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ORIGINAL PAPER

Bounded birationality and isomorphism problems are computable

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Abstract
Let X, Y be two irreducible subvarieties of the projective space $\mathbb{P}^n$, and $d \geq 1$ an integer number. The main result of this paper is an algorithm to construct **explicitly**, in terms of d and the ideals defining X and Y, a quasi-affine algebraic variety parametrising the set of all birational maps f from X onto Y which can be extended to a self-rational map of $\mathbb{P}^n$ of **algebraic** degree $\leq d$. We also prove similar results for the case f is a dominant rational map, regular morphism, isomorphism or regular embedding. Similar results are valid for varieties over an arbitrary algebraically closed field, and also for maps on non-projective varieties.

Keywords Birationality problem · Explicit bounds · Gröbner bases · Linear projections · Monoids

Mathematics Subject Classification 14Exx

1 Introduction

The main theme in this paper is to show that the existence of rational maps from a given projective variety $X \subset \mathbb{P}^n$ to another projective variety $Y \subset \mathbb{P}^n$ with certain interesting properties (such as birational, dominant, regular, isomorphic or regular embedding) and with **bounded algebraic degree** is computable, that is can be detected by an algorithm whose complexity is explicitly bound. We will treat first the main case of interest, that of birational maps. With some minor modifications, the proofs for the projective varieties also work for non-projective varieties. In particular, (biregular) isomorphisms of bounded degrees between two irreducible, smooth algebraic varieties can be parametrised by an algebraic variety. These results are valid over fields of any characteristic, which is a not-negligible point, given that many results known in

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characteristic zero are not known in positive characteristic. For example, it is still unknown whether resolution of singularities holds in positive characteristic, or to what extent the known results in the Minimal Model Program can be done in positive characteristic.

We note that in general deciding isomorphisms requires much more work than deciding birational maps. Based on the main results in this paper, we propose a rough strategy towards the birationality problem, via Iitaka's fibrations, in the appendix. We use monoids in our proofs since they allow explicit constructions in the proof of Theorem 2.1. Of course, proofs using other approaches are possible, but we feel that our approach is very natural.

Remark From now on, for simplicity, except when specifically mentioned, we will call degree of a map f, its algebraic degree, and not its topological degree. As far as we know, this paper is the first in the literature where a rigorous proof for the concerned results (see the next sections) is given.

2 Birational maps of bounded degrees

This section treats the class of maps, reflected in the title of this paper, which is of main interest to us: birational maps.

The birationality problem asks whether there is a birational map f between two given irreducible algebraic varieties X and Y. It is a fundamental and classical question in algebraic geometry. Having the seminal Gödel theorem that many mathematical questions are undecidable, it is at least psychologically important to ask the question: Is the birationality problem decidable? This question is also practically important. A special case of the birationality problem, the so-called rationality problem, which seeks to check whether a given variety is birationally equivalent to a projective space, has attracted a lot of attention and effort. As far as we know, even this special case is still open in general. To this end, in this paper, we prove that a weaker version of the birationality question, called **bounded** birationality problem, is not only decidable but also even **computable**. That is, there is an algorithm—whose complexity is explicitly bound—to solve the weaker version. As a consequence, the union of all varieties $W(X, Y, d)$ in Theorem 2.1 below, where d runs all over $\mathbb{N}$, is a countable complete set of invariants for the birationality problem. (Note that it is usually the case in mathematics that a property is determined via a countable set of invariants. For example, the Kodaira dimension of a variety X is defined based on the behaviour of the sequence $h^0(X, K_X^{\otimes m})$. Related questions for rationally connected varieties and uniruled varieties, for a smooth projective variety X, are also stated in terms of countable invariants: in the first case it is Mumford's conjecture which concerns the vanishing of all $h^0(X, (T_X^*)^{\otimes m})$, in the second case it is Mori's conjecture which concerns the vanishing of all $h^0(X, K_X^{\otimes m})$. In this aspect, our result can be restated as that the birationality problem for given X and Y is characterised by an explicit countable set of invariants.)

Bounded birationality problem Given X, Y irreducible complex algebraic subvarieties of $\mathbb{P}^n$ (where $n \geq 2$) and a positive integer d. Is there a (bi)rational map

$F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ with $\deg(F) \leq d$ whose restriction to X is a birational map $F|_X : X \dashrightarrow Y$?

Note that any rational map $X \dashrightarrow Y$ is the restriction of a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Moreover, by results in Mella and Polastri (2009) and Ciliberto et al. (2016), whenever $X, Y \subset \mathbb{P}^n$ with $n-2 \geq \dim(X), \dim(Y)$, the existence of a birational map $X \dashrightarrow Y$ is equivalent to the existence of a birational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ whose restriction to X is a birational map $F|_X : X \dashrightarrow Y$. On the other hand, there are cases of $\dim(X) = \dim(Y) = n-1$ for which there are no such birational map extensions F (Mella and Polastri 2012). In the above formulation of the Bounded Birationality Problem, we allow X, Y to be hypersurfaces of $\mathbb{P}^n$.

Remark Since we will fix a projective space $\mathbb{P}^n$ once and for all, we can regard n as a constant for the remaining of this paper.

Define $\mathcal{B}(X, Y, d) := \{F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n : \ F$ is **rational**, $\deg(F) \leq d$, $F|_X$ is birational from X onto $Y\}$. Also, define $\mathcal{B}^+(X, Y, d) := \{F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n : \ F$ is **birational**, $\deg(F) \leq d$, $F|_X$ is birational from X onto $Y\}$. The set $\bigcup_{d=1}^{\infty} \mathcal{B}(X, Y, d)$ is exactly those rational selfmaps of $\mathbb{P}^n$ whose restriction to X is a birational map onto Y. A similar interpretation can be given for $\bigcup_{d=1}^{\infty} \mathcal{B}^+(X, Y, d)$. Note that from results in Mella and Polastri (2012) as mentioned above, if X and Y are hypersurfaces in $\mathbb{P}^n$, it may happen that $\bigcup_{d=1}^{\infty} \mathcal{B}(X, Y, d) \neq \emptyset$ while $\bigcup_{d=1}^{\infty} \mathcal{B}^+(X, Y, d) = \emptyset$. On the other hand, by Mella and Polastri (2009) as mentioned above, if $\dim(X) = \dim(Y) \leq n-2$, then $\bigcup_{d=1}^{\infty} \mathcal{B}(X, Y, d) \neq \emptyset$ if and only if $\bigcup_{d=1}^{\infty} \mathcal{B}^+(X, Y, d) \neq \emptyset$.

The main result in this section is the following. (An accompanying algorithm will also be given.)

Theorem 2.1 *The Bounded Birationality Problem has a solution if and only if at least one among* **explicitly** *constructed C_1 systems of polynomial equations in C_2 variables, each polynomial of degree bounded from above by a constant C_3, has one solution in $\mathbb{C}$. Here C_1, C_2, C_3 and the number of polynomials in each system of equations are explicitly bounded in terms of $X, Y,$ $\deg(F)$ and n.*

In other words, there is an explicitly bounded number of variables $\alpha_1, \ldots, \alpha_N$, an explicitly constructed finite dimensional (generally reducible) variety $W(X, Y, d)$ in variables $\alpha_1, \ldots, \alpha_N$, and a surjective map $\kappa : W(X, Y, d) \to \mathcal{B}(X, Y, d)$. The variety $W(X, Y, d)$ is non-empty if and only if the set $\mathcal{B}(X, Y, d)$ is non-empty.

Similarly, there is an explicitly constructed finite dimensional variety $W^+(X, Y, d)$ and a surjective map $\kappa^+ : W^+(X, Y, d) \to \mathcal{B}^+(X, Y, d)$.

Remark 2.2 From the proof of the theorem, it is easy to see that it is also valid for varieties over an arbitrary algebraically closed field.

We now present a consequence of this result. Denote by $\mathcal{R}(n, d)$ the set $\{F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n : \ \deg(F) \leq d$ and F is **rational**$\}$, and by $\mathcal{R}^+(n, d)$ the set $\{F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n : \ \deg(F) \leq d$ and F is **birational**$\}$. We recall that here we work with Zariski topology, and that a set in a topological space is **locally closed** if it is the intersection of an open and a closed set, and a set is **constructible** if it is a finite union of locally closed sets. For example, $\mathbb{C}^2 \backslash \{(0, 0)\}$ is a constructible set, and it is a union of two algebraic varieties $\mathbb{C}^2 \backslash \{0\} \times \mathbb{C}$ and $\mathbb{C}^2 \backslash \mathbb{C} \times \{0\}$. We recall here the fact that a constructible set in a projective space is parametrised by an algebraic variety. Indeed, such a set

is a finite union of affine varieties, say $A_1, A_2, \ldots, A_N \subset \mathbb{C}^m$. Then we can embed $A_1, \ldots, A_N$ as disjoint algebraic subvarieties $B_1, \ldots, B_N$ of $\mathbb{C}^{Nm}$. The union B of $B_1, \ldots, B_N$ is an algebraic subvariety of $\mathbb{C}^{Nm}$, and the natural map from B to the union of $A_1, \ldots, A_N$, given by $B_i \to A_i$ is the desired surjective morphism.

Corollary 2.3 *The sets $\mathcal{R}(n, d)$ and $\mathcal{R}^+(n, d)$ are algebraic varieties.*

The subset $\mathcal{B}(X, Y, d)$ of $\mathcal{R}(n, d)$ is constructible. In particular, $\mathcal{B}(X, Y, d)$ is parametrised by an affine algebraic variety, that is there is a surjective morphism from an affine algebraic variety onto $\mathcal{B}(X, Y, d)$.

Similarly, the subset $\mathcal{B}^+(X, Y, d)$ of $\mathcal{R}^+(n, d)$ is constructible. In particular, $\mathcal{B}^+(X, Y, d)$ parametrised by an affine algebraic variety, that is there is a surjective morphism from an affine algebraic variety onto $\mathcal{B}^+(X, Y, d)$.

Before giving the detail of the proofs of the above results, we introduce some useful lemmas.

The following result, so-called Andreotti–Bezout inequality in the literature (see e.g. Angeniol 1981), will be used throughout the paper.

Lemma 2.4 *Let $\mathcal{I} =< f_1, \ldots, f_k >$ be an ideal in $\mathbb{P}^N$. Then for $C = 1$ the following holds. If $V \subset \mathbb{P}^N$ is any irreducible component of the reduced variety defined by $\mathcal{I}$, then* $\deg(V) \leq C \deg(f_1) \ldots \deg(f_k)$.

Consequently, we obtain effective upper bounds for the degree of the graph of a map of bounded degree and for degrees of linear projections. For a subvariety V of $\mathbb{P}^n \times \mathbb{P}^n$, we define $\deg(V)$ to be the degree computed with the polarisation $O(1, 1)$.

Lemma 2.5 (1) *Let X, Y be irreducible subvarieties of $\mathbb{P}^n$. Assume that a rational map $f : X \dashrightarrow Y$ is given of degree $\leq d$, that is it is the restriction of a rational self-map of $\mathbb{P}^n$ of degree $\leq d$. Let $\Gamma_f \subset \mathbb{P}^n \times \mathbb{P}^n$ be the graph of f. Then the degree of Γ_f is effectively bounded in terms of d.*

(2) *Let $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the natural projection. Let $Z \subset \mathbb{C}^n$ be an irreducible variety. Then there is a constant $C > 0$, independent of Z, so that* $\deg(\pi(Z)) \leq C \deg(Z)$.

