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ORIGINAL PAPER

# Strongly Gorenstein-projective modules over rings of Morita contexts

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**Abstract**
Let $\Delta_{(0,0)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ be a Morita ring such that the bimodule homomorphisms are zero. In this paper, we give sufficient conditions for a $\Delta_{(0,0)}$-module $(X, Y, f, g)$ to be strongly Gorenstein-projective. Moreover, we describe all strongly Gorenstein-projective modules over the $2 \times 2$ matrix algebra $\mathrm{M}_2(A)$ over $A$.

**Keywords** Strongly Gorenstein-projective modules · Morita rings · Strongly complete projective resolutions · Gorenstein-projective modules

**Mathematics Subject Classification** 16E05 · 16G50

## 1 Introduction

Auslander and Bridger (1969) generalized finitely generated projective modules to modules of Gorenstein dimension zero over two-sided Noetherian rings. After two decades, Enochs and Jenda (1995) generalized it to an arbitrary ring and called it Gorenstein-projective modules. Bennis and Mahdou introduced the notion of strongly Gorenstein-projective modules and showed that a module is Gorenstein-projective if and only if it is a direct summand of a strongly Gorenstein-projective module (Bennis and Mahdou (2007), Theorem 2.7).

Projective modules are strongly Gorenstein-projective modules (the converse is not true in general). Over an algebra of finite global dimension, Gorenstein-projective modules are projective (Enochs and Jenda (2000), Proposition 10.2.3). Gao and Zhang determined all finitely generated strongly Gorenstein-projective modules over upper triangular matrix artin algebras in Gao and Zhang (2009). Mao (2020) explicitly described the structures of strongly Gorenstein-projective, injective and flat modules over formal triangular matrix rings.

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Morita rings have been introduced by Bass (1962). This class of rings contains a lot of good examples of algebra. Gao et al. (2021) studied the monomorphism category and epimorphism category of Morita rings with bimodule morphisms being zero, and characterized the Ringel–Schmidmeier–Simson equivalence between them. Guo (2022) constructed an example of Gorenstein-projective modules over a special class of Morita context rings. Asefa (2022b) gave sufficient conditions for a $\Lambda_{(0,0)}$-module $(X, Y, f, g)$ to be Gorenstein-projective. Asefa (2022a) described all the complete projective resolutions and all finitely generated Gorenstein-projective modules over a Morita ring $\Lambda_{(0,0)}(A, B, M, N)$, by giving the corresponding sufficient and necessary conditions. Gao and Psaroudakis (2017) constructed Gorenstein-projective modules over Morita rings. Green and Psaroudakis (2014) described all Gorenstein-projective modules over a Morita ring $\Delta_{(\phi,\phi)} = \left(\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right)$, where $A$ is Gorenstein algebra. However, strongly Gorenstein-projective modules over a Morita ring have not yet been explicitly described. Therefore, our aim is to explicitly describe strongly Gorenstein-projective modules over a Morita ring. This generalizes strongly Gorenstein-projective modules over formal triangular matrix rings.

## 2 Preliminaries

In this section, we recall some basic definitions and facts that will be used throughout the paper.

Throughout the paper, $A$-Mod denotes the category of left $A$-modules, for a ring $A$. $\mathrm{pd}(M)$ and $\mathrm{fd}(M)$ denote the projective and flat dimensions of an $A$-module $M$ respectively. Following Enochs and Jenda (2000), an $A$-module $M$ is said to be Gorenstein-projective in $A$-Mod if there is an exact sequence of projective modules:

$$\mathscr{P}^{\bullet} := \cdots \longrightarrow P^{-1} \longrightarrow P^{0} \xrightarrow{d^0} P^{1} \longrightarrow \cdots$$

with $\mathrm{Hom}_A(P^{\bullet}, Q)$ exact for any projective $A$-module $Q$ such that $M \cong \mathrm{Ker}\, d^0$. A complex $\mathscr{P}^{\bullet}$ is called a complete projective resolution in $A$-Mod, if $\mathscr{P}^{\bullet}$ is of the form

$$\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

then $M$ is said to be a strongly Gorenstein-projective module, SG-projective for short, and $\mathscr{P}^{\bullet}$ is called a strongly complete projective resolution. Denote by $\mathrm{SGProj}A$ the full subcategory of SG-projective $A$-modules.

Let $A$ and $B$ be two rings, ${}_AN_B$ an $A$-$B$-bimodule, ${}_BM_A$ a $B$-$A$-bimodule, and $\phi : M \otimes_A N \longrightarrow B$ a $B$-$B$-bimodule homomorphism, and $\psi : N \otimes_B M \longrightarrow A$ an $A$-$A$-bimodule homomorphism. Define

$$\Delta_{(\phi,\psi)}(A, B, M, N) := \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix} = \{\left(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\right) \mid a \in A,\ b \in B,\ m \in M,\ n \in N\}.$$

Consider the addition of $\Delta_{(\phi,\psi)}(A, B, M, N)$ as the addition of matrices, and the multiplication is given by

$$\left(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a' & n' \\ m' & b' \end{smallmatrix}\right) = \left(\begin{smallmatrix} aa'+\psi(n\otimes m') & an'+nb' \\ ma'+bm' & bb'+\phi(m\otimes n') \end{smallmatrix}\right).$$

This multiplication of $\Delta_{(\phi,\psi)}(A, B, M, N)$ has associativity if and only if

$$\phi(m \otimes n)m' = m\psi(n \otimes m'), \quad n\phi(m \otimes n') = \psi(n \otimes m)n'$$

for all $m, m' \in M$ and all $n, n' \in N$. In this case, $\Delta_{(\phi,\psi)}(A, B, M, N)$ is a ring, which is called the Morita ring.

For example, if $B = A = M = N$, then we have $\psi = \phi : A \otimes A \longrightarrow A$. By the associativity condition, $\psi = \phi$ if and only if $\psi(1 \otimes 1) = \phi(1 \otimes 1)$. We denote the corresponding Morita ring by $\Delta_{(\phi,\phi)}(A) := \left(\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right)$.

Note that $\phi(1 \otimes 1) = a$ if and only if $a$ is in the center of $A$. There are two kinds of important cases, namely, $\phi(1 \otimes 1) = 1$ and $\phi(1 \otimes 1) = 0$.

If $\phi(1 \otimes 1) = 1$, then the corresponding Morita ring $\Delta_{(\phi,\phi)}(A)$ is just the $2 \times 2$ matrix algebra $\mathrm{M}_2(A)$.

