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Modules with finite reducing Gorenstein dimension

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Abstract
If M is a nonzero finitely generated module over a commutative Noetherian local ring R such that M has finite injective dimension and finite Gorenstein dimension, then it follows from a result of Holm that M has finite projective dimension, and hence a result of Foxby implies that R is Gorenstein. We prove that the same conclusion holds for certain nonzero finitely generated modules that have finite injective dimension and finite reducing Gorenstein dimension, where the reducing Gorenstein dimension is a finer invariant than the classical Gorenstein dimension, in general. Along the way, we also prove new results, independent of the reducing dimensions, concerning modules of finite Gorenstein dimension.

Keywords Gorenstein dimension · Reducing dimensions · Injective dimension

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1 Introduction

Throughout R denotes a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k, and modules over R are assumed to be finitely generated.

Levin and Vascensolos (1968, 2.2) proved that, if R is a Gorenstein ring, then an R-module has finite projective dimension if and only if it has finite injective dimension. Subsequently, Foxby (1977, 4.4) proved a surprising converse: if R admits a nonzero module of finite projective and finite injective dimension, then R must be Gorenstein. Nearly three decades later, Holm (2004, 2.2) improved Foxby's result by considering modules (not necessarily finitely generated) of finite Gorenstein projective dimension. Holm's result (Holm 2004), in the local case, implies that, if M is an R-module of finite injective dimension, then the projective dimension of M equals the Gorenstein dimension of M; see Sect. 2.2. In the local setting, the results of Foxby and Holm from the foregoing discussion can be summarized as the following beautiful theorem:

Theorem 1.1 (Foxby (1977, 4.4) and Holm (2004, 2.2)) *Let R be a local ring and let M be a nonzero R-module such that $\mathrm{id}_R(M) < \infty$. Then the following hold:*

(i) $\text{G-dim}_R(M) = \text{pd}_R(M)$.
(ii) If $\text{G-dim}_R(M) < \infty$*, then R is Gorenstein.*

Araya and Celikbas (2020), motivated by Bergh's study of complexity of modules (Bergh 2007, 2009), introduced and studied the notion of reducing homological dimensions. These homological dimensions have been recently considered in the noncommutative setting by Araya and Takahashi (2022). In general, a module may have infinite, but finite reducing, homological dimension; see Sect. 2.4 and Example 2.5 for the details.

The main purpose of this paper is to consider Theorem 1.1: we investigate whether the conclusion of the theorem holds when the Gorenstein dimension and the projective dimension are replaced with their reducing versions. We prove that the conclusion of the first part of Theorem 1.1 also holds for reducing homological dimensions. Moreover, we are able to extend the conclusion of the second part of the theorem for two distinct classes of modules. More precisely, we prove:

Theorem 1.2 *Let R be a d-dimensional local ring. If M is a nonzero R-module such that $\mathrm{id}_R(M) < \infty$, then the following hold:*

(i) $\text{red-G-dim}_R(M) = \text{red-pd}_R(M)$.
(ii) If $\text{red-G-dim}_R(M) \leq 1$*, or* $\text{red-G-dim}_R(M) < \infty$ *and* $\text{depth}_R(M) \geq d - 1$*, then R is Gorenstein.*

One of the motivations for Theorem 1.2 comes from a result of Araya and Celikbas, which establishes Theorem 1.2(ii) for the case where M is maximal Cohen–Macaulay; see Sect. 2.6. A nontrivial consequence of Theorem 1.2 is that, if R is a one-dimensional local ring and M is a nonzero R-module such that $\mathrm{id}_R(M) < \infty$ and $\text{red-G-dim}_R(M) < \infty$, then R is Gorenstein. Furthermore, for the two-dimensional case, it follows immediately from Theorem 1.2 that:

Corollary 1.3 *Let R be a two-dimensional local ring and let M be a nonzero torsion-free R-module (e.g., M is an ideal of R). If* $\mathrm{id}_R(M) < \infty$ *and* $\textsf{red-G-dim}_R(M) < \infty$, *then R is Gorenstein.*

In view of Corollary 1.3, it seems worth noting that, in general, if a local ring R admits a nonzero module of finite injective dimension, then R must be Cohen-Macaulay, but it is not necessarily Gorenstein, even if the module in question is torsion-free; for example, one can consider the canonical module of the non-Gorenstein local ring $R = k[\![t^3, t^4, t^5]\!]$; see also Roberts (1987) and Roberts (1998, page 113). On the other hand, Peskine and Szpiro (1973) proved that, if R admits a nonzero cyclic module of finite injective dimension, then R must be Gorenstein.

