



A norm functor for quadratic algebras

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Abstract

Given commutative, unital rings \mathcal{A} and \mathcal{B} with a ring homomorphism $\mathcal{A} \rightarrow \mathcal{B}$ making \mathcal{B} free of finite rank as an \mathcal{A} -module, we can ask for a “trace” or “norm” homomorphism taking algebraic data over \mathcal{B} to algebraic data over \mathcal{A} . In this paper we construct a norm functor for the data of a quadratic algebra: given a locally-free rank-2 \mathcal{B} -algebra \mathcal{D} , we produce a locally-free rank-2 \mathcal{A} -algebra $\text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$ in a way that is compatible with other norm functors and which extends a known construction for étale quadratic algebras. We also conjecture a relationship between discriminant algebras and this new norm functor.

Keywords Norm functor · Quadratic algebra · Ring extension · Algebra of finite rank · Étale algebra

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1 Introduction

Given a homomorphism of commutative, unital rings $\mathcal{A} \rightarrow \mathcal{B}$ that makes \mathcal{B} locally free of finite rank n as an \mathcal{A} -module, there is a zoo of “trace” or “norm” operations taking algebraic data over \mathcal{B} to algebraic data of the same type over \mathcal{A} . For example:

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- Given an element $b \in \mathcal{B}$, we can take its *trace* to get an element of \mathcal{A} : working locally so that \mathcal{B} has an \mathcal{A} -basis, represent multiplication by b as the action of a square matrix over \mathcal{A} , and then take the trace of that matrix. This gives us an \mathcal{A} -linear function $\mathrm{Tr}_{\mathcal{B}/\mathcal{A}} : \mathcal{B} \rightarrow \mathcal{A}$.
- Alternatively, we can take the *norm* of b by taking the determinant of the matrix by which b acts. This gives a multiplicative function $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}} : \mathcal{B} \rightarrow \mathcal{A}$.
- Given a line bundle (a locally free rank-1 module) \mathcal{L} over \mathcal{B} , we can take its “norm” to get $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{L}) = \mathrm{Hom}_{\mathcal{A}}(\wedge^n \mathcal{B}, \wedge^n \mathcal{L})$, a line bundle over \mathcal{A} . We can also apply this norm operations to homomorphisms of line bundles $\mathcal{L} \rightarrow \mathcal{L}'$, and furthermore, if an endomorphism $f : \mathcal{L} \rightarrow \mathcal{L}$ is given by multiplication by $b \in \mathcal{B}$, then the endomorphism $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(f)$ of $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{L})$ is multiplication by the ordinary norm of b in \mathcal{A} .
- More generally, Ferrand (1998) shows how to take an arbitrary \mathcal{B} -module \mathcal{M} and construct its “norm” $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ as an \mathcal{A} -module. In the special case that \mathcal{M} is a line bundle this functor agrees with the above definition of $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$, but in general if \mathcal{M} is locally free it does not follow that $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ is locally free.

In each case, there is a group or monoid operation that the trace or norm preserves. The ordinary trace preserves addition: $\mathrm{Tr}_{\mathcal{B}/\mathcal{A}}(b+b') = \mathrm{Tr}_{\mathcal{B}/\mathcal{A}}(b) + \mathrm{Tr}_{\mathcal{B}/\mathcal{A}}(b')$, while the ordinary norm preserves multiplication $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(bb') = \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(b)\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(b')$. On the other hand, the norm for line bundles and other modules is a functor that preserves tensor products up to isomorphism: $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{M} \otimes_{\mathcal{B}} \mathcal{M}') \cong \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{M}) \otimes_{\mathcal{A}} \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{M}')$.

In case the map of schemes $\pi : \mathrm{Spec}(\mathcal{B}) \rightarrow \mathrm{Spec}(\mathcal{A})$ is étale of degree n (meaning that after a faithfully-flat finite-presentation base change it becomes a trivial degree- n cover of the form $\coprod_{i=1}^n \mathrm{Spec}(\mathcal{A}) \rightarrow \mathrm{Spec}(\mathcal{A})$), then for any sheaf \mathcal{G} of abelian groups on the big étale site over $\mathrm{Spec}(\mathcal{A})$ we have a “trace” homomorphism $\pi_* \pi^* \mathcal{G} \rightarrow \mathcal{G}$, for which taking sections over $\mathrm{Spec}(\mathcal{A})$ gives us a homomorphism $\mathcal{G}(\mathcal{B}) \rightarrow \mathcal{G}(\mathcal{A})$. If \mathcal{G} is the additive group $\mathbb{G}_{a,\mathcal{A}}$, represented by the scheme $\mathrm{Spec}(\mathcal{A}[x])$, then this homomorphism is just the ordinary trace map $\mathcal{B} \rightarrow \mathcal{A}$. If \mathcal{G} is the multiplicative group $\mathbb{G}_{m,\mathcal{A}}$ represented by $\mathrm{Spec}(\mathcal{A}[x, x^{-1}])$, then we get the norm map on units $\mathcal{B}^\times \rightarrow \mathcal{A}^\times$. It is not hard to extend the argument in Milne (2016, Lemma V.1.12) to sheaves of commutative monoids, giving us the full multiplicative norm map $\mathcal{B} \rightarrow \mathcal{A}$.

Meanwhile, for the norm of line bundles, one way to think about what is going on in the case of $\mathcal{A} \rightarrow \mathcal{B}$ étale is that line bundles are torsors for \mathbb{G}_m , and so we are applying $H^1(\mathrm{Spec}(\mathcal{A}), \cdot)$ to the “trace” homomorphism $\pi_* \pi^* \mathbb{G}_{m,\mathcal{A}} \rightarrow \mathbb{G}_{m,\mathcal{A}}$, giving a function

$$\begin{aligned} H^1(\mathrm{Spec}(\mathcal{A}), \pi_* \pi^* \mathbb{G}_{m,\mathcal{A}}) &\rightarrow H^1(\mathrm{Spec}(\mathcal{A}), \mathbb{G}_{m,\mathcal{A}}) \\ \text{i.e. } H^1(\mathrm{Spec}(\mathcal{B}), \mathbb{G}_{m,\mathcal{B}}) &\rightarrow H^1(\mathrm{Spec}(\mathcal{A}), \mathbb{G}_{m,\mathcal{A}}) \end{aligned}$$

sending the isomorphism class of a line bundle over \mathcal{B} to the isomorphism class of its norm as a line bundle over \mathcal{A} . This, however, forgets the functorial nature of the norm operation on line bundles. Meanwhile, Ferrand’s extension of the norm functor to all

\mathcal{B} -modules is defined in terms of a universal property ($\text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ is the universal \mathcal{A} -module equipped with a homogeneous degree- n polynomial law $\mathcal{M} \rightarrow \text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$) and does not readily admit a cohomological interpretation even in the étale case.

However, Waterhouse (1987) uses the cohomological method to define a “trace” operation for étale quadratic algebras, since they are parameterized by S_2 -torsors. Explicitly, if $\mathcal{A} \rightarrow \mathcal{B}$ is a rank- n étale algebra in which 2 is a unit, we can describe this “trace” operation as sending each étale quadratic \mathcal{B} -algebra of the form $\mathcal{B}[\sqrt{d}]$, with $d \in \mathcal{B}^\times$, to the étale quadratic \mathcal{A} -algebra $\mathcal{A}[\sqrt{\text{Nm}_{\mathcal{B}/\mathcal{A}}(d)}]$. It is the goal of the present paper to extend this construction to a norm functor from the category of quadratic \mathcal{B} -algebras to the category of quadratic \mathcal{A} -algebras, regardless of whether any of the algebras are étale or whether 2 is a unit. This norm operation is defined for quadratic algebras with a chosen generator in Definition 3.1, made functorial in Proposition 3.9, and extended to arbitrary quadratic algebras in Definition 4.1.

The main theorems about this norm functor for quadratic algebras are:

1. It commutes with base change (Theorem 4.3).
2. It is “transitive” in the sense that for a tower of algebras $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$, taking the norm of a quadratic \mathcal{C} -algebra to get a quadratic \mathcal{B} -algebra, then taking the norm again to get a quadratic \mathcal{A} -algebra, gives the same result as regarding \mathcal{C} as an \mathcal{A} -algebra and applying the norm operation once (Theorem 4.4).
3. It extends the notion of trace of an S_2 -torsor in the case that the algebras are étale (Theorem 4.8).
4. For the unique extension of S_2 -torsor addition to a monoid operation $*$ on all quadratic algebras characterized by Voight (2016), the norm functor is a monoid homomorphism with respect to $*$ (Theorem 4.5).
5. Taking the norm of a quadratic algebra commutes with taking its determinant line bundle and discriminant bilinear form (Theorems 4.6, 4.7).

However, it is still an open question whether this notion of the norm of a quadratic algebra is compatible with the notion of discriminant algebra from Biesel and Gioia (2016) in the sense of Waterhouse (1987); see Conjecture 4.9.

2 Finite-rank and quadratic algebras

We begin with background and notation for concepts related to rank- n algebras in general and quadratic algebras in particular. In the following, all rings and algebras are commutative and unital, and are denoted by calligraphic capital letters: $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \dots$

Definition 2.1 Let \mathcal{A} and \mathcal{B} be rings and $\mathcal{A} \rightarrow \mathcal{B}$ a ring homomorphism, so that \mathcal{B} is an \mathcal{A} -algebra. We say that \mathcal{B} is a *rank- n \mathcal{A} -algebra* if \mathcal{B} is projective of constant rank n as an \mathcal{A} -module; equivalently, if there exist $a_1, \dots, a_k \in \mathcal{A}$ together generating the unit ideal such that each localization \mathcal{B}_{a_i} is isomorphic to $\mathcal{A}_{a_i}^{\oplus n}$ as \mathcal{A}_{a_i} -modules.

Definition 2.2 A *quadratic \mathcal{A} -algebra* \mathcal{D} is an \mathcal{A} -algebra of rank 2. If \mathcal{D} is equipped with a choice of algebra generator x , so that $\mathcal{D} \cong \mathcal{A}[x]/(x^2 - tx + n)$ for some elements $t, n \in \mathcal{A}$ (the *trace* and *norm* of $x \in \mathcal{D}$), then we call \mathcal{D} a *based quadratic*

algebra. As a based quadratic algebra is determined up to unique isomorphism by the ordered pair (t, n) , we follow Loos (2007) in using the notation

$$((t : n)]_{\mathcal{A}} := \mathcal{A}[x]/(x^2 - tx + n)$$

for based quadratic algebras (the subscript ring may be omitted if it is clear from context).

