Beitr Algebra Geom (2024) 65:229–240
https://doi.org/10.1007/s13366-023-00685-z

ORIGINAL PAPER

On purely-maximal ideals and semi-Noetherian power series rings

Nader Ouni[1] · Ali Benhissi[1]

Received: 3 June 2022 / Accepted: 10 January 2023 / Published online: 23 January 2023

Abstract

Tarizadeh and Aghajani conjectured that each purely-prime ideal is purely-maximal (Tarizadeh and Aghajani in Commun Algebra 49(2):824–835, 2021, Conjecture 5.8). We study purely-prime and purely-maximal ideals in rings of the form $A + XS$ (where S is either $B[X]$ or $B[[X]]$), subrings of $A[[X]]$ of the form $A[X] + I[[X]]$ and $A + I[[X]]$ (where A is a subring of a commutative unitary ring B and I an ideal of A) and Nagata's idealization ring. As application, we give necessary and sufficient conditions on each of the aforementioned ring to be semi-Noetherian. We deduce that the power series ring $A[[X]]$ is semi-Noetherian if and only if the ring A is semi-Noetherian. We deduce that Tarizadeh and Aghajani's conjecture holds in each of the aforementioned ring if and only if it holds in the ring A.

Keywords Power series ring · Purely-maximal ideal · Nagata idealization ring

Mathematics Subject Classification 13F25 · 13E05

1 Introduction

Throughout this paper all rings are commutative with identity. Let A be a subring of a ring B and I an ideal of A. The ideal I is said to be pure if for every $a \in I$ there exists $b \in I$ such that $a = ab$ (Borceux and Van den Bossche 1983, p. 141). The ideal I is said to be purely-maximal if it is maximal (under inclusion) in the lattice of proper pure ideals of A (Borceux and Van den Bossche 1983, p. 156). The ideal I is said to be purely-prime if it is proper and if for any pure ideals I_1, I_2 of A with $I_1 \cap I_2 \subseteq I$, then $I_1 \subseteq I$ or $I_2 \subseteq I$ (Borceux and Van den Bossche 1983, p. 156). Borceaux and Van den Bossche had introduced and studied purely-maximal and purely-prime ideals in their book (Borceux and Van den Bossche 1983). Al-Ezeh had studied purely-maximal

✉ Nader Ouni
nader.ouni27@gmail.com

1 Department of Mathematics, Faculty of Sciences of Monastir, 5019 Monastir, Tunisia

ideals of a PF-ring (Al-Ezeh 1988). Recently, Tarizadeh and Aghajani have established new algebraic and topological results on purely-prime ideals (Tarizadeh and Aghajani 2021). Also they called a ring A to be semi-Noetherian if every pure ideal of A is finitely generated (Tarizadeh and Aghajani 2021, p. 834). Tarizadeh and Aghajani asked if each purely-prime ideal is purely-maximal (Tarizadeh and Aghajani 2021, Conjecture 5.8).

The aim of this paper is (I) to study purely-maximal ideals of rings of the form $A + XS$ (where S is either $B[X]$ or $B[[X]]$), subrings of $A[[X]]$ of the form $A[X] + I[[X]]$ and $A + I[[X]]$, Nagata's idealization ring $R \ltimes M$ (where M is an R-module), (II) to give necessary and sufficient conditions so that each of the aforementioned rings is semi-Noetherian and (III) to provide necessary and sufficient conditions so that Tarizadeh and Aghajani's conjecture holds in each of the aforementioned ring.

In the second section, we show that purely-maximal ideals of the ring $\mathcal{R} = A + XS$ are precisely ideals of the form $I\mathcal{R}$ where I ranges over purely-maximal ideals of A (Theorem 2.4). We deduce an analogous of Hilbert's basis theorem for semi-Noetherian ring: the ring $A[[X]]$ (respectively $A[X]$) is semi-Noetherian if and only if A is semi-Noetherian (Corollary 2.7).

In the third section, we show that purely-maximal ideals of the ring $\mathcal{R}(I) = A + I[[X]]$ (respectively $A + I[X]$, $A[X] + I[[X]]$) are precisely ideals of the form $J\mathcal{R}(I)$ where J ranges over purely-maximal ideals of A (Corollary 3.3). We deduce that the ring $\mathcal{R}(I)$ is semi-Noetherian if and only if A is semi-Noetherian (Corollary 3.4).

Let R be a ring and M a module over R. The fourth section is devoted to study purely-maximal ideals of Nagata idealization ring $R \ltimes M$. We show that purely-maximal ideals of Nagata idealization ring $R \ltimes M$ are precisely ideals of the form $I \ltimes IM$ where I ranges over purely-maximal ideals of A (Theorem 4.4). We deduce that the ring $R \ltimes M$ is semi-Noetherian if and only if R is semi-Noetherian (Theorem 4.4).

Every purely-maximal ideal is purely-prime (Borceux and Van den Bossche 1983, Chapter 7–Proposition 26). Tarizadeh and Aghajani noticed that in all known examples each purely-prime ideal is purely-maximal (Tarizadeh and Aghajani 2021, p. 834). So they asked if this is true for all rings or not. We study their question in each of the aforementioned ring. We give necessary and sufficient conditions on each of the aforementioned ring so that Tarizadeh and Aghajani's conjecture holds.

2 Purely-maximal ideals of the ring $A + XS$

For any ring extension $A \subseteq B$ and for any ideal J of B, then the set $A + J = \{a + b : a \in A,\ b \in J\}$ is a subring of B which contains A. Throughout this section, A denotes a subring of a ring B, X an indeterminate over B and $\mathcal{R} = A + XS$ (where S is either $B[X]$ or $B[[X]]$) the subring of S formed by elements whose constant term belongs to A. For each $f \in S$, $f^{(i)}$ denotes the coefficient of X^i in f.

