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ORIGINAL PAPER

SFT modules and ring extensions

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Abstract
Let $A \subseteq B$ be two commutative rings with identity. It is well-known that if A is a Noetherian ring and B is a finitely generated A-module, then B is also a Noetherian ring. In this paper, we want to prove an analogue of the above result for SFT rings. For that, we extend the notion of SFT rings to modules. An A-module M is said to be an SFT module, if for each submodule N of M, there exist $k \geq 1, x_1, \ldots, x_n \in N$ such that for each $a \in (N : M) = \{\alpha \in A,\ \alpha M \subseteq N\}$ and $x \in M$, $a^k x \in \langle x_1, \ldots, x_n \rangle$. First of all, we investigate some properties of SFT modules. In fact, we show that properties of SFT rings can be generalized to SFT modules. In the end of this paper, we give a partial answer of the main question of this work.

Keywords SFT-rings · SFT-modules · Rings extension

Mathematics Subject Classification 13B25 · 13E05 · 13A15

1 Introduction

In this paper, all rings considered are commutative with unit element and all modules are left side and unital modules. For two sets X and Y, the symbol $X \subset Y$ means that X is strictly contained in Y. In Arnold (1973a), Arnold has introduced the concept of SFT (strong finite type) rings as follow, a ring A is called SFT, if for each ideal I of A there exist an integer $k \geq 1$ and a finitely generated ideal $F \subseteq I$ of A such that $x^k \in F$ for every $x \in I$ (in this case I is called an SFT ideal). He also showed that the SFT condition is necessary for the finiteness of the Krull dimension of the power series ring $A[[X]]$. After, Coykendall in Coykendall (2002), has showed that

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this property is not sufficient. In fact, Coykendall gave an example of an SFT ring A such that $dim(A[[X]]) = +\infty$.

Let $A \subseteq B$ be a ring extension such that B is a finitely generated A-module. It is well known that A is a Noetherian ring if and only if B is a Noetherian ring. In this paper, we are interested in the case of SFT rings. In other words, if A is SFT, is the ring B SFT? Inspired by the proof of the Noetherian case, it is natural to think about extending the SFT notion from rings to modules. For that, we define the SFT-modules as a generalization of SFT rings as follow. Let A be a ring and M an A-module. The module M is called SFT, if for each submodule N of M, there exist an integer $k \geq 1$ and a finitely generated submodule $L \subseteq N$ of M such that $a^k m \in L$ for every $a \in (N :_A M)$ and $m \in M$.

First of all, we show that this new notion is a generalization of SFT rings. For instance, we show that a ring A is SFT if and only if it is an SFT A-module (see Example 2.1). Next, we study some basic property of SFT module, for example, the analogue of Cohen Theorem type, the homomorphic image of an SFT module, the product of SFT modules etc... In fact, we show that an A-module M is SFT if and only if PM is an SFT submodule of M for every prime ideal P of A. Among other results, we show that $Hom_A(M, N)$ is an SFT A-module where M and N are two A-modules, provided that M is a free finitely generated A-module and N is an SFT A-module.

This paper is the first one which is devoted to study the sufficient conditions for a finitely generated extension of an SFT ring to be an SFT ring. It is also the first work concerning the natural extention of the SFT notion from rings to modules and study their nice properties. In fact, we show that if $A \subseteq B$ is a ring extension such that B is a finitely generated A-module and A is a zero dimensional SFT ring, then the ring B is SFT. Under the same hypothesis, it is shown that the ring $A + XB[X]$ is SFT (see Theorem 2.23). On the other hand, we give a sufficient condition for an A-module M such that the $A[X_1, \ldots, X_n]$-module $M[X_1, \ldots, X_n]$ (respectively, $A[[X_1, \ldots, X_n]]$-module $M[[X_1, \ldots, X_n]]$) is SFT.

Also, we show that an SFT ring extension B of A is not in general an SFT module. In fact, we give an example of an SFT ring B and a subring A of B which is not an SFT A-module. It is well known that a submodule of a Noetherian module is always a Noetherian module. This result does not hold in the case of SFT module. In other words, a submodule of an SFT module can be a nonSFT module (see Example 2.2).

2 Main results

We start this section by the following definitions and some examples showing the importance of this concept.

2.1 Definitions

Let A be a ring and M an A-module.

1. A submodule N of M is said to be an SFT submodule, if there exist $k \geq 1$, $x_1, \ldots, x_n \in N$ such that for each $a \in (N : M) = \{\alpha \in A,\ \alpha M \subseteq N\}$ and $x \in M$, $a^k x \in \langle x_1, \ldots, x_n \rangle$.
2. We call that M is an SFT A-module if every submodule of M is an SFT submodule.

The following example shows that this notion is a natural extention of the SFT (strong finite type) condition from ring to module.

Example 2.1 1. Let A be a ring and N an ideal of A. Then N is an SFT ideal of A if and only if it is an SFT A-submodule of A. Indeed, suppose that N is an SFT ideal of A. We have $(N : A) = N$. Consequentely, there exist $k \geq 1$ and $x_1, \ldots, x_n \in N$ such that for each $x \in (N : A)$, $x^k \in \langle x_1, \ldots, x_n \rangle$. Let $a \in (N : A) = N$ and $x \in A$. We have $a^k \in \langle x_1 \ldots, x_n \rangle$. Then $a^k x \in \langle x_1, \ldots, x_n \rangle$. Thus N is an SFT A-submodule of A.

