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ORIGINAL PAPER

The dimension-overrings equation and maximal ideals of integral domains

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Abstract
We investigate integral domains with only finitely many overrings and establish several new sharp inequalities relating the cardinality of the set of all overrings, the Krull dimension, and the number of maximal ideals. In fact it is shown that knowing any two of these parameters will induce sharp lower and upper bounds for the third. Similar results for the length of an integral domain are also obtained. In particular, we show that if the number of overrings of an integrally closed domain R is a prime number, then the field of fractions of R has a unique maximal subring.

Mathematics Subject Classification 13B02 · 13B22 · 13B30 · 13E99 · 13F05 · 13F30 · 13G05 · 13H99

1 Introduction

Let R be an integral domain with field of fractions $Frac(R)$ (also denoted $qf(R)$). The set $O(R) = [R, Frac(R)]$, i.e. the set of subrings T of $Frac(R)$ such that $R \subseteq T \subseteq Frac(R)$ is usually called the set of overrings of R. If $O(R)$ is finite then R is called an FO domain (Gilmer 2003). Ring extensions with only finitely many intermediate rings have been named FIP extensions in Dobbs et al. (2005). Several finiteness conditions on the set of intermediate rings of a ring extension have been investigated in Jaballah (2010). If R is an FO integrally closed domain (in its field of fractions $Frac(R)$) and the set of prime ideals of R is explicitly known and ordered by the usual set inclusion, then Corollary 2.4 and Algorithm 2.5 of Jaballah (2005) explain how to compute the number of overrings of R . In the case where no enough information about the set of prime ideals is available, the present work shows that we still have sharp approximations on the number of overrings depending on the number of maximal ideals and the Krull dimension of R. These results will also induce sharp

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approximations of the number of maximal ideals and the dimension. Similar results are also obtained for the length of R. The search for such results had been already started in Ayache and Jaballah (1997), where several approximations for the number and length of chains of intermediate rings in normal pairs, and number and length of chains of overrings of Prüfer domains have been established. The exact number of overrings for PID and FO integrally closed domains was established in Jaballah (1997) and Jaballah (2005), which has been generalized for normal pairs in Ben Nasr and Jaballah (2008) and Ayache and Jarboui (2008). An algorithm listing overrings was obtained in Jaballah (2011). For not integrally closed rings, several results have been recently obtained in Ben Nasr (2016), Ben Nasr and Jaballah (2020), and Jaballah and Jarboui (2020). More approximations and exact results for the number of intermediate rings can be found in Jaballah (2012), Jaballah (2013), and Jaballah (2016).

Several results on the links between the cardinality of $O(R)$ and the Krull dimension of the integral domain R have been obtained by several authors. Several characterizations have been recently obtained by Mimouni for domains R with $|O(R)| = n + \dim R$, when $n \leq 8$ for integrally closed domains, see (Mimouni 2009). We establish in this work the corresponding result for general n. We also establish an upper bound for the number of maximal ideals.

We obtain in Sect. 1 sharp upper bounds for the number of overrings of an FO integrally closed domain in Theorem 1. This will allow to obtain a sharp lower bound for the number of overrings of not necessarily integrally closed domains, Corollary 4. The particular case where the number of overrings is prime will induce the existence of a unique maximal subring of $Frac(R)$, Proposition 5 and Example 6.

In Sect. 2, we obtain several approximations for the number of maximal ideals in Theorem 8, Example 9 and Theorem 10; and the Krull dimension in Corollary 11.

Several inequalities relating the length, the Krull dimension and the number of maximal ideals are obtained in the last section, Proposition 12 and Corollary 13.

All rings in this work are assumed to be commutative with identity. The set of proper prime ideals of the ring R is usually denoted by $Spec(R)$, while $Max(R)$ denotes the set of its maximal ideals. The Krull dimension of R is defined to be the maximal length of chains of prime R-ideals, and is denoted by $\dim(R)$.