If $\phi(1\otimes 1) = 0$, then the corresponding Morita ring will be denoted by $\Delta_{(0,0)}(A) := \left(\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right)$. Thus, the multiplication of $\Delta_{(0,0)}(A)$ is given by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) = \left(\begin{smallmatrix} aa' & ab'+bd' \\ ca'+dc' & dd' \end{smallmatrix}\right).$$

This is, in fact, a new ring in some sense.

The modules over a Morita ring $\Delta_{(\phi,\psi)}$ were described in Green (1982). Let $\mathfrak{M}(\Delta_{(\phi,\psi)})$ be the category whose objects are tuples $(X, Y, f, g)$, where $X \in A$-Mod, $Y \in B$-Mod, and

$$f \in \mathrm{Hom}_B(M \otimes_A X, Y), \quad g \in \mathrm{Hom}_A(N \otimes_B Y, X)$$

such that the following diagrams commute:

$$\begin{tikzcd} N \otimes_B M \otimes_A X \arrow[r, "\mathrm{Id}_N \otimes f"] \arrow[d, "\psi\otimes\mathrm{Id}_X"'] & N \otimes_B Y \arrow[d, "g"] \\ A \otimes_A X \arrow[r, "\cong"] & X \end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd} M \otimes_A N \otimes_B Y \arrow[r, "\mathrm{Id}_M \otimes g"] \arrow[d, "\phi\otimes\mathrm{Id}_Y"'] & M \otimes_A X \arrow[d, "f"] \\ B \otimes_B Y \arrow[r, "\cong"] & Y \end{tikzcd} \tag{2.2}$$

A morphism $(X, Y, f, g) \longrightarrow (X', Y', f', g')$ in $\mathfrak{M}(\Delta_{(\phi,\psi)})$ is a pair $(a, b)$, where $a : X \longrightarrow X'$ is an $A$-homomorphism and $b : Y \longrightarrow Y'$ is a $B$-homomorphism, such that the following diagrams commute:

$$\begin{tikzcd}
M \otimes_A X \arrow[r, "f"] \arrow[d, "\mathrm{Id}_M \otimes a"'] & Y \arrow[d, "b"] \\
M \otimes_A X' \arrow[r, "f'"] & Y'
\end{tikzcd} \tag{2.3}$$

$$\begin{tikzcd}
N \otimes_B Y \arrow[r, "g"] \arrow[d, "\mathrm{Id}_N \otimes b"'] & X \arrow[d, "a"] \\
N \otimes_B Y' \arrow[r, "g'"] & X'
\end{tikzcd} \tag{2.4}$$

The relationship between $\Delta_{(\phi,\psi)}$-Mod and $\mathfrak{M}(\Delta)$ is given via the functor $F : \mathfrak{M}(\Delta) \longrightarrow \Delta_{(\phi,\psi)}$-Mod which is defined on objects $(X, Y, f, g)$ of $\mathfrak{M}(\Delta)$ as follows: $F(X, Y, f, g) = X \oplus Y$ as abelian groups, with a $\Delta_{(\phi,\psi)}$-module structure given by

$$\left(\begin{smallmatrix} a & n \\ m & b \end{smallmatrix}\right)(x, y) = (ax + g(n \otimes y), by + f(m \otimes x))$$

for all $a \in A$, $b \in B$, $n \in N$, $m \in M$, $x \in X$, and $y \in Y$. If $(a, b) : (X, Y, f, g) \to (X', Y', f', g')$ is a morphism in $\mathfrak{M}(\Delta)$ then $F(a, b) = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) : X \oplus Y \to X' \oplus Y'$. Then the functor $F$ turns out to be an equivalence of categories, (see (Green 1982, Theorem 1.5)). From now on we identify the modules over $\Delta_{(\phi,\psi)}$ with the objects of $\mathfrak{M}(\Delta)$.

Let $\Delta_{(\phi,\psi)} = \begin{pmatrix} A & {}_AN_B \\ {}_BM_A & B \end{pmatrix}$ be a Morita ring. Then we have the following facts (see for e.g. Gao and Psaroudakis (2017)).

**Lemma 1** (i) *Let* $(a, b) : (X, Y, f, g) \longrightarrow (X', Y', f', g')$ *be a morphism in* $\Delta_{(\phi,\psi)}$-Mod, $c : \operatorname{Ker} a \hookrightarrow X$ *and* $d : \operatorname{Ker} b \hookrightarrow Y$ *the canonical embedding. Then the kernel of* $(a, b)$ *is the object* $(\operatorname{Ker} a, \operatorname{Ker} b, h, j)$, *where h is induced by the following commutative diagram*:

$$\begin{tikzcd}
M \otimes_A \operatorname{Ker} a \arrow[r, "\mathrm{Id}_M \otimes c"] \arrow[d, dashed, "h"] & M \otimes_A X \arrow[r, "\mathrm{Id}_M \otimes a"] \arrow[d, "f"] & M \otimes_A X' \arrow[d, "f'"] \\
\operatorname{Ker} b \arrow[r, hook, "d"] & Y \arrow[r, "b"] & Y'
\end{tikzcd}$$

*and j is induced by the following commutative diagram*:

$$\begin{tikzcd}
N \otimes_B \operatorname{Ker} b \arrow[r, "\mathrm{Id}_N \otimes d"] \arrow[d, dashed, "j"] & N \otimes_B Y \arrow[r, "\mathrm{Id}_N \otimes b"] \arrow[d, "g"'] & N \otimes_B Y' \arrow[d, "g'"] \\
\operatorname{Ker} a \arrow[r, hook, "c"] & X \arrow[r, "a"] & X'
\end{tikzcd}$$

*Similarly, one can derive a description for the Cokernel of the morphism* $(a, b)$.

(ii) *A sequence of* $\Delta_{(\phi,\psi)}$*-homomorphisms*

$$0 \longrightarrow (X_1, Y_1, f_1, g_1) \xrightarrow{(a,b)} (X_2, Y_2, f_2, g_2) \xrightarrow{(a',b')} (X_3, Y_3, f_3, g_3) \longrightarrow 0$$

*is exact if and only if the sequence of* $A$*-homomorphisms*

$$0 \longrightarrow X_1 \xrightarrow{a} X_2 \xrightarrow{a'} X_3 \longrightarrow 0$$

*is exact in* $A$-Mod*, and the sequence of* $B$*-homomorphisms*

$$0 \longrightarrow Y_1 \xrightarrow{b} Y_2 \xrightarrow{b'} Y_3 \longrightarrow 0$$

*is exact in* $B$-Mod.