Proof (1) We can proceed as follows. Let $x_0, x_1, \ldots, x_n$ be the homogeneous coordinates for the copy of $\mathbb{P}^n$ containing X, and $y_0, y_1, \ldots, y_n$ be the homogeneous coordinates for the copy of $\mathbb{P}^n$ containing Y. Let $F = [F_0 : \ldots : F_n]$ be a rational map from $\mathbb{P}^n$ to itself whose degree is $\leq d$ and whose restriction to X is f. Then F_i's are homogeneous polynomials of degree $\leq d$ in $x_0, \ldots, x_n$. Hence, the graph Γ_F is a component where the intersection of the n hypersurfaces $y_i F_j - y_j F_i = 0$ is **proper**—that is of correct dimension—where the indices run on all i, j for which both F_i and F_j are non-zero. Since the degrees of these hypersurfaces are bounded explicitly in terms of d, and the number of these hypersurfaces is also explicitly bounded, we have that the degree of Γ_F is explicitly bounded, by applying Lemma 2.4. Next, since Γ_f is one component where Γ_F and $X \times Y$ intersect properly, it follows from Lemma 4.1 in Truong (2020) that the degree of Γ_f is also explicitly bounded.

(2) The proof is simple. We choose H a very ample divisor of $\mathbb{P}^{n-1}$. Then $deg(\pi(Z)) = \pi(Z).H^{n-1-k}$, where k is the dimension of $\pi(Z)$. We can choose

a realisation of H so that $Z \cap H$ is contained in $\mathbb{C}^{n-1}$. Then by simple moving lemma, it can be checked that $\pi(Z).H^{n-1-k} \leq C.Z.\pi^*(H^{k-1-l})$, here C is a positive constant independent of Z. The RHS of the previous inequality is bounded in terms of $\deg(Z)$.

□

For the next lemma, we first introduce the notation and some basic properties of monoids, which are important and allow explicit constructions in the proof of Theorem 2.1. For more detail, see for example Sections 1.2 and 1.3 in Ciliberto et al. (2016). We use the notation $(0)_p = (0, \ldots, 0)$ (p times). We also use combinations of these, such as $((0)_p, 1, (0)_q)$ and $((0)_p, (0)_q, 1) = ((0)_{p+q}, 1)$.

A hypersurface $M \subset \mathbb{P}^r$ (where $r \geq 2$) of degree d is a monoid with vertex p if it is **irreducible** and p is a point in M of multiplicity exactly $d-1$. If we choose the coordinates for $\mathbb{P}^r$ so that $p = [0 : \ldots : 0 : 1]$, then the defining equation for M is $f_{d-1}(x_0, \ldots, x_{r-1})x_r + f_d(x_0, \ldots, x_{r-1})$. Here f_{d-1} is a homogeneous polynomial of degree $d-1$ and f_d is a homogeneous polynomial of degree d. The assumption that M is irreducible is equivalent to that $GCD(f_{d-1}, f_d) = 1$.

The projection from p gives rise to a birational map between a monoid $M \subset \mathbb{P}^r$ and the projective space $\mathbb{P}^{r-1}$. More precisely, the projection map is $\pi : M \dashrightarrow \mathbb{P}^{r-1}$ given by $[x_0 : \ldots : x_{r-1} : x_r] \mapsto [x_0 : \ldots : x_{r-1}]$, and the inverse $\pi^{-1} : \mathbb{P}^{r-1} \dashrightarrow M$ is given by

$$[x_0 : \ldots : x_{r-1}] \mapsto [f_{d-1}x_0 : \ldots : f_{d-1}x_{r-1} : -f_d].$$

From these formulas, it is obvious that π maps $M \backslash \{f_{d-1} = f_d = 0\} = M \backslash \{f_{d-1} = 0\}$ isomorphically to its image $\mathbb{P}^{r-1} \backslash \{f_{d-1} = 0\}$. We note also that the indeterminacy set of π is p, and the indeterminacy set of π^{-1} is $\{f_{d-1} = f_d = 0\}$. For a subvariety $p \neq A \subset M$, the cone $C_p(A)$ over A with vertex p is contained in M iff the image $\pi(A \backslash \{p\}) \subset \mathbb{P}^{r-1}$ is in the indeterminacy set of π^{-1}.

Note that a monoid M can have two or more vertices. If a monoid $M \subset \mathbb{P}^r$ has two vertices $[0 : \ldots : 0 : 1]$ and $[0 : \ldots : 0 : 1 : 0]$, then it has a defining equation of the form:

$$\begin{aligned} &f_d(x_0, \ldots, x_{r-2}) + x_{r-1}g(x_0, \ldots, x_{r-2}) \\ &\quad + x_r h_{d-1}(x_0, \ldots, x_{r-2}) + x_r x_{r-1} f_{d-2}(x_0, \ldots, x_{r-2}) = 0. \end{aligned}$$

In this case, if we compose the birational map $\mathbb{P}^{r-1} \dashrightarrow M$ of the projection from one vertex and the projection $M \dashrightarrow \mathbb{P}^{r-1}$ from the other vertex of M, then we obtain a biration map $\mathbb{P}^{r-1} \dashrightarrow \mathbb{P}^{r-1}$. Moreover, we note that in this case the defining polynomial is irreducible iff $GCD(f_d(x_0, \ldots, x_{r-2}) + x_{r-1}g(x_0, \ldots, x_{r-2}), h_{d-1}(x_0, \ldots, x_{r-2}) + x_{r-1} f_{d-2}(x_0, \ldots, x_{r-2})) = 1$ (apply the case of monoids with only one vertex above).

The following effective versions of Lemma 2.1 and 2.2 in Ciliberto et al. (2016) are also important in the proof of Theorem 2.1. The fact that the bound is independent of the vertex p is crucial.

Lemma 2.6 (1) *Let $Z \subset \mathbb{P}^r$, with $r \geq 3$, be an irreducible variety of dimension $1 \leq s \leq r-2$ and let $p \in \mathbb{P}^r$ be such that the projection of Z from p is birational*

to its image. Then for d large enough, depending only on **the degree** *of Z and r, there is a monoid in $\mathbb{P}^r$ of degree d with vertex p, containing Z but not containing the cone $C_p(Z)$ over Z with vertex p.*

(2) *Let $Z \subset \mathbb{P}^r$, with $r \geq 3$, be an irreducible variety of dimension $1 \leq s \leq r-3$, and let $p_1, p_2 \in \mathbb{P}^r$ be distinct points such that the projection of Z from the line going through p_1 and p_2 is birational to its image. Then, with the same number d as in part 1, there is a monoid in $\mathbb{P}^r$ of degree d with vertices p_1 and p_2, containing Z but not containing the cones $C_{p_i}(Z)$ over Z with vertices p_i (for $i = 1, 2$).*

Proof (1) Let $\pi : V \to \mathbb{P}^r$ be the blowup of $\mathbb{P}^r$ at p. Let $Z' \subset V$ be the strict transform of Z. From the proof of Lemma 2.1 in Ciliberto et al. (2016), it suffices to show the following, where E is the exceptional divisor and H is a hyperplane: $h^0(Z', O_{Z'}(d(H-E)))$ is bounded explicitly in terms of d and $(H-E)^s \cdot Z'$. (The latter is smaller than or equal to the degree of Z. To see this claim about the bound for $(H-E)^s \cdot Z'$, we expand (here $C(s, j)$ are binomial numbers)

$$\begin{aligned}(H-E)^s \cdot Z' &= H^s \cdot Z' + \sum_{j=1}^{s} (-1)^j C(s, j) H^{s-j} \cdot E^j \cdot Z' \\ &= \deg(Z) + \sum_{j=1}^{s} (-1)^j C(s, j) H^{s-j} \cdot E^j \cdot Z',\end{aligned}$$

and note that for all $j \geq 1$ the class of E^j is $(-1)^{j-1}$ of a linear subspace in E, and the intersections between linear subspaces of E and Z' are psef. Therefore, $\sum_{j=1}^{s} (-1)^j H^{s-j} \cdot E^j \cdot Z' \leq 0$ and $(H-E)^s \cdot Z' \leq \deg(Z)$.)

More precisely, putting $\delta = (H-E)^s.Z' \geq 1$, we will show the following: There is a polynomial $p_{s,\delta}(d)$ in the variable d, of degree s, whose all coefficients are explicitly bounded in terms of s and δ, whose coefficient of d^s is $\delta/s!$, and so that $h^0(Z', O_{Z'} d(H-E)) \leq p_{s,\delta}(d)$ for all non-negative integers $d \geq 0$.

Note that since the linear system $H-E$ is nef, and is also movable (having no base locus, this can be seen by observing that the strict transform in V of a hyperplane in $\mathbb{P}^r$ containing p is an element of the linear system $|H-E|$), we have a SES

$$0 \to O_{Z'}(d(H-E)) \to O_{Z'}((d+1)(H-E)) \to O_{Z_1}((d+1)(H-E)) \to 0,$$

where $Z_1 = Z' \cap S$ (S is a generic element of $|H-E|$) is a variety of dimension $= \dim(Z') - 1 = s-1$, and $(H-E)^{s-1}.Z_1 = (H-E)^s.Z'$. Therefore, we have a LES

$$\begin{aligned}0 \to H^0(O_{Z'}(d(H-E))) &\xrightarrow{i_1} H^0(O_{Z'}((d+1)(H-E))) \xrightarrow{i_2} H^0(O_{Z_1}((d+1)(H-E))) \\ &\to H^1(O_{Z'}(d(H-E))) \to \dots\end{aligned}$$

From this we obtain

$$\begin{aligned}h^0(O_{Z'}((d+1)(H-E))) &= \dim(ker(i_2)) + \dim(im(i_2)) \\ &= \dim(im(i_1)) + \dim(im(i_2))\end{aligned}$$

$$\leq h^0(O_{Z'}d(H-E)) + h^0(O_{Z_1}(d+1)(H-E)).$$

Summing this over d, we obtain

$$h^0(O_{Z'}d(H-E)) \leq \sum_{j=0}^{d} h^0(O_{Z_1}j(H-E)).$$

Then we have the conclusion by induction on the dimension of Z_1. In fact, the base case of dimension 0 is obvious, in which case we obtain that Z_1 is a union of δ points, and hence $h^0(O_{Z_1}d(H-E)) = \delta$ for all non-negative integers d. If we define by induction the following sequence of polynomials: $q_0(d) = 1$ for all d, and for $m \geq 1$

$$q_m(d) = \sum_{j=0}^{d} q_{m-1}(j),$$

then we have that $q_m(d)$ is a polynomial of degree m in d whose coefficient of d^m is $1/m!$. The first several elements in the sequence are listed below:

$$\begin{aligned} q_0(d) &= 1, \\ q_1(d) &= d+1, \\ q_2(d) &= \frac{d(d+1)}{2} + d + 1. \end{aligned}$$

Moreover, we have from the above arguments that

$$h^0(Z', O_{Z'}d(H-E)) \leq \delta q_s(d)$$

for all non-negative integers d. Therefore, the choice of $p_{s,\delta}(d) = \delta q_s(d)$ satisfies the claim.