Let P and P' be two primes of R such that $P \subset P'$. The prime P' is said to cover P (in $Spec(R)$), and we write P' cov P, if there is no prime Q of R such that $P \subset Q \subset P'$. A function α had been defined on $Spec(R)$ by $\alpha(P) := 1$ if P is a maximal ideal of R, and $\alpha(P) := \Pi_{P' \text{cov } P}(1 + \alpha(P'))$ if P is not maximal. The function α applied on the spectrum of an integrally closed FO domain R gives the number of overrings of R, see Corollary 2.4 of Jaballah (2005). Indeed the number of overrings is given by $|O(R)| = \alpha(\{0\})$, where $\{0\}$ denotes the zero-ideal of R. For a subset I of R, $Z(I)$ is usually defined to be the set of primes P of R such that $I \subseteq P$.

2 Approximating the number of overrings

We start with the main result of this work by establishing sharp upper bounds for the number of overrings, when the dimension and the number of maximal ideals are known. Notice that according to Theorem 1.5 of Gilmer (2003), an integrally closed domain D is an FO domain if and only if D is a finite-dimensional Prüfer domain with finite maximal spectrum.

Theorem 1 *Let R be a Prüfer domain with finite Krull dimension,* $\dim(R)$*, and finite number m of maximal ideals. Then,*

$$\left(2^m + \dim(R) - 1\right) \leq |O(R)| \leq (1 + \dim(R))^m .$$

Proof If $\dim(R) = 1$, then the inequalities are trivially satisfied and are indeed equalities. Let us assume by induction that the inequalities are satisfied for every Prüfer FO domain of dimension d with $1 \leq d \leq n$, and let R be a Prüfer FO domain of dimension $n + 1$. We want to show that the inequalities hold true for R as well. That is,

$$\left(2^m + n\right) \leq |O(R)| \leq (1 + (n + 1))^m .$$

Let $\{P_t | t = 1, \ldots, s\}$ be the set of prime ideals of R covering $\{0\}$. We have then,

$$|O(R)| = \alpha(0) = \Pi_{P \text{ cov } \{0\}}(1 + \alpha(P)) = \Pi_{t=1}^{s}(1 + \alpha(P_t)).$$

The domain R/P_t is a Prüfer domain. Each $\alpha(P_t)$ depends on the set $Z(P_t)$ of prime ideals of R containing P_t. The set $Z(P_t)$ is in bijective correspondence, respecting inclusion, with the spectrum of R/P_t. Then $\alpha(P_t) = \alpha(0_{R/P_t})$. Hence $\alpha(P_t) = |O(R/P_t)|$. Using induction and letting $n_t = \dim(R/P_t)$ and $m_t = |Max(R/P_t)|$, we obtain,

$$\begin{aligned}\left(2^{m_t} - 1 + n_t\right) &\leq \alpha(P_t) = |O(R/P_t)| \\ &\leq (1 + n_t)^{m_t} .\end{aligned}$$

Then,

$$\begin{aligned}\Pi_{t=1}^{s}\left(2^{m_t} + n_t\right) &\leq \Pi_{t=1}^{s}(1 + \alpha(P_t)) \\ &\leq \Pi_{t=1}^{s}(1 + (1 + n_t)^{m_t}).\end{aligned}$$

Each $n_t \leq n$ and $n_{t_0} = n$ for at least one of the indices since $\dim(R) = n + 1$. Hence,

$$\begin{aligned}n + \Pi_{t=1}^{s}(2^{m_t}) &\leq \Pi_{t=1}^{s}(2^{m_t} + n_t) \\ &\leq |O(R)| = \Pi_{t=1}^{s}(1 + \alpha(P_t)) \\ &\leq \Pi_{t=1}^{s}(1 + (1 + n_t)^{m_t}) \\ &\leq \Pi_{t=1}^{s}(1 + (1 + n)^{m_t}) \\ &\leq \Pi_{t=1}^{s}(1 + (1 + n))^{m_t}.\end{aligned}$$

Therefore,

$$n + \Pi_{t=1}^{s}(2^{m_t}) \leq |O(R)| \leq \Pi_{t=1}^{s}(2 + n)^{m_t}.$$

Finally, using the fact that $\sum m_t = m$ as $Spec(R)$ is treed, we obtain,

$$n + 2^m \leq |O(R)| \leq (2+n)^m.$$

This finishes the proof of the claimed result. □

Remark 2 Theorem 1 shows that the Krull dimension or the number of maximal ideals of a Prüfer domain is arbitrarily large, if and only if so is the number of overrings.