For example, $((1 : 0)]_{\mathcal{A}} = \mathcal{A}[x]/(x^2 - x) \cong \mathcal{A} \times \mathcal{A}$ with generating element $(1, 0)$, and $((0 : 0)]_{\mathcal{A}} \cong \mathcal{A}[\varepsilon]/(\varepsilon^2)$ with generator ε . Note, not every quadratic algebra admits a singleton generating set, but every quadratic algebra does so locally, so we can reduce claims about general quadratic algebras to claims about based quadratic algebras. This is how we will construct norms of quadratic algebras: first for based quadratic algebras in Sect. 3, then for general quadratic algebras in Sect. 4.

In Definition 2.2, we referred to the trace and norm of an element. We will rarely need the notion of trace of a general element of a rank- n algebra, but we will make frequent use of elements' norms. Because there are so many notions of “norm” we will be using in this paper, we will reserve the notation $\text{Nm}_{\mathcal{B}/\mathcal{A}}$ for norm functors, applied to mathematical objects such as line bundles and other modules, and soon quadratic algebras. To refer to the ordinary norm function for a rank- n algebra $\mathcal{A} \rightarrow \mathcal{B}$, we use the notation s_n as defined below:

Definition 2.3 Let $\mathcal{A} \rightarrow \mathcal{B}$ be a rank- n algebra. Then there is a canonical function $s_n: \mathcal{B} \rightarrow \mathcal{A}$, called the *norm*, defined as follows:

1. If \mathcal{B} has a basis as an \mathcal{A} -module, and if b is an element of \mathcal{B} , then its norm $s_n(b)$ is the determinant of the map $\mathcal{B} \rightarrow \mathcal{B}$ given by multiplication by b (which is independent of choice of basis).
2. If \mathcal{B} is a general rank- n \mathcal{A} -algebra, and if $b \in \mathcal{B}$, then we can define the norm of b in any localization \mathcal{B}_a that is free as an \mathcal{A}_a -module. These values of $s_n(b)$ in each such \mathcal{A}_a glue to give a single well-defined value for $s_n(b) \in \mathcal{A}$.

Remark 2.4 The norm function $s_n: \mathcal{B} \rightarrow \mathcal{A}$ is not generally a ring homomorphism, but rather has the property that if $a \in \mathcal{A}$ is any element of the base ring, then $s_n(ab) = a^n s_n(b)$. Furthermore, the calculation of $s_n(b)$ commutes with base change, making s_n an example of a *polynomial law* in the sense of Roby (1963). For us, the upshot of s_n being a polynomial law is that it automatically comes with polarized versions $s_{k_1, k_2, \dots, k_m}(b_1, b_2, \dots, b_m)$ for each list of natural numbers $k_1 + \dots + k_m = n$, defined by

$$s_{k_1, k_2, \dots, k_m}(b_1, b_2, \dots, b_m) := \frac{\text{the coefficient of } \lambda_1^{k_1} \lambda_2^{k_2} \dots \lambda_m^{k_m} \text{ in}}{s_n(\lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_m b_m)},$$

the latter of which is calculated via the norm map of the base changed rank- n algebra $\mathcal{A}[\lambda_1, \dots, \lambda_m] \rightarrow \mathcal{B}[\lambda_1, \dots, \lambda_m]$. Phrased another way, the s_{k_1, \dots, k_m} are defined so

that we have the following identity:

$$s_n(\lambda_1 b_1 + \dots + \lambda_m b_m) = \sum_{\substack{k_1, \dots, k_m \in \mathbb{N} \\ k_1 + \dots + k_m = n}} \lambda_1^{k_1} \dots \lambda_m^{k_m} s_{k_1, \dots, k_m}(b_1, \dots, b_m).$$

For example, the coefficients of the characteristic polynomial of an element $b \in \mathcal{B}$ can be expressed in terms of the polarized forms of the norm function:

$$\begin{aligned} s_n(\lambda - b) &= s_n(\lambda 1 + \mu b)|_{\mu=-1} \\ &= \sum_{k=0}^n \lambda^k \mu^{n-k} s_{k, n-k}(1, b) \Big|_{\mu=-1} \\ &= \sum_{k=0}^n \lambda^k (-1)^{n-k} s_{k, n-k}(1, b). \end{aligned}$$

In particular, the *trace* of $b \in \mathcal{B}$ can be calculated as the unique element $\text{Tr}(b) \in \mathcal{A}$ such that, in every localization making \mathcal{B} a free \mathcal{A} -module, we have $\text{Tr}(b)/1$ equal to the trace of the matrix representing multiplication by $b/1$, or in terms of the polynomial law s_n it can be expressed as the polarized form $s_{1, n-1}(b, 1)$.

We now prove some of the basic results about the norm functions s_n that we will need later:

Lemma 2.5 *The norm functions are multiplicative: If $\mathcal{A} \rightarrow \mathcal{B}$ is a rank- n algebra and $b, b' \in \mathcal{B}$, then $s_n(bb') = s_n(b)s_n(b')$ in \mathcal{A} .*

Proof We can check locally, so assume that \mathcal{B} has an \mathcal{A} -basis. Then the $n \times n$ matrix by which multiplication by bb' acts, with respect to this basis, is the product of the matrices by which b and b' act. Taking determinants of both sides, and since the determinant is multiplicative, we obtain $s_n(bb') = s_n(b)s_n(b')$. \square

Lemma 2.6 *The norm functions are transitive: Let $\mathcal{A} \rightarrow \mathcal{B}$ be a rank- n algebra and $\mathcal{B} \rightarrow \mathcal{C}$ be a rank- m algebra, so that \mathcal{C} is also a rank- mn algebra. For all $c \in \mathcal{C}$, we have $s_n(s_m(c)) = s_{mn}(c)$ as elements of \mathcal{A} .*

Proof We can work locally on \mathcal{A} , so assume that \mathcal{B} is free as an \mathcal{A} -module. By Grothendieck (1961, Prop. 6.1.12 on p. 113), since $\mathcal{A} \rightarrow \mathcal{B}$ is finite we can also assume that \mathcal{C} is free as a \mathcal{B} -module by localizing \mathcal{A} . So assume that \mathcal{B} is free of rank n as an \mathcal{A} -module and that \mathcal{C} is free of rank m as a \mathcal{B} -module. Choose an \mathcal{A} -basis $\theta_1, \dots, \theta_n$ for \mathcal{B} and a \mathcal{B} -basis ϕ_1, \dots, ϕ_m for \mathcal{C} .

First consider the $m \times m$ matrix $(M_{ij})_{i,j}$ with elements in \mathcal{B} representing multiplication by c with respect to basis ϕ_1, \dots, ϕ_m , i.e.

$$c\phi_i = \sum_{j=1}^m M_{ij}\phi_j.$$

Then for each $M_{ij} \in \mathcal{B}$, consider the $n \times n$ matrix $(P_{ijk\ell})_{k\ell}$ representing multiplication by M_{ij} with respect to basis $\theta_1, \dots, \theta_n$, i.e.

$$M_{ij}\theta_k = \sum_{\ell=1}^n P_{ijk\ell}\theta_\ell.$$

Then the $mn \times mn$ matrix $(P_{ijk\ell})_{(i,k),(j,\ell)}$, with rows and columns indexed by $\{1, \dots, m\} \times \{1, \dots, n\}$, represents multiplication by c with respect to the \mathcal{A} -basis $\phi_1\theta_1, \phi_2\theta_1, \dots, \phi_m\theta_n$ for \mathcal{C} :

$$\begin{aligned} c\phi_i\theta_k &= \sum_{j=1}^m M_{ij}\phi_j\theta_k \\ &= \sum_{j=1}^m \sum_{\ell=1}^n P_{ijk\ell}\phi_j\theta_\ell \\ &= \sum_{(j,\ell) \in \{1,\dots,m\} \times \{1,\dots,n\}} P_{ijk\ell}\phi_j\theta_\ell. \end{aligned}$$

We can regard $(P_{ijk\ell})_{(i,k),(j,\ell)}$ as an $m \times m$ block matrix, where the $n \times n$ blocks that represent multiplication by M_{ij} all commute with each other (since the elements $M_{ij} \in \mathcal{B}$ all commute, as \mathcal{B} is a commutative algebra). Then by Kovacs et al. (1999), the determinant of $(P_{ijk\ell})_{(i,k),(j,\ell)}$ is equal to the determinant of the $n \times n$ matrix given by formally applying the $m \times m$ determinant formula to the blocks of $(P_{ijk\ell})_{(i,k),(j,\ell)}$. In other words,

$$s_{mn}(c) = s_n(\det((M_{ij})_{i,j})) = s_n(s_m(c)).$$

□

Here is a collection of results about the polarized forms of s_n :

Lemma 2.7 *Let $\mathcal{A} \rightarrow \mathcal{B}$ be a rank- n algebra, let $n = k_1 + \dots + k_m$ be a partition of n , and let $b_1, \dots, b_m \in \mathcal{B}$. Then the following identities hold:*

1. (Reordering) *If $\sigma : \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ is any permutation, then*

$$s_{k_1, \dots, k_m}(b_1, \dots, b_m) = s_{k_{\sigma(1)}, \dots, k_{\sigma(m)}}(b_{\sigma(1)}, \dots, b_{\sigma(m)}).$$

2. (Homogeneity) *If $a \in \mathcal{A}$, then*

$$s_{k_1, \dots, k_m}(ab_1, \dots, b_m) = a^{k_1} s_{k_1, \dots, k_m}(b_1, \dots, b_m).$$

3. (Degeneracy) *If $k_1 = 0$, then*

$$s_{k_1, k_2, \dots, k_m}(b_1, b_2, \dots, b_m) = s_{k_2, \dots, k_m}(b_2, \dots, b_m).$$

4. (Multiplicativity) If $b' \in \mathcal{B}$, then

$$s_{k_1, \dots, k_m}(b'b_1, \dots, b'b_m) = s_n(b')s_{k_1, \dots, k_m}(b_1, \dots, b_m).$$

Proof 1. (Reordering) This follows from comparing the coefficients of $\lambda_1^{k_1} \dots \lambda_m^{k_m} = \lambda_{\sigma(1)}^{k_{\sigma(1)}} \dots \lambda_{\sigma(m)}^{k_{\sigma(m)}}$ in $s_n(\lambda_1 b_1 + \dots + \lambda_m b_m) = s_n(\lambda_{\sigma(1)} b_{\sigma(1)} + \dots + \lambda_{\sigma(m)} b_{\sigma(m)})$.