(2) This follows from (1) as in the proof of Lemma 2.2 in Ciliberto et al. (2016). In fact, let H_1 and H_2 be the hyperplanes in $\mathbb{P}^r$ so that the projection from p_1 maps onto H_1 and the projection from p_2 maps onto H_2. Let $Z_1 \subset H_1$ and $Z_2 \subset H_2$ be images of Z under the mentioned projections. Then $\deg(Z_1), \deg(Z_2) \leq \deg(Z)$, being (a component of) the (proper) intersection between a cone over Z and H_1 (respectively H_2). Since $\dim(Z) \leq r-3$, we have that $\dim(Z_1) \leq \dim(H_1) - 2$ and $\dim(Z_2) \leq \dim(H_2) - 2$. Moreover, if p_1' is the image of p_1 under the projection from p_2, then the projection under p_1' of Z_2 is birational to its image. Similarly for the image p_2' of p_2 under the projection from p_1. We can then apply part 1 to obtain a monoid S_1' in H_1, which may be assumed of the same degree d, with vertex p_2', containing Z_1 but not the cone $C_{p_2'}(Z_1)$. Let S_1 be the cone over S_1' with vertex p_1. Then S_1 is a monoid of the same degree d whose vertices include all points on the line contacting p_1 and p_2', and hence in particular include p_1 and p_2. Moreover, S_1 contains Z but not the cone over Z with vertex p_2. We construct a similar monoid S_2. Then a generic linear combination between S_1 and S_2 gives the desired answer. □

Here is the proof of Theorem 2.1. It gives an algorithm to construct the systems of polynomial equations. For the convenience of the readers, we will present the algorithm explicitly afterwards.

Proof of Theorem 2.1 We first give the conclusion for the set $\mathcal{B}(X, Y, d)$. By a linear change of coordinates, we may assume that none of X, Y belongs to the hyperplane at infinity of $\mathbb{P}^n$. For simplicity, from now on (except when otherwise indicated) we will work with the Zariski open dense sets $\mathbb{C}^k$ of $\mathbb{P}^k$ only.

Step 1 (Forward: here we go step by step from the graph $\Gamma_f \subset \mathbb{P}^n \times \mathbb{P}^n$, assuming that it is known, down to X and Y, each step construct a birational model of Γ_f inside an ambient space of smaller dimension): If $X = Y = \mathbb{P}^n$ then there is nothing to do. Hence we can assume that $\dim(X) = \dim(Y) \leq n-1$. Since $n \geq 2$ and $\dim(\Gamma_f) = \dim(X)$, it follows that $\Gamma_f \subset \mathbb{C}^n \times \mathbb{C}^n$ is of codimension at least 3. Hence, by Lemma 2.6, there is a monoid $M_{0,0} \subset \mathbb{C}^n \times \mathbb{C}^n$ of degree effectively bounded by the degree of Γ_f, containing Γ_f and has two vertices $((0)_{n-1}, 1, (0)_n)$ and $((0)_n, (0)_{n-1}, 1)$. Note that the projections from $((0)_n, (0)_{n-1}, 1)$ and $((0)_{n-1}, 1, (0)_n)$ are birational from Γ_f onto its image. Let $\Gamma_{0,1} \subset \mathbb{C}^n \times \mathbb{C}^{n-1}$ and $\Gamma_{1,0} \subset \mathbb{C}^{n-1} \times \mathbb{C}^n$ be the corresponding images of Γ_f. Their degrees are effectively bounded in terms of the degree of Γ_f (see part 2 of Lemma 2.5) and hence, by part 1 of Lemma 2.5, in terms of d.

Similarly, we can find a monoid $M_{0,1} \subset \mathbb{C}^n \times \mathbb{C}^{n-1}$ of degree effectively bounded by the degree of $\Gamma_{0,1}$ and has the vertex $((0)_n, (0)_{n-2}, 1)$. Again, note that the projection from the point $((0)_n, (0)_{n-2}, 1)$ is birational from $\Gamma_{0,1}$ to its image $\Gamma_{0,2} \subset \mathbb{C}^n \times \mathbb{C}^{n-2}$. Similarly, we construct $M_{1,0} \subset \mathbb{C}^{n-1} \times \mathbb{C}^n$ and $\Gamma_{2,0} \subset \mathbb{C}^{n-2} \times \mathbb{C}^n$.

Repeating the above process, we obtain monoids $M_{0,j} \subset \mathbb{C}^n \times \mathbb{C}^{n-j}$, $M_{j,0} \subset \mathbb{C}^{n-j} \times \mathbb{C}^n$ and the birational images of $\Gamma_f = \Gamma_{0,0}$: they are $\Gamma_{0,j} \subset \mathbb{C}^n \times \mathbb{C}^{n-j}$ and $\Gamma_{j,0} \subset \mathbb{C}^{n-j} \times \mathbb{C}^n$. Note that $\Gamma_{0,n} = X$ and $\Gamma_{n,0} = Y$.

Step 2 Now to prove Theorem 2.1, we **reverse** the above argument, and proceed from what we know (that is $\Gamma_{0,n} = X$ and $\Gamma_{n,0} = Y$) step by step, each step constructing a system of equations parametrising the $\Gamma_{j,0}$ and $\Gamma_{0,j}$ in Step 1.

We do not know the graph Γ_f and consequently, the monoids $M_{0,j}$ and $M_{j,0}$ above. Also we do not know the intermediate $\Gamma_{j,0}$ and $\Gamma_{0,j}$, except that $\Gamma_{0,n} = X$ and $\Gamma_{n,0} = Y$.

We do, however, **know** that the degrees of the monoids $M_{0,j}$ and $M_{j,0}$ are effectively bounded. We also know that these monoids are in fixed variables $x_1, \ldots, x_n, y_1, \ldots, y_{n-j}$ and $x_1, \ldots, x_{n-j}, y_1, \ldots, y_n$. Therefore, we can parametrise them in terms of their coefficients. The number of these coefficients is effectively bounded.

Now we start the process of writing out the equations promised in the theorem. We will start from X, and then go step by step through all $\Gamma_{0,j}$ and $\Gamma_{j,0}$, at each step add some polynomial equations, and end up at Y. At that stage we have the needed polynomial systems.

We start from $X = \Gamma_{0,n} \subset \mathbb{C}^n$ and want to go up to $\Gamma_{0,n-1} \subset \mathbb{C}^n \times \mathbb{C}$. We do not know about $\Gamma_{0,n-1}$, but we know that $\Gamma_{0,n-1} \subset M_{0,n-1}$ and X is the birational image of $\Gamma_{0,n-1}$ via the projection from the point $((0)_n, 1)$. We next show that this can be described in terms of some polynomial equations.

Using the homogeneous coordinates, the monoid $M_{0,n-1}$ is given by an equation

$$f_{0,n-1}(x_0,\ldots,x_n,y_1) = f_{0,n-1,1}(x_0,\ldots,x_n)y_1 + f_{0,n-1,2}(x_0,\ldots,x_n) = 0.$$

We can dehomogenise to get an equation in affine coordinates $x_1,\ldots,x_n,y_1$. Knowing the bound on the degree of $f_{0,n-1}$, we know how many parameters (which are the coefficients of the monomials in the equation for the given monoid) we will need.

From Lemma 2.6, we see that X is the birational image of some $\Gamma_{0,n-1} \subset M_{0,n-1}$ under the projection from $((0)_n, 1)$ exactly when $X \not\subset \{f_{0,n-1,1} = f_{0,n-1,2} = 0\}$. We note that in $M_{0,n-1}$, **set theoretically** the locus $\{f_{0,n-1,1} = f_{0,n-1,2} = 0\}$ is the same as the hypersurface $\{f_{0,n-1,1} = 0\}$. Provided this condition is satisfied, then $\Gamma_{0,n-1}$ will be the **strict transform** in $M_{0,n-1}$ of X, under the inverse of the projection map π. Hence $\Gamma_{0,n-1}$ is contained in the total inverse image $\pi^{-1}(X) \cap M_{0,n-1}$. If X is defined by an ideal $I(X) = \{g_1,\ldots,g_m\}$, then the total inverse image of X is given by the ideal $\{g_1,\ldots,g_m, f_{0,n-1,1}y_1 + f_{0,n-1,2}\}$. However, this set usually is bigger than what we want (the variety $\Gamma_{0,n-1}$), and it will make later computations and arguments harder. So, one **key idea** here is to consider only the preimage $H_{0,n-1}$ of the Zariski open set $X\backslash\{f_{0,n-1,1} = 0\}$ of X, which is given by the ideal $\{g_1,\ldots,g_m, f_{0,n-1,1}y_1 + f_{0,n-1,2}, 1 - t_{0,n-1}f_{0,n-1,1}\}$. (Using the common trick, we added a variable $t_{0,n-1}$.) This latter set $H_{0,n-1}$ is a Zariski open dense set of the hypothetical $\Gamma_{0,n-1}$, and it is isomorphic (to see this, note that the projection map from $((0)_n, 1)$ is an isomorphism between $M_{0,n-1}\backslash\{f_{0,n-1} = 0\}$ onto its image) to the Zariski dense open set $X\backslash\{f_{0,n-1,1} = 0\}$ of X, provided that $X\backslash\{f_{0,n-1,1} = 0\}$ is non-empty and the following conditions on the monoid are satisfied.

- The first condition is that both $f_{0,n-1,1}$ and $f_{0,n-1,2}$ are non-zero polynomials. The condition that $f_{0,n-1,1}$ is a non-zero polynomial is already taken care (provided $H_{0,n-1}$ is non-empty) by the equation $1-t_{0,1}f_{0,n-1,1} = 0$ in the defining equations for $H_{0,n-1}$. For the condition that $f_{0,n-1,2}$ is non-zero, we need only that at least one of the coefficients a_I of some monomial x^I is non-zero, and this condition can be again described using the trick of adding one new variable s_I so that $1 - s_I a_I = 0$. So an explicitly bounded number of such equations will cover our case.
- The second condition is that the monoid $M_{0,n-1}$ should be **irreducible**, which is the same as that $GCD(f_{0,n-1,1}, f_{0,n-1,2}) = 1$. We will show that this condition is also taken care by the equation $1 - t_{0,n-1}f_{0,n-1,1} = 0$ already given above. In fact, assume that $GCD(f_{0,n-1,1}, f_{0,n-1,2}) = h_{0,n-1}$. Then, by the first condition, we can write $f_{0,n-1,1} = h_{0,n-1}w_{0,n-1,1}$ and $f_{0,n-1,2} = h_{0,n-1}w_{0,n-1,2}$ where $GCD(w_{0,n-1,1}, w_{0,n-1,2}) = 1$. Then, the equation

$$0 = 1 - t_{0,n-1}f_{0,n-1,1} = 1 - t_{0,n-1,1}h_{0,n-1,1}w_{0,n-1,1}$$

  of $H_{0,n-1}$ implies that in fact $H_{0,n-1}$ is contained in the set $\{h_{0,n-1,1} \neq 0\}$, and hence is contained in the irreducible monoid $w_{0,n-1,1}y_1 + w_{0,n-1,2} = 0$.