Remark 3 The inequalities of Theorem 1 contain several interesting particular cases.

1. If $m = 1$, then all terms of the inequalities are equal to $|O(R)| = 1 + \dim(R)$. This is the case of valuation domains of finite dimension.
2. If $\dim(R) = 1$, then all terms are equal to $|O(R)| = 2^m$. This is the case of semi-local one-dimensional Prüfer domains.

For integral domains that are not necessary integrally closed, we have the following result.

Corollary 4 *Let R be an FO integral domain with finite Krull dimension,* $\dim(R)$*, and finite number m of maximal ideals. Then*

$$|O(R)| \geq \left(2^m + \dim(R) - 1\right).$$

Proof Let R' be the integral closure of R and $m' = |Max(R')|$ the number of maximal ideals of R'. We have,

$$|O(R)| \geq |O(R')| \geq (2^{m'} + \dim(R') - 1) \geq \left(2^m + \dim(R) - 1\right).$$

□

The case where the number of overrings is a prime number is particularly interesting and is presented in the next result and example. We say that T is a maximal subring of R if $T \subset R$ and there is no subring S of R such that $T \subset S \subset R$, i.e. $[T, R] = \{T, R\}$. We will show in the next result that when the number of overring is prime, then $Frac(R)$ has a unique maximal subring.

Proposition 5 *If R is an FO integrally closed domain, such that $|O(R)|$ is a prime number, then R has a unique height* 1 *prime ideal which is comparable to all prime ideals of R, and $Frac(R)$ has a unique maximal subring which is comparable to all overrings of R.*

Proof Since $|O(R)| = \alpha(\{0\}) = \Pi_{P \text{ cov } \{0\}}(1 + \alpha(P))$ is a prime number, then there is a unique term in the product $\Pi_{P \text{ cov } \{0\}}(1 + \alpha(P))$ and $\alpha(\{0\}) = 1 + \alpha(P)$, where P is the only prime containing $\{0\}$. Hence P is contained in all non-zero primes of R. Therefore R_P contains all localizations R_Q for non-zero primes Q of R. Then R_P is contained in $R_0 = Frac(R)$ and contains all other overrings as they are intersections of such localizations R_Q since R is a Prüfer domain. This means that R_P is the unique maximal subring of $Frac(R)$. □

Example 6 Let R be a Prüfer domain whose spectrum is isomorphic to a Y-graph as in Fig. 1 below. Such an integral domain exists by Theorem 3.1 of Lewis (1973). As a concrete example we can define R with the following pullback construction of commutative rings, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ is the field of rationals.

$$\begin{array}{ccc} R \simeq \mathbb{Z}_{2\mathbb{Z}} \cap \mathbb{Z}_{3\mathbb{Z}} + x\mathbb{Q}[x]_{(x)} & \rightarrow & \mathbb{Z}_{2\mathbb{Z}} \cap \mathbb{Z}_{3\mathbb{Z}} \\ \downarrow & & \downarrow \\ \mathbb{Q}[x]_{(x)} & \rightarrow & \mathbb{Q}[x]_{(x)}/x\mathbb{Q}[x]_{(x)} \simeq \mathbb{Q} \end{array}$$