2. (Homogeneity) The quantity $s_{k_1, \dots, k_m}(ab_1, \dots, b_m)$ is the coefficient of $\lambda_1^{k_1} \dots \lambda_m^{k_m}$ in

$$s_n(a\lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_m b_m) = \sum_{\substack{\ell_1, \dots, \ell_m \in \mathbb{N} \\ \ell_1 + \dots + \ell_m = n}} (a\lambda_1)^{\ell_1} \lambda_2^{\ell_2} \dots \lambda_m^{\ell_m} s_{\ell_1, \dots, \ell_m}(b_1, \dots, b_m),$$

so the coefficient of $\lambda_1^{k_1} \lambda_2^{k_2} \dots \lambda_m^{k_m}$ is $a^{k_1} s_{k_1, \dots, k_m}(b_1, \dots, b_m)$, as desired.

3. (Degeneracy) If $k_1 = 0$, then we are looking for the coefficient of $\lambda_2^{k_2} \dots \lambda_m^{k_m}$ in $s_n(\lambda_1 b_1 + \dots + \lambda_m b_m)$, which since it does not involve λ_1 is unchanged if we base change to give λ_1 a concrete value. In particular, we can base change along $\mathcal{A}[\lambda_1, \dots, \lambda_m] \rightarrow \mathcal{A}[\lambda_2, \dots, \lambda_m] : \lambda_1 \mapsto 0$, giving us the coefficient of $\lambda_2^{k_2} \dots \lambda_m^{k_m}$ in $s_n(\lambda_2 b_2 + \dots + \lambda_m b_m)$, namely, $s_{k_2, \dots, k_m}(b_2, \dots, b_m)$.

4. (Multiplicativity) We have that $s_{k_1, \dots, k_m}(b'b_1, \dots, b'b_m)$ is the coefficient of $\lambda_1^{k_1} \dots \lambda_m^{k_m}$ in

$$s_n(\lambda_1 b' b_1 + \dots + \lambda_m b' b_m) = s_n(b')s_n(\lambda_1 b_1 + \dots + \lambda_m b_m)$$

by Lemma 2.5. The coefficient of $\lambda_1^{k_1} \dots \lambda_m^{k_m}$ is therefore $s_n(b')s_{k_1, \dots, k_m}(b_1, \dots, b_m)$. □

Now we consider the various senses in which rank- n algebras over \mathcal{A} form not just a set of isomorphism classes, but a category. In general only considering isomorphisms of \mathcal{A} -algebras is too restrictive, but arbitrary homomorphisms are too ill-behaved; the right notion for us is that of a *universally norm-preserving* homomorphism:

Definition 2.8 Let \mathcal{A} be a ring and $\mathcal{B}, \mathcal{B}'$ two \mathcal{A} -algebras of rank n . An \mathcal{A} -algebra homomorphism $\mathcal{B} \rightarrow \mathcal{B}'$ is called *norm-preserving* if, together with the norm functions s_n , the resulting triangle of functions

$$\begin{array}{ccc} \mathcal{B} & \rightarrow & \mathcal{B}' \\ & \searrow & \swarrow \\ & \mathcal{A} & \end{array}$$

commutes. We say that the homomorphism is *universally norm-preserving* if it is norm-preserving and remains so after arbitrary base change.

Example 2.9 If \mathcal{A} is a nonzero ring, then the \mathcal{A} -algebra homomorphism $\mathcal{A}^2 \rightarrow \mathcal{A}^2$ sending (a, a') to (a, a) is not norm-preserving, since $(1, 0)$ has norm 0 and $(1, 1)$ has norm 1. Isomorphisms are always universally norm-preserving, but the converse is not

always true: for example, the inclusion of quadratic \mathbb{Z} -algebras $\mathbb{Z}[\sqrt{8}] \hookrightarrow \mathbb{Z}[\sqrt{2}]$ is universally norm-preserving but not an isomorphism.

Lemma 2.10 *Let \mathcal{A} be a ring and $((t : n)]_{\mathcal{A}}$ and $((t' : n')]_{\mathcal{A}}$ be two based quadratic algebras. For a homomorphism $f : ((t' : n')]_{\mathcal{A}} \rightarrow ((t : n)]_{\mathcal{A}}$ sending $x \mapsto ux + c$, the following are equivalent:*

1. f is universally norm-preserving,
2. f is norm-preserving,
3. $t' = ut + 2c$ and $n' = u^2n + uct + c^2$.

Proof (1) \implies (2) is trivial. For (2) \implies (3), suppose that f is norm-preserving. Then the norms of $x \in ((t' : n']$ and $f(x) = ux + c \in ((t : n)]_{\mathcal{A}}$ are equal; namely

$$\begin{aligned} n' &= s_2(ux + c) \\ &= s_2(ux) + s_{1,1}(ux, c) + s_2(c) \\ &= u^2s_2(x) + ucs_{1,1}(x, 1) + c^2s_2(1) \\ &= u^2n + uct + c^2. \end{aligned}$$

Now note that in any quadratic algebra, the trace of any element d is $s_{1,1}(d, 1) = s_2(d + 1) - s_2(d) - 1$, so norm-preserving maps are also trace-preserving. Therefore the traces of $x \in ((t' : n']$ and $f(x) = ux + c \in ((t : n)]_{\mathcal{A}}$ are also equal:

$$\begin{aligned} t' &= s_{1,1}(ux + c, 1) \\ &= us_{1,1}(x, 1) + cs_{1,1}(1, 1) \\ &= ut + 2c. \end{aligned}$$

Finally, for (3) \implies (1) suppose that $t' = ut + 2c$ and $n' = u^2n + uct + c^2$, so that $p_{ux+c}(\lambda) = \lambda^2 - t'\lambda + n'$ is the characteristic polynomial of $ux + c \in ((t : n]]$. But for any element b of any rank- n \mathcal{A} -algebra \mathcal{B} , the homomorphism $\mathcal{A}[x]/(p_b(x)) \rightarrow \mathcal{B}$ sending x to b is universally norm-preserving by Biesel and Gioia (2016, Prop. 7.1). \square

Remark 2.11 To specify a universally norm-preserving homomorphism $((t' : n')] \rightarrow ((t : n]]$, it suffices to find an element $ux + c \in ((t : n]]$ with trace $ut + 2c = t'$ and norm $u^2n + uct + c^2 = n'$. Then the characteristic polynomial of $ux + c$ is $p_{ux+c}(\lambda) = \lambda^2 - t'\lambda + n'$, so because $p_{ux+c}(ux + c) = 0$ we get an algebra homomorphism $((t' : n')] = \mathcal{A}[x]/(x^2 - t'x + n') \rightarrow ((t : n]]$ sending $x \mapsto ux + c$, which is automatically universally norm-preserving by Lemma 2.10. For the homomorphism to be an isomorphism, it is equivalent to require that u be a unit in \mathcal{A} .

3 Norms of based quadratic algebras

We will now be considering rank- n algebras $\mathcal{A} \rightarrow \mathcal{B}$ and the relationships between quadratic algebras over \mathcal{A} and over \mathcal{B} . As a general convention for ease of context, we

will tend to use lowercase letters like s, t, m, u, c, \dots for elements of \mathcal{A} , and capital letters like S, T, M, U, C, \dots for elements of \mathcal{B} .

As a bit of motivation for the definition of the norm of a quadratic algebra, suppose we have a rank- n algebra $\mathcal{A} \rightarrow \mathcal{B}$ and a based quadratic \mathcal{B} -algebra $((T : N)]_{\mathcal{B}}$. We want the discriminant of the norm of $((T : N)]$ to be the norm of the discriminant of $((T : N)]_{\mathcal{B}}$, namely $s_n(T^2 - 4N)$. So if we have $\text{Nm}_{\mathcal{B}/\mathcal{A}}((T : N)]_{\mathcal{B}} = ((t : m)]_{\mathcal{A}}$, we had better have the identity

$$t^2 - 4m = s_n(T^2 - 4N).$$

Is there a canonical way to write $s_n(T^2 - 4N)$ as a square minus a multiple of four? Yes: we can use the polarized forms of s_n to expand out $s_n(T^2 - 4N)$:

$$\begin{aligned} s_n(-4N + T^2) &= \sum_{k=0}^n s_{k,n-k}(-4N, T^2) \\ &= \sum_{k=0}^n (-4)^k s_{k,n-k}(N, T^2) \\ &= s_n(T^2) + \sum_{k=1}^n (-4)^k s_{k,n-k}(N, T^2), \end{aligned}$$

since $s_{0,n}(N, T^2) = s_n(T^2)$ by Lemma 2.7 (degeneracy),

$$= [s_n(T)]^2 - 4 \left[\sum_{k=1}^n (-4)^{k-1} s_{k,n-k}(N, T^2) \right],$$

where we have used Lemma 2.5 to rewrite $s_n(T^2)$ as $s_n(T)^2$. If we then let $t = s_n(T)$ and $m = \sum_{k=1}^n (-4)^{k-1} s_{k,n-k}(N, T^2)$, we have $t^2 - 4m = s_n(T^2 - 4N)$ as desired, and $((t : m)]_{\mathcal{A}}$ is a good candidate for the norm of $((T : N)]_{\mathcal{B}}$. That is indeed the definition we will use for the norm of a based quadratic algebra:

Definition 3.1 Let \mathcal{B} be a rank- n \mathcal{A} -algebra, and let $((T : N)]_{\mathcal{B}}$ be a based quadratic \mathcal{B} -algebra. Then we define the *norm* of $((T : N)]_{\mathcal{B}}$ to be the based quadratic \mathcal{A} -algebra

$$\text{Nm}_{\mathcal{B}/\mathcal{A}}((T : N)]_{\mathcal{B}} := \left(\left(s_n(T) : \sum_{k=1}^n (-4)^{k-1} s_{k,n-k}(N, T^2) \right) \right]_{\mathcal{A}}.$$

Remark 3.2 Noting that over the polynomial ring $\mathcal{A}[\lambda]$ we have

$$s_n(\lambda N + T^2) = \sum_{k=0}^n \lambda^k s_{k,n-k}(N, T^2),$$

and therefore

$$\sum_{k=1}^n \lambda^{k-1} s_{k,n-k}(N, T^2) = \frac{s_n(\lambda N + T^2) - s_n(T^2)}{\lambda},$$

we will sometimes write the norm of $((T : N)]_{\mathcal{B}}$ as

$$\text{Nm}_{\mathcal{B}/\mathcal{A}}((T : N)]_{\mathcal{B}} = \left(\left(s_n(T) : \frac{s_n(\lambda N + T^2) - s_n(T^2)}{\lambda} \right) \Big|_{\lambda=-4} \right) \Big|_{\mathcal{A}}$$

to denote that the norm entry can be calculated as that fraction in $\mathcal{A}[\lambda]$ and then evaluated in \mathcal{A} along the map $\mathcal{A}[\lambda] \rightarrow \mathcal{A} : \lambda \mapsto -4$. We will often work with this kind of expression, since it makes some identities clearer, but there is also a danger: it is tempting, in expressions of the form $\frac{p(c,\lambda)}{\lambda} \Big|_{\lambda=c}$ where the numerator is some polynomial in λ that involves c in some of its coefficients, to think “Since we are setting $\lambda = c$ at the end anyway, we might as well replace the c with λ to get $\frac{p(\lambda,\lambda)}{\lambda} \Big|_{\lambda=c}$.” However, if c is a zerodivisor these quantities may not be equal; for example, working over $\mathbb{Z}/6\mathbb{Z}$ we have

$$\frac{(\lambda + 2)(\lambda + 3)}{\lambda} \Big|_{\lambda=2} = (\lambda + 5)|_{\lambda=2} = 1,$$

but

$$\frac{(\lambda + \lambda)(\lambda + 3)}{\lambda} \Big|_{\lambda=2} = 2(\lambda + 3)|_{\lambda=2} = 4.$$

However, if upon replacing c with another formal variable ν , the polynomial $p(\nu, \lambda)$ is still divisible by λ , say as $\frac{p(\nu,\lambda)}{\lambda} = q(\nu, \lambda)$, then we have

$$\frac{p(c, \lambda)}{\lambda} \Big|_{\lambda=c} = q(c, c) = \frac{p(\lambda, \lambda)}{\lambda} \Big|_{\lambda=c}.$$

We will therefore only use the technique of replacing extra c 's with λ 's when the divisibility by λ remains even if c is replaced by a formal indeterminate.