Conversely, assume that N is an SFT A-submodule of A. Then there exist $k \geq 1$, $x_1, \cdots, x_n \in N$ such that for every $a \in (N : A)$ and $x \in A$, $a^k x \in \langle x_1, \ldots, x_n \rangle$. In particular, for $x = 1$, we obtain $a^k \in \langle x_1, \ldots, x_n \rangle$ for all $a \in (N : A) = N$. Hence N is an SFT ideal of A.

2. Using (1), we see that a ring A is an SFT ring if and only if it is an SFT A-module.

Example 2.2 1. Let A be a ring and M be an A-module. If M is an SFT A-module, then M is finitely generated. Indeed, there exist $k \geq 1$ and $x_1, \ldots, x_n \in M$ such that for every $a \in (M : M) = A$ and $x \in M$, $a^k x \in \langle x_1, \ldots, x_n \rangle$. In particular, if $a = 1$, we get $x = 1^k x \in \langle x_1, \ldots, x_n \rangle$ for each $x \in M$. Therefore, $M = \langle x_1, \ldots, x_n \rangle$.
2. A submodule of an SFT-module is not necessary an SFT-module. Indeed, let K be a field, $X = \{X_n,\ n \geq 1\}$ a family of indeterminates over K, $A = M = K[X]/\langle X \rangle^2$ and $N = \langle \bar{X}_n,\ n \geq 1 \rangle$. The only prime ideal of A is N which satisfies $N^2 = \{0\}$. Thus A is an SFT-ring. By Example 2.1, the A-module M is SFT. As the A-module N is not finitely generated, by (1) of Example 2.2, it is not an SFT A-module.
3. A finitely generated submodule of an A-module M is an SFT submodule. Indeed, let $N = \langle x_1, \ldots, x_n \rangle$ be a finitely generated submodule of M. Then for each $a \in (N : M)$ and $x \in M$, $ax \in (N : M)M \subseteq N = \langle x_1 \ldots, x_n \rangle$. This shows that every Noetherian module is an SFT module.

The analogue of the Cohen's Theorem type is a natural question in this concept. This allows us to consider this question in the beginning of our paper.

Lemma 2.2 *Let A be a ring, M a finitely generated A-module and N a maximal element among the non-SFT submodules of M. Then $P = (N : M)$ is a prime ideal of A.*

Proof Since M is finitely generated, then $N \neq M$ (by (3) of Example 2.2) and so $P \neq A$. Assume that P is not a prime ideal of A. Then there exist $a, b \in A \setminus P$ such that $ab \in P$. Let $L = N + aM$. Since $aM \nsubseteq N$, we have $N \subset L$. By maximality of N, the submodule L is SFT. Thus there exist $k \geq 1,\ x_1, \ldots, x_n \in N,\ m_1, \ldots, m_n \in M$ such that for every $\alpha \in (L : M)$ and $x \in M$, $\alpha^k x \in \langle x_1 + am_1, \ldots, x_n + am_n \rangle$.

Now, set $K = (N : a) = \{x \in M,\ ax \in N\}$. It is clear that K is a submodule of M. As $b \notin P$, there exists $x \in M$ such that $bx \notin N$, but $abx \in PM \subseteq N$. Therefore,

$bx \in K$. Hence $N \subset K$. Again by maximality of N, K is an SFT submodule of M. Consequentely, there exist $l \geq 1$, $y_1, \ldots, y_r \in K$ such that for each $\beta \in (K : M)$ and $y \in M$, $\beta^l y \in \langle y_1, \ldots, y_r \rangle$.
Since $N \subset L$, then $P \subseteq (L : M)$. Let $\alpha \in P$ and $x \in M$. We have $\alpha^k x = \sum_{i=1}^{n} \alpha_i(x_i + am_i)$ for some $\alpha_1, \ldots, \alpha_n \in A$. Thus $a\sum_{i=1}^{n} \alpha_i m_i = \alpha^k x - \sum_{i=1}^{n} \alpha_i x_i \in N$Consequentely, there exist $l \geq 1$, $y_1, \ldots, y_r \in K$ such that for each $\beta \in (K : M)$ and $y \in M$, $\beta^l y \in \langle y_1, \ldots, y_r \rangle$.
Since $N \subset L$, then $P \subseteq (L : M)$. Let $\alpha \in P$ and $x \in M$. We have $\alpha^k x = \sum_{i=1}^{n} \alpha_i(x_i + am_i)$ for some $\alpha_1, \ldots, \alpha_n \in A$. Thus $a\sum_{i=1}^{n} \alpha_i m_i = \alpha^k x - \sum_{i=1}^{n} \alpha_i x_i \in N$. Hence $\sum_{i=1}^{n} \alpha_i m_i \in (N : a)$. As $N \subset K$, we have $P \subseteq (K : M)$. Therefore, $\alpha^l \left(\sum_{i=1}^{n} \alpha_i m_i\right) = \sum_{j=1}^{r} \beta_j y_j$ where $\beta_1, \ldots, \beta_r \in A$. Hence $\sum_{i=1}^{n} \alpha_i m_i \in (N : a)$. As $N \subset K$, we have $P \subseteq (K : M)$. Therefore, $\alpha^l (\sum_{i=1}^{n} \alpha_i m_i) = \sum_{j=1}^{r} \beta_j y_j$ where $\beta_1, \ldots, \beta_r \in A$. Hence

