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ORIGINAL PAPER

Nearly Kaehler manifolds admitting a closed conformal vector field

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Abstract
We study a nearly Kaehler manifold M admitting a closed conformal vector field V, and obtain three results under the following assumptions (i) V is almost analytic, (ii) M has real dimension > 6, is complete and strictly nearly Kaehler, and (iii) M is complete strictly nearly Kaehler of global constant type.

Keywords Nearly Kaehler manifold · Closed conformal vector field · First Chern class · Global constant type

Mathematics Subject Classification 53C21 · 58J60 · 53C25

1 Introduction

An almost Hermitian manifold is a real $2n$-dimensional smooth manifold M with a (1,1) tensor field J and a Riemannian metric g such that $J^2 = -I$ and $g(JX, JY) = g(X, Y)$ for arbitrary vector fields X, Y on M. If J is integrable, i.e. the Nijenhuis tensor N of J vanishes, then M is a Hermitian manifold.

An almost Hermitian manifold is said to be an almost Kaehler manifold if the fundamental 2-form Ω defined by $\Omega(X, Y) := g(X, JY)$ is closed, i.e. $d\Omega = 0$.

An almost Kaehler manifold whose underlying almost complex structure J is integrable, is known as a Kaehler manifold. An almost Hermitian manifold is Kaehler if and only if $\nabla J = 0$, where ∇ is the Levi–Civita connection of g.

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A nearly Kaehler manifold is an almost Hermitian manifold satisfying

$$(\nabla_X J)Y + (\nabla_Y J)X = 0. \tag{1.1}$$

A nearly Kaehler manifold is one of the 16 classes of almost Hermitian manifolds described by the Gray–Hervella classification (Gray and Hervella 1980). For details on nearly Kaehler manifolds and related results, we refer to Gray (1970) and Chen (2011).

A smooth vector field V on a Riemannian manifold (M, g) with Riemannian metric g is said to be a conformal vector field if it satisfies

$$L_V g = 2ag, \tag{1.2}$$

where L_V is the Lie derivative operator along V, and a is a smooth function (conformal scale function) on M. In particular, if a is constant, then V is homothetic, and if $a = 0$, then V is Killing. Tanno and Weber (1969) obtained several conditions under which a compact Riemannian manifold admits a closed conformal vector field. Kaehler manifolds carrying a conformal vector field have been studied by Deshmukh (2011). We also recall that a vector field V on an almost complex manifold is said to be almost analytic if $L_V J = 0$. In particular, an almost analytic vector field on a Kaehler manifold is called analytic (the real part of a holomorphic vector field). Details can be found in Yano (1965). For a closed conformal vector field on Kaehler manifolds, we state the following result of Goldberg (1964).

Theorem 1.1 (Goldberg) *A closed conformal vector field V on a Kaehler manifold M is homothetic and analytic.*

Intrigued by this result, and bearing in mind the facts that a nearly Kaehler manifold M with $dim(M) < 6$ is Kaehlerian (Gray 1969), and for $dim(M) = 6$, a strictly nearly Kaehler manifold is positively Einstein (Gray 1976), we study a nearly Kaehler manifold M with $dim(M) \geq 6$ admitting a closed conformal vector field V, and obtain the following results: (i) if V is almost analytic, then it is homothetic, also, in addition if M is complete, then M is isometric to the complex Euclidean space $\mathbf{C}^n$. (ii) If M has real dimension > 6, is complete and strictly nearly Kaehler, then it has a non-vanishing first Chern class. (iii) If M is complete strictly nearly Kaehler of global constant type, then it is isometric to a 6-sphere.

Our study of a nearly Kaehler manifold with a closed conformal vector field is mainly motivated by the fact that the unit sphere S^6 carries a strictly (i.e. non-Kaehler) nearly Kaehler structure inherited from the Cayley division algebra (Ejiri 1981) *and admits many closed conformal vector fields*. To illustrate the last part of the foregoing statement, let N be a unit normal vector on S^6 in the Euclidean space R^7 with Euclidean metric $<>$, then for any constant vector field C on R^7, its restriction to S^6 can be decomposed as $C = V + fN$, where $f =< C, N >$ is a smooth function and V turns out to be a gradient (hence closed) conformal vector field on S^6, with the conformal scale function $-f$.

It is worth pointing out that nearly Kaehler manifolds with conformal Killing forms were studied by Naveira and Semmelmann in Naveira and Semmelmann (2020).

2 Closed conformal vector fields on nearly Kaehler manifolds

First, we prove the following result.