For any ideals I and J of A, then $I \subseteq J$ if and only if $I\mathcal{R} \subseteq J\mathcal{R}$.

Lemma 2.1 *If J is a proper pure ideal of $\mathcal{R}$, then $J \subseteq I\mathcal{R}$ for some proper pure ideal I of A.*

Proof Let $I = \{f^{(0)}, f \in J\}$ be the ideal of A consisting of constant terms of elements of J. For $\mathcal{R} = A + XB[[X]]$, since invertible elements of $\mathcal{R}$ are elements whose constant terms are invertible in A, I is proper. For the other case $\mathcal{R} = A + XB[X]$, suppose that $I = A$. Then $1 = f^{(0)}$ and $f = fg$ for some $f, g \in J$. It follows that $g^{(0)} = 1$ and $f^{(1)} = f^{(1)}g^{(0)} + f^{(0)}g^{(1)}$. So $g^{(1)} = 0$. By induction, $g^{(i)} = 0$ for all $i \geq 1$ and so $1 = g \in J$, a contradiction. Then I is proper.

Let $a \in I$ and $f \in J$ be such that $a = f^{(0)}$. Let $g \in J$ be such that $f = fg$. Thus $a = ag^{(0)}$ and $g^{(0)} \in I$. Then I is a pure ideal of A. Let $f, g \in J$ such that $f = fg$. Thus $f^{(0)} = f^{(0)}g^{(0)} \in aA$ (where $a = g^{(0)} \in I$) and $f^{(1)} = f^{(0)}g^{(1)} + f^{(1)}g^{(0)} \in aB$. By induction, $f^{(i)} \in aB$ for all $i \geq 1$. So $f = ah \in I\mathcal{R}$ for some $h \in \mathcal{R}$. □

Lemma 2.2 *For any ring map $f : R \longrightarrow R'$, if I is a pure ideal of R then its extension IR' is a pure ideal of R'. If moreover, f is injective and I is proper then IR' is proper.*

Proof The first assertion is easy and well known (Borceux and Van den Bossche 1983, Chapter 7–Lemma 60). For the second, first note that for any finite subset $\{a_1, \ldots, a_n\} \subseteq I$ there exists some $b \in I$ such that $a_i = a_i b$ for all i. Now if $1 \in IR'$ then we may write $1 = \Sigma_{i=1}^{n} r'_i f(a_i)$ where $r'_i \in R'$ and $a_i \in I$ for all i. So there exists some $b \in I$ such that $a_i = a_i b$ for all i. It follows that $f(1) = 1 = f(b)$ and so $1 = b \in I$, a contradiction. □

Lemma 2.3 *Let I be an ideal of A.*

1. *$I\mathcal{R}$ is a proper pure ideal of $\mathcal{R}$ if and only if I is a proper pure ideal of A*
2. *$I\mathcal{R}$ is a purely-prime (respectively purely-maximal) ideal of $\mathcal{R}$ if and only if I is a purely-prime (respectively purely-maximal) ideal of A.*

Proof 1. The implication "$\Leftarrow$" follows from Lemma 2.2. Assume that $I\mathcal{R}$ is a pure ideal of $\mathcal{R}$ and let $a \in I$. There exists $f \in I\mathcal{R}$ such that $a = af$. Thus $f^{(0)} \in IA = I$ and $a = af^{(0)}$.

2. Assume that I is a purely-maximal ideal of A. By (1), $I\mathcal{R}$ is proper and pure. Let J be a proper and pure ideal of $\mathcal{R}$ such that $I\mathcal{R} \subseteq J$. By Lemma 2.1, $J \subseteq J'\mathcal{R}$ for some proper pure ideal J' of A. Thus $I \subseteq J'$ and so $I = J'$. Then $I\mathcal{R} = J$. Conversely, assume that $I\mathcal{R}$ is purely-maximal. Thus I is proper and pure by (1). Let I' be a proper and pure ideal of A such that $I \subseteq I'$. Thus $I\mathcal{R} \subseteq I'\mathcal{R}$ which is a proper pure ideal of $\mathcal{R}$ by (1). So $I\mathcal{R} = I'\mathcal{R}$. Then $I = I'$.

Assume that I is a purely-prime ideal of A. Let J_1, J_2 be two proper pure ideals of $\mathcal{R}$ such that $J_1 J_2 \subseteq I\mathcal{R}$. By Lemma 2.1, $J_i \subseteq I_i\mathcal{R}$ for some proper pure ideal I_i of R (where I_i is the set of constant terms of elements of J_i). Thus $I_1 I_2 \subseteq I$ and so $I_i \subseteq I$ for some i. Then $J_i \subseteq I_i\mathcal{R} \subseteq I\mathcal{R}$ for some i. Conversely, assume that $I\mathcal{R}$ is purely-prime and let I_1, I_2 be two pure ideals of A such that $I_1 I_2 \subseteq I$. Since each $I_i\mathcal{R}$ is pure and $I_1\mathcal{R}I_2\mathcal{R} \subseteq I_1 I_2\mathcal{R} \subseteq I\mathcal{R}$, $I_i\mathcal{R} \subseteq I\mathcal{R}$ for some i. Then $I_i \subseteq I$. □

Theorem 2.4 *Purely-maximal ideals of the ring $\mathcal{R}$ are precisely $I\mathcal{R}$ where I ranges over purely-maximal ideals of A. In particular, the ring $\mathcal{R}$ is semi-Noetherian if and only if the ring A is semi-Noetherian.*