$$\alpha^{l+k} x = \sum_{i=1}^{n} (\alpha^l \alpha_i) x_i + \sum_{j=1}^{r} \beta_j (a y_j) \in \langle x_1, \ldots, x_n, a y_1, \ldots, a y_r \rangle$$

with $x_1, \ldots, x_n, ay_1, \ldots, ay_r \in N$. Thus N is an SFT submodule of M: absurd. Hence P is a prime ideal of A. □

Theorem 2.3 *Let A be a ring and M a finitely generated A-module. Then M is an SFT module if and only if for each prime ideal P of A, PM is an SFT submodule of M.*

Proof $'' \Longleftarrow''$ Assume that M is not an SFT-module. Then the set $\mathcal{F}$ of all non-SFT submodules of M is not empty. Let $(N_\alpha)_{\alpha \in \Lambda}$ be a totally ordered family of $(\mathcal{F}, \subseteq)$ and $N = \bigcup_{\alpha \in \Lambda} N_\alpha$. It is clear that N is a submodule of M. Supposons that N is an SFT submodule of M. Then there exist $k \geq 1$ and $x_1, \ldots, x_n \in N$ such that for each $a \in (N : M)$ and $x \in M$, $a^k x \in \langle x_1, \ldots, x_n \rangle$. Since the family $(N_\alpha)_{\alpha \in \Lambda}$ is totally ordered, there exists $\alpha \in \Lambda$ such that $x_1, \ldots, x_n \in N_\alpha$. On the other hand, $N_\alpha \subseteq N$, thus $(N_\alpha : M) \subseteq (N : M)$. Hence $a^k x \in \langle x_1, \ldots, x_n \rangle$ for every $a \in (N_\alpha : M)$ and $x \in M$, which contradicts the fact that N_α is not an SFT submodule of M. Therefore, $N \in \mathcal{F}$. It follows that $(\mathcal{F}, \subseteq)$ is inductive. By Zorn's Lemma, $(\mathcal{F}, \subseteq)$ has a maximal element N. By Lemma 2.2, $P = (N : M)$ is a prime ideal of A. By hypothesis, there exist $k \geq 1$ and $x_1, \ldots, x_n \in PM$ such that $a^k x \in \langle x_1, \ldots, x_n \rangle$ for every $a \in (PM : M)$ and $x \in M$. As $P \subseteq (PM : M) \subseteq (N : M) = P$, we have $P = (PM : M)$. Consequently, N is an SFT submodule of M: contradiction with the choice of N. Hence M is an SFT A-module.

Note that if we consider a ring A as an A-module in Theorem 2.3, we get exactely the analogue of Cohen theorem type of SFT ring. □

Example 2.3 Let A be an SFT ring. Then the A-module $A \times A$ is SFT. Indeed, let P be a prime ideal of A. There exist $k \geq 1$ and $a_1, \ldots, a_n \in P$ such that $a^k \in \langle a_1, \ldots, a_n \rangle$

for every $a \in P$. Let $a \in (P(A \times A) : A \times A)$. Then $a(A \times A) \subseteq P(A \times A)$. In particular, $(a, a) \in P \times P$. It follows that $a \in P$. Therefore, $(P(A \times A) : A \times A) = P$. Now, let $a \in P$ and $(x, y) \in A \times A$. Set $a^k = \sum_{i=1}^{n} \alpha_i a_i$ where $\alpha_1, \ldots, \alpha_n \in A$. Thus

$$a^k(x, y) = (a^k x, 0) + (0, a^k y) = \sum_{i=1}^{n} x\alpha_i(a_i, 0) + \sum_{i=1}^{n} y\alpha_i(0, a_i)$$

with $(a_i, 0) = a_i(1, 0) \in P(A \times A)$, $(0, a_i) = a_i(0, 1) \in P(A \times A)$ and $x\alpha_i, y\alpha_i \in A$ for each $1 \leq i \leq n$. Hence $P(A \times A)$ is an SFT A-submodule of $A \times A$. By Theorem 2.3, $A \times A$ is an SFT A-module.

The following proposition gives a generalization of Example 2.3.

