a rank one valuation ring and $\overline{D_1} \neq V$.

We now state and prove our first main result.

Theorem 2.8 *Let R be an integral domain, which is not a field, and K be its field of fractions. Then the following conditions are equivalent:*

(i) *$\overline{R}$ is the unique maximal overring of R,*
(ii) *R is a local i-domain and $dim R = dim \overline{R} = 1$.*

Proof (i) $\Longrightarrow$ (ii): Suppose $\overline{R}$ is the unique maximal overring of R, then by Lemma 2.7, $\overline{R}$ is a rank one valuation domain. Therefore R is a local i-domain, by Proposition 2.3.1. It follows that $\dim R = \dim \overline{R} = 1$.

(ii) $\Longrightarrow$ (i): We first prove that $\overline{R}$ is a comparable overring of R. Since $\overline{R}$ is a valuation domain, therefore $R \subset K$ is a P-extension. By (Dobbs and Noomen 2022, Theorem 2.5) it is enough to prove that each $T \subset K$ is Prüfer-closed for any $T \in [R, \overline{R}[$. Since $\overline{R}$ is local and $T \subset \overline{R}$, therefore T is local as well, say, with maximal ideal N. Since T is local and $(T, \widehat{T}_K)$ is a normal pair, it follows that $\widehat{T}_K = T_P$ for some prime ideal P of T (Davis 1973, Theorem 1). As $\dim(T) = \dim(\overline{R}) = 1$, so $P = (0)$ or $P = N$. If $P = 0$, then $\widehat{T}_K = T_P = K$, so $\overline{R} \subset K = \widehat{T}_K$. The extension $T \subset \overline{R}$ is integral and $(T, \widehat{T}_K)$ is a normal pair, it follows that $T = \overline{R}$, which is a contradiction. Therefore $P = N$ and $T = T_N = \widehat{T}_K$, i.e. $T \subset K$ is Prüfer-closed. Since $\overline{R}$ is a rank one valuation domain, therefore $\overline{R}$ is the unique maximal overring of R. This completes the proof of the theorem. □

Next we characterize the domains, among the class of Prüfer domains, possessing the unique maximal overring.

Theorem 2.9 *Let R be a Prüfer domain. Then R has the unique maximal overring of R if and only if there exists a nonzero minimal divided prime ideal of R.*

Proof Let R_1 be the unique maximal overring of R. By (Ayache 2022, Proposition 9), R_1 is a valuation domain and its maximal ideal, say, N is a nonzero divided prime ideal of R. We claim that N is the nonzero minimal prime ideal of R. For if N_1 is a nonzero prime ideal of R such that $N_1 \subset N$, then R_{N_1} is a valuation domain containing R_1. But R_1 is the unique maximal overring of R, therefore $R_1 = R_{N_1}$. It follows that $N = N_1$, this establishes the claim. Conversely, let R be a Prüfer domain, then each proper overring T of R is an intersection of localizations of R, say, $T = \bigcap_i R_{P_i}$, where $P_i \in \mathrm{Spec}(R)$. If P is the nonzero minimal divided prime ideal of R, then for each i, $P \subseteq P_i$, thus $R_{P_i} \subseteq R_P$. Therefore $T \subset R_P$, that is R_P is the unique maximal overring of R. This completes the proof of the theorem. □

3 Ring extensions pinched at the integral closure

Let $R \subseteq S$ be an extension. It was proved in (Dobbs et al. 2015, Theorem 3.9) that $R \subseteq S$ satisfies FCP if and only if $R(X) \subseteq S(X)$ satisfies FCP, where $R(X)$ is the *Nagata Ring* of R. The Nagata ring $R(X)$ of R is defined to be the ring of fractions of $R[X]$ w.r.t. a staurated multiplicative closed set $\Sigma_R = \{p(X) \in R[X] |\ p(X)$ is a primitive polynomial$\}$. The analogous result for FIP extensions is bit more subtle. It turns out that the subintegral part $R \subseteq^+_S S$ is the only obstruction for $R(X) \subseteq S(X)$ having FIP. In an attempt to prove the FIP result G. Picavet and M. Picavet-L'Hermitte were led to the definition of *arithmetic extensions* (Picavet and Picavet-L'Hermitte 2016).