Proof By Lemma 2.3, if I is a purely-maximal ideal of A, then $I\mathcal{R}$ is a purely-maximal ideal of $\mathcal{R}$. Let J be a purely-maximal ideal of $\mathcal{R}$ and $I = \{f^{(0)}, f \in J\}$. By

Lemma 2.1, I is a proper pure ideal of A and $J \subseteq I\mathcal{R}$. By Lemma 2.3, $I\mathcal{R}$ is proper and pure and so $J = I\mathcal{R}$ and I is purely-maximal. It was shown that a ring is semi-Noetherian if and only if each purely-maximal ideal is finitely generated (Tarizadeh and Aghajani 2021, Theorem 6.2). Assume that the ring A is semi-Noetherian and let J be a purely-maximal ideal of $\mathcal{R}$. Then $J = I\mathcal{R}$ for some purely-maximal ideal I of A. Since I is a finitely generated ideal of A, J is a finitely generated ideal of $\mathcal{R}$. Conversely, assume that the ring $\mathcal{R}$ is semi-Noetherian and let I be a purely-maximal ideal of A. Then $I\mathcal{R} = (f_1, \ldots, f_n)$ is a finitely generated ideal of $\mathcal{R}$ and so using Lemma 2.1 and since $I\mathcal{R}$ is purely-maximal ideal, we obtain that $I = (f_1^{(0)}, \ldots, f_n^{(0)})$. Hence every purely-maximal ideal of A is finitely generated. □

Tarizadeh and Aghajani noticed that in all known rings each purely-prime ideal is purely maximal (Tarizadeh and Aghajani 2021). So, they asked if this fact holds for any ring. We study their question on the ring $\mathcal{R}$. Any ideal I contains a largest pure ideal (i.e., the sum of all pure ideals contained in I), denoted $\nu(I)$ (see Tarizadeh and Aghajani 2021, p. 825) (also denoted I° in (Borceux and Van den Bossche 1983, Proposition 8, p. 147).

Lemma 2.5 *Let $A \subseteq B$ be a ring extension and J be a purely-prime ideal of B. Then $\nu(J \cap A)$ is a purely-prime ideal of A.*

Proof By Borceux and Van den Bossche (1983, Chapter 7–Lemma 62). □

The following shows that Tarizadeh and Aghajani's conjecture holds in the ring $\mathcal{R}$ if and only if it holds in the ring A.

Corollary 2.6 *Every purely-prime ideal of $\mathcal{R}$ is purely maximal if and only if every purely-prime ideal of A is purely maximal.*

Proof Assume that every purely-prime ideal of $\mathcal{R}$ is purely-maximal and let I be a purely-prime ideal of A. By Lemma 2.3, $I\mathcal{R}$ is a purely-prime ideal of $\mathcal{R}$. Thus $I\mathcal{R}$ is purely-maximal by hypothesis. Again by Lemma 2.3, I is purely-maximal. Conversely, assume that every purely-prime ideal of A is purely-maximal and let J be a purely-prime ideal of $\mathcal{R}$. By Lemma 2.5, $\nu(J \cap A)$ is a purely-prime ideal of A. By hypothesis, $\nu(J \cap A)$ is a purely-maximal ideal of A and so $\nu(J \cap A)\mathcal{R}$ is a purely-maximal ideal of $\mathcal{R}$ by Theorem 2.4. Since $\nu(J \cap A) \subseteq J$, $\nu(J \cap A)\mathcal{R} \subseteq J$ and so $\nu(J \cap A)\mathcal{R} = J$. Hence J is a purely-maximal ideal of $\mathcal{R}$. □

Corollary 2.7
1. *Purely-maximal ideals of the ring $A[[X]]$ (respectively $A[X]$) are precisely $IA[[X]]$ (respectively $I[X]$) where I ranges over purely-maximal ideals of A.*
2. *The ring $A[[X]]$ (respectively $A[X]$) is semi-Noetherian if and only if the ring A is semi-Noetherian.*
3. *Every purely-prime ideal of $A[[X]]$ (respectively $A[X]$) is purely maximal if and only if every purely-prime ideal of A is purely maximal.*

Proof If we take $A = B$ then $A + XB[X] = A[X]$ and $A + XB[[X]] = A[[X]]$. Now the assertion is easily deduced from Theorem 2.4 and Corollary 2.6. □

Every Noetherian ring is semi-Noetherian, but the converse is not true (see Tarizadeh and Aghajani 2021, p. 834). In the following we give other examples of a semi-Noetherian ring which is not Noetherian.

Example 2.8 It is well known that if A is an integral domain or a local ring, then the only pure ideals of A are zero and A and so A is semi-Noetherian. Let $\mathbb{Z}$ be the ring of integers, $(X_i)_{i\geq 1}$ a countably set of (algebraically independent) indeterminates over $\mathbb{Z}$ and $A = \mathbb{Z}[X_1, X_2, \ldots]$, then A a semi-Noetherian ring that is not Noetherian. Hence the ring $\mathbb{Z}[X_1, X_2, \ldots][X]$ (also $\mathbb{Z}[X_1, X_2, \ldots][[X]]$) is semi-Noetherian by Corollary 2.7 and is not Noetherian by Hilbert's basis theorem. Also the ring $\mathbb{Z} + XS$ (where S is either $\mathbb{Q}[X]$ or $\mathbb{Q}[[X]]$) is semi-Noetherian but is not Noetherian (see Hizem 2009, Proposition 2.1).