Proposition 2.4 *Let A be a ring, M and N two A-modules. If the A-modules M and N are SFT, so is the A-module product $M \times N$.*

Proof Let P be a prime ideal of A. We have $J = (P(M \times N) : M \times N) = (PM : M) \bigcap (PN : N)$. Indeed, let $a \in J$. Then $a(M \times N) \subseteq P(M \times N) = PM \times PN$. It yields that $aM \subseteq PM$ and $aN \subseteq PN$. Which shows that $J \subseteq (PM : M) \bigcap (PN : N)$. Conversely, let $a \in (PM : M) \bigcap (PN : N)$ and $(x, y) \in M \times N$. Then $a(x, y) = (ax, ay) \in PM \times PN = P(M \times N)$.
By hypothesis, there exist $k_1, k_2 \geq 1$ two integers, $x_1, \ldots, x_n \in PM$ and $y_1, \ldots, y_r \in PN$ such that $a^{k_1} x \in \langle x_1, \ldots, x_n \rangle$ and $b^{k_2} y \in \langle y_1, \ldots, y_r \rangle$ for every $a \in (PM : M)$, $b \in (PN : N)$, $x \in M$ and $y \in N$. Set $k = k_1 + k_2$. Let $a \in J$ and $(x, y) \in M \times N$. Thus $a^k x = a^{k_2} a^{k_1} x = a^{k_2} \sum_{i=1}^{n} \alpha_i x_i$ and $a^k y = a^{k_1} a^{k_2} y = a^{k_1} \sum_{i=1}^{r} \beta_i y_i$ where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_r \in A$. Hence

$$a^k(x, y) = a^k(x, 0) + a^k(0, y) = \sum_{i=1}^{n} (a^{k_2}\alpha_i)(x_i, 0) + \sum_{i=1}^{r} (a^{k_1}\beta_i)(0, y_i).$$

It follows that

$$a^k(x, y) \in \langle (x_1, 0), \ldots, (x_n, 0), (0, y_1), \ldots, (0, y_r) \rangle$$

with $(x_1, 0), \ldots, (x_n, 0), (0, y_1), \ldots, (0, y_r) \in P(M \times N)$. Therefore, $P(M \times N)$ is an SFT submodule of $M \times N$. Hence $M \times N$ is an SFT-module.

The next Corollary is an easy consequence of Example 2.1 and Proposition 2.4.

Corollary 2.5 *1. Let A be an SFT ring. Then the A-module A^n is SFT for every integer $n \geq 1$.*
2. Let A be a ring, $M_1, \ldots, M_n$ be a finite number of SFT A-modules. Then $M_1 \times M_2 \times \ldots \times M_n$ is an SFT A-module.

Proposition 2.6 *The homomorphic image of an SFT-module is also an SFT-module.*

Proof Let A be a ring and $\phi : M \longrightarrow N$ be a surjective homomorphism of A-modules. Assume that M is an SFT A-module. Let P be a prime ideal of A and $L = \phi^{-1}(PN)$. We have L is an SFT submodule of M. Then there exist $k \geq 1$ and $x_1, \ldots, x_n \in L$ such that $a^k x \in \langle x_1, \ldots, x_n \rangle$ for every $a \in (L : M)$ and $x \in M$.

We are going to show that $(L : M) = (PN : N)$. Let $a \in (L : M)$. Then $aM \subseteq L$. Therefore, $aN = a\phi(M) = \phi(aM) \subseteq \phi(L) = PN$. Consequently, $a \in (PN : N)$. Conversely, let $a \in (PN : N)$. We have $\phi(aM) = a\phi(M) = aN \subseteq PN$. It yields that $aM \subseteq \phi^{-1}(PN) = L$. It follows that $a \in (L : M)$. Therefore, $(L : M) = (PN : N)$. Now, let $a \in (PN : N)$ and $y \in N$. There exists $x \in M$ such that $y = \phi(x)$. We have $a^k x \in \langle x_1, \ldots, x_n \rangle$. Thus $a^k y = a^k \phi(x) = \phi(a^k x) \in \langle \phi(x_1), \ldots, \phi(x_n) \rangle$ with $\phi(x_1), \ldots, \phi(x_n) \in \phi(L) = PN$. Hence PN is an SFT submodule of N. By Theorem 2.3, N is an SFT A-module.

As a natural application of Proposition 2.6, we get the following Example. □

Example 2.4 Let A be a ring and M an SFT A-module. Then for each submodule N of M, M/N is an SFT A-module.

Now, we are going to give a characterization of SFT modules over an SFT ring. It is well known that any finitely generated module over a Noetherian ring is a Notherian module. In fact, if M is an A-module such that A is a Noetherian ring then M is Noetherian if and only if M is finitely generated. We show a similar result in the SFT case.

Corollary 2.7 *Let A be an SFT ring and M be an A-module. Then M is an SFT A-module if and only if it is finitely generated.*

Proof $'' \Longrightarrow''$ By (1) of Example 2.2. $'' \Longleftarrow''$ By Example 2.1, the A-module A is SFT. Let $\{x_1, \ldots, x_n\} \subseteq M$ be such that $M = \langle x_1, \ldots, x_n \rangle$ and

$$\begin{aligned} \phi : A^n &\longrightarrow M \\ (a_1, \ldots, a_n) &\longmapsto \sum_{i=1}^{n} a_i x_i \end{aligned}.$$

It is clear that ϕ is a surjective A-homomorphism of modules. Thus $M \simeq A^n/ker(\phi)$. By Corollary 2.5, the A-module A^n is SFT and by Example 2.4, $A^n/ker(\phi)$ is an SFT A-module. Hence M is an SFT A-module.