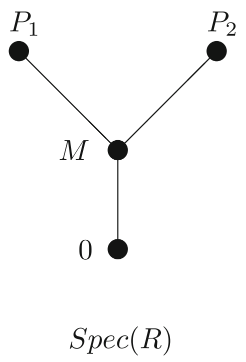
The domain $\mathbb{Z}_{2\mathbb{Z}} \cap \mathbb{Z}_{3\mathbb{Z}}$ is Prüfer as it is an overring of $\mathbb{Z}$. Therefore the domain $\mathbb{Z}_{2\mathbb{Z}} \cap \mathbb{Z}_{3\mathbb{Z}} + x\mathbb{Q}[x]_{(x)}$ is also a Prüfer domain by Theorem 2.1 of Bastida and Gilmer (1973). The spectrum of $\mathbb{Z}_{2\mathbb{Z}} \cap \mathbb{Z}_{3\mathbb{Z}} + x\mathbb{Q}[x]_{(x)}$ is

$$\{\{0\}, M = x\mathbb{Q}[x]_{(x)}, P_1 = M + 2\left(\mathbb{Z}_{2\mathbb{Z}} \cap \mathbb{Z}_{3\mathbb{Z}}\right), P_2 = M + 3\left(\mathbb{Z}_{2\mathbb{Z}} \cap \mathbb{Z}_{3\mathbb{Z}}\right)\}.$$

Using Corollary 2.4 of Jaballah (2005), we obtain that the number of overrings is $\alpha(\{0\}) = 5$. It is clear that the ideal M is the unique height 1 prime ideal of R. The set of overrings is ordered as in Fig. 2 below. Hence the overring R_M is the unique maximal subring of $Frac(R)$.

Given an integer $n \geq 3$, we would like to know whether there exist a domain R such that $|O(R)| = n + \dim R$, see (Mimouni 2009). For the integrally closed case and $n = 2$ we have the following result.

Fig. 1 Spec(R) ordered by inclusion



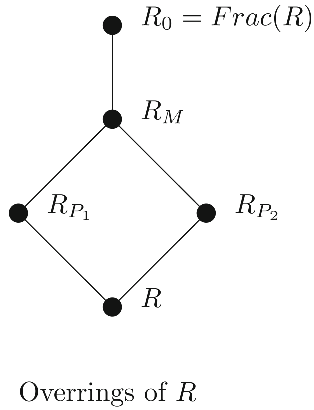


Fig. 2 A unique maximal subring

Remark 7 There is no integrally closed domain R of finite dimension such that $|O(R)| = 2 + \dim(R)$.

Proof We necessarily have $|O(R)| \geq 1 + \dim(R)$. We also know that $|O(R)| = 1 + \dim(R)$ if and only if R is a valuation domain by Theorem 7 of Mimouni and Samman (2003). Now if R has at least two maximal ideals, then let $P_0 = \{0\} \subset P_1 \subset \ldots \subset P_d = M$ be a chain of prime ideals of length $d = \dim(R)$, where M is a maximal ideals of R. Also let N be a second maximal ideal. Then R has at least $d + 3$ overrings, namely $R_{P_0} \supset R_{P_1} \supset \ldots \supset R_{P_d} = R_M$, R_N and $R_N \cap R_M$. Hence $|O(R)| \geq 3 + \dim(R)$. Therefore there is no integrally closed domain R such that $|O(R)| = 2 + \dim(R)$. □

3 Maximal ideals, dimension and overrings

Motivated by Theorem 2.5 of (Mimouni 2009), where it is shown that if R is an integral domain with finite Krull dimension, dim R, n an integer with $n \geq 2$, and $|O(R)| = n + dimR$, then $|Max(R)| \leq n - 1$. We give in this section sharp approximations for the number of maximal ideals.