We denote by $\psi_X$ the following composition:

$$\begin{tikzcd} N \otimes_B M \otimes_A X \arrow[r, "\psi\otimes \mathrm{Id}_X"] \arrow[rr, bend left, "\psi_X"] & A \otimes_A X \arrow[r, "\cong"] & X \end{tikzcd}$$

i.e., $\psi_X := c_X(\psi \otimes \mathrm{Id}_X) : N \otimes_B M \otimes_A X \longrightarrow X$, where $c_X : A \otimes_A X \longrightarrow X$ is the canonical $A$-isomorphism.

We denote by $\phi_Y$ the following composition:

$$\begin{tikzcd} M \otimes_A N \otimes_B Y \arrow[r, "\phi\otimes \mathrm{Id}_Y"] \arrow[rr, bend left, "\phi_Y"] & B \otimes_B Y \arrow[r, "\cong"] & Y \end{tikzcd}$$

i.e., $\phi_Y := c_Y(\phi \otimes \mathrm{Id}_Y) : M \otimes_A N \otimes_B Y \longrightarrow Y$, where $c_Y : B \otimes_B Y \longrightarrow Y$ is the canonical $B$-isomorphism.

We now recall the functors given in Green and Psaroudakis (2014).

1. The functor $\mathrm{T}_A : A\text{-Mod} \longrightarrow \Delta_{(\phi,\psi)}\text{-Mod}$ is given by $\mathrm{T}_A(X) := (X, M \otimes_A X, \mathrm{Id}_{M\otimes_A X}, \psi_X)$ for any object $X$ in $A$-Mod.
2. The functor $\mathrm{T}_B : B\text{-Mod} \longrightarrow \Delta_{(\phi,\psi)}\text{-Mod}$ is given by $\mathrm{T}_B(Y) := (N \otimes_B Y, Y, \phi_Y, \mathrm{Id}_{N\otimes_B Y})$ for any object $Y$ in $B$-Mod.
3. The functor $\mathrm{U}_A : \Delta_{(\phi,\psi)}\text{-Mod} \longrightarrow A\text{-Mod}$ is given by $\mathrm{U}_A(X, Y, f, g) := X$ for any object $(X, Y, f, g)$ in $\Delta_{(\phi,\psi)}$-Mod.
4. The functor $\mathrm{U}_B : \Delta_{(\phi,\psi)}\text{-Mod} \longrightarrow B\text{-Mod}$ is given by $\mathrm{U}_B(X, Y, f, g) := Y$ for any object $(X, Y, f, g)$ in $\Delta_{(\phi,\psi)}$-Mod.
5. Let $X$ be any object in $A$-Mod, then we denote by $\epsilon_X : N \otimes_B \mathrm{Hom}_A(N, X) \longrightarrow X$ the map $A$-module given by involution. The functor $\mathrm{H}_A : A\text{-Mod} \longrightarrow \Delta_{(\phi,\psi)}\text{-Mod}$ is given by $\mathrm{H}_A(X) := (X,\ \mathrm{Hom}_A(N, X),\ \mathrm{Hom}_A(N, \psi_X) \circ \delta'_{M\otimes_A X},\ \epsilon_X)$ for any object $X$ in $A$-Mod.

6. Let $Y$ be any object in $B$-Mod, then we denote by $\epsilon_Y : M \otimes_A \mathrm{Hom}_B(M, Y) \longrightarrow Y$ the map $B$-module given by involution. The functor $\mathrm{H}_B : B\text{-Mod} \longrightarrow \Delta_{(\phi,\psi)}$-Mod is given by $\mathrm{H}_B(Y) := (\mathrm{Hom}_B(M, Y),\ Y,\ \epsilon_Y,\ \mathrm{Hom}_B(M, \phi_Y) \circ \delta_{N \otimes_B Y})$ for any object $Y$ in $B$-Mod.
7. The functor $\mathrm{Z}_A : A\text{-Mod} \longrightarrow \Delta_{(\phi,\psi)}$-Mod is defied by $\mathrm{Z}_A(X) := (X, 0, 0, 0)$ for any object $X$ in $A$-Mod. The functor $\mathrm{Z}_B : B\text{-Mod} \longrightarrow \Delta_{(\phi,\psi)}$-Mod can be similarly defined.

For $\Delta_{(\phi,\phi)}(A) = \left(\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right)$, we will use special notations for the functors $\mathrm{T}_A$ and $\mathrm{T}_B$.

The functor $\mathrm{T}_1 : A\text{-Mod} \longrightarrow \Delta_{(\phi,\phi)}$-Mod is given by $\mathrm{T}_1(X) := (X,\ A \otimes_A X,\ \mathrm{Id}_{A \otimes_A X},\ \phi_X)$ for any object $X \in A$-Mod, and for an $A$-homomorphism $a : X \longrightarrow X'$, $\mathrm{T}_1(a) := (a, a)$.

The functor $\mathrm{T}_2 : A\text{-Mod} \longrightarrow \Delta_{(\phi,\phi)}$-Mod is given by

$$\mathrm{T}_2(X) := (A \otimes_A X,\ X,\ \phi_X,\ \mathrm{Id}_{A \otimes_A X})$$

for any object $X \in A$-Mod, and for an $A$-homomorphism $b : X \longrightarrow X'$, $\mathrm{T}_2(b) := (b, b)$.

The following result gives more information about the above functors.

**Proposition 2** ((Green and Psaroudakis 2014, Prop. 2.4)) *Let* $\Delta_{(\phi,\psi)} = \begin{pmatrix} A & {}_A N_B \\ {}_B M_A & B \end{pmatrix}$ *be a Morita ring. Then*
(i) *The functors* $\mathrm{T}_A$, $\mathrm{T}_B$, $\mathrm{H}_A$, *and* $\mathrm{H}_B$ *are fully faithful.*
(ii) *The four pairs* $(\mathrm{T}_A, \mathrm{U}_A)$, $(\mathrm{T}_B, \mathrm{U}_B)$, $(\mathrm{U}_A, \mathrm{H}_A)$, *and* $(\mathrm{U}_B, \mathrm{H}_B)$ *are adjoint pairs of functors.*
(iii) *The functors* $\mathrm{U}_A$ *and* $\mathrm{U}_B$ *are exact.*

