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ORIGINAL PAPER

Rotational surfaces in a 3-dimensional normed space

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Abstract
We study rotational surfaces with constant Minkowski Gaussian curvature and rotational surfaces with constant Minkowski mean curvature in a 3-dimensional normed space with rotationally symmetric norm. We have a generalization of the catenoid, pseudo-sphere and Delaunay surfaces.

Keywords Rotational surface · Normed space · Birkhoff orthogonal · Birkhoff-Gauss map · Minkowski Gaussian curvature · Minkowski mean curvature

Mathematics Subject Classification 53A35 · 53A10 · 52A15 · 52A21 · 46B20

1 Introduction

It is interesting to generalize differential geometry of curves and surfaces in Euclidean spaces to that in normed spaces, or generally, in gauge spaces (cf. Balestro et al. 2019a, b, 2020a, b, c, d, 2021; Busemann 1950; Guggenheimer 1965), where how to compensate for the lack of the notion of angle is the problem, and the notion of Birkhoff orthogonality plays an important role. For surfaces in 3-dimensional normed spaces, the notions of Birkhoff-Gauss map, Minkowski Gaussian curvature and Minkowski mean curvature are particularly important (cf. Balestro et al. 2019b, 2020c, d, 2021).

In this paper, we study rotational surfaces in a 3-dimensional normed space with rotationally symmetric norm, in particular, rotational surfaces with constant Minkowski Gaussian curvature and rotational surfaces with constant Minkowski mean curvature.

This paper is organized as follows. In Sect. 2, following Balestro et al. (2020c), we recall some basic facts on surfaces in 3-dimensional normed spaces. In Sect. 3, we give a basic computation for rotational surfaces in a 3-dimensional normed space with rotationally symmetric norm. In Sect. 4, we consider rotational minimal surfaces in the

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3-dimensional normed space. In Sect. 5, we discuss rotational surfaces with non-zero constant Minkowski Gaussian curvature in the 3-dimensional normed space. In Sect. 6, we study rotational surfaces with non-zero constant Minkowski mean curvature in the 3-dimensional normed space, which can be seen as a generalization of the Delaunay surfaces (Delaunay 1841).

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2 Preliminaries

In this section, following Balestro et al. (2020c), we recall some basic facts on surfaces in 3-dimensional normed spaces.

Let $(\mathbb{R}^3, \|\cdot\|)$ be a 3-dimensional normed space whose unit ball B and unit sphere S are defined by

$$B = \{x \in \mathbb{R}^3; \|x\| \leq 1\}, \quad S = \{x \in \mathbb{R}^3; \|x\| = 1\}.$$

In the following, we assume that S is smooth and strictly convex, that is, S is a smooth surface and S contains no line segment.

Remark. We do not assume that S has positive Euclidean Gaussian curvature as in Balestro et al. (2020c), because we treat the case where S has points with zero Euclidean Gaussian curvature.

Let v be a non-zero vector in $\mathbb{R}^3$ and Π be a plane in $\mathbb{R}^3$. We say that v is Birkhoff orthogonal to Π (denoted by $v \dashv_B \Pi$) if the tangent plane of S at $v/\|v\|$ is Π.

Let M be a surface immersed in $(\mathbb{R}^3, \|\cdot\|)$. Let T_pM be the tangent plane of M at $p \in M$. There exists a vector $\eta(p) \in S$ such that $\eta(p) \dashv_B T_pM$, which gives a local smooth map $\eta : U \subset M \to S$ called the Birkhoff-Gauss map. It can be global if and only if M is orientable. We define the Minkowski Gaussian curvature K and the Minkowski mean curvature H of M at p by

$$K = \det\left(d\eta_p\right), \quad H = \frac{1}{2}\operatorname{trace}\left(d\eta_p\right).$$

We say that M is flat if $K = 0$ identically, and minimal if $H = 0$ identically.

A surface which is homothetic to the unit sphere S is called a Minkowski sphere. A Minkowski sphere has positive constant Minkowski Gaussian curvature and non-zero constant Minkowski mean curvature.

3 Rotational surfaces

Let

$$S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | \left(x_1^2 + x_2^2\right)^m + x_3^{2m} = 1\right\}$$

where m is a positive integer. It is given by rotating $x_1^{2m} + x_3^{2m} = 1$ around x_3-axis. Then there exists a norm $\| \cdot \|$ on $\mathbb{R}^3$ whose unit sphere is the above S. Set

$$\Phi(x_1, x_2, x_3) := \left(x_1^2 + x_2^2\right)^m + x_3^{2m}.$$

Throughout this paper, we consider this 3-dimensional normed space $(\mathbb{R}^3, \| \cdot \|)$. The case where $m = 1$ is the Euclidean case. We assume that $m \geq 2$ in the following.