Definition 3.1 An extension $R \subseteq S$ is called an *arithmetic extension* if $R_M \subseteq S_M$ is a λ-extension for each $M \in \mathrm{Supp}_R(S/R)$.

In (Picavet and Picavet-L'Hermitte 2016, Proposition 5.2 (b)) it was proved that if $R \subseteq S$ satisfies FMC and is a λ-extension, then $|\mathrm{MSupp}_R(S/R)| = 1$. This gives rise to a natural question: Let $R \subseteq S$ be a ring extension satisfying FMC, what additional conditions one must impose so that the converse of (Picavet and Picavet-L'Hermitte 2016, Proposition 5.2 (b)) holds true? In the following we shall explore the aforementioned question.

Proposition 3.1.1 *Let $R \subset S$ be an extension and $R \subset \overline{R} \subset S$. Then the following conditions are equivalent:*

(i) *For each* $R \subseteq J \subset \overline{R}$ *such that* $J \subset \overline{R}$ *is a minimal ring extension, one has* $(J : \overline{R}) \subseteq M$ *for each* $M \in MSupp_{\overline{R}}(S/\overline{R})$,
(ii) *For each* $R \subseteq J \subset \overline{R}$ *such that* $J \subset \overline{R}$ *is a minimal ring extension, one has* $|MSupp_J(S/J)| = 1$.

Proof (i) $\Longrightarrow$ (ii): Suppose that $J \in [R, \overline{R}[$ such that $J \subset \overline{R}$ is a minimal ring extension. If N is the crucial ideal of $J \subset \overline{R}$, then $N \in \mathrm{MSupp}_J(S/J)$. We claim that N is the only maximal ideal of J in $\mathrm{MSupp}_J(S/J)$. For if $M_1 \in \mathrm{MSupp}_J(S/J)$, since $\overline{R}$ is integral over J, so by lying over property of integral extensions, there exists $M \in \mathrm{Max}(\overline{R})$ such that $M \cap J = M_1$ and $J_{M_1} = \overline{R}_{M_1}$. The last equality follows from the facts that $M_1 \neq N$ and $J \subset \overline{R}$ is a minimal. By (Dobbs et al. 2012, Lemma 2.4), we have $J_{M_1} = \overline{R}_{M_1} = \overline{R}_M$ and $S_{M_1} = S_M$. We now claim that $M \in \mathrm{MSupp}_{\overline{R}}(S/\overline{R})$, suppose on contrary $M \notin \mathrm{MSupp}_{\overline{R}}(S/\overline{R})$. Then $J_{M_1} = \overline{R}_{M_1} = \overline{R}_M = S_M = S_{M_1}$, from these equalities it follows that $M_1 \notin \mathrm{MSupp}_J(S/J)$. This contradiction establishes the claim that $M \in \mathrm{MSupp}_{\overline{R}}(S/\overline{R})$. By (i) the conductor $(J : \overline{R}) = N \subseteq M$, whence $N \subseteq M_1$. Since N is a maximal ideal of J, therefore $N = M_1$. This completes the first half of the proposition.