Theorem 8 *Let R be an integrally closed domain with finite Krull dimension $d \geq 1$, and let n be a positive integer. If $|O(R)| = n + d$, then the following statements hold true:*

1. *The number of maximal ideals is finite and satisfies the inequalities:*

$$\log_{d+1}(n + d) \leq |Max(R)| \leq \log_2(n + 1).$$

2. *R has exactly $m = \log_{d+1}(n+d)$ maximal ideals if and only if $n = (d+1)^m - d$ and R is a Prüfer domain with spectrum isomorphic to a tree consisting of m chains each containing $d + 1$ elements and meeting only at their respective minimal elements.*
3. *R has exactly $m = \log_2(n + 1)$ maximal ideals if and only if $n = 2^m - 1$ and R is a Prüfer domain with spectrum isomorphic to a tree consisting of a chain containing*

d elements and connected at the top element with m additional edges to each of the m maximal ideals.

Proof (1) Let d be the dimension of R, and let m be the number of maximal ideals. Then by Theorem 1 we have,

$$2^m + d - 1 \leq |O(R)| = n + d.$$

Hence, $2^m \leq n + 1$, and $m \leq \log_2(n + 1)$. On the other hand and using the same theorem, we obtain,

$$n + d = |O(R)| \leq (d + 1)^m.$$

Therefore,

$$m \geq \log_{d+1}(n + d).$$

Finally $\log_{d+1}(n + d) \leq m \leq \log_2(n + 1)$ as required.

(2) If the number of maximal ideals is $m = \log_{d+1}(n+d)$, then $n+d = (d+1)^m = |O(R)|$. But $|O(R)| = \alpha(\{0\})$ by Corollary 2.4 of Jaballah (2005). $\alpha(\{0\}) = (d+1)^m$ is only possible for $Spec(R)$ isomorphic to a tree consisting of m chains each containing $d + 1$ elements and meeting only at their respective minimal elements. On the other hand if $Spec(R)$ is isomorphic to such a tree, we have $\alpha(\{0\}) = (d + 1)^m = |O(R)|$ as required.

(3) If the number of maximal ideals is $m = \log_2(n + 1)$, then $n = 2^m - 1$ and $|O(R)| = n + d = 2^m + d - 1$. But $|O(R)| = \alpha(\{0\})$ by Corollary 2.4 of Jaballah (2005). Then $\alpha(\{0\}) = 2^m + d - 1$ is only possible for $Spec(R)$ isomorphic to a tree consisting of a chain containing d elements and connected at the top with m additional edges to each of the m maximal ideals.. On the other hand if $Spec(R)$ isomorphic to such a tree, we have $\alpha(\{0\}) = 2^m + d - 1 = |O(R)|$ as required. □

If $d = 1$ in part (1) of Theorem 8, then the number of maximal ideals is necessarily $|Max(R)| = m = \log_2(n + 1)$. Hence $n = 2^m - 1$ and $|O(R)| = 1 + n = 2^m$. Therefore if $d = 1$, then the only possible values for $|O(R)|$ are powers of 2.

To give examples related to the previous results, recall that for each finite tree with a unique minimal element, there exists a Prüfer domain whose spectrum ordered by the usual set inclusion is isomorphic to the given tree, see Theorem 3.1 of Lewis (1973). We give a concrete example that satisfies part (3) of Theorem 8.

Example 9 Let $p_1, p_2, \ldots, p_m$ be m distinct prime numbers, consider the domain $T = \mathbb{Z}_{p_1\mathbb{Z}} \cap \mathbb{Z}_{p_2\mathbb{Z}} \cap \ldots \cap \mathbb{Z}_{p_m\mathbb{Z}}$ and define R with the following pullback construction of commutative rings:

$$\begin{array}{ccc} R \simeq T + x\mathbb{Q}[x]_{(x)} & \rightarrow & T \\ \downarrow & & \downarrow \\ \mathbb{Q}[x]_{(x)} & \rightarrow & \mathbb{Q}[x]_{(x)}/x\mathbb{Q}[x]_{(x)} \simeq \mathbb{Q} \end{array}$$

The integral domain T is a PID with $Spec(T) = \{\{0\}, p_1T, \ldots, p_mT\}$, which is ordered as in Fig. 3 below. The domain T is Prüfer as it is an overring of $\mathbb{Z}$. By Theorem 2.1 of Bastida and Gilmer (1973), we can see that the integral domain $T + x\mathbb{Q}[x]_{(x)}$ is also a Prüfer domain with a spectrum consisting of the prime ideals:

$$\{0\},\ M = x\mathbb{Q}[x]_{(x)},\ \text{and}\ P_i = M + p_iT,\ 1 \leq i \leq m.$$