The proof of Theorem 1.2 is given in Sect. 2; the proof of the theorem makes use of several preliminary results, which are recorded in Sect. 2 and are proved in Sect. 3.

2 Preliminaries and the proof of Theorem 1.2

In this section we prove our main result, namely Theorem 1.2. Along the way, we record several definitions and preliminary results that are used in the proof, as well as an argument for Theorem 1.1.

2.1 (**Syzygy module**) Let M be an R-module. For a positive integer i, we denote by $\Omega^i M$ the i-th *syzygy* of M, namely, the image of the i-th differential map in a minimal free resolution of M. As a convention, we set $\Omega^0 M = M$.

2.2 (**Gorenstein dimension** Auslander and Bridger (1969)) Let R be a local ring and let M be an R-module. Then M is said to be *totally reflexive* provided that $M \cong M^{**}$ and $\mathrm{Ext}^i_R(M, R) = 0 = \mathrm{Ext}^i_R(M^*, R)$ for all $i \geq 1$.

The infimum of n for which there exists an exact sequence $0 \to X_n \to \cdots \to X_0 \to M \to 0$ such that each X_i is totally reflexive is called the *Gorenstein dimension* of M. If M has Gorenstein dimension n, we write $\textsf{G-dim}_R(M) = n$. Therefore, M is totally reflexive if and only if $\textsf{G-dim}_R(M) \leq 0$, where by convention $\textsf{G-dim}_R(0) = -\infty$.

In the proof of Theorem 1.2, we use the fact that the category of modules of finite Gorenstein dimension is closed under taking direct summands; see, for example, (Christensen 2000, 1.1.10(c)). □

We use the following result of Ischebeck in the proof of Theorem 1.2:

2.3 (Ischebeck 1969, 2.6) Let R be a local ring and let M and N be nonzero R-modules. If $\mathrm{id}_R(N) < \infty$, then it follows that $\mathrm{depth}(R) - \mathrm{depth}_R(M) = \sup\{n \mid \mathrm{Ext}^n_R(M, N) \neq 0\}$.

2.4 (**Reducing dimensions** (Araya and Celikbas 2020, 2.1)) Let M be an R-module, and let $\mathbb{I}$ be a *homological invariant* of R-modules, for example, $\mathbb{I} = \textsf{pd}$ or $\mathbb{I} = \textsf{G-dim}$.

We write $\textsf{red-}\mathbb{I}(M) < \infty$ provided that there exists a sequence of R-modules $K_0, \ldots, K_r$, positive integers $a_1, \ldots, a_r, b_1, \ldots, b_r, n_1, \ldots, n_r$, and short exact sequences of the form

$$0 \to K_{i-1}^{\oplus a_i} \to K_i \to \Omega^{n_i} K_{i-1}^{\oplus b_i} \to 0 \tag{2.4.1}$$

for each $i = 1, \dots r$, where $K_0 = M$ and $\mathbb{I}(K_r) < \infty$. If a sequence of modules as in (2.4.1) exists, then we call $\{K_0, \dots, K_r\}$ a *reducing* $\mathbb{I}$*-sequence* of M.

The *reducing invariant* $\mathbb{I}$ of M is defined as follows:

$$\text{red-}\mathbb{I}(M) = \inf\{r \in \mathbb{N} \cup \{0\} : \text{there is a reducing } \mathbb{I} - \text{sequence } K_0, \dots, K_r \text{ of } M\}.$$

We set, $\text{red-}\mathbb{I}(M) = 0$ if and only if $\mathbb{I}(M) < \infty$. □

Next we recall an example from Araya and Celikbas (2020) which shows that the reducing homological dimensions are finer than regular homological dimensions.

Example 2.5 (Araya and Celikbas 2020, 2.3) Let $R = k[x, y]/(x, y)^2$. Then $\mathsf{pd}_R(k) = \infty = \mathsf{G\text{-}dim}_R(k)$, but we have that $\mathsf{red\text{-}G\text{-}dim}_R(k) = 1 = \mathsf{red\text{-}pd}_R(k)$.