(ii) $\Longrightarrow$ (i): Suppose $J \in [R, \overline{R}]$ such that $J \subset \overline{R}$ is a minimal ring extension and set $N = (J : \overline{R})$. Then N is the crucial maximal ideal of the minimal ring extension $J \subset \overline{R}$. Therefore $N \in \mathrm{MSupp}_J(S/J)$. Let $M \in \mathrm{MSupp}_{\overline{R}}(S/\overline{R})$ be such that $M \cap J = M^{'} \neq N$. Since $|\mathrm{MSupp}_J(S/J)| = 1$, it follows that $J_{M'} = \overline{R}_{M'} = S_{M'}$. By (Dobbs et al. 2012, Lemma 2.4), there exists a unique $M_1 \in \mathrm{Spec}(\overline{R})$ which lies over $M^{'}$ and for any $\overline{R}$-algebra S, we have by uniqueness, $M_1 = M$ and $\overline{R}_M \subset S_M = S_{M'} = \overline{R}_{M'} = J_{M'}$. One can easily show that $S_M = S_{M'} = J_{M'} \subset \overline{R}_M$, whence $S_M = \overline{R}_M$, which is a contradiction as $M \in \mathrm{MSupp}_{\overline{R}}(S/\overline{R})$. Thus $M \cap J = N$ for all $M \in \mathrm{MSupp}_{\overline{R}}(S/\overline{R})$, hence $N \subset M$ for each $M \in \mathrm{MSupp}_{\overline{R}}(S/\overline{R})$. □

In (Nasr and Zeidi 2017, Theorem 2.7) M. Ben Nasr and N. Zeidi characterized extensions $R \subseteq S$ of integral domains satisfying FCP for which the integral closure $\overline{R}$ of R in S is comparable to all the intermediate rings in $[R, S]$. The preceding result was generalized to ring theoretic settings by Dobbs and Jarboui in (Dobbs and Noomen 2022, Corollary 2.9). Below we prove another equivalent condition for the integral closure $\overline{R}$ to be comparable with all the intermediate rings in $[R, S]$ thereby extending Dobbs and Jarboui's result (Dobbs and Noomen 2022, Corollary 2.9).

Theorem 3.2 *Let* $R \subset S$ *be a ring extension that satisfies FCP and* $R \subset \overline{R} \subset S$. *Then the following conditions are equivalent:*

(i) $[R, S] = [R, \overline{R}] \cup [\overline{R}, S]$.
(ii) *For each* $R \subseteq J \subset \overline{R}$ *such that* $J \subset \overline{R}$ *is a minimal ring extension, one has* $(J : \overline{R}) \subseteq M$ *for each* $M \in MSupp_{\overline{R}}(S/\overline{R})$.
(iii) *For each* $R \subseteq J \subset \overline{R}$ *such that* $J \subset \overline{R}$ *is a minimal ring extension, one has* $|MSupp_J(S/J)| = 1$.

Proof The equivalence (i) $\Longleftrightarrow$ (ii) is proved in (Dobbs and Noomen 2022, Corollary 2.9), and (ii) $\Longleftrightarrow$ (iii) follows from Proposition 3.1.1. □

In Gilmer and Heinzer (1967) Gilmer and Heinzer asked which domains possess the unique minimal overring. The following corollary introduces a new equivalent condition in the characterization of rings possessing the unique minimal overring.

Corollary 3.2.1 *Let $R \subset S$ be a ring extension that satisfies FCP and $R \subset \overline{R} \subset S$. Then the following conditions are equivalent:*

(i) *$\overline{R}$ is the unique minmal overring of R,*

(ii) *R is the maximal non-integrally closed subring of S,*

(iii) *$R \subset \overline{R}$ is minimal and $|MSupp_R(S/R)| = 1$.*

Proof The equivalence (i) $\iff$ (ii) is proved in (Jarboui 2022, Theorem 4.1), and (i) $\iff$ (iii) follows from Theorem 3.2. □

Our next result provides a partial answer to the question: for which S-overring J of R the condition $|\mathrm{MSupp}_J(S/J)| = 1$ is satisfied?