Theorem 2.1 *Let V be a non-parallel closed conformal vector field on a real $2n$-dimensional ($2n > 2$) nearly Kaehler manifold M.*

- (i) *If V is almost analytic (i.e. $L_V J = 0$), then it is homothetic.*
- (ii) *In addition, if M is complete, then M is isometric to the complex Euclidean space $\mathbf{C}^n$.*
- (iii) *If V is homothetic, then it is almost analytic.*

Proof As V is closed, the conformal Eq. (1.2) assumes the simple form

$$\nabla_X V = aX, \tag{2.1}$$

where a is a smooth function on M. Using Eq. (2.1) and the definition $R(Y, X)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z$ we compute

$$R(Y, X)V = (Ya)X - (Xa)Y. \tag{2.2}$$

Taking its inner product with an arbitrary vector field Z gives

$$g(R(Y, X)V, Z) = (Ya)g(X, Z) - (Xa)g(Y, Z).$$

As the curvature tensor is symmetric in the first and second pair of arguments, the above equation provides

$$g(R(V, Y)Z, W) = (Za)g(W, Y) - (Wa)g(Z, Y). \tag{2.3}$$

Let us recall the following property of a nearly Kaehler manifold (Gray (1970) used a different sign convention for the curvature tensor):

$$g(R(X, Y)JZ, JW) - g(R(X, Y)Z, W) = g((\nabla_X J)Y, (\nabla_Z J)W). \tag{2.4}$$

Substituting V for X in the above equation and using Eq. (2.3) gives

$$\begin{aligned} g((\nabla_V J)Y, (\nabla_Z J)W) = {} & ((JZ)a)g(Y, JW) - ((JW)a)g(Y, JZ) \\ & -(Za)g(Y, W) + (Wa)g(Y, Z). \end{aligned} \tag{2.5}$$

Also, we note that

$$\begin{aligned} (L_V J)X &= L_V JX - JL_V X = \nabla_V JX - \nabla_{JX} V - J(\nabla_V X - \nabla_X V) \\ &= (\nabla_V J)X - \nabla_{JX} V + J\nabla_X V. \end{aligned}$$

Using Eq. (2.1) in the preceding equation we find that

$$L_V J = \nabla_V J. \tag{2.6}$$

which shows that V is almost analytic iff $\nabla_V J = 0$.

Now, let us assume that V is almost analytic. Then Eq. (2.5) gives

$$((JZ)a)g(Y, JW) - ((JW)a)g(Y, JZ) - (Za)g(Y, W) + (Wa)g(Y, Z) = 0. \tag{2.7}$$

Factoring out W from the preceding equation we immediately get

$$-((JZ)a)JY + g(Y, JZ)JDa - (Za)Y + g(Y, Z)Da = 0, \tag{2.8}$$

where D is the gradient operator. Contracting Eq. (2.8) at Y and noting that $tr.J = 0$ we get

$$(2 - 2n)Za = 0. \tag{2.9}$$

Since $2n > 2$, it follows that a is a constant, i.e. V is homothetic. This proves part (i).

Next, using Eq. (2.1) we have $D|V|^2 = 2aV$. Differentiating this equation along an arbitrary vector field X yields

$$\nabla_X D|V|^2 = 2a^2 X. \tag{2.10}$$

Now, if M is complete, then by our hypothesis, $a \neq 0$, and so, by a result of Tashiro (1965), Eq. (2.10) implies that M is isometric to the Euclidean space E^{2n}. That is, M is isometric to the flat $\mathbf{C}^n$. This proves part (ii). To prove (iii), we assume that V is homothetic i.e. a is constant, then from Eq. (2.5) we have

$$g((\nabla_V J)Y, (\nabla_Z J)W) = 0. \tag{2.11}$$

Setting $Z = V$ and $W = Y$ in the preceding equation we have $|(\nabla_V J)Y|^2 = 0$, i.e. $\nabla_V J = 0$. This shows by virtue of Eq. (2.6) that V is almost analytic. This completes the proof.

Next, we recall (Gray 1976) that a 6-dimensional strictly nearly Kaehler manifold has vanishing first Chern class. In Gray (1970), for a compact nearly Kaehler manifold M, the first Chern class γ_1 of M is given by

$$\gamma_1(X, Y) = -\frac{1}{2\pi}\sum_{i=1}^{n}\left\{(g(R(X, Y)e_i, Je_i) + \frac{1}{2}g((\nabla_X J)e_i, J(\nabla_Y J)e_i)\right\} \tag{2.12}$$

for arbitrary vector fields X, Y on M, and $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ is a local J-adapted orthonormal $2n$-frame on M. We now state the following results from Euh and Sekigawa (2011) and Nagy (2002) respectively, that would be used in proving our next results. □

Theorem S1: Let (M, J, g) be a compact, irreducible strictly nearly Kaehler manifold. Then, M is Einstein iff the first Chern class of M vanishes.

Theorem S2: Let (M, J, g) be a complete, strictly nearly Kaehler manifold. Then, the following hold:

- (i) If g is not an Einstein metric, the canonical Hermitian connection has reduced holonomy.
- (ii) The metric g has positive Ricci curvature, hence M is compact with finite fundamental group.
- (iii) The scalar curvature of the metric g is a strictly positive constant.