Example 2.5 Let A be an SFT ring and I be an ideal of A. Then I is an SFT A-module if and only if it is a finitely generated ideal of A.

Theorem 2.8 *Let A be a ring, M an A-module and N a submodule of M. If the A-modules N and M/N are SFT, so is M.*

Proof Let P be a prime ideal of A, $L = PM$ and $\bar{L} = \{\bar{x} \in M/N,\ x \in L\}$. By hypothesis, there exist $k \geq 1$ and $x_1, \ldots, x_n \in L$ such that for each $a \in (\bar{L} : M/N)$ and $\bar{x} \in M/N$, $a^k \bar{x} \in \langle \bar{x}_1, \ldots, \bar{x}_n \rangle$. On the other hand, since the A-module N is SFT, there exist $l \geq 1$ and $y_1, \ldots, y_r \in L \bigcap N$ such that for all $b \in (L \bigcap N : N)$ and $y \in N$, $b^l y \in \langle y_1, \ldots, y_r \rangle$.

Let $a \in (L : M)$ and $x \in M$. We have $aM \subseteq L$, it follows that $a(M/N) \subseteq \bar{L}$. Thus $a \in (\bar{L} : M/N)$. Therefore, $a^k\bar{x} = \sum_{i=1}^{n}\alpha_i\bar{x}_i$ where $\alpha_1, \dots, \alpha_n \in A$. Which shows that $a^k x - \sum_{i=1}^{n}\alpha_i x_i \in L\bigcap N$. As $aN \subseteq (aM)\bigcap N \subseteq L\bigcap N$, we have $a \in (L\bigcap N : N)$. Hence

$$a^l(a^k x - \sum_{i=1}^{n}\alpha_i x_i) = \sum_{j=1}^{r}\beta_j y_j$$

where $\beta_1, \dots, \beta_r \in A$. Thus

$$a^{k+l}x = \sum_{i=1}^{n}(a^l\alpha_i)x_i + \sum_{j=1}^{r}\beta_j y_j.$$

Consequently, L is an SFT A-submodule of M. By Theorem 2.3, the A-module M is SFT. □

Proposition 2.9 *Let A be a ring, M an A-module and $N = \bigcap_{P\in spec(A)} PM$. If the A-module M/N is SFT and N is finitely generated, then the A-module M is SFT.*

Proof Let $x_1, \dots, x_n \in N$ be such that $N = \langle x_1, \dots, x_n\rangle$ and P be a prime ideal of A. Set $L = PM$. It is clear that $N \subseteq L$. Then there exist $k \geq 1$ and $y_1, \dots, y_r \in L$ such that for every $a \in (L/N : M/N)$ and $y \in M$, we have $a^k\bar{y} \in \langle\bar{y}_1, \dots, \bar{y}_r\rangle$. Now, let $a \in (L : M)$ and $x \in M$. We have $aM \subseteq L$. Then $a(M/N) \subseteq L/N$. Therefore, $a \in (L/N : M/N)$. Thus $a^k\bar{x} = \sum_{i=1}^{r}\beta_i\bar{y}_i$ where $\beta_1, \dots, \beta_r \in A$. Consequently, $a^k x - \sum_{i=1}^{r}\beta_i y_i \in N$. Hence

$$a^k x = \sum_{i=1}^{r}\beta_i y_i + \sum_{j=1}^{n}\alpha_j x_j$$

where $\alpha_1, \dots, \alpha_n \in A$. As $N \subseteq L$, we get $x_1, \dots, x_n \in L$. It yields that L is an SFT submodule of M. By Theorem 2.3, the A-module M is SFT.

Corollary 2.10 *Let A be a ring. If the A-module $A/Nil(A)$ is SFT and the ideal $Nil(A)$ is finitely generated, then the ring A is SFT.*

Example 2.6 The converse of Corollary 2.10 is false. Indeed, let $\{X_n,\ n \geq 1\}$ be a family of indeterminates over a field K and $A = K[X_n,\ n \geq 1]/\langle X_n,\ n \geq 1\rangle^2$. The only prime ideal of A is $M = \langle\bar{X}_n,\ n \geq 1\rangle$. Since $M^2 = \{0\}$, then the ring A is SFT. But $Nil(A) = M$ is not finitely generated.

Remark 2.11 Let A be a ring and M an A-module. By Corollary 2.5, if M is an SFT A-module, so is the A-module M^n for every integer $n \geq 1$.

For an A-module M, we recall that for each integer $k \geq 1$, we have $Hom_A(A^k, M) \simeq M^k$. Using this interesting isomorphism, we get the following result.

Proposition 2.12 *Let A be a ring, M a free finitely generated A-module and N an SFT A-module. Then $Hom_A(M, N)$ is an SFT A-module.*

Proof Since M is a free finitely generated A-module, there exists $k \geq 1$ such that $A^k \simeq M$. Thus $Hom_A(M, N) \simeq Hom_A(A^k, N) \simeq N^k$. By Remark 2.11, the A-module N^k is SFT. Hence the A-module $Hom_A(M, N) \simeq N^k$ is SFT.