Remark 2.9 Semi-Noetherian ring satisfies the ascending chain condition on pure ideals, i.e., every ascending sequence of pure ideals of A stops. In fact, let $I_1 \subset I_2 \subset \cdots$ be an increasing sequence of pure ideals of A. Let $I = \cup_{n\in\mathbb{N}^*} I_n$, then I is a pure ideal of A. So there exist $a_1, \ldots, a_k \in I$ such that $I = (a_1, \ldots, a_k)$. There exist $s \in \mathbb{N}^*$ such that $a_1, \ldots, a_k \in I_s$. So $I = I_s$. Hence, for any $n \geq s,\ I_n = I_s$.

Example of a ring which is not semi-Noetherian (we construct a strictly ascending chain of pure ideals).

Example 2.10 Let $A = \prod_{\lambda\in\Lambda} K_\lambda$ where Λ is an infinite set and each K_λ is a field. Let $(\lambda_n)_{n\geq 1}$ be an infinite countably subset of Λ and $e_n \in A$ whose λ-component is 1 if $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$ and 0 elsewhere. Thus e_n is idempotent and $e_n = e_n e_{n+1}$ and so $e_1 A \subset e_2 A \subset \cdots$ is a strictly ascending chain of pure ideals of A. By Remark 2.9, the ring A is not semi-Noetherian. We can show this fact otherwise: the direct sum ideal $\oplus_{\lambda\in\Lambda} K_\lambda$ of A is a pure ideal which is not finitely generated, because the index set Λ is infinite. Hence an infinite product of fields is never semi-Noetherian. We also have: if R is an infinite Boolean ring, then R is not a semi-Noetherian ring. Indeed: In a Boolean ring R, each element is idempotent and so each ideal is pure. It follows that R is semi-Noetherian if and only if R is Noetherian. But it is well known that a Boolean ring is Noetherian if and only if it is finite.

Each ideal in the given chain in the previous example, is principal (so finitely generated). The following gives an example of a strictly ascending chain of pure ideals which are not finitely generated.

Example 2.11 Consider the open real interval $(0, 1) \subseteq \mathbb{R}$ and the polynomial ring $\mathbb{F}_2[X_i : i \in (0, 1)]$ modulo the ideal H generated by elements of the form $X_r - X_r X_t$ with $0 < r < t < 1$. For each sub-open interval Ω of $(0, 1)$, let $J(\Omega)$ be the ideal of A generated by $(x_r)_{r\in\Omega}$ where x_r is the class of X_r modulo H. For each $r \in \Omega$, there exists $r' \in \Omega$ such that $r < r'$ and so $x_r = x_r x_{r'}$. Thus $J(\Omega)$ is a pure ideal of A. For each positive integer n, let $\Omega_n = (0, 1 - \frac{1}{2^n})$. Since $\Omega_n \subset \Omega_{n+1}$, $J(\Omega_n) \subseteq J(\Omega_{n+1})$. We claim that $J(\Omega_n) \neq J(\Omega_{n+1})$. Indeed: if not, then $x_s \in J(\Omega_n)$ for some $s \in \Omega_{n+1} - \Omega_n$. Let so $r_1, \ldots, r_k \in \Omega$ and $a_1, \ldots, a_k \in A$ such that $x_s = x_{r_1} a_1 + \cdots + x_{r_k} a_k$. We can assume that $r_1 < \cdots < r_k$. Since $x_{r_i} = x_{r_i} x_{r_k}$, $x_s = x_r a$ for some $a \in A$ where $r = r_k$. Thus $X_s - X_r f \in H$ for some $f \in \mathbb{F}_2[(X_r)_{0<r<1}]$. Let so $s_1, t_1, \ldots, s_m, t_m \in (0, 1)$ and $g_1, \ldots, g_m \in \mathbb{F}_2[(X_r)_{0<r<1}]$ such that $X_s - X_r f = (X_{s_1} - X_{s_1} X_{t_1}) g_1 + \cdots + (X_{s_m} - X_{s_m} X_{t_m}) g_m$ and $s_i < t_i$. Taking the linear form in each side, we have $X_s = X_{s_1} c_1 + \cdots + X_{s_m} c_m$ where $c_i \in \mathbb{F}_2$ is the constant term of g_i. Thus $m = 1$ and $s = s_1$. It follows that $X_s - X_r f = (Xs - X_s X_{t_1}) g_1$. Since $r \neq s$ and X_s is a prime element of $\mathbb{F}_2[(X_r)_{0<r<1}]$, $f \in X_s \mathbb{F}_2[(X_r)_{0<r<1}]$ and so $1 - X_r f = (1 - X_{t_1}) g_1$. Let $\varphi : \mathbb{F}_2[(X_r)_{0<r<1}] \longrightarrow \mathbb{F}_2[(X_r)_{0<r<1}]$ the homomorphism of rings that maps X_r to zero. If $t_1 \neq r$, then $1 = (1 - X_{t_1}) \varphi(g)$ and so $1 - X_{t_1}$ is invertible in $\mathbb{F}_2[(X_r)_{0<r<1}]$ which is absurd because the only invertible element of $\mathbb{F}_2[(X_r)_{0<r<1}]$ is 1. Thus $s < t_1 = r$ and so $s \in \Omega_n$, a contradiction. Hence $J(\Omega_1) \subset J(\Omega_2) \subset \cdots$ is a strictly ascending chain of pure ideals of A. Hence the ring A is not semi-Noetherian. Assume that $J(\Omega)$ is finitely generated for some open sub-interval Ω of $(0, 1)$ of the form $\Omega = (0, \alpha)$. Let so $r_1, \ldots, r_k \in \Omega$ such that $J(\Omega) = x_{r_1} A + \cdots + x_{r_n} A$. We can assume that $r_1 < \cdots < r_n$. Thus $J(\Omega) = x_r A$ where $r = r_n$ because $x_{r_i} = x_{r_i} x_{r_n}$. Let $s \in (0, 1)$ such that $r < s < \alpha$. So $x_s \in J(\Omega)$ and thus $x_s = x_r a$ for some $a \in A$. Thus $X_s - X_r f \in H$ for some $f \in \mathbb{F}_2[(X_r)_{0<r<1}]$. With a similar argument as above, we have $s < r$ which is absurd. Hence $J(\Omega)$ is not finitely generated.