**Lemma 3** *Let* $\Delta_{(0,0)}$ *be Morita ring.*

1. (Krylov and Tuganbaev 2010, Theorem 7.3) *A left* $\Delta_{(0,0)}$*-module* $(P, Q, f, g)$ *is projective if and only if* $(P, Q, f, g) = \mathrm{T}_A(X) \oplus \mathrm{T}_B(Y) = (X, M \otimes_A X, 1, 0) \oplus (Y, N \otimes_B Y, Y, 0, 1)$ *for some projective left A-module X and projective left B-module Y.*
2. (Müller 1987, Corollary 2.2) *A left* $\Delta_{(0,0)}$*-module* $(I, J, f, g)$ *is injective if and only if* $(I, J, f, g) = \mathrm{H}_A(X) \oplus \mathrm{H}_B(Y) = (X, \mathrm{Hom}_A(N, X), 0, \epsilon_X) \oplus (\mathrm{Hom}_B(M, Y), Y, \epsilon_Y, 0)$ *for some injective left A-module X and injective left B-module Y.*

## 3 Strongly Gorenstein-projective modules

The aim of this section is to explicitly describe strongly Gorenstein-projective modules over a Morita ring $\Delta_{(0,0)}(A, B, M, N) = \begin{pmatrix} A & {}_A N_B \\ {}_B M_A & B \end{pmatrix}$.

The following lemmas are required in order to prove the main theorem of the paper.

**Lemma 4** *Let A be a ring and M a B-A-bimodule with a finite flat dimension. If a complex of flat A-modules* $\mathscr{F}^\bullet$ *is exact, then the sequence* $M \otimes_A \mathscr{F}^\bullet$ *is also exact.*

**Lemma 5** *Let $\Delta_{(0,0)}$ be a Morita ring with zero bimodule homomorphisms. Then*

1. (Gao and Psaroudakis 2017, Lemma 3.8) *For each $X \in A$-Mod and each $Y \in B$-Mod, we have the following exact sequences in $\Delta_{(0,0)}$-Mod:*

$$0 \longrightarrow \mathrm{Z}_B(M \otimes_A X) \longrightarrow \mathrm{T}_A(X) \longrightarrow \mathrm{Z}_A(X) \longrightarrow 0$$

*and*

$$0 \longrightarrow \mathrm{Z}_A(N \otimes_B Y) \longrightarrow \mathrm{T}_B(Y) \longrightarrow \mathrm{Z}_B(Y) \longrightarrow 0.$$

2. (Gao and Psaroudakis 2017, Lemma 3.9) *For all $X, X' \in A$-Mod and $Y, Y' \in B$-Mod, we have the following isomorphisms:*

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(X) \oplus \mathrm{T}_B(Y), \mathrm{Z}_A(X')) \cong \mathrm{Hom}_A(X, X')$$

*and*

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(X) \oplus \mathrm{T}_B(Y), \mathrm{Z}_B(Y')) \cong \mathrm{Hom}_B(Y, Y').$$

The following result provides sufficient conditions for the functors $\mathrm{T_A} : A$-Mod $\longrightarrow \Delta_{(0,0)}$-Mod and $\mathrm{T_B} : B$-Mod $\longrightarrow \Delta_{(0,0)}$-Mod to preserve strongly Gorenstein-projective modules.

**Proposition 6** *(1) Assume that $M_A$ has a finite flat dimension and that ${}_AN$ has a finite projective dimension. If $X$ is a strongly Gorenstein-projective $A$-module, then $\mathrm{T}_A(X)$ is a strongly Gorenstein-projective $\Delta_{(0,0)}$-module.*
*(2) Assume that $N_B$ has a finite flat dimension and that ${}_BM$ has a finite projective dimension. If $Y$ is a strongly Gorenstein-projective $B$-module, then $T_B(Y)$ is a strongly Gorenstein-projective $\Delta_{(0,0)}$-module.*

***Proof*** We only prove (1). The assertion (2) can be similarly proved. Since $X$ is a strongly Gorenstein-projective, there is an exact sequence of projective $A$-modules:

$$\mathscr{P}^\bullet : \cdots \xrightarrow{d} P \xrightarrow{d} P \xrightarrow{d} P \xrightarrow{d} \cdots$$

such that $X \cong \mathrm{Ker}\, d$, and $\mathrm{Hom}_A(\mathscr{P}^\bullet, Q)$ exact for any projective $A$-module $Q$. Since $M_A$ has a finite flat dimension, by Lemma 4, the sequence $M \otimes_A \mathscr{P}^\bullet$ is exact. Hence, we get the exact sequence of projective $\Delta_{(0,0)}$-modules:

$$\mathrm{T}_A(\mathscr{P}^\bullet) : \cdots \xrightarrow{(d,1\otimes d)} \mathrm{T}_A(P) \xrightarrow{(d,1\otimes d)} \mathrm{T}_A(P) \xrightarrow{(d,1\otimes d)} \mathrm{T}_A(P) \xrightarrow{(d,1\otimes d)} \cdots$$

such that $\mathrm{T}_A(X) \cong \mathrm{Ker}(d, 1 \otimes d)$. It is now left to show that $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(\mathscr{P}^\bullet), (X', Y', f', g'))$ is exact for any projective $\Delta_{(0,0)}$-module $(X', Y', f', g')$. From Lemma 3, we see that it suffices to show the exactness of $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(\mathscr{P}^\bullet), \mathrm{T}_A(P))$ and $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(\mathscr{P}^\bullet), \mathrm{T}_B(Q))$ for any projective $A$-module $P$, and any projective $B$-module $Q$. By Proposition 2, the functor $\mathrm{T}_A$ is fully faithful. Thus,

$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(\mathscr{P}^\bullet), \mathrm{T}_A(P)) \cong \mathrm{Hom}_A(\mathscr{P}^\bullet, P)$. Hence $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(\mathscr{P}^\bullet), \mathrm{T}_A(P))$ is exact because $\mathrm{Hom}_A(\mathscr{P}^\bullet, P)$ is exact. Since $(T_A, U_A)$ are adjoint pairs, we have

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(\mathscr{P}^\bullet), \mathrm{T}_B(Q)) \cong \mathrm{Hom}_A(\mathscr{P}^\bullet, N \otimes_B Q).$$

A module $N \otimes_B Q$ has a finite projective dimension because it is isomorphic to a direct summand of direct sums of copies of $N$. Since $\mathscr{P}^\bullet$ is a strongly complete $A$-projective resolution, the complex $\mathrm{Hom}_A(\mathscr{P}^\bullet, N \otimes_B Q)$ is exact(see (Holm 2004, Proposition 2.3)). Thus, $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(\mathscr{P}^\bullet), \mathrm{T}_B(Q))$ is exact. Hence, $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathrm{T}_A(\mathscr{P}^\bullet), (X', Y', f', g'))$ is exact for any projective $\Delta_{(0,0)}$-module $(X', Y', f', g')$. Therefore, $\mathrm{T}_A(X)$ is a strongly Gorenstein-projective $\Delta_{(0,0)}$-module. □

In the following result, we give sufficient conditions for a $\Delta_{(0,0)}$-module $(X, Y, f, g)$ to be strongly Gorenstein-projective.