Example 2.7 The hypothesis $''$ the A module M is finitely generated$''$ is important. Indeed, let $A = \mathbb{Z}$ and $M = \mathbb{Z}^{(\mathbb{N}\setminus\{0\})}$. It is clear that M is a free A-module with basis $\{e_n,\ n \geq 1\}$ where $e_n(k) = \delta_{n,k}$. Since M is not a finitely generated A-module, then by Example 2.1, $Hom_A(M, A) \simeq M$ is not an SFT A-module.

Corollary 2.13 *Let A be an SFT ring and M be a free finitely generated A-module. Then for each $k \geq 1$, the A-module $Hom_A(M, A^k)$ is SFT.*

Our next goal is to study the transfer of the SFT property from an A-module M to the ring A. In other words, if an A-module M is SFT, under what condition the ring A is SFT ?

Theorem 2.14 *Let A be a ring and M a free A-module. If M is an SFT A-module, then the ring A is SFT.*

Proof As M is an SFT A-module, by Example 2.1, it is finitely generated. On the other hand, M is a free A-module, then there exists $k \geq 1$ such that $A^k \simeq M$. Thus A-module A^k is SFT. By Proposition 2.6, the A-module A is SFT, and by Example 2.1, the ring A is SFT.

Remark 2.15 Let A be a ring, M and N two A-modules. Then the A-module $M \times N$ is SFT if and only if the A-modules M and N are SFT, it suffices to use Propositions 2.4 and 2.6. Which shows that for every integer $k \geq 1$, the product ring A^k is SFT if and only if the ring A is SFT if and only if the A-module A is SFT.

Let $A \subseteq B$ be a ring extension. It is clear that the ring B is SFT does not imply that the A-module B is SFT. As a counterexample one can take the ring extension $\mathbb{Z} \subseteq \mathbb{Q}$. Since the $\mathbb{Z}$-module $\mathbb{Q}$ is not finitely generated, by Example 2.2, the $\mathbb{Z}$-module $\mathbb{Q}$ is not SFT. In the next theorem, we show that the only condition missed is that the A-module B is finitely generated, which is a necessary and sufficient condition to get this result.

Theorem 2.16 *Let $A \subseteq B$ be a ring extension. Assume that the A-module B is finitely generated. If the ring B is SFT, then the A-module B is SFT.*

Proof Let $b_1, \ldots, b_n \in B$ be such that $B = \langle b_1, \ldots, b_n\rangle A$ and P a prime ideal of A. The ideal PB of B is SFT. Then there exist $k \geq 1$ and $x_1, \ldots, x_r \in PB$ such that for every $a \in PB$, $a^k \in \langle x_1, \ldots, x_r\rangle B$. Let $a \in (PB :_A B)$ and $b \in B$. Set $b = \sum_{i=1}^{n} \beta_i b_i$ where $\beta_1, \ldots, \beta_n \in A$. As $a \in (PB :_A B)$, we have $aB \subseteq PB$, which shows that $a \in PB$. Hence $a^k = \sum_{i=1}^{r} \alpha_i x_i$ where $\alpha_1, \ldots, \alpha_r \in B$. For each $i \in \{1, \ldots, r\}$, let $\alpha_i = \sum_{j=1}^{n} \gamma_{i,j} b_j$, where $\gamma_1, \ldots, \gamma_n \in A$. Therefore,

$$a^k = \sum_{i=1}^{r} \Big(\sum_{j=1}^{n} \gamma_{i,j} b_j \Big) x_j = \sum_{i=1}^{r} \sum_{j=1}^{n} \gamma_{i,j} (b_j x_i).$$

Thus

$$a^k b = \sum_{i=1}^{r} \sum_{j=1}^{n} \gamma_{i,j} \big(b_j x_i\big) \sum_{l=1}^{n} \beta_l b_l = \sum_{i=1}^{r} \sum_{j=1}^{n} \sum_{l=1}^{n} \big(\gamma_{i,j} \beta_l\big) \big(b_l b_j x_i\big).$$

Consequently, $a^k b \in \langle b_l b_j x_j,\ 1 \le i \le r,\ 1 \le j, l \le n \rangle A \subseteq PB$. By Theorem 2.3, the A-module B is SFT.

Note that by combining Theorem 2.16 and Example 2.2, we have the following equivalence for a ring extension $A \subseteq B$ where B is an SFT ring: The A-module B is SFT if and only if the A-module B is finitely generated. In this last result (the same for Theorem 2.16) we do not assume any condition for the ring A. Now, we give a sufficient condition to the descend of the SFT property from a ring B to a subring A of B.

Corollary 2.17 *Let $A \subseteq B$ be a ring extension. Suppose that B is a finitely generated free A-module. If the ring B is SFT, so is A.*

Proof Since B is an SFT ring and a finitely generated A-module, by Theorem 2.16, the A-module B is SFT. By Theorem 2.14, as B is a free finitely generated SFT A-module the ring A is SFT.

Let M be an A-module. We recall that we have the two following extensions of M, the polynomial extension $M[X]$ and power series extension $M[[X]]$ induced by the natural addition and multiplication. Our goal now is to study the transfer of SFT property between the A-module M and the $A[X]$-module $M[X]$ (resp. the $A[[X]]$-module $M[[X]]$). To prove our next result, we need to recall that for every ideal I of A, we have the equality $(IM)[X] = I[X]M[X]$ but this equality is not true in general in the case of power series module see Anderson and Kang (1998).