If I is an ideal of a ring A, then by Lemma 2.3(1), IR is a pure ideal of R if and only if I is a pure ideal of A where R denotes either $A[X]$ or $A[[X]]$. In particular, $I[X]$ $(= IA[X])$ is pure if and only if I is pure. Always $IA[[X]] \subseteq I[[X]]$ but $I[[X]]$ is not necessarily equal to $IA[[X]]$ and it is proved that the equality $IA[[X]] = I[[X]]$ holds if each countably subset of I is contained in a finitely generated ideal of R contained in I (see Gilmer and Heinzer 1968, p. 386). If I is a finitely generated ideal of A, then it is easy to see that $I[[X]] = IA[[X]]$. It is well know that $I[[X]] = IA[[X]]$ for each ideal of I of A if and only if A is Noetherian (Arnold et al. 1977, Proposition 1.2). In the following result we study the purity of the ideal $I[[X]]$ in $A[[X]]$.

Lemma 2.12 *Let I be an ideal of A. The ideal $I[[X]]$ is pure if and only if I is pure and $I[[X]] = IA[[X]]$.*

Proof The implication "$\Leftarrow$" is obvious, because the extension of every pure ideal is pure. Assume that $I[[X]]$ is a pure ideal of $A[[X]]$ and let $a \in I$. There exists $f \in I[[X]]$ such that $a = af$. So $a = af^{(0)}$ and $f^{(0)} \in I$. Then I is pure. By Lemma 2.1, $I[[X]] \subseteq IA[[X]]$ and so $I[[X]] = IA[[X]]$. □

Aghajani proved that a ring A is von-Neumann regular if and only if every ideal of A is pure (Aghajani 2022).

Proposition 2.13 *Let A be a ring. The ideal $I[X]$ of $A[X]$ is pure for all ideal I of A if and only if A is a von-Neumann regular ring.*

Proof By Lemma 2.3 and Aghajani (2022, Corollary 3.5). □

If every maximal ideal of a ring R is pure, then R is absolutely flat (von-Neumann regular) ring (Tarizadeh 2022, Proposition 2.7). Note that in a von-Neumann regular ring A, not only every ideal I is pure, even it is generated by a set of idempotents, because if $a \in I$ then $a = a^2 b$ for some $b \in A$, then clearly $ab \in I$ is an idempotent and $a = a(ab)$. It is also easy to see that in any ring, a finitely generated ideal which is generated by a set of idempotents is a principal ideal. Thus a von-Neumann ring is Noetherian if and only if it is a principal ideal ring (PIR). If A is a von-Neumann regular and Noetherian ring, then clearly $IA[[X]] = I[[X]]$ is a pure ideal of $A[[X]]$ for each ideal I of A. It follows that:

Theorem 2.14 *For a ring A the following assertions are equivalent.*

(i) *A is a finite product of fields.*
(ii) *A is a von-Neumann regular and Noetherian.*
(iii) *A is a von-Neumann regular and PIR.*
(iv) *For each ideal I of A, $I[[X]]$ is a pure ideal of $A[[X]]$.*
(v) *For each maximal ideal M of A, $M[[X]]$ is a pure ideal of $A[[X]]$.*

Proof $(i) \Leftrightarrow (ii)$ Straightforward. By the above comment, we have $(ii) \Rightarrow (iii) \Rightarrow (iv)$. The implication $(iv) \Rightarrow (v)$ is clear. So it remains to prove the implication $(v) \Rightarrow (ii)$: By hypothesis, every maximal ideal M of A is pure, thus by (Tarizadeh 2022, Proposition 2.7), A is von-Neumann regular. Also by Lemma 2.12, for each maximal ideal M of A we have $M[[X]] = MA[[X]]$. Recall that a module N over a ring R is called a $(\star)$-module if each countably generated submodule of N is contained in a finitely generated submodule of N (Arnold et al. 1977, p. 649). An ideal I of A is called a $(\star)$-ideal of A if I is a $(\star)$-module (Arnold et al. 1977, p. 648). By Arnold et al. (1977, Theorem 2.3), it suffices to show that each prime ideal of A is a $(\star)$-ideal of A. It is well known that each prime ideal of a von-Neumann regular ring is maximal. Let so M be a maximal ideal of A and countable subset $\{a_0, a_1, a_2, \ldots\}$ of M. Let $f = \Sigma_{k \geq 0} a_k X^k$. Since $f \in M[[X]] = MA[[X]]$, we may write $f = \Sigma_{i=1}^{n} b_i f_i$ where $b_i \in M$ and $f_i \in A[[X]]$ for all i. It follows that for each $k \geq 0$ $a_k = f^{(k)} = \Sigma_{i=1}^{n} b_i f_i^{(k)} \in (b_1, \ldots, b_n)$. and $(b_1, \ldots, b_n) \subseteq M$. Thus M is a $(\star)$-ideal of A. Therefore by Arnold et al. (1977, Proposition 1.2), A is Noetherian ring. □

3 Purely-maximal ideals of the rings $A + I[[X]]$ and $A[X] + I[[X]]$

Let I be an ideal of A. This section is devoted to study purely-maximal ideals of the ring $A + I[[X]] = \{f \in A[[X]]$ such that $f^{(i)} \in I$ for all $i \geq 1\}$, the ring $A + I[X]$ and the ring $A[X] + I[[X]] = \{f \in A[[X]] : \exists n \geq 1, \forall i \geq n, \ f^{(i)} \in I\}$ which is a subring of $A[[X]]$ containing $A[X]$. For more informations about rings of the form $A[X]+I[[X]]$, readers are referred to Chang and Toan (2021) and Hizem (2009). It is clear that $A+I[X] \subseteq A+I[[X]] \subseteq A[X]+I[[X]] \subseteq A[[X]]$. Let $\mathcal{R}(I) = A+I[X]$ or $A + I[[X]]$ or $A[X] + I[[X]]$.