Let M be a surface in $(\mathbb{R}^3, \| \cdot \|)$ which is rotational around x_3-axis, and is parametrized by

$$f(u, v) = (\alpha(u) \cos v, \alpha(u) \sin v, \beta(u))$$

where $\alpha > 0$, $\alpha' \neq 0$ and $\beta' \neq 0$. Then

$$f_u = \left(\alpha' \cos v, \alpha' \sin v, \beta'\right), \quad f_v = (-\alpha \sin v, \alpha \cos v, 0).$$

The Birkhoff-Gauss map $\eta = \eta(u, v)$ is characterized by the condition

$$(\mathrm{grad}(\Phi))_\eta = \left(\frac{\partial \Phi}{\partial x_1}(\eta), \frac{\partial \Phi}{\partial x_2}(\eta), \frac{\partial \Phi}{\partial x_3}(\eta)\right) = \mu f_u \times f_v,$$

where μ is a positive function and $\times$ denotes the standard cross product in $\mathbb{R}^3$. Then we can get

$$\eta = A^{-\frac{1}{2m}} \left(-(\beta')^{\frac{1}{2m-1}} \cos v, -(\beta')^{\frac{1}{2m-1}} \sin v, (\alpha')^{\frac{1}{2m-1}}\right)$$

where

$$A := \left(\alpha'\right)^{\frac{2m}{2m-1}} + \left(\beta'\right)^{\frac{2m}{2m-1}}.$$

We can compute that

$$\eta_u = -\frac{1}{2m-1} A^{-\frac{2m+1}{2m}} (\alpha')^{-\frac{2m-2}{2m-1}} (\beta')^{-\frac{2m-2}{2m-1}} (\alpha' \beta'' - \alpha'' \beta') f_u \tag{3.1}$$

and

$$\eta_v = -\frac{1}{\alpha} A^{-\frac{1}{2m}} (\beta')^{\frac{1}{2m-1}} f_v. \tag{3.2}$$

Thus we have

$$K = \frac{1}{(2m-1)\alpha} A^{-\frac{m+1}{m}} (\alpha')^{-\frac{2m-2}{2m-1}} (\beta')^{-\frac{2m-3}{2m-1}} (\alpha' \beta'' - \alpha'' \beta') \tag{3.3}$$

and

$$H = -\frac{1}{2(2m-1)\alpha} A^{-\frac{2m+1}{2m}} (\beta')^{-\frac{2m-2}{2m-1}} \times \left\{ \alpha(\alpha')^{-\frac{2m-2}{2m-1}} (\alpha'\beta'' - \alpha''\beta') + (2m-1)A\beta' \right\}. \tag{3.4}$$

Letting $\beta(u) = u$, we have

$$\eta_u = \frac{1}{2m-1} \left((\alpha')^{\frac{2m}{2m-1}} + 1 \right)^{-\frac{2m+1}{2m}} (\alpha')^{-\frac{2m-2}{2m-1}} \alpha'' f_u, \tag{3.5}$$

$$\eta_v = -\frac{1}{\alpha} \left((\alpha')^{\frac{2m}{2m-1}} + 1 \right)^{-\frac{1}{2m}} f_v, \tag{3.6}$$

$$K = -\frac{1}{(2m-1)\alpha} \left((\alpha')^{\frac{2m}{2m-1}} + 1 \right)^{-\frac{m+1}{m}} (\alpha')^{-\frac{2m-2}{2m-1}} \alpha'', \tag{3.7}$$

and

$$H = \frac{1}{2(2m-1)\alpha} \left((\alpha')^{\frac{2m}{2m-1}} + 1 \right)^{-\frac{2m+1}{2m}} \times \left\{ \alpha(\alpha')^{-\frac{2m-2}{2m-1}} \alpha'' - (2m-1) \left((\alpha')^{\frac{2m}{2m-1}} + 1 \right) \right\}. \tag{3.8}$$

By (3.7), we see that $K = 0$ if and only if $\alpha'' = 0$. So we have the following.