Iterating this argument, we can go back further and introduce some explicitly bounded number of systems of equations (each time at one monoid), to go back from $X \subset \mathbb{C}^n$ to some $H_{0,1} \subset \mathbb{C}^n \times \mathbb{C}^{n-1}$. By a similar argument (but a bit more

complicated, see below), we go back to some $H_{0,0} \subset \mathbb{C}^n \times \mathbb{C}^n$. $H_{0,0}$ is then a Zariski dense open set of the hypothetical graph Γ_f. The difference between $H_{0,0}$ and the other $H_{0,j}$ (where $1 \leq j \leq n-1$) is that while the other $H_{0,j}$'s belong to monoids with only one vertex, $H_{0,0}$ belongs to a monoid with 2 vertices. The equation for such a monoid is

$$\begin{aligned} & f_{0,0,1}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) + x_n g_{0,0}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) \\ & \quad + y_n h_{0,0}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) + x_n y_n f_{0,0,2}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) \\ & = f_{0,0,1} + x_n g_{0,0} + y_n (h_{0,0} + x_n f_{0,0,2}) \\ & = f_{0,0,1} + y_n h_{0,0} + x_n (g_{0,0} + y_n f_{0,0,2}). \end{aligned}$$

Here the last two equalities express the same monoid regarded as a monoid of either one of the two vertices in concern. As before, $H_{0,0}$ is not contained in either of the bad sets of the projections from these two vertices, that is $H_{0,0} \not\subset \{f_{0,0,1} + x_n g_{0,0} = 0\} \cup \{f_{0,0,1} + y_n h_{0,0} = 0\}$. As before, we can consider only the complement in of the latter set, which can be described in terms of polynomial equations: $1 - t_{0,0}(f_{0,0,1} + x_n g_{0,0})(f_{0,0,1} + y_n h_{0,0}) = 0$, where $t_{0,0}$ is a new variable.

Now, we see that the graph Γ_f, and hence the birational map f, exists iff the following two conditions are satisfied:

- Condition 1: At least one among the many systems of equations which we produced for $H_{0,0}$ in the above has a non-empty solution set. (Each such solution corresponds to one $H_{0,0}$, but the correspondence may not be 1-to-1.) If this is the case, then from the construction it is obvious that $H_{0,0}$ is birational to X. The latter means exactly that $H_{0,0}$ is the graph of a rational map from X into $\mathbb{P}^n$.
- Condition 2: For at least one system of equations in Condition 1 which has a non-empty solution set, the corresponding $H_{0,0}$ is birational to Y via the second projection $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$. This means exactly that $H_{0,0}$ is the graph of a birational map from X, and the image of that map is Y.

Towards Condition 2, we go step by step, using monoids on the Y-side: $M_{0,0}$, $M_{1,0}$ and so on.

We can apply the same argument employed when we went up from $X = \Gamma_{0,n}$ to $\Gamma_{0,n-1}$. Here we describe how to go from $H_{0,0} \subset \mathbb{C}^n \times \mathbb{C}^n$ to $H_{1,0} \subset \mathbb{C}^n \times \mathbb{C}^n$ (important note: $H_{1,0}$ is still in $\mathbb{C}^n \times \mathbb{C}^n$), the latter being isomorphic to a dense set of the image of $H_{0,0}$ in $\mathbb{C}^{n-1} \times \mathbb{C}^n$ under the projection from $(1, (0)_{n-1}, (0)_n)$. (Note **the difference** here is that we do not need to compute the image of $H_{0,0}$ under the concerned linear projection, which is quite complicated—for example not Zariski closed—in particular when we have parameters in the defining equations.) This $H_{1,0}$ is isomorphic to its image via the projection from the point $(1, (0)_{n-1}, (0)_n)$, and its image is contained in $M_{1,0}$. We will, as when going up from X to $\Gamma_{0,n-1}$, consider only a Zariski open set of $H_{0,0}$ lying inside the monoid $M_{0,0}$ where the projection is an isomorphism to its image. We also need to check that the image is contained in the monoid $M_{1,0}$, and this can be done by checking that for all polynomials h in a defining ideal of $M_{1,0}$, the hypersurface $h = 0$ (now considered as a hypersurface in

$\mathbb{C}^n \times \mathbb{C}^n$ instead of $\mathbb{C}^{n-1} \times \mathbb{C}^n$) contains the variety $H_{0,0}$. (In effect, we are checking that the preimage under the projection from $(1, (0)_{n-1}, (0)_n)$ of $M_{1,0}$ contains as a set the variety $H_{0,0}$). We will show that this can be done by adding some polynomials, as below in the check of whether the image of $H_{n,0}$ belongs to Y.

At the end of this induction process, we get some $H_{n,0} \subset \mathbb{C}^n \times \mathbb{C}^n$ which is birational to $H_{0,0}$ and hence to X. We need only to check that the image of $H_{n,0}$ is a subset of Y (since then, because $H_{n,0}$ and Y have the same dimension, and $H_{n,0}$ is birational to its image, we get that $H_{n,0}$ is birational equivalent to Y, and are done).

Now, we proceed to show that whether the image of $H_{0,0}$ under the projection to the second factor $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ belongs to Y is determined by some polynomial equations. This is the same as requiring the preimage of Y under this projection contains $H_{n,0}$. This reduces to the following question: Let $H_{n,0}$ be defined by an ideal $\{h_1, \ldots, h_k\}$ in variables $x_1, \ldots, x_n, y_1, \ldots, y_n, s_I$ (depending on parameters as well). Let h be a polynomial (which we can think as an element in the ideal of Y). We need to show that whether the set $\{h_1 = \ldots = h_k = 0\}$ is contained in $h = 0$ is described by some polynomial equations.

To this end, we use the following common method. Working in a ring $\mathbb{C}[w_1, \ldots, w_N]$. The set $\{h_1 = \ldots = h_k = 0\}$ is contained in $h = 0$ iff in $\mathbb{C}[w_1, \ldots, w_N, a]$, where a is a new variable, the function 1 is in the ideal $< h_1, \ldots, h_k, 1 - ah >$. For the readers' convenience, we recall here the classical argument. By Hilbert's Nullstellensatz, $\{h_1 = \ldots = h_k = 0\}$ is contained in $\{h = 0\}$ iff there is some positive integer j so that in $\mathbb{C}[w_1, \ldots, w_N]$ the polynomial h^j is in the ideal $< h_1, \ldots, h_k >$. Then in the polynomial ring $\mathbb{C}[w_1, \ldots, w_N, a]$

$$1 = (1 - a^j h^j) + a^j h^j = (1 - ah)(1 + ah + a^2 h^2 + \cdots + a^{j-1} h^{j-1}) + a^j h^j$$

is in the ideal $< h_1, \ldots, h_k, 1-ah >$.) By effective Hilbert's Nullstellensatz (Hermann 1926; Brownawell 1987; Kollár 1998), there will be polynomials $\tau_1, \ldots, \tau_k, \tau \in \mathbb{C}[w_1, \ldots, w_N, a]$ with explicitly bounded degrees so that

$$1 \equiv \tau_1 h_1 + \cdots + \tau_k h_k + \tau(1 - ah)$$

in the ring $\mathbb{C}[w_1, \ldots, w_N, a]$. Note that $\tau_1, \ldots, \tau_k$ will have coefficients not yet determined, and we determine them by balancing the coefficients of the polynomials on the two sides of the above identity. We then get a system of polynomial equations in the coefficients of the concerned polynomials, which we need to have at least a solution.

Combining the above steps, we have several systems of polynomials in the parameter spaces (the coefficients of the monoids $M_{i,j}$ and the coefficients of the polynomials τ_i), at least one among them has a solution if we have a birational map of degree $\leq d$ from X to Y. Conversely, any such a solution will provide us with a birational map from X to Y (even though may be of degrees much bigger than d).

The only thing remains to check is whether the above constructed birational maps (corresponding to these solutions) are of degree $\leq d$. That is, it remains for us to check whether at least one of these birational maps from $X \dashrightarrow Y$ comes from a global rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of degree $\leq d$ actually. We show that this can be detected by some polynomial equations and proceed as follows. That a birational map $X \dashrightarrow Y$

which we constructed above comes from a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is the same as that the variety $H_{0,0} \subset \mathbb{C}^n \times \mathbb{C}^n$ constructed above is contained in the graph Γ_F. Writing in the affine set $\mathbb{C}^n \times \mathbb{C}^n$ (recalling our convention from the beginning of this proof), the ideal of Γ_F is $< y_1 h(x_1, \ldots, x_n) - g_1(x_1, \ldots, x_n), \ldots, y_n h(x_1, \ldots, x_n) - g_n(x_1, \ldots, x_n) >$, where $g_1, \ldots, g_n$ and h are polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ of degree $\leq d$. Again, these polynomials are not yet determined, but we can parametrise them in terms of their coefficients, and the number of these coefficients are efficiently bounded by the degree d. We can check whether Γ_F contains $H_{0,0}$ by using the procedure above in checking whether the image of X is contained in Y. This creates new polynomial equations. We also need to check that X is not contained in the indeterminacy locus of F, which is the same as $H_{0,0} \backslash \{h(x_1, \ldots, x_n) = 0\}$ is non-empty. Adding a new variable t, the latter is the same as the following conclusion: the system of polynomial equations $\{H_{0,0}, 1 - th(x_1, \ldots, x_n) = 0\}$ has at least one solution.

We add all of these new equations into the equations we already constructed above. Therefore, by taking the zero set of these systems of polynomial equations, we have a finite dimensional variety $W(X, Y, d)$ (depending on the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ as well as coefficients of the involved monoids and some other parameters, as stipulated above) together with the surjective map κ as in the theorem. A priori, two distinct points in $W(X, Y, d)$ may give rise to the same birational map $X \dashrightarrow Y$. We can also use Gröbner bases (Gunther 1941; Buchberger 1965) to eliminate the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ from $W(X, Y, d)$ to obtain systems of equations in the parameters only, and hence obtain a (generally reducible) variety $\widetilde{W}(X, Y, d)$ of finite dimension in parameters. It is clear that the parameters which give us birational maps between X and Y is a dense subset $\widetilde{W_0}(X, Y, d)$ of the variety $\widetilde{W}(X, Y, d)$.

To finish off, we give the proof for the conclusion concerning the set $\mathcal{B}^+(X, Y, d)$. Our starting point is the equations in the previous paragraph. We only need to add the equations which the coefficients of a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ in the above must satisfy in order for F to be a birational map. This can be done similarly to the above proof, with the following modifications. Here we construct a monoid $M'_{0,0}$ for Γ_F with **only one vertex** $(1, (0)_{n-1}, (0)_n)$. This is because we only need to check that the projection to the second factor $\mathbb{C}^n \times \mathbb{C}^n$ will give us a birational map, and hence we do not need to consider the vertex $((0)_n, (0)_{n-1}, 1)$ as before. Since Γ_F is of codimension $n \geq 2$ in $\mathbb{C}^n \times \mathbb{C}^n$, we can apply Lemma 2.6. We need to add the equations which check that the monoid $M'_{0,0}$ contains Γ_F. Then we can follow the argument we gave for $H_{0,0}$, until we reach the isomorphic image $H'_{n-1,0}$ in $\mathbb{C} \times \mathbb{C}^n$ of a Zariski open set of Γ_F. At this step, we cannot apply Lemma 2.6, since the dimension of Γ_F is n and the dimension of $\mathbb{C} \times \mathbb{C}^n$ is $n + 1$. To be able to finish this last step, we simply add one more dimension so that the assumptions in Lemma 2.6 are satisfied, concerning $\mathbb{C} \times \mathbb{C}^n$ as a hyperplane $w = 0$ in a bigger space $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}_w$. Then we can construct a monoid in $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}_w$ for $H'_{n-1,0}$, and then the proof is finished by checking that $\mathbb{C}^n = \mathbb{C}^n \times \{w = 0\} \subset \mathbb{C}^n \times \mathbb{C}_w$ contains the image of $H'_{n-1,0}$ under the projection from the point $(1, (0)_n, w = 0)$.