Proposition 2.18 *Let A be a ring and M an A-module.*

1. *If the $A[X]$-module $M[X]$ is SFT, then the A-module M is SFT.*
2. *If the $A[[X]]$-module $M[[X]]$ is SFT, then the A-module M is SFT.*

Proof (1) Let P be a prime ideal of A. Then $P[X]$ is a prime ideal of $A[X]$. If $a \in (PM :_A M)$, then $aM \subseteq PM$. Thus $aM[X] \subseteq (PM)[X] = P[X]M[X]$. It follows that $a \in (P[X]M[X] :_{A[X]} M[X])$. On the other hand, there exist $k \ge 1$ and $f_1, \ldots, f_n \in P[X]M[X]$ such that for each $f \in (P[X]M[X] :_{A[X]} M[X])$ and $g \in M[X]$, we have $f^k g \in \langle f_1, \ldots, f_n \rangle A[X]$. It yields that for every $a \in (PM :_A M) \subseteq (P[X]M[X] :_{A[X]} M[X])$ and $x \in M \subseteq M[X]$, we get $a^k x \subseteq \langle f_1, \ldots, f_n \rangle A[X]$. Therefore, $a^k x \in \langle f_1(0), \ldots, f_n(0) \rangle A$. By Theorem 2.3, the A-module M is SFT.

(2) Let P be a prime ideal of A. If $a \in (PM :_A M)$, then $aM \subseteq PM$. Hence $aM[[X]] \subseteq (PM)[[X]]$. Therefore, $a \in ((PM)[[X]] :_{A[[X]]} M[[X]])$. By hypothesis, there exist $k \geq 1$ and $f_1, \ldots, f_n \in (PM)[[X]]$ such that for each $f \in ((PM)[[X]] :_{A[[X]]} M[[X]])$ and $g \in M[[X]]$, we have $f^k g \in \langle f_1, \ldots, f_n \rangle A[[X]]$. Thus for each $a \in (PM :_A M) \subseteq ((PM)[[X]] :_{A[[X]]} M[[X]])$ and $x \in M \subseteq M[[X]]$, we have $a^k x \subseteq \langle f_1, \ldots, f_n \rangle A[[X]]$. Consequentely, $a^k x \in \langle f_1(0), \ldots, f_n(0) \rangle A$ where $f_i(0)$ is the constant term of f_i for $i = 1, \ldots, n$. By Theorem 2.3, the A-module M is SFT.

Proposition 2.19 *Let A be a one dimensional integrally closed domain and M an A-module. If the ring A is SFT, then the $A[X]$-module $M[X]$ is SFT if and only if the A-module M is finitely generated.*

Proof $'' \Longrightarrow''$ By Proposition 2.18, the A-module M is SFT and by Example 2.2, the A-module M is finitely generated. $'' \Longleftarrow''$ Since A is a one dimensional integrally closed SFT domain, By (Park 2019, Theorem 2.5), the ring $A[X]$ is SFT. On the other hand, the A-module M is finitely generated. It follows that the $A[X]$-module $M[X]$ is finitely generated. By Corollary 2.7, the $A[X]$-module $M[X]$ is SFT.

Proposition 2.20 *Let A be an SFT Prüfer domain and M an A-module. Then the $A[[X_1, \ldots, X_n]]$-module $M[[X_1, \ldots, X_n]]$ is SFT if and only if the A-module M is finitely generated.*

Proof $'' \Longrightarrow''$ By Proposition 2.18, the A-module M is SFT and by Example 2.2, the A-module M is finitely generated. $'' \Longleftarrow''$ Since A is an SFT Prüfer domain, by (Kang and Park 2009, Proposition 10) the ring $A[[X_1, \ldots, X_n]]$ is SFT. On the other part, the A-module M is finitely generated. It yields that the $A[[X_1, \ldots, X_n]]$-module $M[[X_1, \ldots, X_n]]$ is finitely generated. By Corollary 2.7, the $A[[X_1, \ldots, X_n]]$-module $M[[X_1, \ldots, X_n]]$ is SFT.

Note that by the same way in the proof of Proposition 2.20, we can show that if A is an SFT Prüfer domain and M an A-module, then we have the following equivalence: the $A[X_1, \ldots, X_n]$-module $M[X_1, \ldots, X_n]$ is SFT if and only if the A-module M is finitely generated. Now, we are going to study the SFT stability via the power series extension of a module M over an APVD A. For that, we recall that an integral domain A is called an almost pseudo-valuation domain (or for short APVD) if it is a quasi local domain with maximal ideal P, and there is a valuation overring of A in which P is primary ideal.

Theorem 2.21 *Let A be an APVD (almost pseudo-valuation domain) with maximal ideal P and M a finitely generated A-module. If the ring $(P : P) = \{x \in qf(A),\ xP \subseteq P\}$ is SFT, then the $A[[X_1, \ldots, X_n]]$-module $M[[X_1, \ldots, X_n]]$ is SFT.*

Proof First, we recall that the set $(P : P)$ defined above is always an overring of A (i.e. a ring between A and its quotient field). By (Khalifa and Benhissi 2014, Theorem 2.3) the $A[[X_1, \ldots, X_n]]$ is SFT. By Corollary 2.7, the $A[[X_1, \ldots, X_n]]$-module $M[[X_1, \ldots, X_n]]$ is SFT.