Example 3.3 Let \mathcal{A} be a ring in which 2 is a unit, such as a number field. Then by completing the square, all based quadratic algebras over \mathcal{A} can be written in the form $\mathcal{A}[\sqrt{d}] = \mathcal{A}[x]/(x^2 - d) = ((0 : -d)]_{\mathcal{A}}$. What are the norms of based quadratic algebras of this type? If we have a rank- n algebra $\mathcal{A} \rightarrow \mathcal{B}$ (such as a degree- n extension of number fields), and a quadratic \mathcal{B} -algebra $\mathcal{B}[\sqrt{D}]$, then its norm is the quadratic \mathcal{A} -algebra

$$\text{Nm}_{\mathcal{B}/\mathcal{A}}((0 : -D)]_{\mathcal{B}} = \left(\left(s_n(0) : \frac{s_n(\lambda(-D) + (0)^2) - s_n((0)^2)}{\lambda} \right) \Big|_{\lambda=-4} \right) \Big|_{\mathcal{A}}$$

$$\begin{aligned}
 &= \left(\left(0 : \frac{s_n(-\lambda D)}{\lambda} \Big|_{\lambda=-4} \right) \right]_{\mathcal{A}} \\
 &= \left(\left(0 : (-1)^n \lambda^{n-1} s_n(D) \Big|_{\lambda=-4} \right) \right]_{\mathcal{A}} \\
 &= ((0 : -4^{n-1} s_n(D))]_{\mathcal{A}} \\
 &= \mathcal{A}[\sqrt{4^{n-1} s_n(D)}] = \mathcal{A}[2^{n-1} \sqrt{s_n(D)}] \\
 &\cong \mathcal{A}[\sqrt{s_n(D)}]
 \end{aligned}$$

since 2 is a unit in \mathcal{A} . So the norm of $\mathcal{B}[\sqrt{D}]$ is $\mathcal{A}[\sqrt{s_n(D)}]$.

Example 3.4 At the other extreme, suppose that we have a rank- n algebra $\mathcal{A} \rightarrow \mathcal{B}$ in which $2 = 0$, and let \mathcal{D} be a quadratic Artin-Schreier extension of \mathcal{B} :

$$\mathcal{D} = \mathcal{B}[x]/(x^2 - x + N) = ((1 : N)]_{\mathcal{B}}.$$

Then the norm of \mathcal{D} is the quadratic \mathcal{A} -algebra

$$\begin{aligned}
 \text{Nm}_{\mathcal{B}/\mathcal{A}}((1 : N)]_{\mathcal{B}} &= \left(\left(s_n(1) : \sum_{k=1}^n (-4)^{k-1} s_{k,n-k}(N, 1^2) \right) \right]_{\mathcal{A}} \\
 &= ((1 : s_{1,n-1}(N, 1))]_{\mathcal{A}} \text{ since } (-4)^{k-1} = 0 \text{ for } k > 1 \\
 &= ((1 : \text{Tr}(N))]_{\mathcal{A}}.
 \end{aligned}$$

Therefore the norm of $\mathcal{B}[x]/(x^2 - x + N)$ is $\mathcal{A}[x]/(x^2 - x + \text{Tr}(N))$.

Next we have a few examples that begin to show how this norm operation is related to the monoid operation $*$ on quadratic algebras in Voight (2016), which is defined locally for based algebras by

$$\begin{aligned}
 \mathcal{A}[x]/(x^2 - sx + m) * \mathcal{A}[x]/(x^2 - tx + n) &:= \\
 \mathcal{A}[x]/(x^2 - (st)x + (mt^2 + ns^2 - 4mn)), & \tag{1}
 \end{aligned}$$

i.e. $((s : m)] * ((t : n)] := ((st : mt^2 + ns^2 - 4mn)]$. This monoid operation is (up to isomorphism) commutative and associative, with a two-sided identity given by the split algebra $((1 : 0)]_{\mathcal{A}} \cong \mathcal{A} \times \mathcal{A}$ and a zero (absorbing) element given by the degenerate algebra $((0 : 0)]_{\mathcal{A}} \cong \mathcal{A}[\varepsilon]/(\varepsilon^2)$.

Example 3.5 If $\mathcal{A} \rightarrow \mathcal{B}$ is an arbitrary rank- n algebra, then we have $\text{Nm}_{\mathcal{B}/\mathcal{A}}((1 : 0)]_{\mathcal{B}} = ((1 : 0)]_{\mathcal{A}}$ and $\text{Nm}_{\mathcal{B}/\mathcal{A}}((0 : 0)]_{\mathcal{B}} = ((0 : 0)]_{\mathcal{A}}$. In other words, the norm of the split quadratic algebra $\mathcal{B} \times \mathcal{B}$ is again the split algebra $\mathcal{A} \times \mathcal{A}$, and the norm of the degenerate quadratic algebra $\mathcal{B}[\varepsilon]/(\varepsilon^2)$ is the degenerate algebra $\mathcal{A}[\varepsilon]/(\varepsilon^2)$. This is part of what it means for the norm functor on quadratic algebras to be a homomorphism of monoids-with-zero under $*$.

We will prove in Theorem 4.5 that the norm functor is truly a monoid homomorphism, and there we will use the following example showing how the monoid operation $*$ is actually a special case of the norm:

Example 3.6 Given two based quadratic \mathcal{A} -algebras $((S : M]_{\mathcal{A}}$ and $((T : N]_{\mathcal{A}}$, we can consider their ordinary cartesian product as a based quadratic $\mathcal{A} \times \mathcal{A}$ -algebra

$$((S : M]_{\mathcal{A}} \times ((T : N]_{\mathcal{A}} \cong (((S, T) : (M, N)]_{\mathcal{A} \times \mathcal{A}}.$$

Since $\mathcal{A} \rightarrow \mathcal{A} \times \mathcal{A}$ is an algebra of rank $n = 2$, we can take the norm of this quadratic $\mathcal{A} \times \mathcal{A}$ algebra to get a quadratic \mathcal{A} -algebra:

$$\text{Nm}(((S, T) : (M, N)]) = ((s_2((S, T)) : s_{1,1}((M, N), (S, T)^2) - 4s_2((M, N))])$$

To evaluate these entries we look at the matrices by which (S, T) and (M, N) act on $\mathcal{A} \times \mathcal{A}$ with respect to the standard basis $\{(1, 0), (0, 1)\}$. They act by diagonal matrices:

$$(S, T): \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \quad (M, N): \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

so we have $s_2((S, T)) = \det \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} = ST$ and $s_2((M, N)) = \det \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} = MN$.

We also need $s_{1,1}((M, N), (S, T)^2)$, which is the coefficient of $\lambda\mu$ in $s_2(\lambda(M, N) + \mu(S, T)^2)$, i.e. the coefficient of $\lambda\mu$ in

$$\begin{aligned} \det \begin{pmatrix} \lambda M + \mu S^2 & 0 \\ 0 & \lambda N + \mu T^2 \end{pmatrix} &= (\lambda M + \mu S^2)(\lambda N + \mu T^2) \\ &= \lambda^2 MN + \lambda\mu(MT^2 + NS^2) + \mu^2 S^2 T^2, \end{aligned}$$

so $s_{1,1}((M, N), (S, T)^2) = MT^2 + NS^2$. Therefore the norm of $((S : M] \times ((T : N]$ is

$$\begin{aligned} \text{Nm}(((S, T) : (M, N)]) &= ((s_2((S, T)) : s_{1,1}((M, N), (S, T)^2) - 4s_2((M, N))]) \\ &= ((ST : MT^2 + NS^2 - 4MN)], \end{aligned}$$

which is the composite quadratic algebra $((S : M] * ((T : N]$ defined by the monoid operation $*$ in Eq. (1). So the norm of a product of two quadratic algebras is their composite under the monoid operation $*$. Similarly, the norm of the quadratic $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$ -algebra $((S : M] \times ((T : N] \times ((U : P]$ would be the quadratic \mathcal{A} -algebra $((S : M] * ((T : N] * ((U : P]$, and so on.