**Theorem 7** *Assume that* $\mathrm{fd}(M_A) < \infty$, $\mathrm{fd}(N_B) < \infty$, $\mathrm{pd}({}_AM) < \infty$ *and* $\mathrm{pd}({}_AN) < \infty$. *Let* $(X, Y, f, g)$ *be a* $\Delta_{(0,0)}$*-module such that* $M \otimes_A \mathrm{Coker}g \cong \mathrm{Im}f$ *and* $N \otimes_B \mathrm{Coker}f \cong \mathrm{Im}g$. *Then* $(X, Y, f, g)$ *is a strongly Gorenstein-projective* $\Delta_{(0,0)}$*-module if the following conditions hold:*

1. $\mathrm{Coker}g$ *is a strongly Gorenstein-projective left* $A$*-module, i.e., there exists a strongly complete projective resolution* $\cdots \xrightarrow{k} P \xrightarrow{k} P \xrightarrow{k} P \xrightarrow{k} \cdots$ *with* $\mathrm{Coker}g \cong \mathrm{Ker}k$.
2. $\mathrm{Coker}f$ *is a strongly Gorenstein-projective left* $B$*-module, i.e., there exists a strongly complete projective resolution* $\cdots \xrightarrow{h} Q \xrightarrow{h} Q \xrightarrow{h} Q \xrightarrow{h} \cdots$ *with* $\mathrm{Coker}f \cong \mathrm{Ker}h$.
3. *There exist* $\rho : X \longrightarrow N \otimes_B Q$ *and* $\nu : P \longrightarrow X$ *such that* $\rho i_1 = \mathrm{Id}_N \otimes i$, $\pi_1 \nu = \delta$ *and* $\mathrm{Ker}\begin{pmatrix} k & 0 \\ \rho\nu & \mathrm{Id}_N\otimes h \end{pmatrix} = \mathrm{Im}\begin{pmatrix} k & 0 \\ \rho\nu & \mathrm{Id}_N\otimes h \end{pmatrix}$, *where* $i : \mathrm{Coker}f \longrightarrow Q$ *and* $i_1 : N \otimes_B \mathrm{Coker}f \longrightarrow X$ *are monomorphisms,* $\delta : P \longrightarrow \mathrm{Coker}g$ *and* $\pi_1 : X \longrightarrow \mathrm{Coker}g$ *are epimorphisms and* $\mathrm{Ker}\begin{pmatrix} k & 0 \\ \rho\nu & \mathrm{Id}_N\otimes h \end{pmatrix} \in \mathrm{End}(P \oplus N \otimes_B Q)$.
4. *There exist* $\varepsilon : Y \longrightarrow M \otimes_A P$ *and* $\theta : Q \longrightarrow Y$ *such that* $\varepsilon i_2 = \mathrm{Id}_M \otimes \gamma$, $\pi_2\theta = \omega$ *and* $\mathrm{Ker}\begin{pmatrix} rmId_M\otimes k & \varepsilon\theta \\ 0 & h \end{pmatrix} = \mathrm{Im}\begin{pmatrix} \mathrm{Id}_M\otimes k & \varepsilon\theta \\ 0 & h \end{pmatrix}$, *where* $\gamma : \mathrm{Coker}g \longrightarrow P$ *and* $i_2 : M \otimes_A \mathrm{Coker}g \longrightarrow Y$ *are monomorphisms,* $\omega : Q \longrightarrow \mathrm{Coker}f$ *and* $\pi_2 : Y \longrightarrow \mathrm{Coker}f$ *are epimorphisms and* $\begin{pmatrix} \mathrm{Id}_M\otimes k & \varepsilon\theta \\ 0 & h \end{pmatrix} \in \mathrm{End}(M \otimes_A P \oplus Q)$.

***Proof*** By (1), there is a strongly complete projective resolution in $A$-Mod:

$$\mathscr{P}^\bullet : \cdots \xrightarrow{k} P \xrightarrow{k} P \xrightarrow{k} P \xrightarrow{k} \cdots \tag{3.1}$$

with $\mathrm{Ker}k \cong \mathrm{Coker}g$. Write $\gamma : \mathrm{Coker}g \longrightarrow P$ to be the obvious monomorphism and $\delta : P \longrightarrow \mathrm{Coker}g$ the obvious epimorphism such that $\gamma\delta = k$. Since $\mathrm{fd}(M_A) < \infty$, the sequence

$$\cdots \xrightarrow{\mathrm{Id_M}\otimes k} M \otimes_A P \xrightarrow{\mathrm{Id_M}\otimes k} M \otimes_A P \xrightarrow{\mathrm{Id_M}\otimes k} M \otimes_A P \xrightarrow{\mathrm{Id_M}\otimes k} \cdots \tag{3.2}$$

is exact.

By (2), there is a strongly complete projective resolution in $B$-Mod:

$$\mathscr{Q}^\bullet : \cdots \xrightarrow{h} Q \xrightarrow{h} Q \xrightarrow{h} Q \xrightarrow{h} \cdots \tag{3.3}$$

with $\mathrm{Ker}h \cong \mathrm{Coker}f$. Write $i : \mathrm{Coker}f \longrightarrow Q$ to be the obvious monomorphism and $\omega : Q \longrightarrow \mathrm{Coker}f$ the obvious epimorphism such that $i\omega = h$. Since $\mathrm{fd}(N_B) < \infty$, the sequence

$$\cdots \xrightarrow{\mathrm{Id_N}\otimes h} N \otimes_B Q \xrightarrow{\mathrm{Id_N}\otimes h} N \otimes_B Q \xrightarrow{\mathrm{Id_N}\otimes h} N \otimes_B Q \xrightarrow{\mathrm{Id_N}\otimes h} \cdots \tag{3.4}$$

is exact.