Addendum Note that by a linear change of coordinates, we can assume that X and Y are not contained in any coordinate hyperplane. If in Step 2 we consider the smaller

Zariski open subset $X\backslash\{f_{0,n-1,1}f_{0,n-1,2}=0\}$, which corresponds to considering the ideal

$$\{g_1,\ldots,g_m,\ f_{0,n-1,1}y_1+f_{0,n-1,2},\ 1-t_{0,n-1}f_{0,n-1,1}f_{0,n-1,2}\},$$

then we can reduce the number of systems of equations regarding $\Gamma_{0,n-1}$ to 1. Consequently, this reduces the number C_1 in the conclusion of Theorem 2.1. Note, however, that in doing so we increase (double) the degree of the polynomials involved and consequently the number C_3.

The following algorithm is a spin-off of the proof of Theorem 2.1. Note that the constants C_1, C_2 here are not the same as in the theorem.

Algorithm. Detecting the existence of bounded birational maps between given varieties.

Input. Two irreducible subvarieties X and Y of $\mathbb{P}^n$ of the same dimension, and a positive integer d.

Output. An answer Yes or No to the question of whether there are birational maps $f: X \dashrightarrow Y$ which are restrictions of rational maps $F: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ whose degree $\deg(F)$ is $\leq d$.

- Step 1: Use a linear change of coordinates so that both X and Y do not belong to any coordinate hyperplane of $\mathbb{P}^n$.
- Step 2: Compute the explicit constant $C_1 = C_1(d) > 0$ (from part 2 of Lemma 2.5) with the following property: Whenever $Z \subset \mathbb{C}^n \times \mathbb{C}^n$ is an irreducible variety of degree $\leq d$, and $\pi_j : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^j$ is the natural linear projection (where $0 \leq j \leq n$), then $\deg(\pi_j(Z)) \leq C_1$.
- Step 3: With respect to C_1 (considered as the degree of Z in the statement of Lemma 2.6), compute the explicit constant C_2 (considered as the number d in the statement of Lemma 2.6).
- Step 4: Now we start to construct the systems of polynomial equations. We start with $S_{0,n} = \{g_1,\ldots,g_m\}$, where $g_1,\ldots,g_m$ is a basis for the ideal $I(X)$ defining X in $\mathbb{C}^n$.
- Step 5: For each $j = 1, 2, \ldots, n-1$, we construct the system $S_{0,n-j}$ as follows.

  The variables of $S_{0,n-j}$ are the union of the variables of $S_{0,n-j+1}$, the coefficients of $f_{0,n-j,1}$ and $f_{0,n-j,2}$, and an extra variable $t_{0,n-j}$. Here, $f_{0,n-j,1}$ is a general polynomial of degree $C_2 - 1$ in variables $x_1,\ldots,x_n, y_1,\ldots,y_{j-1}$, and $f_{0,n-j,2}$ is a general polynomial of degree C_2 in variables $x_1,\ldots,x_n, y_1,\ldots,y_{j-1}$.

  The equations of $S_{0,n-j}$ are the union of the equations of $S_{0,n-j+1}$ and 2 extra equations: $f_{0,n-j,1}y_j + f_{0,n-j,2}$ and $1 - t_{0,n-j}f_{0,n-j,1}f_{0,n-j,2}$. (These two extra equations represent certain Zariski open dense sets of the involved monoids.)
- Step 6: We now construct a system of polynomials $S_{0,0}$, which is more special than the $S_{0,j}$'s in Step 5.

  The variables of $S_{0,0}$ are the union of the variables of $S_{0,1}$, the coefficients of $f_{0,0,1}$, $g_{0,0}$, $h_{0,0}$ and $f_{0,0,2}$, and two extra variables $t_{0,0,1}$ and $t_{0,0,2}$. Here $f_{0,0,1}$, $g_{0,0}$, $h_{0,0}$

and $f_{0,0,2}$ are all general polynomials in unknowns $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$, of degrees C_2, $C_2 - 1$, $C_2 - 1$ and $C_2 - 2$ respectively.

The equations of $S_{0,0}$ are the union of the equations of $S_{0,1}$ and 3 extra equations: $f_{0,0,1} + x_n g_{0,0} + y_n h_{0,0} + x_n y_n f_{0,0,2} = 0$, $1 - t_{0,0,1}(f_{0,0,1} + x_n g_{0,0})(h_{0,0} + x_n f_{0,0,2}) = 0$ and $1 - t_{0,0,2}(f_{0,0,1} + y_n h_{0,0})(g_{0,0} + y_n f_{0,0,2}) = 0$.

This $S_{0,0}$—in case non-empty—represents certain non-empty Zariski open sets of graphs of rational maps from X to $\mathbb{C}^n$. Each $S_{0,n-j}$ in Step 5 then represents the image of $S_{0,0}$ under the natural linear projections $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^j$. All $S_{0,n-j}$—when non-empty—are birational to X.

- Step 7: We now continue to add more equations and variables. For each $j = 1, 2, \ldots, n$ we construct the system $S_{j,0}$ as follows.

The variables of $S_{j,0}$ are the union of the variables of $S_{j-1,0}$, the coefficients of $f_{j,0,1}$ and $f_{j,0,2}$, two extra variables $t_{j,0}$ and a, together with coefficients of other polynomials $\tau_{j,1,0}, \ldots, \tau_{j,1,m(j)}$ and $\tau_{j,2,0}, \ldots, \tau_{j,2,m(j)}$. Here $f_{j,0,1}$ is a general polynomial of degree $C_2 - 1$ in variables $y_1, \ldots, y_n, x_{n-j-1}, \ldots, x_1$, and $f_{j,0,2}$ is a general polynomial of degree C_2. Here $m(j) =$ the number of polynomials in $S_{j-1,0}$, and $\tau_{j,1,0}, \ldots, \tau_{j,1,m(j)}$ and $\tau_{j,2,0}, \ldots, \tau_{j,2,m(j)}$ are general polynomials in the variables of $S_{j,0}$ and a. The degrees of $\tau_{j,1,0}, \ldots, \tau_{j,1,m(j)}$ and $\tau_{j,2,0}, \ldots, \tau_{j,2,m(j)}$ are determined from the effective Hilbert's Nullstellensatz, related to the two extra equations in the next paragraph.

The equations of $S_{j,0}$ are the union of the polynomials $h_1, \ldots, h_{m(j)}$ of $S_{j-1,0}$ and 2 extra equations:

$$-1 + \tau_{j,1,0}(1 - a(f_{j,0,1}x_j + f_{j,0,2})) + \tau_{j,1,1}h_1 + \cdots \tau_{j,1,m(j)}h_{m(j)} = 0,$$
$$-1 + \tau_{j,2,0}(1 - a(1 - t_{j,0}f_{j,0,1}f_{j,0,2})) + \tau_{j,2,1}h_1 + \cdots \tau_{j,2,m(j)}h_{m(j)} = 0.$$

The above 2 equations check that $\{h_1 = \cdots = h_m = 0\}$ belongs to both the monoid $(f_{j,0,1}x_j + f_{j,0,2}) = 0$ and the set $f_{j,0,1}f_{j,0,2} \neq 0$ where the projection from the monoid has good properties. Recall from the classical fact used in the proof of Theorem 2.1 that to check whether $\{h_1 = \cdots = h_m = 0\}$ belongs to the set $\{h = 0\}$, we add a new variable a and use the effective Hilbert Nullstellensatz to reduce the question to the existence of polynomials $\tau, \tau_1, \ldots, \tau_m$ (in the variables of the polynomials $h_1, \ldots, h_m, h$ and the new variable a) of effectively bounded degrees so that $1 = \tau(1 - ah) + \tau_1 h_1 + \cdots + \tau_m h_m$.

At the end of Step 7, we obtain a system of polynomial equations $S_{n,0}$—which, in case non-empty—represents certain non-empty Zariski open subset of the images of X under birational maps.

- Step 8: We are now ready to construct the final system of polynomial equations S.

The variables of S are the union of the variables of $S_{n,0}$, the coefficients of $F_0, F_1, \ldots, F_n$, an extra variable a, together with the coefficients of polynomials $\tau_{i,j,k}$ coming from effective Hilbert Nullstellensatz (more detail below). Here $F_0, F_1, \ldots, F_n$ are general polynomials of degree $\leq d$ in variables $x_1, \ldots, x_n$.

The equations of S are the union of equations in $S_{n,0}$ and the ones coming from effective Hilbert Nullstellensatz when we want to check that the set defined by $S_{n,0}$ is contained in the set $\{y_1F_0 - F_1 = 0, \ldots, y_nF_0 - F_n = 0, 1 - aF_0 = 0\}$ and Y. The number of these extra equations equals the sum of $n+1$ (which is the number of polynomials in $\{y_1F_0 - F_1 = 0, \ldots, y_nF_0 - F_n = 0, 1 - aF_0 = 0\}$) and the number of a basis of the ideal defining Y. Each of these polynomials will contain the set $S_{n,0}$ and leads to one equation as we mentioned in Step 7.

This S represents the fact that the birational maps induced by solutions of $S_{n,0}$ are restrictions of rational selfmaps of $\mathbb{P}^n$ degrees $\leq d$, and maps X onto Y.

- Step 9: Now we can use Gröebner bases to solve the system of polynomial equations S. If S has at least one solution, then the output of the algorithm is Yes, that is there is a birational map of degree $\leq d$ from X onto Y. If, on the contrary, S has no solution, then the output of the algorithm is No.

Now we give the proof of Corollary 2.3.

Proof of Corollary 2.3 That $\mathcal{R}(n,d)$ is an algebraic variety is easy to see. The last paragraph in the proof of Theorem 2.1 shows that $\mathcal{R}^+(n,d)$ is an algebraic variety and can be explicitly constructed.

The proof of Theorem 2.1 shows that $W(X,Y,d)$ is a subvariety of $\mathbb{C}^N \times \mathcal{R}(n,d)$ for some integer N, and the map $\kappa : W(X,Y,d) \to \mathcal{B}(X,Y,d)$ is simply the restriction to $W(X,Y,d)$ of the projection $\mathbb{C}^N \times \mathcal{R}(n,d) \to \mathcal{R}(n,d)$. Therefore, by Chevalley's theorem, the subset $\mathcal{B}(X,Y,d)$ of $\mathcal{R}(n,d)$ is constructible, and $W(X,Y,d)$ is a parameterization for $\mathcal{B}(X,Y,d)$.

The proof for the subset $\mathcal{B}^+(X,Y,d)$ of $\mathcal{R}^+(n,d)$ is similar.

Remark 2.7 We list some final remarks in this section.

- For a **given** rational map $f : X \dashrightarrow Y$, it has been known for some decades that Gröbner bases can be used to check whether it is a birational map (see e.g. Shannon and Sweedler 1988 for the special case when Y is a projective space). However, as far as we know, our algorithm in the proof of Theorem 2.1 is the first one to treat the case of determining the set of all birational maps from X to Y, which need to deal with maps whose coefficients are **undetermined**. This poses difficulties which do not arise in the case of explicit maps.
- While the bounds in the proof of Theorem 2.1 are explicit, they are quite big. For example, the degree bound in the effective Hilbert Nullstellensatz is exponential, and the degree bound for the graph Γ_f is polynomial. Therefore, more refinements of the given algorithm are needed before it can be used for practical examples. Also, with the ever improvement in computer hardwares and softwares, we hope that in a not far future the theoretical aspects of results in this paper can be put into practice.