Proposition 3.2.1 *If $R \subset T \subseteq \overline{R}$ is such that $MSupp_T(S/T) = \{M\}$ and $M \cap J$ is the crucial maximal ideal of the minimal extension $J \subset T$, where $R \subseteq J \subset T$, then $|MSupp_J(S/J)| = 1$.*

Proof Since $M \in \mathrm{MSupp}_T(S/T)$, therefore $T_M \subset S_M$. Suppose $J \subset T$ is a minimal ring extension, where $R \subset T \subseteq \overline{R}$, then it follows that $M \cap J = N$ is the crucial maximal ideal of $J \subset T$. Thus $N \in \mathrm{MSupp}_J(S/J)$. We now claim that N is the only maximal ideal of J contained in $\mathrm{MSupp}_J(S/J)$. For if $N^{'} \in \mathrm{MSupp}_J(S/J)$, then $J_{N'} = T_{N'}$, because $N^{'} \neq N$. By (Dobbs et al. 2012, Lemma 2.4), there exists the unique $M^{'} \in \mathrm{Max}(T)$ lying over $N^{'}$ and satisfying $T_{N'} = T_{M'}$, and $S_{N'} = S_{M'}$. Since $N \neq N^{'}$, therefore $M^{'} \neq M$. As $\mathrm{MSupp}_T(S/T) = \{M\}$ and $M^{'} \neq M$, so $T_{M'} = S_{M'}$. Whence $S_{N'} = T_{N'} = J_{N'}$, which is a contradiction as $N^{'} \in \mathrm{MSupp}_J(S/J)$. This completes the proof. □

In (Nasr and Zeidi 2017, Corollary 2.3) it is proved that if the extension $R \subset S$ satisfies FCP and the integral closure $\overline{R}$ of R in S is local, then $[R, S] = [R, \overline{R}] \cup [\overline{R}, S]$. Our next result generalizes (Nasr and Zeidi 2017, Corollary 2.3).

Corollary 3.2.2 *If $R \subset \overline{R} \subset S$ satisfies FCP and $MSupp_{\overline{R}}(S/\overline{R}) = \{M\}$ such that $M \cap J$ is the crucial maximal ideal of the minimal extension $J \subset \overline{R}$ for each $J \in [R, S]$, then $[R, S] = [R, \overline{R}] \cup [\overline{R}, S]$.*

Proof By Proposition 3.2.1 $|\mathrm{MSupp}_J(S/J)| = 1$. The conclusion now easily follows from Theorem 3.2. □

Remark 3.3 Let $R \subset \overline{R} \subset S$ be an extension such that $R \subset \overline{R}$ is minimal with the crucial maximal ideal N and $\overline{R} \subset S$ is minimal with the crucial maximal ideal M. If $M \cap R \neq N$, then by Crosswise Exchange Lemma (Dobbs et al. 2012, Lemma 2.7), there exists $U \in [R, S]$ such that $R \subset U$ is minimal with the crucial maximal ideal $M \cap R$. Therefore $\mathrm{MSupp}_R(S/R) = \{M \cap R, N\}$. Thus it is not possible to remove the hypothesis that $M \cap J$ is the crucial maximal ideal of the minimal extension $J \subset T$.

Corollary 3.3.1 *If R is an integral domain and K is its field of fractions such that $R \subset K$ satisfies FCP and $R \neq \overline{R}$, then the following conditions are equivalent:*

(i) $[R, K] = [R, \overline{R}] \cup [\overline{R}, K]$.
(ii) *Each intermediate ring $J \neq \overline{R}$ between R and $\overline{R}$ is local.*
(iii) *For each $R \subseteq J \subset \overline{R}$ with $J \subset \overline{R}$ is minimal one has $|MSupp_J(K/J)| = 1$.*

Proof (i) $\Longleftrightarrow$ (ii) follows from (Nasr and Zeidi 2017, Corollary 2.14). (i) $\Longleftrightarrow$ (iii) follows from Theorem 3.2. □

Remark 3.4 In general (iii) need not imply (ii). In (Ayache 2013, Example 29) Ayache showed that there exists an FCP extension $R \subset S$ such that $[R, S] = \{R\} \cup [\overline{R}, S]$, but R is not local. Since $\overline{R}$ is the unique minimal overring of R, so by Theorem 3.2 $|\mathrm{MSupp}_R(S/R)| = 1$. This completes the remark.