The integral domain R is a Prüfer domain with a spectrum obtained by gluing $Spec(T)$ over $Spec(Q[x]_{(x)})$ and is ordered as in Fig. 3. The number of overrings is:

$$|O(R)| = \alpha(\{0\}) = 1 + 2^m.$$

We also have as in part (3) of Theorem 8:

$$\begin{aligned}\log_2(n+1) &= \log_2(|O(R)| - d + 1)\\ &= \log_2(1 + 2^m - 2 + 1)\\ &= m\\ &= |Max(R)|.\end{aligned}$$

The next result shows that the right hand inequality in part (1) of Theorem 8 is still valid for not necessarily integrally closed domains.

Theorem 10 *Let R be an integral domain with finite Krull dimension d and n a positive integer. If $|O(R)| = n + d$, then the following statements hold true:*

1. *The number of maximal ideals is finite and satisfies the inequality:*

$$|Max(R)| \leq \log_2(n+1).$$

2. *R has exactly $m = \log_2(n+1)$ maximal ideals if and only if $n = 2^m - 1$ and R is a Prüfer domain with spectrum isomorphic to a tree consisting of a chain containing d elements and connected at the top with m additional edges to each of the m maximal ideals.*

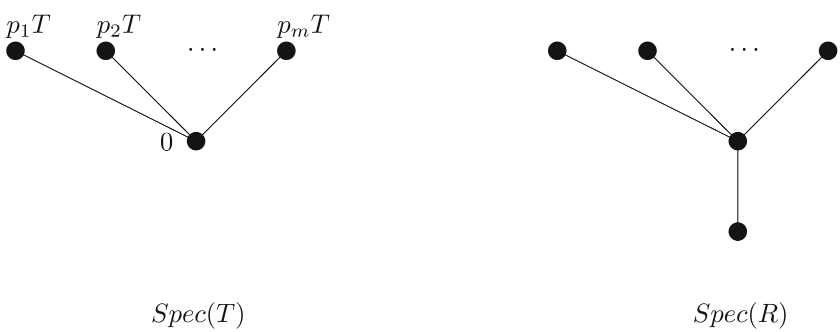


Fig. 3 Maximal ideals and overrings

Proof (1) Let R' be the integral closure of R. Let $d = \dim R$. We have $d = \dim R = \dim R'$ and $|O(R)| = n + d \geq |O(R')|$. Therefore $|O(R')| = n' + d$ for some positive integer $n' \leq n$. Using Theorem 8 we obtain:

$$|Max(R)| \leq |Max(R')| \leq \log_2(n' + 1) \leq \log_2(n + 1).$$

(2) The equality $|Max(R)| = \log_2(n + 1)$ would imply the equality:

$$|Max(R)| = |Max(R')| = \log_2(n' + 1) = \log_2(n + 1).$$

Therefore necessarily $n' = n$ and $|O(R')| = |O(R)|$. Therefore $R' = R$ and R is integrally closed. Therefore the statement follows directly from Theorem 8. □

The following results gives lower and upper bounds for the Krull dimension, when the number of overrings and the number maximal ideals are known.