- As the Addendum in the proof of Theorem 2.1 shows, there are many choices of such $W(X, Y, d)$ and κ. We know that whatever the choice of $W(X, Y, d)$ is, its image (as a set) by κ is always $\mathcal{B}(X, Y, d)$. By Corollay 2.3, $\mathcal{B}(X, Y, d)$ is an algebraic variety. Hence, we can choose $\mathcal{B}(X, Y, d)$ as a canonical choice for all such $W(X, Y, d)$'s. Using the recent result (Harris et al. 2019) on computing images of polynomial maps, we would again be able to explicitly describe $\mathcal{B}(X, Y, d)$ in terms of X, Y and d. The same consideration can also be applied to $\mathcal{B}^+(X, Y, d)$. If we are interested only in finding an answer to the Bounded Birational Problem, we may alternatively proceed in the following simpler manner. We start with any of the $W(X, Y, d)$ given in Theorem 2.1. Then elimination (by using for example Gröbner bases) gives us explicitly a closed subvariety $\mathcal{C}(X, Y, d)$ of $\mathcal{R}(n, d)$. While $\mathcal{C}(X, Y, d)$ may not be reduced, its reduced structure is exactly $\overline{\mathcal{B}(X, Y, d)}$. Again, the reduced structure of $\mathcal{C}(X, Y, d)$ can be explicitly constructed using Gröbner bases. In general, this $\overline{\mathcal{B}(X, Y, d)}$ may be bigger than $\mathcal{B}(X, Y, d)$, and hence a specific point in $\overline{\mathcal{B}(X, Y, d)}$ may not give rise to a birational map from X to Y. However, still we have that $\mathcal{B}(X, Y, d) \neq \emptyset$ if and only if $\overline{\mathcal{B}(X, Y, d)} \neq \emptyset$. Therefore, $\overline{\mathcal{B}(X, Y, d)}$ can still be used to answer the Bounded Birationality Problem, and from the above analysis we may regard $\overline{\mathcal{B}(X, Y, d)}$ as another canonical choice concerning this aspect.

3 Variants

In this section we prove similar results for other interesting classes of maps (such as dominant rational maps, regular morphisms, isomorphisms or regular embeddings). The proofs will combine the ideas in the previous sections together with some additional ingredients to resolve new difficulties associated with the particular case at hand. As before, we work with an algebraically closed field $\mathbb{K}$ of arbitrary characteristic, unless specifically stated otherwise. We fix from now on two irreducible subvarieties $X, Y \subset \mathbb{P}^n$, and a positive integer d.

We also show that some minor modifications prove the same results for maps on affine varieties. Then we deduce from this the validity of all the results for maps on arbitrary algebraic varieties.

3.1 Dominant rational maps of bounded degrees

We consider the following question.

Question 3.1 (*Question (A)*) Is there a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, of degree $\leq d$, so that $F|_X$ is a dominant rational map onto Y?

In the case where $X = \mathbb{P}^k$ a projective space, we have the unirationality problem, which has attracted a lot of interest.

Theorem 3.2 *The above Question (A) is computable.*

Proof We will give an algorithm, whose complexity is explicitly bounded, to solve this question.

We write the maps $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ needed to find in terms of their coefficients.

First, to check that X is not contained in the indeterminacy locus F and that F maps X to Y, we can proceed as in the proof of Theorem 2.1. This way we get some systems of polynomial equations.

We now proceed to check the condition that F maps X onto Y. To this end, we analyse what happens in the opposite case. So, assume for a moment that the map F does not map X onto Y. This means that there is a proper subvariety W of Y so that the projection of Γ_f is W. We observe that, by part 2 of Lemma 2.5, the degree of W is bounded in terms of that of Γ_f. Then since $\mathbb{P}^n \times \mathbb{P}^n$ can be embedded into $\mathbb{P}^{n^2+2n}$, we can use a trick of Mumford (1969) that **set-theoretically** W is generated by polynomials of degrees explicitly bounded in the degree of W (in fact, these can be chosen as cones over W with vertex a linear subspace of $\mathbb{P}^n$, and hence of degree $=$ $\deg(W)$), to find a proper hypersurface Z of explicitly bounded degree of $\mathbb{P}^n$ so that $Y \not\subset Z$, $Z \cap Y$ contains $\pi(\Gamma_f)$. We now work on the affine Zariski open set $\mathbb{C}^n \times \mathbb{C}^n$ of $\mathbb{P}^n \times \mathbb{P}^n$, and interpret the previous sentence in terms of polynomial equations. Since the degree of Z is explicitly bounded, we can as before parameterise it in terms of an explicitly bounded number of coefficients. As in the proof of Theorem 2.1, we can express the condition that $Z \cap Y$ contains $\pi(\Gamma_f)$ which we call (E).

Now we let (E') to be the union of (E) and the polynomial equations expressing the condition that the hypersurfaces Z in the above paragraph contain Y. Then the variety defined by (E') is a subvariety of the variety defined by (E). The maps F which map X onto Y will be parametrised by the complement of (E') in (E), and hence are parametrised by a finite union of algebraic varieties. Hence to check that there are dominant rational maps F from X onto Y is the same as checking that the set of solutions to (E) is strictly bigger than the set of solutions to (E'), which is a decidable problem, by using for example Gröbner bases. □

3.2 Regular morphisms of bounded degrees

We now consider the following question.

Question 3.3 (*Question (B)*) Is there a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, of degree $\leq d$, so that $F|_X$ is a regular morphism into Y?

Note that while the indeterminacy set of $F = [F_0 : F_1 : \ldots : F_n]$, as a rational map from $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$, is simply the set $\mathcal{I}(F) := \{F_0 = \cdots = F_n = 0\}$, determining the indeterminacy set of the restriction of F to a subvariety X is not an easy task. In particular, it is not true that the indeterminacy set of $F|_X$ is always $\mathcal{I}(F) \cap X$. A classical example is the following. Let $X = \{xz = y^2\} \subset \mathbb{P}^2$. The rational map $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $[x : y : z] \mapsto [x : y]$ has $\mathcal{I}(F) = [0 : 0 : 1] \in X$. However, $F|_X$ is a regular morphism. In fact, this follows from the fact that we have $[x : y] = [y : z]$ on X, and at least one among these two represents a genuine point in $\mathbb{P}^1$. One can also check that $[x : y]$ and $[y : z]$ are the only representatives (up to

multiplicative factors) of $F|_X$, and the indeterminacy set of $[y : z]$ is $[1 : 0 : 0] \in X$. Thus for any $G : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with $G|_X = F|_X$ we have that $\mathcal{I}(G) \cap X \neq \emptyset$.

For a given rational map F, Simis (2004) computed the indeterminacy set of $F|_X$ in terms of some algebras associated to F. Hence by checking that these algebras give rise to empty indeterminacy sets, we arrive at a necessary and sufficient condition for $F|_X$ to be regular. However, as far as we know, no criterion has been given in the literature for the case where the map F is not explicitly given. The main result we give here is that provided X is a smooth projective variety, then Question (B) is computable.

Example 3.4 Before stating the main result, let us explain the main idea via the example $X = \{xz = y^2\} \subset \mathbb{P}^2$ and $F[x : y : z] = [x : y]$ above. Let $\pi : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the projection to the first factor. We saw that for $f = F|_X$, the graph Γ_f is not the intersection between Γ_F and $\pi^{-1}(X)$ (which is $\{([x : y : z], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 : xz = y^2,\ xv = yu\}$), but it is an irreducible component of this intersection. Therefore, we can add in several polynomials to obtain the correct ideal for Γ_f. In this case, it turns out that we need only to add one more polynomial, of degree 2. More precisely

$$\Gamma_f = \{([x : y : z], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 :\ xz = y^2,\ xv = yu,\ yv = zu\}.$$

To check that f is a regular morphism is the same as checking that the projection π maps Γ_f isomorphically to X. Since π is a birational morphism on Γ_f, it suffices to check that the **relative tangent spaces** (defined by taking with respect to variables in $\mathbb{P}^1$ of the defining equations for Γ_f) are 0 at every point of Γ_f. (In fact, we can work on a chart, say $\mathbb{C}^2_{x,y} \times \mathbb{C}_u$. If f is regular, then by Hilbert's Nullstellensatz in this chart Γ_f is generated by $u - g(x, y) = 0$ and the ideal for X, which when taking derivative with respect to u will have rank 1 as claimed. Then if $g_1, \ldots, g_m$ are another set of generators for the ideal of Γ_f in this chart, we will get the same answer, which is 1. Hence the dimension of the relative tangent space, which is the corank of the above matrix, is 0. Conversely, if the relative tangent spaces are 0 for every point on Γ_f, this will give us that the projection π is étale. Hence, by using Zariski's Main Theorem, since π is also projective and birational, it is an isomorphism. Alternatively, we observe that any fibre of π must be finite, and hence by cohomological reason has the same number of points, which is 1.) This can be checked by working with the $(2 + 1)(1 + 1) = 6$ coordinate charts $\mathbb{C}^2 \times \mathbb{C}$ of $\mathbb{P}^2 \times \mathbb{P}^1$. For example, in one such chart, corresponding with $z = 1$ and $v = 1$, we obtain

$$\begin{aligned} X &= \{(x, y) \in \mathbb{C}^2 :\ x = y^2\}, \\ \Gamma_f &= \{((x, y), u) \in \mathbb{C}^2 \times \mathbb{C} :\ x = y^2,\ x = yu,\ y = u\}. \end{aligned}$$

The ideal of Γ_f is generated by $x - y^2, x - yu, y - u$, and hence the dimension of the relative tangent space is the corank of the matrix we obtain by taking the first derivatives of the above polynomials with respect to u: which consecutively are $0, -y, -1$. Since the rank of this matrix is 1 everywhere on Γ_f, we conclude that the relative tangent space has dimension 0 everywhere, as desired.

Remark 3.5 The point of view in Example 3.4 is that to check whether f is a regular morphism, we do not need to compute the indeterminacy set of f. Instead, we only need to check that Γ_f is isomorphic to X, and this in turn is the same as checking that the projection $\Gamma_f \to X$ is étale. The special form of the projection makes computations less complicated and is more flexible, allowing us to deal with maps F not explicitly given.

In Example 3.4, we see that the important factor for being able to arrive at an explicit algorithm is that the number of polynomials needed, as well as their degrees, to add into $\Gamma_F \cap \pi^{-1}(X)$—where $\pi : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ is the projection to the first factor—to obtain the ideal for Γ_f (recall that $f = F|_X$), should be explicitly bounded in terms of d and X. (Note that we will need to deal with the case where f is undetermined, and so we do not know anything about Γ_f except that it is contained in $\Gamma_F \cap \pi^{-1}(X)$, its degree is explicitly bounded and it should be isomorphic to X. Hence if X is smooth then Γ_f is also smooth.) The following lemma (see Theorem 10 in Blanco et al. (2004)) is important in this aspect. (For the convenience of the readers, a non-optimal version of this lemma—which is enough for the conclusions of Theorems 3.7 and 3.8—can be proven based solely on Mumford (1969).)

Lemma 3.6 *Let $V \subset \mathbb{A}^n$ be a smooth equi-dimensional algebraic variety and set $m := (n-\dim(V))(1+\dim(V))$. There exist polynomials $f_1, \ldots, f_m$ in $\mathbb{K}[x_1, \ldots, x_n]$ with degrees bounded by $\deg(V)$ such that $I(V) = (f_1, \ldots, f_m)$.*

Now we can state the main result of this subsection.