In (Picavet and Picavet-L'Hermitte 2016, Proposition 5.2(b)) G. Picavet and M. Picavet-L'Hermitte proved that, if $R \subseteq S$ satisfies FMC and $R \subseteq S$ is a λ-extension, then $|\mathrm{MSupp}_R(S/R)| = 1$. It is natural to ask under what conditions the converse holds? Our next result answers this question.

Theorem 3.5 *Let $R \subset S$ be an FMC extension. Then the following conditions are equivalent:*

(i) *$R \subset S$ is a λ-extension,*
(ii) *$MSupp_R(S/R) = \{M\}$ and $R_M \subset S_M$ is a λ-extension.*

Proof (ii) $\Longrightarrow$ (i): Suppose (i) is not true, then there exist $U, V \in [R, S]$ such that neither $U \not\subset V$ nor $V \not\subset U$. If $U \not\subset V$, then via globalization $U_{N_1} \not\subset V_{N_1}$ for some $N_1 \in \mathrm{Max}(R)$. It follows that $R_{N_1} \subset U_{N_1} \subseteq S_{N_1}$. Since the first inclusion is proper so we have $N_1 \in \mathrm{MSupp}_R(S/R)$. Arguing similarly we get $V_{N_2} \not\subset U_{N_2}$ for some $N_2 \in \mathrm{Max}(R)$, and thus $N_2 \in \mathrm{MSupp}_R(S/R)$. By (ii) it follows that $N_1 = N_2 = M$. As $R_M \subset S_M$ is a λ-extension, therefore either $U_M \subset V_M$ or $V_M \subset U_M$, but this contradicts the choice of N_1 and N_2. Thus either $U \subset V$ or $V \subset U$.

(i) $\Longrightarrow$ (ii): The natural map $[R, S] \to [R_Q, S_Q]$ is surjective for each $Q \in \mathrm{Spec}(R)$, therefore if $R \subset S$ is a λ-extension it follows easily that $R_Q \subset S_Q$ is a λ-extension as well for each $Q \in \mathrm{Spec}(R)$. The second part of the implication $|\mathrm{MSupp}_R(S/R)| = 1$ follows from (Picavet and Picavet-L'Hermitte 2016, Proposition 5.2(b)). □

Proposition 3.5.1 *Let $R \subset S$ be an integrally closed Δ-extension and $|MSupp_R(S/R)| = 1$, then the extension $R \subset S$ is a λ-extension.*

Proof By (Knebusch and Zhang 2002, Theorem 1.7, Page 88) the extension $R \subset S$ is Prüfer and therefore (R, S) is a normal pair. Hence (R_M, S_M) is a normal pair for each maximal ideal M of R. Thus $[R_M, S_M]$ is linearly ordered and so $R_M \subseteq S_M$ is a λ-extension by (Knebusch and Zhang 2002, Scholium 10.4, Page 147) and (Knebusch and Zhang 2002, Theorem 3.1, Page 187). Rest of the proof is now similar to the implication (ii) $\Longrightarrow$ (i) in Theorem 3.5. □

In (Dobbs and Noomen 2022, Theorem 3.9) it is proved that if $R \subseteq S$ is an integrally closed FCP extension, then $R \subseteq S$ is a λ-extension if and only if $\mathrm{Supp}_R(S/R)$ is totally ordered by inclusion. In the following we extend the preceding theorem by providing an additional equivalent condition.