Moreover, if M is an R-module, then it follows that $\mathsf{red\text{-}G\text{-}dim}_R(M) \leq \infty = \mathsf{G\text{-}dim}_R(M)$ if and only if $M \cong R^{\oplus \alpha} \oplus k^{\oplus \beta}$ for some integers $\alpha \geq 0$ and $\beta \geq 1$. □

Note that Example 2.5 also shows that the reducing homological dimension of a module is not always bounded by the depth of the ring in question. One can also check (Araya and Celikbas 2020, 2.7) for an example of a two-dimensional Cohen-Macaulay ring R and a maximal Cohen-Macaulay R-module M such that $\mathsf{red\text{-}pd}_R(M) = \mathsf{red\text{-}G\text{-}dim}_R(M) < \infty = \mathsf{G\text{-}dim}_R(M) = \mathsf{pd}_R(M)$.

The following result, due to Araya and Celikbas (2020), is used in the proof of Theorem 1.2:

2.6 (Araya and Celikbas 2020, 3.3(iii)) Let R be a local ring and let M be a (nonzero) maximal Cohen–Macaulay R-module. If $\mathrm{id}_R(M) < \infty$ and $\mathsf{red\text{-}G\text{-}dim}_R(M) < \infty$, then R is Gorenstein. □

The proof of Theorem 1.2 relies upon some preliminary results, namely upon Propositions 2.7, 2.8 and 2.9, which are stated next.

Proposition 2.7 *Let R be a local ring and let M be an R-module. Assume, whenever X is a totally reflexive R-module, one has $\mathsf{Ext}^i_R(X, M) = 0$ for all $i \geq 1$. Then it follows that $\mathsf{red\text{-}G\text{-}dim}_R(M) = \mathsf{red\text{-}pd}_R(M)$.*

Proposition 2.8 *Let R be a local ring and let $0 \to M^{\oplus a} \to K \to \Omega^n M^{\oplus b} \to 0$ be a short exact sequence of R-modules, where $a \geq 1$, $b \geq 1$, and $n \geq 0$ are integers. If $\mathsf{G\text{-}dim}_R(K) < \infty$, then, for each $i \geq 1$, there exists a short exact sequence of R-modules $0 \to M^{\oplus a_i} \to Y_i \to \Omega^{r_i} M^{\oplus b_i} \to 0$, where $\mathsf{G\text{-}dim}_R(Y_i) < \infty$, $r_i = 2^i(n+1) - 1$, $a_i = a^{2^i}$ and $b_i = b^{2^i}$.*

Proposition 2.9 *Let R be a local ring and let M be an R-module. Assume $x \in \mathfrak{m}$ is a non zero-divisor on R and M. If $\{K_0, \dots, K_r\}$ is a reducing $\mathsf{G\text{-}dim}$-sequence of M, then $\{\overline{K_0}, \dots, \overline{K_r}\}$ is a reducing $\mathsf{G\text{-}dim}$-sequence of $\overline{M}$ over $\overline{R}$, where $\overline{M} = M/xM$. Therefore, $\mathsf{red\text{-}G\text{-}dim}_{R/xR}(M/xM) \leq \mathsf{red\text{-}G\text{-}dim}_R(M)$.*

Next we exploit Propositions 2.7, 2.8 and 2.9 and prove Theorem 1.2; we defer the proofs of these preliminary propositions until Sect. 3.

Proof of Theorem 1.2 We start by noting that, since $\mathrm{id}_R(M) < \infty$, R is a Cohen–Macaulay ring; see Bruns and Herzog (1993, 9.6.2 and 9.6.4(ii)).

Part (i) follows immediately from Proposition 2.7: as $\mathrm{id}_R(M) < \infty$, we have that $\mathsf{Ext}^j_R(X, M) = 0$ for all $j \geq 1$ for each totally reflexive R-module X; see Sect. 2.3.