Next we show that the norm operation is invariant under base change and is transitive along towers of algebras:

Theorem 3.7 *The norm operation on based quadratic algebras commutes with base change: Let $\mathcal{A} \rightarrow \mathcal{B}$ be a rank- n algebra, and let $((T : N])_{\mathcal{B}}$ be a based quadratic \mathcal{B} -algebra. If \mathcal{C} is any \mathcal{A} -algebra, then $\mathcal{C} \rightarrow \mathcal{C} \otimes_{\mathcal{A}} \mathcal{B}$ is another rank- n algebra, and*

$$\text{Nm}_{(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{B})/\mathcal{C}}((T : N]) \cong \mathcal{C} \otimes_{\mathcal{A}} \text{Nm}_{\mathcal{B}/\mathcal{A}}((T : N]).$$

Proof Since s_n is a polynomial law, it and its polarized forms commute with base change, so the two based quadratic \mathcal{C} -algebras agree exactly. \square

Theorem 3.8 *The norm operation on based quadratic algebras is transitive: If $\mathcal{A} \rightarrow \mathcal{B}$ is a rank- n algebra and $\mathcal{B} \rightarrow \mathcal{C}$ is a rank- m algebra, and if $((T : N])_{\mathcal{C}}$ is a based quadratic \mathcal{C} -algebra, then $\text{Nm}_{\mathcal{C}/\mathcal{A}}((T : N])_{\mathcal{C}} \cong \text{Nm}_{\mathcal{B}/\mathcal{A}}(\text{Nm}_{\mathcal{C}/\mathcal{B}}((T : N])_{\mathcal{C}})$ as based quadratic \mathcal{A} -algebras.*

Proof We check that the traces and norms of the canonical generators of $\text{Nm}_{\mathcal{C}/\mathcal{A}}((T : N])$ and $\text{Nm}_{\mathcal{B}/\mathcal{A}}\text{Nm}_{\mathcal{C}/\mathcal{B}}((T : N])$ agree. First, the traces of $x \in \text{Nm}_{\mathcal{B}/\mathcal{A}}\text{Nm}_{\mathcal{C}/\mathcal{B}}((T : N])$ and $x \in \text{Nm}_{\mathcal{C}/\mathcal{A}}((T : N])$ are $s_n(s_m(T)) = s_{mn}(T)$. Second, the norm of $x \in \text{Nm}_{\mathcal{B}/\mathcal{A}}\text{Nm}_{\mathcal{C}/\mathcal{B}}((T : N])$ is

$$\left. \frac{s_n \left(\mu \left[\frac{s_m(\lambda N + T^2) - s_m(T^2)}{\lambda} \right]_{\lambda=-4} + s_m(T)^2 \right) - s_n(s_m(T)^2)}{\mu} \right|_{\mu=-4},$$

in which, since we are setting μ equal to -4 anyway, we might as well set λ equal to μ instead of -4 :

$$\begin{aligned} &= \left. \frac{s_n \left(\mu \left[\frac{s_m(\lambda N + T^2) - s_m(T^2)}{\lambda} \right]_{\lambda=\mu} + s_m(T)^2 \right) - s_n(s_m(T)^2)}{\mu} \right|_{\mu=-4} \\ &= \left. \frac{s_n \left(\mu \left[\frac{s_m(\mu N + T^2) - s_m(T^2)}{\mu} \right] + s_m(T)^2 \right) - s_n(s_m(T)^2)}{\mu} \right|_{\mu=-4} \\ &= \left. \frac{s_n \left([s_m(\mu N + T^2) - s_m(T^2)] + s_m(T)^2 \right) - s_n(s_m(T)^2)}{\mu} \right|_{\mu=-4} \\ &= \left. \frac{s_{mn}(\mu N + T^2) - s_{mn}(T^2)}{\mu} \right|_{\mu=-4}, \end{aligned}$$

which is the norm of x in $\text{Nm}_{\mathcal{C}/\mathcal{A}}((T : N])$, as desired. \square

Theorem 3.8 may be understood as a kind of functoriality of the norm map with respect to the underlying algebras. In order to extend the norm operation to all quadratic algebras, we will also need a functoriality with respect to isomorphisms of based

quadratic algebras. In Proposition 3.9 we define how the norm map acts on general norm-preserving homomorphisms of quadratic algebras, and in Proposition 3.11 we prove that this norm map preserves composition of homomorphisms. The combinatorial argument at the heart of Proposition 3.9 is the key result that makes this norm functor well-defined.

Proposition 3.9 *Let \mathcal{B} be a rank- n \mathcal{A} -algebra, and let $((T : N])$ and $((T' : N'])$ be based quadratic \mathcal{B} -algebras. If we have an isomorphism (resp. norm-preserving homomorphism) of \mathcal{B} -algebras*

$$f: ((T' : N']) \rightarrow ((T : N]) \\ : x \mapsto Ux + C,$$

then we also have an isomorphism (resp. norm-preserving homomorphism)

$$\text{Nm}_{\mathcal{B}/\mathcal{A}}(f): \text{Nm}_{\mathcal{B}/\mathcal{A}}((T' : N']) \rightarrow \text{Nm}_{\mathcal{B}/\mathcal{A}}((T : N]) \\ : x \mapsto s_n(U)x + \sum_{k=1}^n 2^{k-1} s_{k,n-k}(C, UT). \quad (2)$$

We will also write the constant term above as $\frac{s_n(\lambda C + UT) - s_n(UT)}{\lambda} \Big|_{\lambda=2}$, to indicate that it is the image under the map $\mathcal{A}[\lambda] \rightarrow \mathcal{A} : \lambda \mapsto 2$ of the element $(s_n(\lambda C + UT) - s_n(UT))/\lambda$.

Proof We prove the more general claim in the case that f is a norm-preserving homomorphism, noting that if f is an isomorphism, then U is a unit in \mathcal{B} , so $s_n(U)$ is a unit in \mathcal{A} and $\text{Nm}_{\mathcal{B}/\mathcal{A}}(f)$ is an isomorphism too.

First let $\tilde{T} = UT$ and $\tilde{N} = U^2N$. We can then decompose the norm-preserving map $((T' : N']) \rightarrow ((T : N])$ into two maps $((T' : N']) \rightarrow ((\tilde{T} : \tilde{N}])$ sending $x \mapsto x + C$ and $((\tilde{T} : \tilde{N}]) \rightarrow ((T : N])$ sending $x \mapsto Ux$. We will show immediately after this proposition that the norm operation on norm-preserving homomorphisms is functorial; it therefore suffices to show that Eq. (2) provides a norm-preserving homomorphism between the norm algebras in the cases that $C = 0$ or $U = 1$.

First consider the case $C = 0$, so that we are considering the map $((UT : U^2N]) \rightarrow ((T : N])$ sending $x \mapsto Ux$. We must show that the map $\text{Nm}((UT : U^2N]) \rightarrow \text{Nm}((T : N])$ sending $x \mapsto s_n(U)x$ is also norm-preserving. Indeed, the trace of $x \in \text{Nm}((UT : U^2N])$ is $s_n(UT) = s_n(U)s_n(T)$, which is also the trace of $s_n(U)x \in \text{Nm}((T : N])$. And their norms match as well: the norm of $x \in \text{Nm}((UT : U^2N])$ is

$$\sum_{k=1}^n (-4)^{k-1} s_{k,n-k}(U^2N, (UT)^2) = \sum_{k=1}^n (-4)^{k-1} s_{k,n-k}(U^2N, U^2T^2) \\ = \sum_{k=1}^n (-4)^{k-1} s_n(U^2) s_{k,n-k}(N, T^2)$$

$$= s_n(U)^2 \sum_{k=1}^n (-4)^{k-1} s_{k,n-k}(N, T^2),$$

which is the norm of $s_n(U)x \in \text{Nm}((T : N])$.

Now we consider the case $U = 1$ but $C \neq 0$, so that we are considering the norm-preserving map $((T + 2C : N + CT + C^2]) \rightarrow ((T : N])$ sending $x \mapsto x + C$, and we wish to show that the map $\text{Nm}((T + 2C : N + CT + C^2]) \rightarrow \text{Nm}((T : N])$ sending $x \mapsto x + \sum_{k=1}^n 2^{k-1} s_{k,n-k}(C, T)$ is also norm-preserving. The trace of x in $\text{Nm}((T + 2C : N + CT + C^2])$ is

$$\begin{aligned} s_n(T + 2C) &= \sum_{k=0}^n s_{k,n-k}(2C, T) \\ &= \sum_{k=0}^n 2^k s_{k,n-k}(C, T) \text{ by homogeneity} \\ &= s_n(T) + 2 \sum_{k=1}^n 2^{k-1} s_{k,n-k}(C, T) \end{aligned}$$

which equals the trace of $x + \sum_{k=1}^n 2^{k-1} s_{k,n-k}(C, T)$ in $\text{Nm}((T : N])$ as desired.

Comparing norms will be more difficult; let us start by considering the norm of x in $\text{Nm}((T' : N']) = \text{Nm}((T + 2C : N + CT + C^2])$, which is

$$\begin{aligned} &\left. \frac{s_n(\mu(N + CT + C^2) + (T + 2C)^2) - s_n((T + 2C)^2)}{\mu} \right|_{\mu=-4} \\ &= \left. \frac{s_n(\mu N + \mu CT + \mu C^2 + T^2 + 4CT + 4C^2) - s_n(T^2 + 4CT + 4C^2)}{\mu} \right|_{\mu=-4} \end{aligned}$$

If we replace each instance of 4 in the numerator by a new variable ν , then the result is still divisible by μ and so we may set $\nu = -\mu$ without changing the final value of the quotient:

$$\begin{aligned} &= \left. \frac{s_n(\mu N + \mu CT + \mu C^2 + T^2 - \mu CT - \mu C^2) - s_n(T^2 - \mu CT - \mu C^2)}{\mu} \right|_{\mu=-4} \\ &= \left. \frac{s_n(\mu N + T^2) - s_n(T^2 - \mu CT - \mu C^2)}{\mu} \right|_{\mu=-4}. \end{aligned}$$

It will be convenient to evaluate the result instead by first replacing μ with $-\lambda^2$ and then setting $\lambda = 2$, so all together the norm of x in $\text{Nm}((T' : N'])$ is

$$\left. \frac{s_n(T^2 + \lambda^2 CT + \lambda^2 C^2) - s_n(T^2 - \lambda^2 N)}{\lambda^2} \right|_{\lambda=2}.$$

Now let us consider the norm of $x + \frac{s_n(\lambda C + T) - s_n(T)}{\lambda} \Big|_{\lambda=2}$ in $\text{Nm}((T : N]) = \left(\left(s_n(T) : \frac{s_n(\mu N + T^2) - s_n(T^2)}{\mu} \Big|_{\mu=-4} \right) \right)$, which is

$$\frac{s_n(\mu N + T^2) - s_n(T^2)}{\mu} \Big|_{\mu=-4} + s_n(T) \frac{s_n(\lambda C + T) - s_n(T)}{\lambda} \Big|_{\lambda=2} + \left(\frac{s_n(\lambda C + T) - s_n(T)}{\lambda} \Big|_{\lambda=2} \right)^2$$

which we can expand as a single expression in λ as

$$= \left(\begin{array}{c} s_n(T^2) - s_n(T^2 - \lambda^2 N) \\ + \lambda s_n(\lambda C T + T^2) - \lambda s_n(T^2) \\ + s_n(\lambda^2 C^2 + 2\lambda C T + T^2) - 2s_n(\lambda C T + T^2) + s_n(T^2) \end{array} \right) / \lambda^2 \Big|_{\lambda=2}.$$

The difference between the norm of x in $\text{Nm}((T' : N'])$ and the norm of its image in $\text{Nm}((T : N])$ is therefore

$$\left(\begin{array}{c} s_n(T^2) - s_n(T^2 + \lambda^2 C T + \lambda^2 C^2) \\ + \lambda s_n(\lambda C T + T^2) - \lambda s_n(T^2) \\ + s_n(\lambda^2 C^2 + 2\lambda C T + T^2) - 2s_n(\lambda C T + T^2) + s_n(T^2) \end{array} \right) / \lambda^2 \Big|_{\lambda=2},$$

a quantity which we will show vanishes, so that the two norms are actually equal and the homomorphism $\text{Nm}((T' : N']) \rightarrow \text{Nm}((T : N])$ is actually norm-preserving.