We prove the following result:

Theorem 2.2 *A complete strictly nearly Kaehler manifold M of real dimension > 6 and admitting a closed conformal vector field V has a non-vanishing first Chern class. If, in addition, M is irreducible, then M cannot be Einstein.*

Proof We prove this by contradiction. Since a complete strictly nearly Kaehler manifold is compact (Theorem S2) the first Chern class of M is given by (2.12). Suppose that the first Chern class of M vanishes. Then from (2.12) we have

$$2\sum_{i=1}^{n} g(R(X,Y)e_i, Je_i) = -\sum_{i=1}^{n} g((\nabla_X J)e_i, J(\nabla_Y J)e_i)) \tag{2.13}$$

Putting $X = V$ yields,

$$2\sum_{i=1}^{n} g(R(V,Y)e_i, Je_i) = -\sum_{i=1}^{n} g((\nabla_V J)e_i, J(\nabla_Y J)e_i)) \tag{2.14}$$

The use of (2.2) reduces (2.14) to

$$2\left[\sum_{i=1}^{n} g(Da, e_i)g(Y, Je_i) - \sum_{i=1}^{n} g(Da, Je_i)g(Y, e_i)\right] = -\sum_{i=1}^{n} g((\nabla_V J)e_i, J(\nabla_Y J)e_i)). \tag{2.15}$$

Substituting $Y = JV$, using $(\nabla J)J = -J(\nabla J)$ and the definition $(\nabla_X J)Y = -(\nabla_Y J)X$ of a nearly Kaehler structure, we get

$$2Va = -\sum_{i=1}^{n} g((\nabla_V J)e_i, (\nabla_V J)e_i)), \tag{2.16}$$

Next, substituting $Z = V$ in (2.5) and then contracting the resulting equation with respect to Y and W yields

$$(2-2n)Va = \sum_{i=1}^{n} g((\nabla_V J)e_i, (\nabla_V J)e_i)) + \sum_{i=1}^{n} g((\nabla_V J)Je_i, (\nabla_V J)Je_i). \tag{2.17}$$

Taking into account that $(\nabla J)J = -J(\nabla J)$ and that g is Hermitian, we use Eq. (2.16) in conjunction with (2.17) in order to get $(2n-6)Va = 0$. As $2n > 6$, $Va = 0$, and hence from (2.16) we obtain $\nabla_V J = 0$. Hence $L_V J = 0$, i.e. V is almost analytic which is a contradiction, because if M is complete, then by Theorem 2.1, M is flat and hence Kaehler. Hence the first Chern class of M cannot vanish.

Now, if M is irreducible, then it follows that M cannot be Einstein (Theorem S1). This completes the proof.

Next, we recall from Gray (1970), that a nearly Kaehler manifold is said to be of global constant type if

$$|(\nabla_X J)Y|^2 = \alpha[|X|^2|Y|^2 - (g(X,Y))^2 - (g(JX,Y))^2] \tag{2.18}$$

where α is a constant function. We now classify a complete strictly nearly Kaehler manifold of global constant type (in which case α is a positive constant) admitting a closed conformal vector field. Precisely, we establish the following result characterizing a 6-sphere. □

Theorem 2.3 *Let (M, g, J) be a complete strictly nearly Kaehler manifold M of global constant type admitting a closed conformal vector field V. Then M is isometric to a 6-sphere.*

Proof Substituting $X = Z = V$ and $W = Y$ in (2.4) gives

$$g(R(V,Y)JV, JY) - g(R(V,Y)V, Y) = g((\nabla_V J)Y, (\nabla_V J)Y). \tag{2.19}$$

The use of (2.3) and (2.18) in (2.19) gives

$$< Y, JV > g(JDa, Y) - (Va)|Y|^2 + < Y, V > Ya = \alpha[|V|^2|Y|^2 - (g(V,Y))^2 - (g(JV,Y))^2]. \tag{2.20}$$

As Y is arbitrary, we choose Y to be orthogonal to both V and JV. The use of this reduces (2.20) to

$$\alpha = -\frac{(Va)}{|V|^2}. \tag{2.21}$$

on any open dense subset $\mathcal{U}$ of M, on which $V \neq 0$.

Hence, $\frac{(Va)}{|V|^2}$ is constant on $\mathcal{U}$. Now replacing Y with V in Eq. (2.3) gives

$$g(Da, Z)g(V, W) = g(Da, W)g(Z, V). \tag{2.22}$$

Putting $Z = V$ and factoring out W yields

$$Da = \frac{(Va)}{|V|^2} V = -\alpha V, \tag{2.23}$$

on $\mathcal{U}$. But the zeroes of V are discrete points (see Ros and Urbano 1998). So, by continuity, $Da = -\alpha V$ on M. Differentiating it along an arbitrary vector field X and using (2.1), we obtain

$$\nabla_X Da = -\alpha a X. \tag{2.24}$$

We note here that a is non-constant, because if it were a constant, then Eq. (2.21) would imply $\alpha = 0$, which in turn, in view of equation (2.18), would imply that M is Kaehler, a contradiction. As M is complete and α is positive, by Obata's theorem (Obata 1965), M is isometric to a sphere. But the only sphere that has a nearly Kaehler structure is the 6-sphere (Gray 1969). This completes the proof. □

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Declarations

Competing interests On behalf of all the authors, Rahul Poddar states that there is no conflict of interest.

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