Next we assume $\mathsf{red\text{-}G\text{-}dim}_R(M) \leq 1$, and show that R is Gorenstein. It follows from Sect. 2.4 that there exists a short exact sequence of R-modules $0 \to M^{\oplus a} \to K \to \Omega^n M^{\oplus b} \to 0$, where a, b, n are positive integers and $\mathsf{G\text{-}dim}_R(K) < \infty$. Then, by Proposition 2.8, we have a short exact sequence of R-modules $0 \to M^{\oplus a_i} \to Y \to \Omega^{r_i} M^{\oplus b_i} \to 0$, where $i \gg 0$, $\mathsf{G\text{-}dim}_R(Y) < \infty$, and $r_i \geq d$. Therefore, $\Omega^{r_i} M^{\oplus b_i}$ is maximal Cohen–Macaulay. Now, as $\mathrm{id}_R(M^{\oplus a_i}) < \infty$, we see that $\mathsf{Ext}^1_R(\Omega^{r_i} M^{\oplus b_i}, M^{\oplus a_i}) = 0$; see Sect. 2.3. So the short exact sequence $0 \to M^{\oplus a_i} \to Y \to \Omega^{r_i} M^{\oplus b_i} \to 0$ splits, and hence $M^{\oplus a_i}$ occurs as a direct summand of Y. This shows that $\mathsf{G\text{-}dim}_R(M) < \infty$ and hence R is Gorenstein; see Sect. 2.2 and Theorem 1.1.

Now, to complete the proof of part (ii), we assume $\mathsf{depth}_R(M) \geq d - 1$, and proceed by induction on d to show that R is Gorenstein. There is nothing to prove if $d = 0$; see Sect. 2.6.

Assume $d \geq 2$ and that the claim is true when $d = 1$. Since R is a Cohen-Macaulay ring and $\mathsf{depth}_R(M) \geq 1$, there exists an element $x \in \mathfrak{m}$ which is a non zero-divisor on both R and M. Then it follows by Proposition 2.9 that $\mathsf{red\text{-}G\text{-}dim}_{R/xR}(M/xM) \leq \mathsf{red\text{-}G\text{-}dim}_R(M) < \infty$. Therefore, since $\mathrm{id}_{R/xR}(M/xM) < \infty$ and $\mathsf{depth}_{R/xR}(M/xM) \geq d - 2$, we conclude by the induction hypothesis that R/xR is Gorenstein, i.e., R is Gorenstein. Therefore, it suffices to prove the case where $d = 1$.

Assume $d = 1$ and choose a reducing $\mathsf{G\text{-}dim}$-sequence $\{K_0, \dots, K_r\}$ of M.

Claim: For each $i = 0, \dots, r$, we have that $K_i \cong M^{\oplus c_i} \oplus L_i$ for some R-module L_i and for some integer $c_i \geq 1$ such that L_i is either zero or maximal Cohen-Macaulay.

Proof of the claim: We proceed by induction on i. If $i = 0$, then, since $K_0 = M$, we pick $L_0 = 0$ and $c_0 = 1$. So we assume $i \geq 1$. Then, by the induction hypothesis, we have that $K_{i-1} \cong M^{\oplus c_{i-1}} \oplus L_{i-1}$ for some R-module L_{i-1} and for some integer $c_{i-1} \geq 1$, where L_{i-1} is either zero or maximal Cohen–Macaulay. Now we consider the following pushout diagram, where the middle horizontal short exact sequence follows by the definition of reducing Gorenstein dimension; see Sect. 2.4.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M^{\oplus c_{i-1}a_i} & = & M^{\oplus c_{i-1}a_i} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & K_{i-1}^{\oplus a_i} & \to & K_i & \to & \Omega^{n_i} K_{i-1}^{\oplus b_i} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & L_{i-1}^{\oplus a_i} & \to & L_i & \to & \Omega^{n_i} K_{i-1}^{\oplus b_i} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since L_{i-1} is either zero or maximal Cohen-Macaulay, we see from the bottom horizontal short exact sequence that L_i is either zero or maximal Cohen-Macaulay. In either case, since $\mathrm{id}_R(M) < \infty$, it follows by Sect. 2.3 that $\mathsf{Ext}^1_R(L_i, M) = 0$. This implies that the middle vertical short exact sequence splits, yields the isomorphism $K_i \cong L_i \oplus M^{\oplus c_{i-1}a_i}$, and proves the claim.

Now, by the claim established above, M is a direct summand of K_r. Then, since $\mathsf{G\text{-}dim}_R(K_r) < \infty$, we conclude that $\mathsf{G\text{-}dim}_R(M) < \infty$; see Sect. 2.2. Therefore, Theorem 1.1 shows that R is Gorenstein, and this completes the proof of the theorem. □

3 Proofs of the preliminary propositions

This section is devoted to the proofs of Propositions 2.7, 2.8 and 2.9. We start by preparing a lemma.