Using this new notion, we return to our main purpose which is studying the finitely ring extension of an SFT ring. This new concept allows us to prove the next Proposition.

Proposition 2.22 *Let $A \subseteq B$ be a ring extension. Assume that A is an SFT ring with characteristic p a prime number and B is a finitely generated multiplication A-module. Then the ring B is SFT.*

Proof By Corollary 2.7, the A-module B is SFT. Let I be an ideal of B. By hypothesis, $I = (I :_A B)B$. On the other hand, there exist $k \geq 1$ and a finitely generated A-submodule F of I such that $a^k x \in F$ for every $a \in I :_A B$ and $x \in B$. Take $r \geq 1$ such that $p^r \geq k$. It is clear that for each $a \in (I :_A B)$ and $x \in B$, $a^{p^r} x \in F$. As $I = (I :_A B)B$, each element $x \in I$ is of the form $x = \sum_{\text{finite}} a_i b_i$ where $a_i \in (I :_A B)$ and $b_i \in B$. Hence $x^{p^r} = \sum_{\text{finite}} a_i^{p^r} b_i^{p^r} \in F \subseteq FB$ where FB is a finitely generated ideal of B contained in I. Therefore, I is an SFT ideal of B. Thus B is an SFT ring.

Our purpose in the next theorem, is to give a sufficient condition to a ring extension $A \subseteq B$ so that the ring $A + XB[X]$ is SFT.

Theorem 2.23 *Let $A \subseteq B$ be a ring extension such that A is a zero dimensional SFT ring and B is a finitely generated A-module. Then the ring $A + XB[X]$ is SFT.*

Proof Let $\{b_1, \dots, b_n\}$ be a generator family of the A-module B, $\{Y_1, \dots, Y_n\}$ a family of indeterminates over A and $\phi : A[X, Y_1, \dots, Y_n] \longrightarrow B[X]$ the A-homomorphism of rings satisfies $\phi(X) = X$ and for every $1 \leq i \leq n$, $\phi(Y_i) = b_i X$. It is clear that

$$\phi(A[X, Y_1, \dots, Y_n]) \subseteq A + XB[X].$$

Conversely, let $f = \sum_{i=0}^{k} a_i X^i \in A + XB[X]$. For $1 \leq i \leq k$, set $a_i = \sum_{j=1}^{n} \alpha_{i,j} b_j$ where $\alpha_{i,j} \in A$ for every $1 \leq i \leq k$ and $1 \leq j \leq n$. Then

$$\begin{aligned} f &= a_0 + \sum_{i=1}^{k}\sum_{j=1}^{n}(\alpha_{i,j} b_j) X^i = a_0 + \sum_{i=1}^{k}\sum_{j=1}^{n} \alpha_{i,j}(b_j X) X^{i-1} \\ &= \phi(a_0 + \sum_{i=1}^{k}\sum_{j=1}^{n}\big(\alpha_{i,j} Y_j) X^{i-1}\big). \end{aligned}$$

Hence $f \in \phi(A[X, Y_1, \dots, Y_n])$. Therefore, $A + XB[X] \simeq A[X, Y_1, \dots, Y_n]/ker(\phi)$. By (Park 2019, Corollary 2.2), the ring $A[X, Y_1 \dots, Y_n]$ is SFT. Thus $A + XB[X]$ is SFT.

In Gabelli (2006), the author has shown that if A is an integral domain with quotient field K, then the ring $A + XK[X]$ is an SFT Prüfer doamin if and only if the ring A is an SFT Prüfer domain. In particular, the ring $\mathbb{Z} + X\mathbb{Q}[X]$ is an SFT Prüfer domain see (Gabelli 2006, Corollary 4.5). Her result shows that there exists an extension of rings $A \subseteq B$ such that the ring $A + XB[X]$ is SFT and the A-module B is not finitely generated. □

Corollary 2.24 *Let $A \subseteq B$ be a ring extension such that A is a zero dimensional SFT ring and B is a finitely generated A-module. Then the ring B is SFT.*

Proof Let I be an ideal of B. By Theorem 2.23, the ideal $XI[X]$ of the ring $A+XB[X]$ is SFT. Thus there exist $k \geq 1$ and $f_1, \ldots, f_n \in XI[X]$ such that $g^k \in \langle f_1, \ldots, f_n \rangle$ for every $g \in XI[X]$. It follows that $a^k X^k \in \langle f_1, \ldots, f_n \rangle$ for each $a \in I$. Hence $a^k \in F = c(f_1)_B + \cdots + c(f_n)_B$ for every $a \in I$ where $F \subseteq I$ is a finitely generated ideal of B. Consequentely, the ring B is SFT.

Note that we can prove the result of Corollary 2.24 by the fact that B is isomorphic to some quotient of the polynomial ring with finitely many indeterminates over A, and using (Park 2019, Corollary 2.2). □

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