Theorem 3.6 *Let $R \subset S$ be an integrally closed FMC ring extension. Then the following conditions are equivalent:*

(i) *$R \subset S$ is a λ-extension,*
(ii) *$Supp_R(S/R)$ is totally ordered by inclusion,*
(iii) *$|Msupp_R(S/R)| = 1$.*

Proof By (Dobbs et al. 2012, Theorem 6.3), an integrally closed extension $R \subset S$ satisfies FMC if and only if it satisfies FCP. The equivalence of (i) and (ii) is proved in (Dobbs and Noomen 2022, Theorem 3.9). We now prove equivalence of (i) and (iii). The implication (i) $\Rightarrow$ (iii) follows from (i) $\Rightarrow$ (ii) of Theorem 3.5. (iii) $\Rightarrow$ (i): Let $R \subset S$ be an integrally closed FMC ring extension, then it is a normal pair by (Dobbs et al. 2012, Theorem 6.3). By (Picavet and Picavet-L'Hermitte 2021, Proposition 4.1), $R \subset S$ is an integrally closed Δ-extension, and now the result follows from Proposition 3.5.1. □

Since the class of λ-extensions is strictly smaller than the class of extensions having a fixed ring comparable to every other intermediate ring (i.e. extensions pinched at some intermediate ring), the preceding result leads naturally to the following question: when is a normal pair pinched at some intermediate ring?

Theorem 3.7 *Let (R, S) be a normal pair of integral domains. Then the following conditions are equivalent:*

(i) *There exsits $P \in C = \{P | PS = S, P \in Spec(R)\}$ such that every prime ideal Q of R is comparable to P under inclusion.*
(ii) *(a) $R_P \subset S$ and $[R, S]$ is pinched at R_P,*
(b) $R_P \subset R_Q$ for each $Q \in Spec(R)$ with $R_Q \notin [R, S]$.

Proof (i) $\implies$ (ii): It follows by (Ayache and Jaballah 1997, Lemma 2.9) that $S = \bigcap R_{P_i}$ for each $P_i \notin C$. If for each $P_i \notin C$, $P_i \subset P$, then $R_P \subset R_{P_i}$ whence $R_P \subset S$. Otherwise, there exists some $P_i \notin C$ such that $P \subset P_i$. Then $R_{P_i} \subset R_P$ and $S = \bigcap R_{P_i} \subset R_{P_i} \subset R_P$. By (Ayache and Jaballah 1997, Lemma 2.9) it follows that $P \notin C$ which contradicts the hypothesis. Therefore $R_P \subset S$. If $T \in [R, S]$, then $T = \bigcap R_{P_i}$. Repeating the similar argument as above one can prove that either $R_P \subset T$ or $T \subset R_P$. Now suppose there is a $Q \in \mathrm{Spec}(R)$ such that $R_Q \notin [R, S]$. By (i) either $Q \subset P$ or $P \subset Q$. If $P \subset Q$, then $R_Q \subset R_P$ then it follows from above that $R_Q \in [R, S]$, which is a contradiction, thus one must have $R_P \subset R_Q$. This completes the first half of the proof.

(ii) $\implies$ (i): We have $R \subset S = \bigcap R_{P_i}$, where $P_i \notin C$. Thus $R_P \subset R_{P_i}$ for each i, which entails that $P_i \subset P$ for all $P_i \notin C$. Let $Q \in C$. Then $S \not\subset R_Q$. Thus, either $R_Q \subset S$ or $R_Q \notin [R, S]$. If $R_Q \subset S$ then either $P \subset Q$ or $Q \subset P$ since $[R, S]$ is pinched at R_P and if $R_Q \notin [R, S]$ then by (2), $R_P \subset R_Q$ thus $Q \subset P$. □

Now we generalize the main result of Gilbert (6, Proposition 2.8, Chapter II) which says that if $R \subset T$ is a λ-extension, then the number of nonzero factors in the direct product decomposition of T is exactly two. This result plays a key role in Gilbert's alternative proof of Ferrand and J Olivier's characterization of minimal ring extensions Ferrand and Olivier (1970).