3.1 If X is a totally reflexive module over a local ring R and $n \geq 1$ is an integer, then the nth *cosyzygy* of X, denoted by $\Omega^{-n}X$, is defined to be the image of the R-dual map ∂_n^* of the n-th differential map in a minimal free resolution of X^*. Note that the cosyzygy $\Omega^{-n}X$ is totally reflexive and $\Omega^n\Omega^{-n}X \cong X$.

Lemma 3.2 *Let R be a local ring and let M be an R-module. Assume, for each totally reflexive R-module X, we have that $\mathsf{Ext}^i_R(X, M) = 0$ for all $i \geq 1$. If X is a totally reflexive R-module and $j \geq 0$, then it follows that $\mathsf{Ext}^i_R(X, \Omega^j M) = 0$ for all $i \geq 1$.*

Proof Let $i \geq 1$ and $j \geq 1$ be integers. Then the following isomorphisms hold:

$$\mathsf{Ext}^i_R(X, \Omega^j M) \cong \mathsf{Ext}^{i+j}_R(\Omega^{-j}X, \Omega^j M) \cong \mathsf{Ext}^i_R(\Omega^{-j}X, M).$$

Here, the first isomorphism is due to the fact $\Omega^j\Omega^{-j}X \cong X$; the second one follows since $\Omega^{-j}X$ is totally reflexive so that $\mathsf{Ext}^s_R(\Omega^{-j}X, R) = 0$ for all $s \geq 1$. As $\mathsf{Ext}^i_R(\Omega^{-j}X, M)$ vanishes in view of the hypothesis, the claim follows. □

In passing we record:

Corollary 3.3 *Let R be a local ring and let M be an R-module such that $\mathrm{id}_R(M) < \infty$. Then, for each integer $j \geq 0$ and each totally reflexive R-module X, it follows that $\mathsf{Ext}^i_R(X, \Omega^j M) = 0$ for all $i \geq 1$.*

Proof The claim is an immediate consequence of Sect. 2.3 and Lemma 3.2. □

Corollary 3.4 *Let R be a local ring and let M be an R-module. Assume, for each totally reflexive R-module X, it follows that $\mathsf{Ext}^j_R(X, M) = 0$ for all $j \geq 1$. Assume further there are short exact sequences of R-modules of the form $0 \to K_{i-1}^{\oplus a_i} \to K_i \to \Omega^{n_i} K_{i-1}^{\oplus b_i} \to 0$ for $i = 1, \ldots r$, where $K_0 = M$ and $r, a_1, \ldots, a_r, b_1, \ldots, b_r, n_1, \ldots, n_r$ are all positive integers. Then, for each totally reflexive R-module X and for each $i = 0, \ldots, r$, it follows that $\mathsf{Ext}^j_R(X, K_i) = 0$ for all $j \geq 1$.*

Proof We proceed by induction on i.

If $i = 0$, then $K_0 = M$, and so there is nothing to prove. Let i be an integer with $1 \leq i \leq r$ and assume, for each totally reflexive R-module X, we have that $\mathsf{Ext}^j_R(X, K_{i-1}) = 0$ for all $j \geq 1$.

Next consider the following short exact sequence that exists by the hypothesis:

$$0 \to K_{i-1}^{\oplus a_i} \to K_i \to \Omega^{n_i} K_{i-1}^{\oplus b_i} \to 0. \tag{3.2.1}$$

Let Y be a totally reflexive R-module and let $j \geq 1$. Then (3.2.1) yields the following exact sequence:

$$\mathsf{Ext}^j_R(Y, K_{i-1}^{\oplus a_i}) \to \mathsf{Ext}^j_R(Y, K_i) \to \mathsf{Ext}^j_R(Y, \Omega^{n_i} K_{i-1}^{\oplus b_i}). \tag{3.2.2}$$

As $\mathsf{Ext}^j_R(Y, K_{i-1}^{\oplus a_i})$ vanishes by the induction hypothesis, to complete the induction argument, it suffices to observe the vanishing of $\mathsf{Ext}^j_R(Y, \Omega^{n_i} K_{i-1})$. However this follows by Lemma 3.2. □

Next we use Corollary 3.4 and prove Proposition 2.7:

Proof of Proposition 2.7: Note that, if $\mathsf{red\text{-}G\text{-}dim}_R(M) = \infty$, then it follows that $\mathsf{red\text{-}pd}_R(M) = \infty$. Hence, to prove the proposition, it suffices to assume $\mathsf{red\text{-}G\text{-}dim}_R(M) < \infty$.