First, note that we can rearrange the terms in the numerator to obtain

$$\left(\begin{array}{c} (2 - \lambda)s_n(T^2) + (\lambda - 2)s_n(\lambda C T + T^2) \\ + s_n(2(\lambda C T) + (T^2 + \lambda^2 C^2)) - s_n(\lambda(\lambda C T) + (T^2 + \lambda^2 C^2)) \end{array} \right) / \lambda^2 \Big|_{\lambda=2},$$

and the numerator does vanish when we set $\lambda = 2$, which is compelling but not reason enough to prove that it still vanishes after dividing by λ^2 and then setting $\lambda = 2$. (For example, over $\mathbb{Z}/4\mathbb{Z}$ we have $\frac{(\lambda-2)(\lambda+2)}{\lambda^2} \Big|_{\lambda=2} = 1$, not 0.) However, if we can show that the numerator is divisible by $\lambda^2(\lambda - 2)$, then it will definitely still vanish after dividing by λ^2 and setting $\lambda = 2$. We will show that this is indeed the case by dividing formally by $\lambda - 2$ and showing that the result is still divisible by λ^2 . We have

$$\begin{aligned} & \frac{\left(\begin{array}{c} (2 - \lambda)s_n(T^2) + (\lambda - 2)s_n(\lambda C T + T^2) \\ + s_n(2(\lambda C T) + (T^2 + \lambda^2 C^2)) - s_n(\lambda(\lambda C T) + (T^2 + \lambda^2 C^2)) \end{array} \right)}{\lambda - 2} \\ &= -s_n(T^2) + s_n(\lambda C T + T^2) - \frac{s_n(\lambda(\lambda C T) + (T^2 + \lambda^2 C^2)) - s_n(2(\lambda C T) + (T^2 + \lambda^2 C^2))}{\lambda - 2} \\ &= -s_n(T^2) + s_n(\lambda C T + T^2) - \frac{\sum_{k=0}^n s_{k,n-k}(\lambda(\lambda C T), T^2 + \lambda^2 C^2) - s_{k,n-k}(2(\lambda C T), T^2 + \lambda^2 C^2 - 2)}{\lambda - 2} \\ &= -s_n(T^2) + s_n(\lambda C T + T^2) - \sum_{k=0}^n \lambda^k s_{k,n-k}(C T, T^2 + \lambda^2 C^2) \frac{\lambda^k - 2^k}{\lambda - 2}. \end{aligned}$$

To decide whether this quantity is divisible by λ^2 , we work in $\mathcal{A}[\lambda]/(\lambda^2)$ and see whether the result is zero. Modulo λ^2 , we have $-s_n(T^2) + s_n(\lambda C T + T^2) =$

$\lambda s_{1,n-1}(CT, T^2)$, and in the sum, the $k = 0$ term vanishes anyway and the $k > 1$ terms all have factors of $\lambda^k = 0$. Therefore the total becomes

$$\begin{aligned} &\equiv \lambda s_{1,n-1}(CT, T^2) - \lambda s_{1,n-1}(CT, T^2 + \lambda^2 C^2) \frac{\lambda - 2}{\lambda - 2} \pmod{\lambda^2} \\ &\equiv \lambda s_{1,n-1}(CT, T^2) - \lambda s_{1,n-1}(CT, T^2) \pmod{\lambda^2} \\ &= 0. \end{aligned}$$

Therefore the difference between the norm of x in $\text{Nm}((T' : N'])$ and its image in $\text{Nm}((T : N])$ really does vanish, so the map $\text{Nm}((T' : N']) \rightarrow \text{Nm}((T : N])$ preserves both the trace and norm of x , and is therefore a norm-preserving homomorphism. \square

Example 3.10 For example, let $\mathcal{A} \rightarrow \mathcal{B}$ be a rank- n algebra and let $f : \mathcal{B}^2 \rightarrow \mathcal{B}^2$ be the “swap” map of quadratic algebras \mathcal{B} -algebras sending $(b, b') \mapsto (b', b)$. (In terms of based quadratic algebras, this is the map $((1 : 0])_{\mathcal{B}} \rightarrow ((1 : 0])_{\mathcal{B}}$ sending $x \mapsto -x + 1$.) If n is odd, then $\text{Nm}_{\mathcal{B}/\mathcal{A}}(f) : \mathcal{A}^2 \rightarrow \mathcal{A}^2$ is also the swap map; if n is even it is the identity map.

Proof Applying the norm map to $f : ((1 : 0])_{\mathcal{B}} \rightarrow ((1 : 0])_{\mathcal{B}} : x \mapsto -x + 1$, we obtain

$$\begin{aligned} \text{Nm}_{\mathcal{B}/\mathcal{A}}(f) : x &\mapsto s_n(-1)x + \frac{s_n(\lambda - 1) - s_n(-1)}{\lambda} \Big|_{\lambda=2} \\ &= (-1)^n x + \frac{(\lambda - 1)^n - (-1)^n}{(\lambda - 1) - (-1)} \Big|_{\lambda=2} \\ &= (-1)^n x + (\lambda - 1)^{n-1} + (-1)(\lambda - 1)^{n-2} + \dots + (-1)^{n-1} \Big|_{\lambda=2} \\ &= (-1)^n x + 1 - 1 + \dots + (-1)^{n-1}. \end{aligned}$$

Now if n is even, this works out to $x \mapsto 1x + 0$, the identity map. And if n is odd, this becomes $-x + 1$, the swap map $\mathcal{A}^2 \rightarrow \mathcal{A}^2$. \square

Proposition 3.11 *The assignment $f \mapsto \text{Nm}_{\mathcal{B}/\mathcal{A}}(f)$ in Proposition 3.9 is functorial: if we have a commuting triangle of univerrally norm-preserving homomorphisms between quadratic \mathcal{B} -algebras*

$$\begin{array}{ccc} & ((T' : N']) & \\ \nearrow & & \searrow \\ ((T'' : N'']) & \longrightarrow & ((T : N]) \end{array}$$

then the resulting triangle of norm-preserving \mathcal{A} -algebra homomorphisms

$$\begin{array}{ccc} & \text{Nm}_{\mathcal{B}/\mathcal{A}}((T' : N']) & \\ \nearrow & & \searrow \\ \text{Nm}_{\mathcal{B}/\mathcal{A}}((T'' : N'']) & \longrightarrow & \text{Nm}_{\mathcal{B}/\mathcal{A}}((T : N]) \end{array}$$

also commutes.

Proof Let us suppose that the homomorphisms $((T'' : N''] \rightarrow ((T' : N'] \rightarrow ((T : N])$ are defined as follows:

$$\begin{aligned} ((T'' : N''] \rightarrow ((T' : N'] \rightarrow ((T : N]) \\ x \mapsto Ux + C \\ x \mapsto Vx + D \end{aligned}$$

so that the composite homomorphism is

$$\begin{aligned} ((T'' : N''] \rightarrow ((T : N]) \\ x \mapsto V(Ux + C) + D \\ = VUx + (VC + D) \end{aligned}$$

To check that the norm operation is functorial, we must check that the composite of the two norms of homomorphisms

$$\begin{aligned} \text{Nm}((T'' : N''] \rightarrow \text{Nm}((T' : N'] \rightarrow \text{Nm}((T : N]) \\ x \mapsto s_n(U)x + \frac{s_n(\lambda C + UT) - s_n(UT)}{\lambda} \Big|_{\lambda=2} \\ x \mapsto s_n(V)x + \frac{s_n(\lambda D + VT') - s_n(VT')}{\lambda} \Big|_{\lambda=2} \end{aligned}$$

compose to equal the norm of the composite homomorphism

$$\begin{aligned} \text{Nm}((T'' : N''] \rightarrow \text{Nm}((T : N]) \\ x \mapsto s_n(VU)x + \frac{s_n(\lambda(VC + D) + VUT) - s_n(VUT)}{\lambda} \Big|_{\lambda=2} \end{aligned}$$

The actual composite expands as

$$\begin{aligned} x \mapsto s_n(V) \left(s_n(U)x + \frac{s_n(\lambda C + UT) - s_n(UT)}{\lambda} \Big|_{\lambda=2} \right) + \frac{s_n(\lambda D + VT') - s_n(VT')}{\lambda} \Big|_{\lambda=2} \\ = s_n(VU)x + \frac{s_n(\lambda VC + VUT) - s_n(VUT)}{\lambda} \Big|_{\lambda=2} \\ + \frac{s_n(\lambda D + VUT + 2VC) - s_n(VUT + 2VC)}{\lambda} \Big|_{\lambda=2} \end{aligned}$$

where we have used the multiplicity of s_n and the fact that $T' = UT + 2C$ since the original homomorphism $((T' : N'] \rightarrow ((T : N])$ is norm-preserving (see Lemma 2.10). We can combine the two constant terms into one larger sum (replacing a couple

of extra 2’s by λ , since they will be set equal to 2 again anyway and the divisibility by λ does not depend on them already equaling 2) as follows:

$$\begin{aligned} & \left. \frac{s_n(\lambda VC + VUT) - s_n(VUT)}{\lambda} \right|_{\lambda=2} + \left. \frac{s_n(\lambda D + VUT + 2VC) - s_n(VUT + 2VC)}{\lambda} \right|_{\lambda=2} \\ &= \left. \frac{s_n(\lambda VC + VUT) - s_n(VUT) + s_n(\lambda D + VUT + \lambda VC) - s_n(VUT + \lambda VC)}{\lambda} \right|_{\lambda=2} \\ &= \left. \frac{s_n(\lambda D + VUT + \lambda VC) - s_n(VUT)}{\lambda} \right|_{\lambda=2} \end{aligned}$$

So in all, the composite map of norms $\text{Nm}((T'' : N'']) \rightarrow \text{Nm}((T' : N']) \rightarrow \text{Nm}((T : N])$ sends

$$x \mapsto s_n(VU)x + \left. \frac{s_n(\lambda(VC + D) + VUT) - s_n(VUT)}{\lambda} \right|_{\lambda=2}$$

which is exactly the norm of the composite map $((T'' : N'']) \rightarrow ((T' : N']) \rightarrow ((T : N])$ sending $x \mapsto VUx + (VC + D)$, as desired. \square

To show that the norm operation on general quadratic algebras is transitive, we will also need transitivity for the norm operation on homomorphisms:

Proposition 3.12 *The norm map on homomorphisms is also transitive: if $\mathcal{A} \rightarrow \mathcal{B}$ is a rank- n algebra, and $\mathcal{B} \rightarrow \mathcal{C}$ is a rank- m algebra, and $f : ((T' : N']) \rightarrow ((T : N])$ is a norm-preserving homomorphism of based quadratic \mathcal{C} -algebras, then*

$$\text{Nm}_{\mathcal{B}/\mathcal{A}}(\text{Nm}_{\mathcal{C}/\mathcal{B}}(f)) = \text{Nm}_{\mathcal{C}/\mathcal{A}}(f).$$

Proof Let us suppose that $f : ((T' : N']) \rightarrow ((T : N])$ sends $x \mapsto Ux + C$. Then on the one hand, we have

$$\text{Nm}_{\mathcal{C}/\mathcal{B}}(f) : x \mapsto s_m(U)x + \left. \frac{s_m(\lambda C + UT) - s_m(UT)}{\lambda} \right|_{\lambda=2}$$

so, since $x \in \text{Nm}_{\mathcal{C}/\mathcal{B}}((T : N])$ has trace $s_m(T)$, we have that $\text{Nm}_{\mathcal{B}/\mathcal{A}}(\text{Nm}_{\mathcal{C}/\mathcal{B}}(f))$ sends

$$\begin{aligned} & x \mapsto s_n(s_m(U))x + \left. \frac{s_n \left(\mu \left. \frac{s_m(\lambda C + UT) - s_m(UT)}{\lambda} \right|_{\lambda=2} + s_m(U)s_m(T) \right) - s_n(s_m(U)s_m(T))}{\mu} \right|_{\mu=2} \\ &= s_{mn}(U)x + \left. \frac{s_n \left(\mu \left. \frac{s_m(\lambda C + UT) - s_m(UT)}{\lambda} \right|_{\lambda=\mu} + s_m(UT) \right) - s_n(s_m(UT))}{\mu} \right|_{\mu=2} \end{aligned}$$

$$\begin{aligned}
 &= s_{mn}(U)x + \left. \frac{s_n \left(\mu \frac{s_m(\mu C + UT) - s_m(UT)}{\mu} + s_m(UT) \right) - s_{mn}(UT)}{\mu} \right|_{\mu=2} \\
 &= s_{mn}(U)x + \left. \frac{s_n(s_m(\mu C + UT) - s_m(UT) + s_m(UT)) - s_{mn}(UT)}{\mu} \right|_{\mu=2} \\
 &= s_{mn}(U)x + \left. \frac{s_{mn}(\mu C + UT) - s_{mn}(UT)}{\mu} \right|_{\mu=2},
 \end{aligned}$$

which is exactly where $Nm_{\mathcal{C}/\mathcal{A}}(f)$ sends x . □

4 Norms of quadratic algebras

We now have all the results we need in order to define the norm of a general quadratic algebra and prove that it is well-defined:

Definition 4.1 Given a rank- n \mathcal{A} -algebra \mathcal{B} , we define the norm of a general quadratic \mathcal{B} -algebra \mathcal{D} as follows:

1. Choose elements $a_1, \dots, a_k \in \mathcal{A}$, together generating the unit ideal, such that each \mathcal{D}_{a_i} is free as a \mathcal{B}_{a_i} -module.
2. Choose generators for each \mathcal{D}_{a_i} to obtain isomorphisms $\mathcal{D}_{a_i} \cong ((T_i : N_i)]_{\mathcal{B}_{a_i}}$.
3. Over overlaps $\mathcal{A}_{a_i a_j}$, use Proposition 3.9 to convert the isomorphisms $((T_i : N_i)]_{\mathcal{B}_{a_i a_j}} \cong \mathcal{D}_{a_i a_j} \cong ((T_j : N_j)]_{\mathcal{B}_{a_i a_j}}$ into isomorphisms

$$Nm_{\mathcal{B}_{a_i a_j}/\mathcal{A}_{a_i a_j}}((T_i : N_i)]_{\mathcal{B}_{a_i a_j}} \cong Nm_{\mathcal{B}_{a_i a_j}/\mathcal{A}_{a_i a_j}}((T_j : N_j)]_{\mathcal{B}_{a_i a_j}}.$$

4. Use these isomorphisms to glue the norms of the based quadratic \mathcal{A}_{a_i} -algebras $Nm_{\mathcal{B}_{a_i}/\mathcal{A}_{a_i}}((T_i : N_i)]$ into a single quadratic \mathcal{A} -algebra, called $Nm_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$.

Lemma 4.2 *The construction of $Nm_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$ in Definition 4.1 is well-defined.*

Proof There are two things to check: that the norms of the based quadratic algebras $Nm_{\mathcal{B}_{a_i}/\mathcal{A}_{a_i}}((T_i : N_i)]$ do, in fact, glue together to yield a quadratic \mathcal{A} -algebra, and that the resulting algebra is independent (up to isomorphism) of the choices of the a_i and generators for \mathcal{D}_{a_i} .

First we check that the quadratic algebras do glue; this is an example of faithfully flat descent. We claim that the cocycle condition is satisfied; namely, over triple overlaps $\mathcal{A}_{a_i a_j a_k}$, the following triangle of isomorphisms commutes:

$$\begin{array}{ccc}
 & Nm((T_j : N_j)] & \\
 \nearrow & & \searrow \\
 Nm((T_i : N_i)] & \longrightarrow & Nm((T_k : N_k)]
 \end{array}$$

Then there must be an \mathcal{A} -module which we call $Nm_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$, equipped with isomorphisms $Nm_{\mathcal{B}/\mathcal{A}}(\mathcal{D})_{a_i} \cong Nm_{\mathcal{B}_{a_i}/\mathcal{A}_{a_i}}((T_i : N_i)]$ and unique up to unique isomorphism. This \mathcal{A} -module automatically inherits the property of being locally free of rank 2, as

well as the structure of an \mathcal{A} -algebra from the unit and multiplication maps for each of its localizations, so $\text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$ is a quadratic \mathcal{A} -algebra as desired.

Now we check that the claim is true: the triangle does commute by Proposition 3.11 because the corresponding triangle of quadratic $\mathcal{B}_{a_i a_j a_k}$ -algebra homomorphisms commutes, since the isomorphisms between them come from chosen isomorphisms with $\mathcal{D}_{a_i a_j a_k}$:

$$\begin{array}{ccc} & ((T_j : N_j]) & \\ \nearrow & & \searrow \\ ((T_i : N_i]) & \longrightarrow & ((T_k : N_k]) \end{array}$$

Second, we check independence of the choices of the a_i and generators for \mathcal{D}_{a_i} : suppose that we had a second collection of elements a'_1, a'_2, \dots and isomorphisms $\mathcal{D}_{a'_i} \cong ((T'_i : N'_i])$. Then we could combine the two collections into one and construct a common gluing of all the $\text{Nm}((T_i : N_i])$ and $\text{Nm}((T'_i : N'_i])$. In particular, the result will be isomorphic to the gluings obtained from each family separately, so the norm is independent of choice of generator. \square

We can now boost our results on the norm operation for based quadratic algebras to this new, more general setting:

Theorem 4.3 *The norm operation on general quadratic algebras commutes with base change: Let $\mathcal{A} \rightarrow \mathcal{B}$ be a rank- n algebra, and let \mathcal{D} be a quadratic \mathcal{B} -algebra. If \mathcal{C} is any \mathcal{A} -algebra, then $\mathcal{C} \rightarrow \mathcal{C} \otimes_{\mathcal{A}} \mathcal{B}$ is another rank- n algebra, and $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}$ is a quadratic $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{B}$ -algebra with*

$$\text{Nm}_{(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{B})/\mathcal{C}}(\mathcal{C} \otimes_{\mathcal{A}} \mathcal{D}) = \mathcal{C} \otimes_{\mathcal{A}} \text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}).$$

Proof We already know by Theorem 3.7 that this holds locally, so we need only check that taking the norms of the gluing isomorphisms also commutes with base change. But the gluing isomorphisms are also built out of polarized forms of the polynomial law s_n , so this is again automatic. \square

Theorem 4.4 *The norm operation is transitive: If $\mathcal{A} \rightarrow \mathcal{B}$ is a rank- n algebra and $\mathcal{B} \rightarrow \mathcal{C}$ is a rank- m algebra, and if \mathcal{D} is a quadratic \mathcal{C} -algebra, then $\text{Nm}_{\mathcal{C}/\mathcal{A}}(\mathcal{D}) \cong \text{Nm}_{\mathcal{B}/\mathcal{A}}(\text{Nm}_{\mathcal{C}/\mathcal{B}}(\mathcal{D}))$.*

Proof We have already shown in Theorem 3.8 that this holds for norms of based quadratic algebras; it remains to show that the isomorphisms used to glue together the norms of based localizations also agree, but this holds by Proposition 3.12. \square

We can now use transitivity of the norm operation to show that $\text{Nm}_{\mathcal{B}/\mathcal{A}}$ is a homomorphism with respect to the monoid operation $*$ on quadratic algebras:

Theorem 4.5 *The norm operation is a homomorphism with respect to the monoid operation $*$ on isomorphism classes of quadratic algebras: If $\mathcal{A} \rightarrow \mathcal{B}$ is a rank- n algebra and \mathcal{D} and \mathcal{D}' are quadratic \mathcal{B} -algebras, then $\text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D} * \mathcal{D}') \cong \text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}) * \text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}')$.*

Proof Consider the tower of algebras $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{B}^2$ and the quadratic \mathcal{B}^2 -algebra $\mathcal{D} \times \mathcal{D}'$. We can take its norm to produce the quadratic \mathcal{A} -algebra

$$\begin{aligned} \mathrm{Nm}_{\mathcal{B}^2/\mathcal{A}}(\mathcal{D} \times \mathcal{D}') &\cong \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathrm{Nm}_{\mathcal{B}^2/\mathcal{B}}(\mathcal{D} \times \mathcal{D}')) \\ &\cong \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D} * \mathcal{D}'). \end{aligned}$$

On the other hand, we can also factor the algebra map $\mathcal{A} \rightarrow \mathcal{B}^2$ as $\mathcal{A} \rightarrow \mathcal{A}^2 \rightarrow \mathcal{B}^2$, for which transitivity of the norm operation gives

$$\begin{aligned} \mathrm{Nm}_{\mathcal{B}^2/\mathcal{A}}(\mathcal{D} \times \mathcal{D}') &\cong \mathrm{Nm}_{\mathcal{A}^2/\mathcal{A}}(\mathrm{Nm}_{\mathcal{B}^2/\mathcal{A}^2}(\mathcal{D} \times \mathcal{D}')) \\ &\cong \mathrm{Nm}_{\mathcal{A}^2/\mathcal{A}}(\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}) \times \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}')) \\ &\cong \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}) * \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}'). \end{aligned}$$