Theorem 3.8 *Let $R \subset T$ be an extension having a comparable T-overring R_1 and suppose that $T = \prod T_i$, where there are at least two factors. Let $\pi_j : T \to T_j$ be the canonical projection and let $I_j = ker\ (\pi_j) \cap R_1$ for each j. Suppose $I_j + I_k \neq R_1$ for each pair j, k of indices. Then the product $\prod T_j$ contains exactly two factors.*

Proof Let R_1 be a comparable T-overring of R. Suppose the conclusion of the theorem does not hold and pick three pairwise distinct indices i, j and k. Define

$$A = \left\{t \in T | \text{there exists } r \in R \text{ such that } \pi_i(t) = \pi_i(r) \text{ and } \pi_j(t) = \pi_j(r)\right\}.$$

Clearly $R \subset A \subset T$. Since R_1 is a comparable overring of R, therefore either $A \subset R_1$ or $R_1 \subset A$. If $A \subset R_1$, set

$$B = \{t \in T | \text{there exists } r \in R_1 \text{ such that } \pi_i(t) = \pi_i(r) \text{ and } \pi_k(t) = \pi_k(r)\},$$

then $A \subset R_1 \subset B$. For an arbitrary element $s \in R$, let t be an element of T with $\pi_k(s)$ as its k-th coordinate and 0 everywhere else. Since $\pi_i(t) = \pi_j(t) = 0$, we have $t \in A$, and also in B. By construction there exists $r_1 \in R_1$ such that $\pi_i(t) = \pi_i(r_1) = 0$ and $\pi_k(r_1) = \pi_k(s)$. It follows that $r_1 \in I_i$ and $s - r_1 \in I_k$. Then $s = r + (s - r) \in I_i + I_k$. As s is an arbitrary element of R, we must have $1 \in I_i + I_k$, and thus $I_i + I_k = R_1$, the intended contradiction.

If $R_1 \subset A$, then we set

$$B = \{t \in R_1 | \text{there exists } r \in R \text{ such that } \pi_i(t) = \pi_i(r) \text{ and } \pi_k(t) = \pi_k(r)\}.$$

Now we have $B \subset R_1 \subset A$. Arguing exactly as above, we again get a contradiction. Thus there are only two factors in the direct product decomposition of T. For the last conclusion $R \neq T$ see the page (6, Page 26). □

Our next result is of independent interest. Almost all the results on the unique minimal overring for general ring extensions are proved by assuming some kind of finiteness hypothesis (e.g. FCP). Below we are able to partially characterize the local rings which are not Prüfer-closed and have the unique minimal overring without assuming any finiteness hypothesis.

Theorem 3.9 *Let (R, M) be a local ring, $R \subset S$ be a ring extension of integral domains and $R \neq \widehat{R}_S$. If R possesses the unique minimal overring $R_1 \subseteq \widehat{R}_S$ in S, then there exists the unique nonmaximal prime ideal P of R which properly contains every other nonmaximal prime ideal of R. The converse holds if R is a QQR-domain and $S \subseteq qf(R)$.*

Proof Let R_1 be the unique minimal overring of R in S and contained in $\widehat{R}_S$. By (Knebusch and Zhang 2002, Theorem 5.2, Page 47) (R, R_1) is a normal pair, so there exists a divided prime ideal $P \in \text{Spec}(R) - \{M\}$ such that $R_1 = R_P$. Now we claim that the prime ideal P has the property stated above. To this end suppose the contrary, so there exists $P_1 \in \text{Spec}(R) - \{M\}$ such that $P \subset P_1 \subset M$. Then $R \subset R_{P_1} \subset R_P$, which contradicts the minimality of $R \subset R_P$.