Assume $\mathsf{red\text{-}G\text{-}dim}_R(M) = r < \infty$ and let $\{K_0, \ldots, K_r\}$ be a reducing $\mathsf{G\text{-}dim}$ sequence of M. Then, since $\mathsf{G\text{-}dim}_R(K_r) < \infty$, we consider the finite projective dimension hull of K_r (Auslander and Buchweitz 1989, 1.1), i.e., a short exact sequence of R-modules of the form $0 \to K_r \to P \to X \to 0$, where $\mathsf{pd}_R(P) < \infty$ and X is totally reflexive. Note that Corollary 3.4 implies that $\mathsf{Ext}^1_R(X, K_r) = 0$. Therefore, the finite projective hull of K_r splits and hence $\mathsf{pd}_R(K_r) < \infty$. This shows that $\mathsf{red\text{-}pd}_R(M) \leq r$. As, in general, we have that $\mathsf{red\text{-}G\text{-}dim}_R(M) \leq \mathsf{red\text{-}pd}_R(M)$, the claim of the proposition follows. □

Remark 3.5 It is worth noting that there are examples of local rings R and modules M over R with $\mathsf{red\text{-}G\text{-}dim}_R(M) < \infty = \mathsf{pd}_R(M)$ and $\mathsf{Ext}^i_R(X, M) = 0$ for each totally reflexive R-module X and each $i \geq 1$. For example, if R is as in Example 2.5 and $M = k$, then each totally reflexive R-module is free so that $\mathsf{Ext}^i_R(X, M) = 0$ for all $i \geq 1$ and $\mathsf{pd}_R(M) = \infty$.

The next two results are used for the proof of Proposition 2.8; the first one, Sect. 3.6, is well-known, but we include it for completeness. The second one, 3.7, is a special case of (Takahashi 2006, 3.1) and plays an important role for the proof of Proposition 2.8.

3.6 (Dao and Takahashi 2015, 2.2) Let R be a local ring and let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules.

(i) There is an exact sequence $0 \to \Omega C \to F \oplus A \to B \to 0$, where F is a free R-module.
(ii) If $n \geq 0$ is an integer, then there is an exact sequence $0 \to \Omega^n A \to G \oplus \Omega^n B \to \Omega^n C \to 0$, where G is a free R-module.

3.7 Let R be a commutative ring. If $0 \to L \to X \to N \to 0$ is a short exact sequence of R-modules such that $\mathsf{G\text{-}dim}_R(X) < \infty$ and $L \cong G \oplus Y$ for some free R-module G. Then there exists a short exact sequence of R-modules $0 \to Y \to A \to N \to 0$, where $\mathsf{G\text{-}dim}_R(A) < \infty$; see (Takahashi 2006, 3.1).

Next is the proof of the second proposition:

Proof of Proposition 2.8 We make use of 3.6 with the exact sequence $0 \to M^{\oplus a} \to K \to \Omega^n M^{\oplus b} \to 0$ and obtain the exact sequences

$$0 \to \Omega^{n+1} M^{\oplus b} \to F \oplus M^{\oplus a} \to K \to 0 \tag{2.8.1}$$

and

$$0 \to \Omega^{n+1} M^{\oplus a} \to G \oplus \Omega^{n+1} K \to \Omega^{2n+1} M^{\oplus b} \to 0, \tag{2.8.2}$$

where F and G are free R-modules.

By taking the direct sum of a copies of the short exact sequence in (2.8.1) and the direct sum of b copies of the short exact sequence in (2.8.2), we obtain the following short exact sequences:

$$0 \to \Omega^{n+1} M^{\oplus ab} \xrightarrow{\alpha} F^{\oplus a} \oplus M^{\oplus a^2} \to K^{\oplus a} \to 0 \tag{2.8.3}$$

$$0 \to \Omega^{n+1} M^{\oplus ab} \xrightarrow{\beta} G^{\oplus b} \oplus \Omega^{n+1} K^{\oplus b} \to \Omega^{2n+1} M^{\oplus b^2} \to 0 \tag{2.8.4}$$