Proposition 3.1 *A rotational surface in* $(\mathbb{R}^3, \|\cdot\|)$ *parametrized by*

$$f(u, v) = (\alpha(u)\cos v, \alpha(u)\sin v, u)$$

where $\alpha > 0$, $\alpha' \neq 0$ *is flat if and only if it is a circular cone.*

4 Rotational minimal surfaces

Let $(\mathbb{R}^3, \|\cdot\|)$ be the 3-dimensional normed space as in Sect. 3. Let M be a rotational surface in $(\mathbb{R}^3, \|\cdot\|)$ parametrized by

$$f(u, v) = (\alpha(u)\cos v, \alpha(u)\sin v, u)$$

where $\alpha > 0$ and $\alpha' \neq 0$.

By (3.8), the rotational surface M is minimal if and only if

$$\alpha(\alpha')^{-\frac{2m-2}{2m-1}} \alpha'' - (2m-1) \left((\alpha')^{\frac{2m}{2m-1}} + 1 \right) = 0. \tag{4.1}$$

From the Eq. (4.1), we have

$$\frac{2m-1}{\alpha} = \frac{\alpha''}{(\alpha')^2 + (\alpha')^{\frac{2m-2}{2m-1}}}.$$

Multiplying by $2\alpha'$ we have

$$2(2m-1)\frac{\alpha'}{\alpha} = \frac{((\alpha')^2)'}{(\alpha')^2 + (\alpha')^{\frac{2m-2}{2m-1}}},$$

and

$$2(2m-1)\log\alpha = \int \frac{((\alpha')^2)'}{(\alpha')^2 + (\alpha')^{\frac{2m-2}{2m-1}}} du.$$

Letting

$$(\alpha')^{\frac{2}{2m-1}} =: Z$$

for the right-hand side, we have

$$\begin{aligned} 2\log\alpha &= \int \frac{Z^{m-1}}{Z^m+1} dZ = \frac{1}{m}\log\left(Z^m+1\right) + c_1 \\ &= \frac{1}{m}\log\left((\alpha')^{\frac{2m}{2m-1}} + 1\right) + c_1 \end{aligned}$$

for a constant c_1. Then

$$\frac{d\alpha}{du} = \pm\frac{1}{c_2^{2m-1}}\left(\alpha^{2m} - c_2^{2m}\right)^{\frac{2m-1}{2m}}$$

for a positive constant c_2, and

$$u(\alpha) = \pm\int_{c_2}^{\alpha} \frac{c_2^{2m-1}}{\left(\rho^{2m} - c_2^{2m}\right)^{\frac{2m-1}{2m}}} d\rho + c_3$$

for a constant c_3, where $\alpha > c_2$. Here we note that since

$$0 < \frac{2m-1}{2m} < 1,$$

the above integral converges and

$$\lim_{\alpha\to c_2}\int_{c_2}^{\alpha} \frac{c_2^{2m-1}}{\left(\rho^{2m} - c_2^{2m}\right)^{\frac{2m-1}{2m}}} d\rho = 0.$$

Then we have the following.

Theorem 4.1 *A rotational surface in* $(\mathbb{R}^3, \|\cdot\|)$ *given by*

$$\bar{f}(\alpha, v) = (\alpha\cos v, \alpha\sin v, u(\alpha))$$

where $\alpha > 0$ *is minimal if and only if*

$$u(\alpha) = \pm\int_{c_2}^{\alpha} \frac{c_2^{2m-1}}{\left(\rho^{2m} - c_2^{2m}\right)^{\frac{2m-1}{2m}}} d\rho + c_3$$

for constants $c_2 > 0$ *and* c_3*, where* $\alpha > c_2$*.*

Now, let us set

$$u_{\pm}(\alpha) := \pm\int_{c_2}^{\alpha} \frac{c_2^{2m-1}}{\left(\rho^{2m} - c_2^{2m}\right)^{\frac{2m-1}{2m}}} d\rho + c_3$$

for constants $c_2 > 0$ and c_3, where $\alpha > c_2$, and we consider the behavior of the graph of $u_{\pm}(\alpha)$. Since $m \geq 2$,

$$\lim_{\alpha\to\infty}\int_{c_2}^{\alpha} \frac{c_2^{2m-1}}{\left(\rho^{2m} - c_2^{2m}\right)^{\frac{2m-1}{2m}}} d\rho = d_1$$

for some positive value d_1. So we have

$$\lim_{\alpha\to c_2} u_{\pm}(\alpha) = c_3, \quad \lim_{\alpha\to\infty} u_{+}(\alpha) = c_3 + d_1, \quad \lim_{\alpha\to\infty} u_{-}(\alpha) = c_3 - d_1.$$