Conversely, suppose R has the unique nonmaximal prime ideal P with the property stated in the statement. By (Knebusch and Zhang 2002, Theorem 5.2, Page 47) the pair $(R, \widehat{R}_S)$ is normal, so there exists a prime ideal P_1 such that $\widehat{R}_S = R_{P_1}$. Since P satisfies the property stated in theorem, we must have $P_1 \subset P$. It follows then $R \subset R_P \subseteq \widehat{R}_S$ and $R \subset R_P$ is a minimal ring extension. If $T \in [R, S]$, since R is a QQR-domain, T is an intersection of quotient rings of R. By (Gilmer 1992, Corollary 5.5), T is an intersection of localizations of R. If Q is an arbitrary nonmaximal prime ideal of R, it follows from the hypothesis that $R_P \subseteq R_Q$, whence $R_P \subseteq T$. This completes the proof of the theorem. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

References

Anderson, D., Dobbs, D.E., Huckaba, J.: On seminormal overrings. Comm. Algebra **10**, 1421–1448 (1982)
Ayache, A.: A constructive study about the set of intermediate rings. Comm. Algebra **41**, 4637–4661 (2013)
Ayache, A.: When is a fixed ring comparable to all overrings? J. Algebra Appl. **21**(03), 2250050 (2022)
Ayache, A., Jaballah, A.: Residually algebraic pairs of rings. Math. Z. **225**, 49–65 (1997)
Bastida, E., Gilmer, R.: Overrings and divisorial ideals of rings of the form $D + M$, The Michi. Math. J. **20**, 79–95 (1973)
Davis, E.D.: Overrings of commutative rings III: Normal Pairs. Trans. Am. Math. Soc. **182**, 175–185 (1973)
Dobbs, D.E., Noomen, J.: Prüfer-closed extensions and FCP λ-extensions of commutative rings. Pales. J. Math. **11**(3), 362–378 (2022)
Dobbs, D.E., Picavet, G., Picavet-L'Hermitte, M.: Characterizing the ring extensions that satisfy FIP or FCP. J. Algebra **371**, 391–429 (2012)
Dobbs, D.E., Picavet, G., Picavet-L'Hermitte, M.: Transfer results for the FIP and FCP properties of ring extensions. Comm. Algebra **43**, 1279–1316 (2015)
Ferrand, D., Olivier, J.: Homomorphismes Minimaux d'Anneaux. J. Algbera **16**, 461–471 (1970)
Gilbert, M.: Extensions of Commutative Rings With Linearly Ordered Intermediate Rings, Doctoral Dissertation, University of Tennessee-Knoxville
Gilmer, R.: Multiplicative ideal theory, Queenn's Papers in Pure and Applied Mathematics, Vol. 90, (1992)
Gilmer, R., Heinzer, W.J.: Intersections of quotient rings of an integral domain. J. Math. Kyoto Univ. **7–2**, 133–150 (1967)
Gilmer, R., Huckaba, J.: Δ-rings. J. Algebra **28**, 414–432 (1974)

Jarboui, N.: An answer to a question about maximal non-integrally closed subrings. J. Algebra Appl. **21**(04), 2250073 (2022)

Kaplansky, I.: Commutative rings, Revised Edition, The University of Chicago Press

Knebusch, M., Zhang, D.: Manis Valuations and Prüfer Extensions I: A New Chapter in Commutative Algebra, Lecture Notes in Mathematics 1791, Springer (2002)

Nasr, M.B., Zeidi, N.: When is the integral closure comparable to all intermediate rings. Bull. Aust. Math. Soc. **95**, 14–21 (2017)

Papick, I.: Topologically defined classes of Going-Down domains. Trans. Am. Math. Society **219**, 2 (1976)

Picavet, G., Picavet-L'Hermitte, M.: About minimal morphisms, Multiplicative Ideal Theory in Commutative Algebra pp 369–386

Picavet, G., Picavet-L'Hermitte, M.: Some more combinatorics results on Nagata extensions. Pales. J. Math. **5**, 49–62 (2016)

Picavet, G., Picavet-L'Hermitte, M.: FCP Δ-extensions of rings. Arab. J. Math. **10**, 211–238 (2021)