Theorem 3.7 *Assume that X is a smooth projective variety. Then Question (B) is computable.*

Proof Again, checking that a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ gives rise to a rational map from X into Y can be expressed in terms of polynomial equations. (This does not require any condition on X.)

We now proceed to showing that when X is a smooth projective variety, the conditions for that $f = F|_X$ is regular, as a map from X into $\mathbb{P}^n$, can also be expressed in terms of polynomial equations.

As seen from Example 3.4, that f is regular is equivalent to Γ_f is isomorphic to X and hence in particular is smooth. By Lemma 3.6, we can add an explicitly bounded number of polynomials of explicitly bounded degrees and which vanish on $H_{0,0}$ in the proof of Theorem 2.1 (this conditon can be expressed in terms of polynomial equations by the argument in the proof of Theorem 2.1) into the polynomials already generating $\Gamma_F \cap \pi^{-1}(X)$, where $\pi : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ is the projection to the first factor, to obtain generators $h_1, \ldots, h_p$ for some Γ (which we want to be the graph Γ_f). Note that the number p and the degrees of $h_1, \ldots, h_p$ are explicitly bounded.

We want that Γ is exactly Γ_f and moreover it is isomorphic to X. Note that these two conditions are equivalent to that there is one such constructed Γ which is isomorphic to X. In fact, if this is the case, then Γ is in particular irreducible. Since Γ contains $H_{0,0}$ as a **set**, it follows from the fact that Γ is isomorphic to X, that Γ is exactly the graph Γ_f.

Now to check that Γ is isomorphic to X, we need to check that the relative tangent spaces at every point in Γ has dimension 0. (Strictly speaking, this check only shows

that the irreducible component of Γ containing Γ_f as a set is isomorphic to X, but this is enough to have that Γ_f is isomorphic to X as desired. A priori, since we are not given explicitly the map F and the polynomials we add in are also undetermined, there may be other components of Γ besides Γ_f.) We can work with Zariski open sets $\mathbb{C}^n \times \mathbb{C}^n$ of $\mathbb{P}^n \times \mathbb{P}^n$. From the above generators $h_1, \ldots, h_p$ of Γ, which are undetermined, hence need to be parametrised in terms of their coefficients, we compute the derivatives of them with respect to the variables $y_1, \ldots, y_n$ in the second factor of $\mathbb{C}^n \times \mathbb{C}^n$. Call the Jacobian matrix obtained $J_{\Gamma/X}$, and call $M_1, \ldots, M_q$ all the $n \times n$-minors of $J_{\Gamma/X}$. Note that the number q and the degrees of $M_1, \ldots, M_q$ are also explicitly bounded. Then the fact that the relative tangent spaces at every point of Γ has dimension 0 is translated into that the system defined by $M_1 = \cdots = M_q = 0$ has no solution on Γ. As in the proof of Theorem 2.1, this and the effective Hilbert's Nullstellensatz allow us to translate Question B into the question about the existence of solutions to an explicitly constructed variety. □

3.3 Isomorphisms/regular embeddings of bounded degrees

We now consider the following question. Question (C): Is there a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, of degree $\leq d$, so that $F|_X$ is an isomorphism onto Y? We also consider a more general question. Question (C'): Is there a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, of degree $\leq d$, so that $F|_X$ is a regular embedding into Y?

Combining the previous results, we obtain the following result.

Theorem 3.8 *Assume that X is a smooth projective variety. Then Questions (C) and (C') are computable.*

Proof We only need to prove for Question (C').

Using Theorem 2.1, we can check whether there is a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of degree $\leq d$ so that $f = F|_X$ is a birational map into Y. Call Z the image of f. (Here, we do not need to specify Z.)

We can use Theorem 3.7 to check if $F|_X$ is a regular morphism.

It remains to check whether the birational map $g^{-1} : Z \dashrightarrow X$, where $g = f^{-1}$, is also a regular morphism. To this end, we can compute the derivatives of the generators we have for Γ_f in the proof of Theorem 3.7 with respect to variables in the other copy of $\mathbb{P}^n$ containing Y, and denote the resulting matrix by $J_{\Gamma/Z}$. As in the proof of Theorem 3.7, we only need to check that $J_{\Gamma/Z}$ gives rise to that the relative tangent spaces with respect to the projection $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ to the second factor are 0. As before, this can be described in terms of polynomial equations involving minors of $J_{\Gamma/Z}$.

Note that here a priori, we do not know whether Z is smooth or not. To this end, we can add the equations for the condition that Z is smooth: it is the same that at least one of the $(n - \dim(X)) \times (n - \dim(X))$ minors of the derivatives of the generators of $I(Z)$ must be non-zero at every point of Z, which then can be interpreted by effective Hilbert's Nullstellensatz.

At the end, we obtain some explicit systems of polynomial equations, whose solutions parametrise Question (C). □

3.4 Maps on affine varieties

In this subsection we discuss the above results for the cases where varieties concerned are affine. More precisely, considering now $X, Y \subset \mathbb{C}^n$ be closed irreducible varieties, and a positive integer d. We ask whether there is a rational map $F : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ of degree $\leq d$ so that the restriction $f = F|_X$ maps X into Y and is: (i) birational; or (ii) dominant rational; or (iii) regular; or (iv) regular embedding. We also ask the following stronger versions of (iii) and (iv), which are not necessary in the projective setting: whether f is (iii') regular and surjective; or (iv') a regular embedding onto a closed subvariety of Y. We recall that the reason that in the affine setting (iii') and (iv') are actually stronger than (iii) and (iv) respectively is because the image of a closed subvariety of $\mathbb{C}^n$ under a polynomial map is not always a closed subvariety but only constructible (Chevalley's theorem). In the above formulations, we may as well ask for the case where F is a polynomial, in which case the proof will be the same.

We will show that all of the above questions are computable. As usual, the degree of an affine variety is the degree of its closure in the projective space. With this convenience, the proofs of the previous results apply straightly forwardly to questions (i) and (ii), and show that they are computable. For question (iii), the application is almost straightly forward, after we projectivise the varieties and the maps. Then an application of Zariski's main theorem (which says that the preimage of a smooth point in a variety V by a birational morphism is connected, and hence is one point if the fibre has an isolated point) will again give the following. The fact that relative tangent spaces are 0 at every point on the graph implies that the projection π onto the first factor is a regular embedding of the graph Γ_f into X and vice versa. To check that the graph Γ_f is the graph of a regular morphism, we need to check that the projection $\pi : \Gamma_f \to X$ is surjective. This then can be checked as in the end of the proof of (iv') below. Since constructible sets form a Boolean algebra (closed under finite union and complementation), it follows that we can parametrise question (iii) also by algebraic varieties. For question (iv), we use (iii), and then similarly check that the relative tangent spaces of the projection $\Gamma_f \to Y$ are all 0.

Now we briefly show how to solve questions (iii') and (iv'). First we consider question (iii'). By question (iii), we can check whether f is regular and whose image is contained in Y. To check that the image is actually Y and hence solve the stronger question (iii'), it suffices to show that for $\pi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ the projection to the second factor, there is no point $y \in Y$ for which the intersection $\pi^{-1}(y) \cap \Gamma$, here Γ is the one constructed in the proof of Theorem 3.7, is empty. Note that the condition that $\pi^{-1}(y) \cap \Gamma = \emptyset$ can be described in terms of polynomial equations by effective Hilbert Nullstellensatz. Hence, the maps from question (iii) (call their parameter space, in terms of the coefficients of F only, W_1) whose image are not the whole of Y can be parametrised (in terms of the coefficients of F only) by a constructible subset W_2 of W_1. By Harris et al. (2019), W_1 and W_2 would be explicitly constructed from X, Y and d. Then the parameter space for (iii') is the constructible set $W = W_1 \backslash W_2$. In particular, it is a finite union of algebraic varieties.

We end this subsection by giving a proof for the fact that question (iv') is also computable. By question (iv), we can check whether f is a regular embedding into

Y. Denote by $Z = f(X) \subset Y$, which we do not know in advance. We need to have that Z is a closed subvariety of Y. Let $\overline{Z}$ be the closure of Z in Y, then $\overline{Z}$ is a closed subvariety of Y of the same dimension as that of X. As in the proof of Theorem 3.2, the degree of $\overline{Z}$ is explicitly bounded. Since in question iv') we want to show that $f(X)$ is a closed subvariety of Y, it follows that $f(X) = \overline{Z}$, and hence $\overline{Z}$ must also be smooth. Then by Lemma 3.6, the ideal $I(Z)$ is generated by an explicitly bounded number of polynomials of degrees $\leq \deg(\overline{Z})$. (Again, these polynomials are undetermined, but we can parametrise them by their coefficients whose number is explicitly bounded.) We add to these polynomials also a finite set of generators for the variety Y to make sure that $\overline{Z}$ is contained in Y. We also add in the equations for the condition that $\overline{Z}$ is smooth (which should be, if it is to be the image of X under an isomorphism f). Denote by $\pi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ the projection to the second factor. Then as in the previous paragraph, the fact that $f(X) = \overline{Z}$ is the same as that there is no $z \in \overline{Z}$ for which $\pi^{-1}(z) \cap \Gamma = \emptyset$. Hence, as before, among the regular embeddings f of X into Y, those for which $f(X)$ is not a closed subvariety is parametrised by a constructible subset of the variety parametrising the answer to question (iv). Hence, the parameter space for (iv'), which is the difference between these two varieties, is also a constructible set, and is parameterized by an affine variety.

3.5 Maps on arbitrary algebraic varieties

Since any algebraic variety has a finite Zariski open cover by affine varieties, the results in the previous subsection imply that all questions (i), (ii), (iii), (iii'), (iv) and (iv') in the previous subsection are also computable when X and Y are arbitrarily irreducible algebraic varieties. To this end, it is enough to make precise for a rational map $f : X \dashrightarrow Y$, what it means that f has bounded degree. To this end, we can proceed as follows. Let $X_1, \ldots, X_p$ be a Zariski open covering by affine varieties for X, and $Y_1, \ldots, Y_q$ be a Zariski open covering by affine varieties for Y. Assume that X_i and Y_j belong to $\mathbb{C}^n$ for all i, j. Then we say that the degree of f is bounded by a positive integer d if for every pair i, j for which f maps X_i into Y_j, there is a rational map $F_{i,j} : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ of degree $\leq d$ and so that $F_{i,j}|_{X_i} = f|_{X_i}$.

4 Concluding remarks

In this paper we showed that birational maps of bounded degrees between algebraic varieties can be parametrised by some algebraic varieties. We provide an explicit algorithm for implementing on computers. We also proved that similar results hold for other interesting classes of maps, such as biregular isomorphisms. These results are valid for both varieties over $\mathbb{C}$ and over fields of positive characteristic. They provide countable invariants for characterising these maps. Based on these results, together with Iitaka's fibrations, we proposed a rough approach towards solving the birationality problem, computationally and effectively, in general.

An application of the above results is the following. Let X be a smooth algebraic affine curve. Then whether or not X is algebraically embedded in the affine plane $\mathbb{A}^2$

is characterised by a countable set of varieties explicitly constructed from X. In fact, X is algebraically embedded into $\mathbb{A}^3$, and hence we can apply the results in Sect. 3.4. The question of whether there is a finite set of invariants characterising that a smooth affine algebraic curve is algebraically embeddable into $\mathbb{C}^2$ is a long standing, classical open question.