Lemma 3.1 *Let Q be an ideal of A.*

1. *If Q is a pure ideal of A, then $QA[[X]] \cap \mathcal{R}(I) = Q\mathcal{R}(I)$.*
2. *Q is a pure ideal of A if and only if $Q\mathcal{R}(I)$ is a pure ideal of $\mathcal{R}(I)$.*
3. *If J is a proper pure ideal of $\mathcal{R}(I)$, then $J \subseteq J'\mathcal{R}(I)$ for some proper pure ideal J' of A.*

Proof 1. By Lemma 2.2, $QA[[X]] = \{f \in A[[X]]$ such that $f = af$ for some $a \in Q\}$.

2. Assume that Q is a pure ideal of A and let $f \in QA[[X]] \cap \mathcal{R}(I)$. By Lemma 2.3, $QA[[X]]$ is a pure ideal of $A[[X]]$ and $f = bf$ for some $b \in Q$ (see the proof of Lemma 2.2). Thus $QA[[X]] \cap \mathcal{R}(I)$ is a pure ideal of $\mathcal{R}(I)$. Conversely, let $a \in Q \subseteq Q\mathcal{R}(I)$. There exists $f \in Q\mathcal{R}(I)$ such that $a = af$. Thus $a = af^{(0)}$ and $f^{(0)} \in Q$.
3. Let J' be the set of constant terms of elements of J. Clearly, J' is a pure ideal of A. Suppose that $J' = A$ and let $f, g \in J$ such that $1 = f^{(0)}$ and $f = fg$. It follows that $g^{(0)} = 1$ and $f^{(1)} = f^{(1)}g^{(0)} + f^{(0)}g^{(1)}$. So $g^{(1)} = 0$. By induction, $g^{(i)} = 0$ for all $i \geq 1$ and so $1 = g \in J$, a contradiction. Then J' is proper. With a similar argument as in the proof of Lemma 2.1, we show that $J \subseteq J'A[[X]]$. So $J \subseteq J'A[[X]] \cap \mathcal{R}(I) = J'\mathcal{R}(I)$.

□

Theorem 3.2 *Let Q be an ideal of A.*

1. *$Q\mathcal{R}(I)$ is a purely-maximal ideal of $\mathcal{R}(I)$ if and only if Q is a purely-maximal ideal of A.*
2. *$Q\mathcal{R}(I)$ is a purely-prime ideal of $\mathcal{R}(I)$ if and only if Q is a purely-prime ideal of A.*

Proof 1. Assume that Q is a purely-maximal ideal of A. By Lemma 3.1, $Q\mathcal{R}(I)$ is pure ideal of $\mathcal{R}(I)$. Since Q is proper, so is $QA[[X]] \cap \mathcal{R}(I) = Q\mathcal{R}(I)$. Let J be a proper pure ideal of $\mathcal{R}(I)$ such that $Q\mathcal{R}(I) \subseteq J$. By Lemma 3.1, $J \subseteq J'\mathcal{R}(I)$ for some proper pure ideal J' of A. It follows that $Q \subseteq J'$ and so $Q = J'$. Then $J = Q\mathcal{R}(I)$. Conversely, assume that $Q\mathcal{R}(I)$ is purely-maximal of $\mathcal{R}(I)$. By Lemma 3.1, Q is a pure ideal of A. Since $Q\mathcal{R}(I)$ is proper, so is Q. Let P be a pure proper ideal of A such that $Q \subseteq P$. So $Q\mathcal{R}(I) \subseteq P\mathcal{R}(I)$ which is a pure and proper ideal of $\mathcal{R}(I)$. Then $Q\mathcal{R}(I) = P\mathcal{R}(I)$ and so $Q = P$.

2. Assume that Q is a purely-prime ideal of A. Thus $QA[[X]]$ is a purely-prime ideal of $A[[X]]$ by Lemma 2.3. So $\nu(QA[[X]] \cap \mathcal{R}(I))$ is a purely-prime ideal of $\mathcal{R}(I)$ by Lemma 2.5. By Lemma 3.1, $QA[[X]] \cap \mathcal{R}(I) = Q\mathcal{R}(I)$ is pure and so $Q\mathcal{R}(I)$ is purely-prime. Conversely, similar argument as in the proof of Lemma 2.3. □

Corollary 3.3 *Purely-maximal ideals of the ring $\mathcal{R}(I)$ are precisely $Q\mathcal{R}(I)$ where Q ranges over purely-maximal ideals of A.*

Proof Let J be a purely-maximal ideal of $\mathcal{R}(I)$ and Q the set of constant terms of elements of J. By Lemma 3.1, $J \subseteq Q\mathcal{R}(I)$ and Q is a proper pure ideal of A. Again by Lemma 3.1, $Q\mathcal{R}(I)$ is a proper pure ideal of $\mathcal{R}(I)$ and so $J = Q\mathcal{R}(I)$. By Theorem 3.2, Q is purely-maximal. Theorem 3.2 completes the proof. □