Now we take the pushout of the maps α and β from the exact sequences in (2.8.3) and (2.8.4), and obtain the following diagram with with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^{n+1}M^{\oplus ab} & \xrightarrow{\alpha} & F^{\oplus a}\oplus M^{\oplus a^2} & \longrightarrow & K^{\oplus a} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\beta} & & \downarrow & & \| & & \\
0 & \longrightarrow & G^{\oplus b}\oplus\Omega^{n+1}K^{\oplus b} & \longrightarrow & X & \longrightarrow & K^{\oplus a} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{2n+1}M^{\oplus b^2} & = & \Omega^{2n+1}M^{\oplus b^2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Now assume $\mathsf{G}\text{-}\mathsf{dim}_R(K) < \infty$. Then the exact sequence in the middle row in the above diagram implies that $\mathsf{G}\text{-}\mathsf{dim}_R(X) < \infty$. So we use 3.7 for the exact sequence $0 \to F^{\oplus a} \oplus M^{\oplus a^2} \to X \to \Omega^{2n+1}M^{\oplus b^2} \to 0$, and obtain a short exact sequence of the form

$$0 \to M^{\oplus a^2} \to A \to \Omega^{2n+1}M^{\oplus b^2} \to 0, \tag{2.8.5}$$

where $\mathsf{G}\text{-}\mathsf{dim}_R(A) < \infty$. Therefore, setting $Y_1 = A$, we establish the claim for the case where $i = 1$.

Next assume $i \geq 2$. Then, by the induction hypothesis, there exists a short exact sequence of R-modules of the form

$$0 \to M^{\oplus a_{i-1}} \to Y_{i-1} \to \Omega^{r_{i-1}}M^{\oplus b_{i-1}} \to 0, \tag{2.8.6}$$

where $\mathsf{G}\text{-}\mathsf{dim}_R(Y_{i-1}) < \infty$, $r_{i-1} = 2^{i-1}(n+1) - 1$, $a_{i-1} = a^{2^{i-1}}$ and $b_{i-1} = b^{2^{i-1}}$. Hence we can apply the previous process to the short exact sequence in (2.8.6) and obtain a short exact sequence of R-modules $0 \to M^{\oplus a_i} \to Y_i \to \Omega^{r_i}M^{\oplus b_i} \to 0$, where $\mathsf{G}\text{-}\mathsf{dim}_R(Y_i) < \infty$, $r_i = 2^i(n+1) - 1$, $a_i = a^{2^i}$ and $b_i = b^{2^i}$. This completes the induction argument and establishes the proposition. □

Our next aim is to prove Proposition 2.9; for that we proceed and prepare two lemmas. The first one can be found, for example, in (Avramov 1996, 1.2.4).

Lemma 3.8 *Let R be a local ring, M an R-module, and let $x \in \mathfrak{m}$ be an element of R.*

(i) x is a non zero-divisor on R, then x is a non zero-divisor on $\Omega^i_R(M)$ for each $i \geq 1$.

(ii) If x is a non zero-divisor on R and M, then $\Omega^i_R(M)/x\Omega^i_R(M) \cong \Omega^i_{R/xR}(M/xM)$ for each $i \geq 0$.

Lemma 3.9 *Let R be a local ring and let $0 \to K_{i-1}^{\oplus a_i} \to K_i \to \Omega_R^{n_i} K_{i-1}^{\oplus b_i} \to 0$ be short exact sequences of R-modules for $i = 1, \dots r$, where a_i, b_i, n_i are nonnegative integers and r is a positive integer. If $x \in \mathfrak{m}$ is a non zero-divisor on R and on K_0, then x is a non zero-divisor on K_i for each $i = 0, \dots, r$.*

Proof We proceed by induction on i. If $i = 0$, then the claim is just the hypothesis. Hence we assume $i \geq 1$. Then, by the induction hypothesis, it follows that x is a non-zero divisor on K_{i-1}. Thus, tensoring the given short exact sequences by R/xR, for each $i = 1, \dots, r$, we obtain an exact sequence of the form $\mathsf{Tor}_1^R(K_{i-1}^{\oplus a_i}, R/xR) \to \mathsf{Tor}_1^R(K_i, R/xR) \to \mathsf{Tor}_1^R(\Omega_R^{n_i} K_{i-1}^{\oplus b_i}, R/xR)$. This yields $\mathsf{Tor}_1^R(K_i, R/xR) = 0$ and hence shows that x is a non zero-divisor on K_i, as required. □

We are now ready to prove Proposition 2.9:

Proof of Proposition 2.9 There is nothing to prove if $\mathsf{red\text{-}G\text{-}dim}_R(M) = \infty$. Therefore we assume $\mathsf{red\text{-}G\text{-}dim}_R(M) = r < \infty$. Then, by definition, there exist short exact sequences of R-modules

$$0 \to K_{i-1}^{\oplus a_i} \to K_i \to \Omega_R^{n_i} K_{i-1}^{\oplus b_i} \to 0, \tag{2.9.1}$$

where $i = 1, \dots, r$, $K_0 = M$ and $\mathsf{G\text{-}dim}_R(K_r) < \infty$.