Let $\pi_1 : X \longrightarrow \mathrm{Coker}g$ and $\pi_2 : Y \longrightarrow \mathrm{Coker}f$. Consider the following commutative diagram of $A$-modules:

$$\begin{tikzcd}
N \otimes_B M \otimes_A X \arrow[r, "\mathrm{Id}_N\otimes f"] \arrow[d, "\psi\otimes\mathrm{Id}_X"'] & N \otimes_B Y \arrow[r, "\mathrm{Id}_N\otimes\pi_2"] \arrow[d, "g"] & N \otimes_B \mathrm{Coker}f \arrow[r] \arrow[dl, dashed, "i_1"'] & 0 \\
A \otimes_A X \arrow[r, "\cong"] & X & &
\end{tikzcd}$$

Since $\psi = 0$, by the above diagram, there exists a unique $A$-map $i_1 : N \otimes_B \mathrm{Coker}f \longrightarrow X$ such that $g = i_1 \circ (\mathrm{Id}_N \otimes \pi_2)$. From $\mathrm{Im}g \cong N \otimes_B \mathrm{Coker}f$, it follows that $i_1$ is an injective $A$-map. Thus, we get the exact sequence

$$0 \longrightarrow N \otimes_B \mathrm{Coker}f \xrightarrow{i_1} X \xrightarrow{\pi_1} \mathrm{Coker}g \longrightarrow 0. \tag{3.5}$$

Similarly, there exists an exact sequence

$$0 \longrightarrow M \otimes_A \mathrm{Coker}g \xrightarrow{i_2} Y \xrightarrow{\pi_2} \mathrm{Coker}f \longrightarrow 0 \tag{3.6}$$

such that $f = i_2 \circ (\mathrm{Id}_M \otimes \pi_1)$.

By (3), there exist $\rho : X \longrightarrow N \otimes_B Q$ and $\nu : P \longrightarrow X$ such that $\rho i_1 = \mathrm{Id}_N \otimes i$, $\pi_1\nu = \delta$ and $\mathrm{Ker}\begin{pmatrix} k & 0 \\ \rho\nu & \mathrm{Id}_N\otimes h \end{pmatrix} = \mathrm{Im}\begin{pmatrix} k & 0 \\ \rho\nu & \mathrm{Id}_N\otimes h \end{pmatrix}$.

Define $\tau : P \oplus N \otimes_B Q \longrightarrow X$ by

$$\tau(x, y) = \nu(x) + i_1(\mathrm{Id_N} \otimes \omega)(y),\ x \in P,\ y \in N \otimes_B Q$$

and define $\eta : X \longrightarrow P \oplus N \otimes_B Q$ by

$$\eta(z) = (\gamma\pi_1(z), \rho(z)),\ z \in X.$$

Then we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N \otimes_B Q \arrow[r, "\lambda_1"] \arrow[d, "\mathrm{Id}_N \otimes \omega"'] & P \oplus N \otimes_B Q \arrow[r, "\rho_1"] \arrow[d, "\tau"'] & P \arrow[r] \arrow[d, "\delta"'] & 0 \\
0 \arrow[r] & N \otimes_B \mathrm{Coker} f \arrow[r, "i_1"] \arrow[d, "\mathrm{Id}_N \otimes i"'] & X \arrow[r, "\pi_1"] \arrow[d, "\eta"'] & \mathrm{Coker} g \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & N \otimes_B Q \arrow[r, "\lambda_1"] & P \oplus N \otimes_B Q \arrow[r, "\rho_1"] & P \arrow[r] & 0
\end{tikzcd}$$

By (3.2), the first column of the above diagram is exact. Thus, $\mathrm{Id}_N \otimes \omega$ is an epimorphism. Since $\delta$ is also epimorphism, $\tau$ is an epimorphism. Similarly, since $\mathrm{Id}_N \otimes i$ and $\gamma$ are monomorphism, $\eta$ is a monomorphism.

For any $x \in P$ and $y \in N \otimes_B Q$, we have

$$\begin{aligned}
\eta\tau(x, y) &= (\gamma\pi_1(\nu(x) + i_1(\mathrm{Id}_{\mathrm{N}} \otimes \omega)(y)), \rho(\nu(x) + i_1(\mathrm{Id}_{\mathrm{N}} \otimes \omega)(y))) \\
&= (\gamma\pi_1\nu(x) + \gamma\pi_1 i_1(\mathrm{Id}_N \otimes \omega)(y), \rho\nu(x) + \rho i_1(\mathrm{Id}_N \otimes \omega)(y)) \\
&= (\gamma\delta(x) + 0, \rho\nu(x) + (\mathrm{Id}_N \otimes i)(\mathrm{Id}_N \otimes \omega)(y)) \\
&= (\gamma\delta(x), \rho\nu(x) + (\mathrm{Id}_N \otimes i\omega)(y)) \\
&= (k(x), \rho\nu(x) + (\mathrm{Id}_N \otimes h)(y)) \\
&= \left(\begin{smallmatrix} k & 0 \\ \rho\nu & \mathrm{Id}_N \otimes h \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)
\end{aligned}$$

Then $\eta\tau = \left(\begin{smallmatrix} k & 0 \\ \rho\nu & \mathrm{Id}_N \otimes h \end{smallmatrix}\right)$. Hence $\mathrm{Ker}(\eta\tau) = \mathrm{Im}(\eta\tau)$. Thus, we get the following exact sequence in $A$-Mod:

$$\mathcal{W}^\bullet : \cdots \xrightarrow{\eta\tau} P \oplus N \otimes_B Q \xrightarrow{\eta\tau} P \oplus N \otimes_B Q \xrightarrow{\eta\tau} P \oplus N \otimes_B Q \xrightarrow{\eta\tau} \cdots . \tag{3.7}$$

By (4), there exist $\varepsilon : Y \longrightarrow M \otimes_A P$ and $\theta : Q \longrightarrow Y$ such that $\varepsilon i_2 = \mathrm{Id}_M \otimes \gamma$, $\pi_2\theta = \omega$ and $\mathrm{Ker} \left(\begin{smallmatrix} \mathrm{Id}_M \otimes k & \varepsilon\theta \\ 0 & h \end{smallmatrix}\right) = \mathrm{Im} \left(\begin{smallmatrix} \mathrm{Id}_M \otimes k & \varepsilon\theta \\ 0 & h \end{smallmatrix}\right)$.