In the remarks after the proof of Theorem 2.1, we proposed an approach toward the birationality problem. If it is true that X and Y are **not** birationally equivalent, and we want to confirm this, then we can try the following alternative approach. That X and Y are not birationally equivalent is the same as that all the systems $W(X, Y, d)$ $(d = 1, 2, 3, \ldots)$ have no solutions, which by effective Hilbert's Nullstellensatz is the same as having an identity $1 \equiv \sum \tau_i h_i$, where h_i are a generator for the ideal of $W(X, Y, d)$, and all τ_i and h_i have degrees bounded in terms of d and X, Y. The systems $W(X, Y, d + 1)$ contains as a special case the system $W(X, Y, d)$, for all d. Hence if we are able to verify the existence of such an identity for small values of d, we may be able to use induction to deduce the identity for general d.

It is interesting to know if in Theorems 3.7 and 3.8 we can get rid of the smoothness assumptions on the variety X.

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Appendix: A rough strategy towards the birationality problem via Iitaka's fibrations

In this section, we state a rough strategy, based on Theorem 2.1, for a computational approach towards the birationality problem.

We first recall briefly about the fact that the birationality problem for varieties of general type over $\mathbb{C}$ was completely solved about 10 years ago. We thank John Christian Ottem for communicating this remark. Note that the birationality problem for smooth surfaces of general type is **computable**. For any smooth surface of general type X, the linear system $|5K_X|$ is a birational embedding of X into some projective space. Given two smooth projective surfaces of general type X, Y, if $h^0(X, 5K_X) \neq h^0(Y, 5K_Y)$, then they are not birationally equivalent. In the case $h^0(X, 5K_X) = h^0(Y, 5K_Y) = N$, let X' be the birational embedding of X in $\mathbb{P}^N$ using the linear system $|5K_X|$ and Y' be the birational embedding of Y in $\mathbb{P}^N$ using the linear system $|5K_Y|$. If $f : X \dashrightarrow Y$ is birational, then $f^* : H^0(X, 5K_X) \to H^0(Y, 5K_Y)$ is isomorphic, and hence there is a linear map in $\mathbb{P}^N$ which maps X' onto Y'. The latter question is computable. Similarly, the question of whether two smooth complex projective varieties of **general type** in higher dimension are birational equivalent is **decidable**. (Here we use the following distinction between "computable" and "decidable": an algorithm is computable if its complexity is explicitly given, while for a decidable program, we do not know its complexity but we know that it will stop in a finite time.) This follows from the following result in Hacon and McKernan (2006), Takayma (2006), Tsuji (2006) and Tsuji (2007): Let X be a smooth irreducible projective variety of dimension k. Then there is a positive number $r_k > 0$ depending only on k such that the linear system $|r_k K_X|$ is a birational embedding. If $k > 2$, it is not known whether this question is again **computable**, since the above number r_k is not yet explicitly determined.

Note that in the proof of Theorem 2.1, the assumption that the birational map $f : X \dashrightarrow Y$ is the restriction of a rational map $F : X \dashrightarrow Y$ of degree $\leq d$ is needed only to deduce that the degree of the graph Γ_g is explicitly bounded. Therefore, the birationality problem is solved if the following question has an affirmative answer.

Question 5.1 (*Main Question*) Let $X, Y \subset \mathbb{P}^n$ be irreducible varieties, where $n \geq 2$, and assume that there is a birational map $f : X \dashrightarrow Y$. Is there another birational map $g : X \dashrightarrow Y$ so that the degree of the graph Γ_g, viewed as a subvariety of $\mathbb{P}^n \times \mathbb{P}^n$, is explicitly bounded in terms of n and the ideals for X and Y?

A special case when Question 5.1 is answered in the affirmative is when X is a monoid and Y is a projective space, see the description before the proof of Theorem 2.1. The bound in degree is the degree of the monoid. Less trivially, we have the following result.

Proposition 5.2 (1) *To solve Question 5.1, it is sufficient to do it for the case where X and Y are either:*

(a) *Hypersurfaces; or*
(b) *Normal varieties.*

(2) *Over $\mathbb{C}$, Question 5.1 has an affirmative answer when X and Y are varieties of general type.*
(3) *Question 5.1 has an affirmative answer when X and Y are curves.*

Proof (1) (a) Let $m = \dim(X) + 1$, we can always find a linear projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, under which the strict transform X' of X is birational to X and the strict transform Y' of Y is birational to Y. Assume that Question 5.1 is affirmatively

answered for hypersurfaces. Then, provided there is a birational map from X to Y, there will be a birational map $g' : X' \dashrightarrow Y'$ whose graph $\Gamma_{g'} \subset X' \times Y' \subset \mathbb{P}^m \times \mathbb{P}^m$ has explicitly bounded degree in terms of X, Y. Then the graph $\Gamma_g \subset X \times Y \subset \mathbb{P}^n \times \mathbb{P}^n$ of the lifting birational map $g : X \dashrightarrow Y$, which is contained in the intersection between $X \times Y$ and the strict transform of $\Gamma_{g'}$ under the dominant rational map $\pi \times \pi$, also has explicitly bounded degree. Then the proof of Theorem 2.1 provides us with a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, whose degree is explicitly bounded in terms of X, Y, and whose restriction to X is g.

(b) The proof is similar, now we use that any variety has a normalisation.

(2) By the results mentioned above, there is a number $r_k > 0$ depending only on $k = \dim(X) = \dim(Y)$ so that the pluricanonical divisor $r_k K_X$ gives a birational map $X \dashrightarrow M(X) \subset \mathbb{P}^{N_1}$, and similarly $r_k K_Y$ gives a birational map $Y \dashrightarrow M(Y) \subset \mathbb{P}^{N_2}$. From the properties of the canonical divisor, we have that X and Y are birational iff $N_1 = N_2 =: N$ and $M(X)$ is isomorphic to $M(Y)$ via a linear automorphism of $\mathbb{P}^N$. Then, the degree of the graph of the composition $X \dashrightarrow M(X) \to M(Y) \dashrightarrow Y$ is bounded effectively in terms of X and Y.

(3) We consider different cases.

Case 1 X and Y are curves of genus ≥ 2. We can apply the argument in (2).

Case 2 X and Y are elliptic curves. We use that on an elliptic curve the divisor $3p$, where p is any point in the elliptic curve gives rise to an embedding of that elliptic curve as a cubic curve in $\mathbb{P}^2$. Now, two smooth cubic curves in $\mathbb{P}^2$ are isomorphic iff they are so under some linear automorphisms of $\mathbb{P}^2$. Then, the same argument as in (2) completes the proof.

Case 3 X and Y are rational curves. In this case we can use the anti-canonical divisor and argue as in (2).

□

Proposition 5.2 gives some evidence that Question 5.1 should have an affirmative answer. In the remaining of this section, we provide a rough argument, based on Iitaka's fibration, to reduce Question 5.1 to the following two special cases: varieties of Kodaira dimension $-\infty$ or 0. The argument we present here is largely heuristic and relies on further advancement on the understanding of Iitaka's fibrations.

In the remaining of this section we work over $\mathbb{C}$. Iitaka has shown that for a smooth projective variety Z of non-negative Kodaira dimension, for m large enough so that $h^0(Z, mK_Z) \neq 0$, the maps $Z \dashrightarrow \mathbb{P}H^0(Z, mK_Z)^*$ are all birational to one common fibration, now so-called the Iitaka's fibration. Moreover, the Iitaka's fibration has a universal property among fibrations whose general fibers have zero Kodaira's dimension, which is unique up to birational morphisms. Under some assumptions (in particular, including that the general fiber of the Iitaka's fibration has good minimal models), Pacienza (2009) showed that the bound on m is effectively computed in terms of the dimension of Z, the pluricanonical divisors of general fibers, and the Betti numbers of some associated cover over the general fibers. More recently, Birkar and Zhang (2016) (Theorem 1.2 therein) proved the same result unconditionally.

Now let X and Y be two smooth projective varieties of the same Kodaira dimension ≥ 0. By the cited result, there is an effective constant m (in terms of X and Y), so that

both natural maps $\varphi_X : X \dashrightarrow IF(X) \subset \mathbb{P}H^0(X, mK_X)^*$ and $\varphi_Y : Y \dashrightarrow IF(Y) \subset \mathbb{P}H^0(Y, mK_Y)^*$ are both birational to the corresponding Iitaka's fibrations of X and Y. Note that $IF(X)$ and $IF(Y)$ are varieties of general type of the same dimension. By the uniqueness of Iitaka's fibration up to birational maps, and the properties of the pluricanonical divisors mentioned in the proof of part 2 of Proposition 5.2, we deduce that X and Y are birational iff there is a birational map $\psi : X \dashrightarrow Y$ which takes a general fiber of φ_X to a general fiber of φ_Y, and that the induced map $IF(X) \dashrightarrow IF(Y)$ is the restriction of a linear automorphism of $\mathbb{P}H^0(X, mK_X)^* = \mathbb{P}H^0(Y, mK_Y)^*$. Note then that also the degrees of the graphs of the birational maps between the general fibers of φ_X and φ_Y are uniformly bounded in terms of the degrees of intersection between the graph of ψ and the products of the general fibers of the concerned Iitaka's fibrations.

We now assume that Question 5.1 has an affirmative answer for varieties of Kodaira dimensions 0 and $-\infty$. Then we will next argue heuristically that Question 5.1 also has an affirmative answer in the general case. In fact, given X and Y two varieties of the same Kodaira dimension ≥ 0. Let m be the effective constant in Birkar and Zhang (2016), and $\varphi_X : X \dashrightarrow IF(X)$ and $\varphi : Y \dashrightarrow IF(Y)$ be the corresponding Iitaka's fibrations. If $\mathbb{P}H^0(X, mK_X)^*$ and $\mathbb{P}H^0(Y, mK_Y)^*$ are not isomorphic, then we conclude that X and Y are not birational, and nothing else needs to be done. So, we can assume that $\mathbb{P}H^0(X, mK_X)^* = \mathbb{P}H^0(Y, mK_Y)^*$. Then we can construct an algebraic variety parametrising all isomorphisms from $IF(X)$ to $IF(Y)$ which are restrictions of linear automorphisms of $\mathbb{P}H^0(X, mK_X)^* = \mathbb{P}H^0(Y, mK_Y)^*$. Now, using the assumption that Question 5.1 has an affirmative answer for varieties of Kodaira dimension 0, we construct for each general fiber of φ_X an algebraic variety representing all birational maps onto the corresponding fiber of φ_Y (under the isomorphism between $IF(X)$ and $IF(Y)$ mentioned in the previous sentence) whose graphs have degrees bounded effectively in terms of the specific fibers. Now, since the fibers vary algebraically, we expect that there will be a uniform bound on the degrees of the graphs mentioned in the last sentence. Then we expect that these separate varieties, one for each general fiber of φ_X, can be put together to give a variety parametrising birational maps (of a special form, preserving the given Iitaka's fibrations) from X to Y. Combining all the above construction, we see at the same time that Question 5.1 has an affirmative answer.

So, assuming that the heuristic argument in the above paragraph works (which will, as can be easily seen, require a deeper understanding of Iitaka's fibrations), for solving the birationality problem it remains to solve Question 5.1 for varieties of Kodaira dimensions 0 and $-\infty$. For varieties of Kodaira dimension 0, we speculate that some generalisation of the argument for elliptic curves in the proof of part (3) of Proposition 5.2 may be useful. For varieties of Kodaira dimension $-\infty$, it may be useful to look at the pluri-anticanonical divisors $-mK_X$.

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