Tensoring the short exact sequences in (2.9.1) by R/xR, we obtain the following exact sequences

$$\mathsf{Tor}_1^R(\Omega_R^{n_i} K_{i-1}^{\oplus b_i}, \overline{R}) \to \left(\overline{K_{i-1}}\right)^{\oplus a_i} \to \overline{K_i} \to \overline{\left(\Omega_R^{n_i} K_{i-1}^{\oplus b_i}\right)} \to 0, \tag{2.9.2}$$

where $i = 1, \dots, r$ and $\overline{(-)}$ denotes $- \otimes_R R/xR$.

Note that, since x is a non zero-divisor on R, it follows from Lemma 3.8(i), x is a non zero-divisor on $\Omega_R^{n_i} K_{i-1}$ for each $i = 1, \dots, r$. Therefore, it follows that

$$\mathsf{Tor}_1^R(\Omega_R^{n_i} K_{i-1}^{\oplus b_i}, \overline{R}) = 0 \text{ for each } i = 1, \dots r. \tag{2.9.3}$$

Recall that x is a non zero-divisor on K_0. Hence, it follows from Lemma 3.9 that x is a non zero-divisor on K_i for each $i = 0, \dots, r$. Consequently, Lemma 3.8(ii) implies that

$$\overline{\left(\Omega_R^{n_i} K_{i-1}^{\oplus b_i}\right)} \cong \Omega_{\overline{R}}^{n_i} \left(\overline{K_{i-1}}\right)^{\oplus b_i} \text{ for each } i = 1, \dots r. \tag{2.9.4}$$

Now, the exact sequence in (2.9.2), in view of (2.9.3) and (2.9.4), yields the following exact sequence of $\overline{R}$-modules for each $i = 1, \dots, r$.

$$0 \to \left(\overline{K_{i-1}}\right)^{\oplus a_i} \to \overline{K_i} \to \Omega_{\overline{R}}^{n_i} \left(\overline{K_{i-1}}\right)^{\oplus b_i} \to 0. \tag{2.9.5}$$

We know that $\mathsf{G\text{-}dim}_{\overline{R}}(\overline{K_r}) = \mathsf{G\text{-}dim}_R(K_r) < \infty$; see (Christensen 2000, 1.4.5). Therefore, (2.9.5) shows that $\{\overline{K_0}, \dots, \overline{K_r}\}$ is a reducing $\mathsf{G\text{-}dim}$-sequence of $\overline{M}$ over the ring $\overline{R}$. This implies that $\mathsf{red\text{-}G\text{-}dim}_{\overline{R}}(M/xM) \leq r$. □

One can see from the proof of Proposition 2.8 that the fact $\text{G-dim}_{\overline{R}}(\overline{K_r}) = \text{G-dim}_R(K_r)$ is used only once at the end of the argument, and in fact, it suffices to have the inequality $\text{G-dim}_{\overline{R}}(\overline{K_R}) \leq \text{G-dim}_R(K_r)$ for the proposition to hold. Therefore, we finish this section by noting that the proof of Proposition 2.8 yields the following more general result:

Remark 3.10 Let $(R, \mathfrak{m})$ be a local ring and let $\mathbb{I}$ be a homological invariant of R-modules. Assume we have that $\mathbb{I}_{R/xR}(M/xM) \leq \mathbb{I}_R(M)$, in case M is an R-module and $x \in \mathfrak{m}$ is a non zero-divisor on R and M. Then it follows that $\text{red-}\mathbb{I}_{R/xR}(M/xM) \leq \text{red-}\mathbb{I}_R(M)$ in case M is an R-module and $x \in \mathfrak{m}$ is a non zero-divisor on R and M, where red-$\mathbb{I}$ is the reducing invariant of $\mathbb{I}$ defined as in 2.4; see also (Araya and Celikbas 2020, 2.1).

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