Define $\alpha : M \otimes_A p \oplus Q \longrightarrow Y$ by

$$\alpha(x, y) = i_2(\mathrm{Id}_M \otimes \delta)(x) + \theta(y), \ x \in M \otimes_A P, \ y \in Q$$

and define $\beta : Y \longrightarrow M \otimes_A P \oplus Q$ by

$$\beta(z) = (\varepsilon(z), i\pi_2(z)), \ z \in Y.$$

Then we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_A P \arrow[r, "\lambda_2"] \arrow[d, "\mathrm{Id}_M \otimes \delta"'] & M \otimes_A P \oplus Q \arrow[r, "\rho_2"] \arrow[d, "\alpha"'] & Q \arrow[r] \arrow[d, "\omega"'] & 0 \\
0 \arrow[r] & M \otimes_A \mathrm{Coker} g \arrow[r, "i_2"] \arrow[d, "\mathrm{Id}_M \otimes \gamma"'] & Y \arrow[r, "\pi_2"] \arrow[d, "\beta"'] & \mathrm{Coker} f \arrow[r] \arrow[d, "i"'] & 0 \\
0 \arrow[r] & M \otimes_A p \arrow[r, "\lambda_2"] & M \otimes_A P \oplus Q \arrow[r, "\rho_2"] & Q \arrow[r] & 0
\end{tikzcd}$$

By (3.4), the first column of the above diagram is exact. Thus, $\mathrm{Id}_M \otimes \delta$ is an epimorphism. Since $\omega$ is also epimorphism, $\alpha$ is an epimorphism. Similarly, since $\mathrm{Id}_M \otimes \gamma$ and $i$ are monomorphisms, $\beta$ is a monomorphism.

For any $x \in M \otimes_A P$ and $y \in Q$, we have

$$\begin{aligned}
\beta\alpha(x, y) &= (\varepsilon(i_2(\mathrm{Id}_M \otimes \delta)(x) + \theta(y)), i\pi_2(i_2(\mathrm{Id}_M \otimes \delta)(x) + \theta(y))) \\
&= (\varepsilon i_2(\mathrm{Id}_M \otimes \delta)(x) + \varepsilon\theta(y), i\pi_2 i_2(\mathrm{Id}_M \otimes \delta)(x) + i\pi_2\theta(y)) \\
&= ((\mathrm{Id}_M \otimes \gamma)(\mathrm{Id}_M \otimes \delta)(x) + \varepsilon\theta(y), 0 + i\omega(y)) \\
&= ((\mathrm{Id}_M \otimes \gamma\delta)(x) + \varepsilon\theta(y), 0 + i\omega(y)) \\
&= ((\mathrm{Id}_M \otimes k) + \varepsilon\theta(y), h(y)) \\
&= \begin{pmatrix} \mathrm{Id}_M \otimes k & \varepsilon\theta \\ 0 & h \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

Then $\beta\alpha = \begin{pmatrix} \mathrm{Id}_M \otimes k & \varepsilon\theta \\ 0 & h \end{pmatrix}$. Hence $\mathrm{Ker}(\beta\alpha) = \mathrm{Im}(\beta\alpha)$. Thus, we get the following exact sequence in $B$-Mod:

$$\mathcal{U}^\bullet : \cdots \xrightarrow{\beta\alpha} M \otimes_A P \oplus Q \xrightarrow{\beta\alpha} M \otimes_A P \oplus Q \xrightarrow{\beta\alpha} M \otimes_A P \oplus Q \xrightarrow{\beta\alpha} \cdots . \tag{3.8}$$

By Lemma 1(ii), we obtain the following exact sequence in $\Delta_{(0,0)}$-Mod from (3.7) and (3.8):

$$\mathscr{T}^\bullet : \cdots \xrightarrow{(\eta\tau\ \beta\alpha)} \mathrm{T}_A(P) \oplus \mathrm{T}_B(Q) \xrightarrow{(\eta\tau\ \beta\alpha)} \mathrm{T}_A(P) \oplus \mathrm{T}_B(Q) \xrightarrow{(\eta\tau\ \beta\alpha)} \cdots \tag{3.9}$$

with $\mathrm{Ker}(\eta\tau\ \beta\alpha) = (X, Y, f, g)$.

Now we are left to show that $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, (X', Y', f', g'))$ is exact for each projective $\Delta_{(0,0)}$-module $(X', Y', f', g')$. From Lemma 3, we can infer that it suffices to show that $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{T}_A(P))$ and $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{T}_B(Q))$ are exact for each projective $A$-module $P$ and for each projective $B$-module $Q$. By Lemma 5(1), the sequence $0 \longrightarrow \mathrm{Z}_B(M \otimes_A P) \longrightarrow \mathrm{T}_A(P) \longrightarrow \mathrm{Z}_A(P) \longrightarrow 0$ is exact. Since each term in the complex $\mathscr{T}^\bullet$ is a projective $\Delta_{(0,0)}$-module, the sequence

$$0 \longrightarrow \mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{Z}_B(M \otimes_A P)) \longrightarrow \mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{T}_A(P)) \longrightarrow$$

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{Z}_A(P)) \longrightarrow 0 \tag{3.10}$$

is exact. By Lemma 5(2), we have

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{Z}_A(P)) \cong \mathrm{Hom}_A(\mathscr{P}^\bullet, P).$$

The complex $\mathrm{Hom}_A(\mathscr{P}^\bullet, P)$ is exact because $\mathscr{P}^\bullet$ is a complete projective resolution. The complex $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{Z}_A(P))$ is, therefore, exact. Lemma 5(2) also gives us

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{Z}_B(M \otimes_A P)) \cong \mathrm{Hom}_B(\mathscr{Q}^\bullet, M \otimes_A P).$$

To show the exactness of $\mathrm{Hom}_B(\mathscr{Q}^\bullet, M \otimes_A P)$, we know that a $B$-module $M \otimes_A P$ has a finite projective dimension since $M \otimes_A P$ is isomorphic to the direct summand of a direct sum of copies of $M$. Thus, $\mathrm{Hom}_B(\mathscr{Q}^\bullet, M \otimes_A P)$ is exact by (Holm 2004, Proposition 2.3), which implies $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{Z}_B(M \otimes_A P))$ is exact. Hence, from the exact sequence of complexes in (3.10), it follows that the complex $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{T}_A(P))$ is exact. Similarly, the complex

$$\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, \mathrm{T}_B(Q))$$

is exact. Thus, $\mathrm{Hom}_{\Delta_{(0,0)}}(\mathscr{T}^\bullet, (X', Y', f', g'))$ is exact for each projective $\Delta_{(0,0)}$-module $(X', Y', f', g')$. Therefore, a $\Delta_{(0,0)}$-module $(X, Y, f, g)$ is a strongly Gorenstein-projective. □

## 4 The case of $\Delta_{(\phi,\phi)}(A) = \mathrm{M}_2(A)$ with $\phi(1 \otimes 1) = 1$

In this section, we consider the Morita ring $\Delta_{(\phi,\phi)}(A) = \left(\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right)$ with $\phi(1 \otimes 1) = 1$. As a result, $\Delta_{(\phi,\phi)}(A) = \mathrm{M}_2(A)$, the $2 \times 2$ matrix algebra over $A$. We will describe all strongly Gorenstein-projective $\mathrm{M}_2(A)$-modules.