Corollary 3.4 *The ring $\mathcal{R}(I)$ is semi-Noetherian if and only if the ring A is semi-Noetherian.*

Proof This result follows from Theorem 3.2(1), the fact that a ring is semi-Noetherian if and only if each purely-maximal ideal is finitely generated (so is a principal ideal generated by an idempotent) (Tarizadeh and Aghajani 2021, Theorem 6.2), the fact that idempotent elements of $\mathcal{R}(I)$ (also of $A[[X]]$) are precisely idempotent elements of A (by Benhissi 2003) and the fact that if e is an idempotent element of A, then $eA[[X]] \cap \mathcal{R}(I) = e\mathcal{R}(I)$. □

Example 3.5 It was proved that for an ideal I of $\mathbb{Z}$, the ring $\mathbb{Z}[X]+I[[X]]$ is Noetherian if and only if $I = 0$ or $\mathbb{Z}$ (Kosan et al. 2013, Example 14). So for each integer $n \geq 2$, the ring $\mathbb{Z}[X] + n\mathbb{Z}[[X]]$ is semi-Noetherian by Corollary 3.4 but is not Noetherian.

Example 3.6 It was proved for a proper ideal I of a ring A, the ring $A + I[[X]]$ is Noetherian if and only if A is Noetherian and $I^2 = I$ (Hizem 2009, Proposition 2.4). Then the ring $\mathbb{Z} + 2\mathbb{Z}[[X]]$ is semi-Noetherian by Corollary 3.4 but is not Noetherian because $2\mathbb{Z} \neq 4\mathbb{Z}$.

In the following we give an example of a semi-Noetherian ring which is not an integral domain, not local and not Noetherian ring.

Example 3.7 Let $A = \mathbb{Z}/12\mathbb{Z}$, $I = 2\mathbb{Z}/12\mathbb{Z}$ and $\mathcal{R}(I) = A + I[[X]]$. Then by the above Corollary $\mathcal{R}(I)$ is semi-Noetherian ring. Clearly $\mathcal{R}(I)$ is not an integral domain. Since A is not a local ring, then by Hizem and Benhissi (2005, Proposition 1.3), $\mathcal{R}(I)$ is not so. Since $I^2 \neq I$, then by Hizem (2009, Proposition 2.4), $\mathcal{R}(I)$ is not a Noetherian ring.

Tarizadeh and Aghajani's conjecture holds in the ring $\mathcal{R}(I)$ if and only if it holds in the ring A as shows the following:

Corollary 3.8 *Every purely-prime ideal of $\mathcal{R}(I)$ is purely-maximal if and only if every purely-prime ideal of A is purely-maximal.*

Proof Assume that every purely-prime ideal of $\mathcal{R}(I)$ is purely-maximal and let Q be a purely-prime ideal of A. By Theorem 3.2, $Q\mathcal{R}(I)$ is a purely-prime ideal of $\mathcal{R}(I)$. Thus $Q\mathcal{R}(I)$ is purely-maximal by hypothesis. Then Q is a purely-maximal ideal of A. Conversely, let J be a purely-prime ideal of $\mathcal{R}(I)$. By Lemma 2.5, $\nu(J \cap A)$ is purely-prime ideal of A. So $\nu(J \cap A)$ is a purely-maximal ideal of A. By Theorem 3.2, $\nu(J \cap A)\mathcal{R}(I)$ is a purely-maximal ideal of $\mathcal{R}(I)$. Since $\nu(J \cap A)\mathcal{R}(I) \subseteq J$, $\nu(J \cap A)\mathcal{R}(I) = J$. □

4 Purely-maximal ideals of Nagata's idealization ring

Let R be a ring and M be a unitary R-module. We recall that Nagata introduced the ring extension of R called the idealization of M in R, denoted here by $R \times M$, as the R-module $R \oplus M$ endowed with a multiplicative structure defined by:

$$(a, x)(b, y) = (ab, ay + bx) \text{ for all } a, b \in R \text{ and } x, y \in M$$

For more informations on the ring $R \times M$, readers are referred to Anderson and Winders (2009).

Lemma 4.1 *If I is a pure ideal of R, then $IM = \{x \in M$ such that $x = rx$ for some $r \in I\}$.*

Proof Obvious, because if I is a pure ideal then for any finite subset $\{a_1, \ldots, a_n\} \subseteq I$ there exists some $b \in I$ such that $a_i = a_i b$ for all i. See also (Borceux and Van den Bossche 1983, Chapter 7–Proposition 11). □

Lemma 4.2 *The pure ideals of the Nagata idealization ring $R \times M$ are precisely of the form $I \times IM$ where I ranges over the pure ideals of R.*

Proof Assume that I is a pure ideal of R and let $(r, x) \in I \times IM$. There exists $a \in I$ such that $r = ra$. By Lemma 4.1, $x = bx$ for some $b \in I$. Let $c \in I$ such that $a = ac$ and $b = bc$. Since $rc = rac = ra = r$ and $cx = cbx = bx = x$, $(r, x) = (r, x)(c, 0)$ and $(c, 0) \in I \times IM$. Thus $I \times IM$ is pure.

Conversely, Let J be a pure ideal of $R \times M$. Let I be the set of elements $r \in R$ such that $(r, x) \in J$ for some $x \in M$. Clearly I is an ideal of R.
Claim: I is a pure ideal of R. If $r \in I$, then $(r, x) \in J$ for some $x \in M$ and so $(r, x) = (r, x)(a, y)$ for some $(a, y) \in J$. Thus $r = ra$ and $a \in I$.