(In the second line, we have used base-change invariance of the norm operation to construct the norm over each factor separately, and in the third line, we have invoked Example 3.6) Comparing these two expressions for $\mathrm{Nm}_{\mathcal{B}^2/\mathcal{A}}(\mathcal{D} \times \mathcal{D}')$, we obtain $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D} * \mathcal{D}') \cong \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}) * \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}')$, as desired. \square

Our last results are that this notion of norm of quadratic algebras commutes with taking determinant line bundles and discriminant bilinear forms:

Theorem 4.6 *The norm operation commutes with taking determinant line bundles: If $\mathcal{A} \rightarrow \mathcal{B}$ is a rank- n algebra and \mathcal{D} is a quadratic \mathcal{B} -algebra, then $\bigwedge_{\mathcal{A}}^2 \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D}) = \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\bigwedge_{\mathcal{B}}^2 \mathcal{D})$.*

Proof If \mathcal{L} is a line bundle (rank-1 module) over \mathcal{B} , then its norm $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{L})$ is the line bundle obtained by the following process analogous to Definition 4.1:

1. Choose a set of elements $a_i \in \mathcal{A}$ generating the unit ideal such that each localization \mathcal{L}_{a_i} is free as an \mathcal{B}_{a_i} -module.
2. Choose generators for each \mathcal{L}_{a_i} to obtain isomorphisms $\phi_i : \mathcal{L}_{a_i} \cong \mathcal{B}_{a_i}$.
3. Over overlaps $\mathcal{A}_{a_i a_j}$, we have isomorphisms $(\phi_i)_{a_j} \circ (\phi_j^{-1})_{a_i} : \mathcal{B}_{a_i a_j} \cong \mathcal{L}_{a_i a_j} \cong \mathcal{B}_{a_i a_j}$. Each such isomorphism is given by multiplication by a unit $U_{ij} \in \mathcal{B}_{a_i a_j}$; take the norm of U_{ij} to get units $s_n(U_{ij}) \in \mathcal{A}_{ij}$.
4. Glue together free rank-1 \mathcal{A}_{a_i} -modules, using multiplication by $s_n(U_{ij})$ as the isomorphisms on overlaps, to get a single rank-1 \mathcal{A} -module, called $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{L})$.

Now suppose $\mathcal{L} = \bigwedge^2 \mathcal{D}$. We will follow along steps 1 through 4 of Definition 4.1 and see that $\bigwedge^2 \mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$ agrees with the above construction of $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{L})$.

Since the determinant line bundle of a based quadratic algebra $((T : N]_{\mathcal{B}})$ is free of rank 1 with canonical generator $1 \wedge x$, completing steps 1 and 2 of Definition 4.1 also completes steps 1 and 2 of constructing the line bundle norm $\mathrm{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{L})$. For step 3, suppose that for each $i, j \in \{1, \dots, k\}$ we have an isomorphism $((T_i : N_i]_{\mathcal{B}_{a_i a_j}} \cong \mathcal{D}_{a_i a_k} \cong ((T_j : N_j]_{\mathcal{B}_{a_i a_j}}$ sending $x \mapsto U_{ij}x + C_{ij}$. Then taking exterior powers, we have isomorphisms $\bigwedge^2((T_i : N_i]_{\mathcal{B}_{a_i a_j}} \cong \bigwedge^2 \mathcal{D}_{a_i a_k} \cong \bigwedge^2((T_j : N_j]_{\mathcal{B}_{a_i a_j}}$ sending

$1 \wedge x \mapsto U_{ij} \cdot (1 \wedge x)$. This means that multiplication by U_{ij} is also the composite isomorphism $\mathcal{B}_{a_i a_j} \cong \mathcal{L}_{a_i a_j} \cong \mathcal{B}_{a_i a_j}$.

To construct $\text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$, we apply Proposition 3.9 to the isomorphisms $((T_i : N_i) \llbracket_{\mathcal{B}_{a_i a_j}} \cong ((T_j : N_j) \llbracket_{\mathcal{B}_{a_i a_j}}$ sending $x \mapsto U_{ij}x + C_{ij}$ to get isomorphisms

$$\begin{aligned} \text{Nm}((T_i : N_i) \llbracket_{\mathcal{B}_{a_i a_j}} &\cong \text{Nm}((T_j : N_j) \llbracket_{\mathcal{B}_{a_i a_j}} \\ &: x \mapsto s_n(U_{ij})x + \left. \frac{s_n(\lambda C_{ij} + U_{ij}T_j) - s_n(U_{ij}T_j)}{\lambda} \right|_{\lambda=2} \end{aligned}$$

which we use as gluing maps to construct $\text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$. Taking the exterior powers of these maps, we obtain isomorphisms sending $1 \wedge x \mapsto s_n(U_{ij}) \cdot (1 \wedge x)$, meaning that $\bigwedge^2 \text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$ is obtained by gluing together free rank-1 \mathcal{A}_{a_i} -modules with the gluing isomorphisms given by multiplication by $s_n(U_{ij})$. That is exactly the gluing which produces $\text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{L})$, as claimed. \square

Theorem 4.7 *Taking the norm commutes with taking discriminants: if $\mathcal{A} \rightarrow \mathcal{B}$ is a rank- n algebra and \mathcal{D} is a quadratic \mathcal{B} -algebra with discriminant bilinear form $\delta_{\mathcal{D}} : (\bigwedge^2_{\mathcal{B}} \mathcal{D})^{\otimes 2} \rightarrow \mathcal{B}$, then the discriminant bilinear form of the quadratic \mathcal{A} -algebra $\text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$ agrees with the norm of $\delta_{\mathcal{D}}$ as a map of line bundles.*

Proof Note that this only makes sense because by Theorem 4.6 we have an isomorphism $\bigwedge^2 \text{Nm}(\mathcal{D}) = \text{Nm}(\bigwedge^2 \mathcal{D})$, so the norm of the line bundle homomorphism $\delta_{\mathcal{D}} : (\bigwedge^2 \mathcal{D})^{\otimes 2} \rightarrow \mathcal{B}$ can be viewed as a homomorphism $\text{Nm}(\delta_{\mathcal{D}}) : (\bigwedge^2 \text{Nm}(\mathcal{D}))^{\otimes 2} \rightarrow \mathcal{A}$. Then we can check locally that this agrees with $\delta_{\text{Nm}(\mathcal{D})}$, so assume without loss of generality that $\mathcal{D} = (T : N \llbracket$.

With respect to the canonical basis $1 \wedge x$ for $\bigwedge^2 \mathcal{D}$, we have that $\delta_{\mathcal{D}}$ is just multiplication by $T^2 - 4N$. The norm of this map is therefore multiplication by $s_n(T^2 - 4N)$.

On the other hand, with respect to the canonical basis $\delta_{\text{Nm}(\mathcal{D})}$ is multiplication by

$$\begin{aligned} s_n(T)^2 - 4 \left[\frac{s_n(\lambda N + T^2) - s_n(T^2)}{\lambda} \right]_{\lambda=-4} & \\ = s_n(T)^2 + \left[s_n(\lambda N + T^2) - s_n(T^2) \right]_{\lambda=-4} & \\ = s_n(T^2) + s_n(T^2 - 4N) - s_n(T^2) & \\ = s_n(T^2 - 4N), & \end{aligned}$$

which agrees with $\text{Nm}(\delta_{\mathcal{D}})$ as desired. \square

In particular, if a quadratic \mathcal{B} -algebra \mathcal{D} is étale (its discriminant bilinear form is an isomorphism $(\bigwedge^2 \mathcal{D})^{\otimes 2} \xrightarrow{\sim} \mathcal{B}$), then its norm $\text{Nm}_{\mathcal{B}/\mathcal{A}}(\mathcal{D})$ is also an étale quadratic \mathcal{A} -algebra. Since étale quadratic algebras are equivalent to S_2 -torsors, we may ask if the norm operation agrees with the trace map for S_2 -torsors:

Theorem 4.8 *On étale quadratic algebras, the norm operation reduces to the ordinary trace for S_2 -torsors.*

Proof Let $\mathcal{A} \rightarrow \mathcal{B}$ be a rank- n algebra. Since a split étale \mathcal{B} -algebra is isomorphic to $\mathcal{B}^2 \cong ((1 : 0)]_{\mathcal{B}}$, and $\text{Nm}_{\mathcal{B}/\mathcal{A}}((1 : 0)]_{\mathcal{B}} = ((1 : 0)]_{\mathcal{A}} \cong \mathcal{A}^2$, we know that since the norm operation commutes with base change, therefore the norm of any étale quadratic algebra is still étale. It remains to show that the norm of a \mathcal{B} -algebra automorphism of \mathcal{B}^2 is its trace as an automorphism of \mathcal{A}^2 . Any \mathcal{B} -algebra automorphism of \mathcal{B}^2 is locally either the identity map $\mathcal{B}^2 \rightarrow \mathcal{B}^2$ (which is sent to the identity map $\mathcal{A}^2 \rightarrow \mathcal{A}^2$) or the swap map $\mathcal{B}^2 \rightarrow \mathcal{B}^2$ sending $(b, b') \mapsto (b', b)$. By Example 3.10, the norm of the swap map is either the identity (if the rank n is even) or again the swap map (if n is odd). In other words, the norm of the swap map is the swap map composed with itself n times, which is exactly the trace of the swap map, as desired. \square

Finally, we have the following conjectured relationship between norms and discriminant algebras which was the motivation for constructing the norm functor for quadratic algebras in the first place:

Conjecture 4.9 *Let Δ be the discriminant algebra operation of Biesel and Gioia (2016), sending rank- n algebras $\mathcal{A} \rightarrow \mathcal{B}$ to quadratic algebras $\mathcal{A} \rightarrow \Delta_{\mathcal{B}/\mathcal{A}}$. Now let $\mathcal{A} \rightarrow \mathcal{B}$ be a rank- n algebra and $\mathcal{B} \rightarrow \mathcal{C}$ be a rank- m algebra, so that $\mathcal{A} \rightarrow \mathcal{C}$ is also a rank- mn algebra. Is it the case that*

$$\Delta_{\mathcal{C}/\mathcal{A}} \cong \text{Nm}_{\mathcal{B}/\mathcal{A}}(\Delta_{\mathcal{C}/\mathcal{B}}) * \Delta_{\mathcal{B}/\mathcal{A}}^{*m}?$$

It is known that Conjecture 4.9 holds in case $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{B} \rightarrow \mathcal{C}$ are étale; see Waterhouse (1987, Theorem 4) for a proof.

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