From (2.1) and (2.2), it follows that any $\mathrm{M}_2(A)$-module is $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_{(f,g)}$, where $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are $A$-maps, such that $gf = \mathrm{Id}_X = fg$. Thus, we prefer to write any $\mathrm{M}_2(A)$-module as $\left(\begin{smallmatrix} X \\ X \end{smallmatrix}\right)_f$, where $f : X \longrightarrow X$ is an $A$-isomorphism. The action is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+f^{-1}(ny) \\ f(mx)+by \end{pmatrix}.$$

By (2.3) and (2.4), it follows that any $\mathrm{M}_2(A)$-map is of the form

$$\begin{pmatrix} a \\ b \end{pmatrix} : \begin{pmatrix} X \\ X \end{pmatrix}_f \longrightarrow \begin{pmatrix} X' \\ X' \end{pmatrix}_{f'},$$

where $a : X \longrightarrow X'$ and $b : X \longrightarrow X'$ are $A$-maps such that $b = f'af^{-1}$. Thus, for any $\mathrm{M}_2(A)$-module $\left(\begin{smallmatrix} X \\ X \end{smallmatrix}\right)_f$, we have $\mathrm{M}_2(A)$-isomorphism

$$\begin{pmatrix} \mathrm{Id}_X \\ f^{-1} \end{pmatrix} : \begin{pmatrix} X \\ X \end{pmatrix}_f \longrightarrow \begin{pmatrix} X \\ X \end{pmatrix}_{\mathrm{Id}_X}$$

Thus, any $\mathrm{M}_2(A)$-module will be simply written as $\binom{X}{X}$, and the action is given just by the usual multiplication of the matrices. Furthermore, every $\mathrm{M}_2(A)$-map has the form

$$\binom{a}{a} : \binom{X}{X} \longrightarrow \binom{X'}{X'}$$

where $a : X \longrightarrow X'$ is an $A$-map.

In particular, the projective $\mathrm{M}_2(A)$-modules are exactly

$$\mathrm{T}_1(P) = (P,\ P,\ \mathrm{Id}_P,\ \phi_P) \text{ or } \mathrm{T}_2(Q) = (Q,\ Q,\ \phi_Q,\ \mathrm{Id}_Q)$$

where $P$ and $Q$ run over projective $A$-modules. Of course, $\mathrm{T}_2(P) = \mathrm{T}_1(P)$. That is, the projective $\mathrm{M}_2(A)$-modules are exactly $\mathrm{T}_1(P) = \binom{P}{P}$ with the action given by the matrix multiplication, where $P$ ranges over projective $A$-modules. Thus any $\mathrm{M}_2(A)$-map $f : \binom{P}{P} \longrightarrow \binom{P}{P}$ is of the form

$$f := \binom{\alpha}{\alpha} : \binom{P}{P} \longrightarrow \binom{P}{P}$$

where $\alpha : P \longrightarrow P$ ranges over $A$-maps.

We now explicitly describe strongly Gorenstein-projective modules over $\mathrm{M}_2(A)$.

**Proposition 8** *The* $\mathrm{M}_2(A)$*-module* $\binom{K}{K}$ *is a strongly Gorenstein-projective* $\mathrm{M}_2(A)$*-module with the strongly complete* $\mathrm{M}_2(A)$*-projective resolution*

$$\mathscr{X}^\bullet = \cdots \xrightarrow{\binom{\alpha}{\alpha}} \binom{P}{P} \xrightarrow{\binom{\alpha}{\alpha}} \binom{P}{P} \xrightarrow{\binom{\alpha}{\alpha}} \cdots$$

*if and only if $K$ is a strongly Gorenstein-projective $A$-module with the strongly complete $A$-projective resolution*

$$\mathscr{P}^\bullet = \cdots \xrightarrow{\alpha} P \xrightarrow{\alpha} P \xrightarrow{\alpha} \cdots$$

*Moreover, any strongly complete* $\mathrm{M}_2(A)$*-projective resolution and any strongly Gorenstein-projective* $\mathrm{M}_2(A)$*-module is obtained in the above way.*

***Proof*** Suppose that $\binom{K}{K}$ is a strongly Gorenstein-projective $\mathrm{M}_2(A)$-module with the strongly complete $\mathrm{M}_2(A)$-projective resolution

$$\mathscr{X}^\bullet = \cdots \xrightarrow{\binom{\alpha^{-1}}{\alpha^{-1}}} \binom{P^0}{P^0} \xrightarrow{\binom{\alpha^0}{\alpha^0}} \binom{P^1}{P^1} \xrightarrow{\binom{\alpha^1}{\alpha^1}} \cdots .$$

Then we get the exact sequence

$$\mathscr{P}^\bullet = \cdots \xrightarrow{\alpha^{-1}} P^0 \xrightarrow{\alpha^0} P^1 \xrightarrow{\alpha^1} \cdots .$$

Since

$$\mathrm{Hom}_{\mathrm{M}_2(A)}(\mathscr{X}^\bullet, \binom{A}{A}) \cong \mathrm{Hom}_{\mathrm{M}_2(A)}(\mathrm{T}_1(\mathscr{P}^\bullet), \mathrm{T}_1(A)) \cong \mathrm{Hom}_A(\mathscr{P}^\bullet, A)$$

it follows from the exactness of $\mathrm{Hom}_{\mathrm{M}_2(A)}(\mathscr{X}^\bullet, \binom{A}{A})$ that

$$\mathscr{P}^\bullet = \cdots \xrightarrow{\alpha^{-1}} P^0 \xrightarrow{\alpha^0} P^1 \xrightarrow{\alpha^1} \cdots$$

is a complete $A$-projective resolution. Thus, $K$ is a Gorenstein-projective $A$-module with the complete $A$-projective resolution

$$\mathscr{P}^\bullet = \cdots \xrightarrow{\alpha^{-1}} P^0 \xrightarrow{\alpha^0} P^1 \xrightarrow{\alpha^1} \cdots .$$

If $\mathscr{P}^\bullet$ is a complete $A$-projective resolution, then by the same argument $\mathscr{X}^\bullet$ is a complete $\mathrm{M}_2(A)$-projective resolution. □

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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