Let $(r, x), (a, y) \in J$ be such that $(r, x) = (r, x)(a, y)$. Thus $r, a \in I$ and $x = ry+ax \in IM$. Then $J \subseteq I\times IM$. Conversely, let $(r, x) \in I\times IM$. There exists $y \in M$ such that $(r, y) \in J$. Since $x \in IM$, there exists $b \in I$ such that $x = bx$ by Lemma 4.1. Let so $z \in M$ such that $(b, z) \in J$. Then $(0, x) = (b, z)(0, x) \in J$ and so $0\times IM \subseteq J$. Since $y \in IM$, $(r, 0) = (r, y) - (0, y) \in J$. Hence $(r, x) = (r, 0) + (0, x) \in J$. □

Lemma 4.3 *For each ideal I of R, $I \times IM$ is a purely-prime (respectively purely-maximal) ideal of $R\times M$ if and only if I is a purely-prime (respectively purely-maximal) ideal of R.*

Proof Assume that $I \times IM$ is a purely-maximal ideal of $R \times M$. So I is a proper pure ideal of R. Let Q be a proper pure ideal of R such that $I \subseteq Q$. Thus $I\times IM \subseteq Q\times QM$ which is a proper and pure ideal of $R \times M$. So $I \times IM = Q \times QM$ and thus $I = Q$. Conversely, assume that I is a purely-maximal ideal of R and let J be a proper pure ideal of $R \times M$ such that $I \times IM \subseteq J$. By Lemma 4.2, $J = Q \times QM$ for some proper pure ideal Q of R. Thus $I \subseteq Q$ and so $I = Q$. Then $J = I \times IM$.

Assume that $I \times IM$ is a purely-prime ideal of $R \times M$. Clearly I is a pure and proper ideal of R. Let I_1, I_2 be two pure ideals of R such that $I_1 I_2 \subseteq I$. Since $[I_1 \times I_1 M][I_2 \times I_2 M] \subseteq I_1 I_2 \times I_1 I_2 M \subseteq I \times IM$ and each $I_i \times I_i M$ is pure, $I_i \times I_i M \subseteq I \times IM$ for some i and so $I_i \subseteq I$. Conversely, assume that I is a purely-prime ideal of R and let J_1, J_2 be two pure ideals of $R \times M$ such that $J_1 J_2 \subseteq I \times IM$. By Lemma 4.2, each $J_i = I_i \times I_i M$ for some pure ideal I_i of R. Since $J_1 J_2 \subseteq I \times IM$, $I_1 I_2 \subseteq I$ and so $I_i \subseteq I$ for some i. Hence $J_i \subseteq I_i \times I_i M \subseteq I \times IM$ for some i. □

Theorem 4.4 *Purely-maximal ideals of the ring $R \times M$ are precisely $I \times IM$ where I ranges over purely-maximal ideals of R. In particular, the ring $R \times M$ is semi-Noetherian if and only if the ring R is semi-Noetherian.*

Proof By Lemma 4.3, if I is a purely-maximal ideal of R, then $I \times IM$ is a purely-maximal ideal of $R \times M$. Let J be a purely-maximal ideal of $R \times M$. By Lemma 4.2,

$J = I \times IM$ for some proper pure ideal I of R. By Lemma 4.3, I is purely-maximal. Assume that the ring R is semi-Noetherian and let J be a purely-maximal ideal of $R \times M$. Thus $J = I \times IM$ for some purely-maximal ideal I of R. Since I is finitely generated, I is principal generated by an idempotent element e. Since $x = ex$ for each $x \in IM$, $(r, x) = (r, x)(e, 0)$ for each $(r, x) \in I \times IM$. Then J is principal generated by $(e, 0)$. The "only if" part follows from the fact that for each ideal I of R, if the ideal $I \times IM$ is finitely generated, then so is I. □

Example of a semi-Noetherian ring of the form $R \times M$ that is not Noetherian.

Example 4.5 The ring $\mathbb{Z} \times \mathbb{Q}$ is semi-Noetherian by Theorem 4.4 but it is not Noetherian by Anderson and Winders (2009, Theorem 4.8) because $\mathbb{Q}$ is not finitely generated over $\mathbb{Z}$. More generally, for each integral domain (not necessarily Noetherian) R with quotient field $K \neq R$, the ring $R \times K$ is semi-Noetherian but is not Noetherian.

Tarizadeh and Aghajani's conjecture holds in the ring $R \times M$ if and only if it holds in the ring R as shows the following:

Theorem 4.6 *Every purely-prime ideal of* $R \times M$ *is purely maximal if and only if every purely-prime ideal of* R *is purely maximal.*

Proof Assume that every purely-prime ideal of $R \times M$ is purely-maximal and let I be a purely-prime ideal of R. By Lemma 4.3, $I \times IM$ is a purely-prime ideal of $R \times M$. Thus $I \times IM$ is purely-maximal by hypothesis. Again by Lemma 4.3, I is purely-maximal. Conversely, assume that every purely-prime ideal of R is purely-maximal and let J be a purely-prime ideal of $R \times M$. By Lemmas 4.2 and 4.3, $J = I \times IM$ for some purely-prime ideal I of R. Thus I is purely-maximal. Hence J is purely-maximal. □

In spite of the contributions of the present article, the conjecture (Tarizadeh and Aghajani 2021, Conjecture 5.8) is still unsolved.

Acknowledgements The authors would like to thank the referee for several valuable comments which suggested some alternate proofs and additional examples. We have incorporated several of referee's suggestions in the paper and especially Theorem 2.14 is due to the referee. The first author is grateful to Dr. Mohamed Khalifa for valuable discussions.

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