Corollary 11 *Let R be an integrally closed domain with only finitely many overrings, and let $m = |Max(R)|$. Then the Krull dimension* $\dim R$ *of R satisfies the following inequalities:*

$$-1 + \sqrt[m]{|O(R)|} \leq \dim(R) \leq |O(R)| + 1 - 2^m.$$

Proof Using the left inequality of Theorem 1, we have

$$\left(2^m - 1 + \dim(R)\right) \leq |O(R)|.$$

This trivially implies that

$$\dim(R) \leq |O(R)| + 1 - 2^m.$$

Then the required right hand inequality is proven. Now we use the right hand inequality $|O(R)| \leq (1 + \dim(R))^m$ of the same theorem. We obtain

$$\sqrt[m]{|O(R)|} \leq (1 + \dim(R)).$$

This trivially gives the required left hand inequality:

$$-1 + \sqrt[m]{|O(R)|} \leq \dim(R).$$

This concludes the proof for both inequalities. □

4 Length, dimension and number of maximal ideals

The length $l[R, S]$ of the set $[R, S]$ of intermediate rings is defined to be the supremum of the length of chains of intermediate rings, see (Jaballah 1999b). The length $l[R]$ of the ring R is defined to be $l(O(R))$. We can now give sharp bounds for the length and the number of prime ideals of an integral domain.

Proposition 12 *Let R be an integrally closed domain with only finitely many overrings. Let* $m = |Max(R)|$ *and* $d = \dim(R)$*, then the following inequalities are satisfied:*

$$m + d - 1 \leq l[R] = |Spec(R)| - 1 \leq md.$$

Proof Since $l[R] = |Spec(R)| - 1$ by (Jaballah 1999a, Corollary 3.4), let us prove the equivalent inequalities:

$$m + d \leq |Spec(R)| \leq md + 1.$$

It is clear that $m + d \leq |Spec(R)|$. So we need only to prove that $|Spec(R)| \leq md + 1$. If $\dim(R) = 1$, then the inequality is trivially satisfied and is indeed an equality. Let us assume by induction that the inequality is satisfied for every FO integrally closed domain of dimension $d = \dim(R)$ with $1 \leq d \leq n$, and let R be an FO integrally closed domain of dimension $n + 1$. We want to show that

$$|Spec(R)| \leq m(n + 1) + 1.$$

Let $\{P_t | t = 1, \ldots, s\}$ be the set of prime ideals of R covering $\{0\}$. We have then,

$$|Spec(R)| = 1 + \sum_{P \text{ cov } \{0\}} |Z(P)| = 1 + \sum_{t=1}^{s} |Z(P_t)|.$$

The set $Z(P_t)$ of prime ideals is order isomorphic to $Spec(R/P_t)$, where R/P_t is an FO integrally closed domain of dimension $\leq n$. Set $m_t = |Max(R/P_t)|$. So by induction we obtain,

$$|Z(P_t)| \leq m_t(\dim(R/P_t)) + 1 \leq m_t n + 1.$$

We have $\sum_{t=1}^{s} m_t = m$ as $Spec(R)\backslash\{0\}$ is the disjoint union of the $Z(P_t)$ since $Spec(R)$ is treed. Then,

$$|Spec(R)| - 1 = \sum_{t=1}^{s} |Z(P_t)| \leq \sum_{t=1}^{s} (m_t n + 1).$$

We have $s \leq m$ since $Spec(R)$ is treed. Therefore,

$$|Spec(R)| - 1 \leq \sum_{t=1}^{s} (m_t n + 1)$$

$$\leq \left(\sum_{t=1}^{s} m_t\right) n + s = mn + s$$
$$\leq mn + m.$$

That is,

$$|Spec(R)| \leq mn + m + 1 = m(n+1) + 1.$$

This finishes the proof of the claimed result. □

Using the precedent Proposition 12, we can easily obtain the following approximations for the dimension and the number of maximal ideals in terms of the length.

Corollary 13 *Let R be an integrally closed domain with only finitely many overrings. Let $m = |Max(R)|$ and assume that $d = \dim(R)$ is a positive integer. Then the following inequalities are satisfied:*

1. $\frac{l[R]}{d} \leq m \leq l[R] + 1 - d$.
2. $\frac{l[R]}{m} \leq d \leq l[R] + 1 - m$.

For not necessarily integrally closed integral domains, we have the following inequality.

Remark 14 Let R be an integral domain of finite dimension $d = \dim(R)$, and only finitely many m maximal ideals, then,

$$l[R] \geq m + d - 1.$$

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