The function $u_{+}(\alpha)$ is an increasing function and

$$\lim_{\alpha\to c_2} u'_{+}(\alpha) = \infty.$$

Similarly, $u_{-}(\alpha)$ is a decreasing function and

$$\lim_{\alpha\to c_2} u'_{-}(\alpha) = -\infty.$$

Let $\alpha_{+}(u)$ be the inverse function of $u_{+}(\alpha)$. It is an increasing function on (c_3, c_3+d_1) and

$$\lim_{u\to c_3} \alpha_{+}(u) = c_2, \qquad \lim_{u\to c_3+d_1} \alpha_{+}(u) = \infty, \quad \lim_{u\to c_3} \alpha'_{+}(u) = 0.$$

Let $\alpha_-(u)$ be the inverse function of $u_-(\alpha)$. It is a decreasing function on $(c_3 - d_1, c_3)$ and

$$\lim_{u \to c_3} \alpha_-(u) = c_2, \qquad \lim_{u \to c_3 - d_1} \alpha_-(u) = \infty, \qquad \lim_{u \to c_3} \alpha'_-(u) = 0.$$

We define a function $\hat{\alpha}(u)$ on $(c_3 - d_1, c_3 + d_1)$ by

$$\hat{\alpha}(u) = \begin{cases} \alpha_+(u), \ c_3 < u < c_3 + d_1 \\ \alpha_-(u), \ c_3 - d_1 < u < c_3 \, . \\ c_2, u = \qquad c_3 \end{cases}$$

Then $\hat{\alpha}(u)$ is a C^1-function on $(c_3 - d_1, c_3 + d_1)$ such that

$$\hat{\alpha}'(u) = \begin{cases} \alpha'_+(u), \ c_3 < u < c_3 + d_1 \\ \alpha'_-(u), \ c_3 - d_1 < u < c_3 \, . \\ 0, u = \qquad c_3 \end{cases}$$

For $u \in (c_3 - d_1, c_3) \cup (c_3, c_3 + d_1)$, $\hat{\alpha}(u)$ satisfies the Eq. (4.1). Then, noting that $m \geq 2$, we can see that

$$\lim_{u \to c_3} (\hat{\alpha}'(u))^{-\frac{2m-2}{2m-1}} \hat{\alpha}''(u) = \frac{2m-1}{c_2}$$

and

$$\lim_{u \to c_3} \hat{\alpha}''(u) = 0.$$

Thus the function $\hat{\alpha}(u)$ is a C^2-function on $(c_3 - d_1, c_3 + d_1)$. Also by (3.5) and (3.6), we find that the Birkhoff-Gauss map can be C^1-extended for $u \in (c_3 - d_1, c_3 + d_1)$.

Therefore, we have the following.

Theorem 4.2 *Under the notation above, the rotational surface in* $(\mathbb{R}^3, \| \cdot \|)$ *parametrized by*

$$\hat{f}(u, v) = (\hat{\alpha}(u) \cos v, \hat{\alpha}(u) \sin v, u), \quad (u, v) \in (c_3 - d_1, c_3 + d_1) \times [0, 2\pi]$$

is minimal.

Remark. The above surface can be seen as a generalization of the catenoid in the Euclidean 3-space. But we should note that the range of u is bounded.

5 Non-zero constant Gaussian curvature

Let $(\mathbb{R}^3, \|\cdot\|)$ be the 3-dimensional normed space as in Sect. 3. Let M be a rotational surface in $(\mathbb{R}^3, \|\cdot\|)$ parametrized by

$$f(u, v) = (\alpha(u)\cos v, \alpha(u)\sin v, u)$$

where $\alpha > 0$ and $\alpha' \neq 0$.

By (3.7), if K is a non-zero constant, then

$$-\frac{1}{2m-1}\left(\left(\alpha'\right)^{\frac{2m}{2m-1}}+1\right)^{-\frac{m+1}{m}}(\alpha')^{-\frac{2m-2}{2m-1}}\alpha'' = K\alpha. \tag{5.1}$$

Multiplying by $2\alpha'$ we have

$$-\frac{2}{2m-1}\left(\left(\alpha'\right)^{\frac{2m}{2m-1}}+1\right)^{-\frac{m+1}{m}}(\alpha')^{\frac{1}{2m-1}}\alpha'' = K(\alpha^2)'.$$

Integrating it we have

$$\left(\left(\alpha'\right)^{\frac{2m}{2m-1}}+1\right)^{-\frac{1}{m}} = K\alpha^2 + c_1 \ \ (>0)$$

for a constant c_1. Then

$$\frac{d\alpha}{du} = \pm\frac{\left\{1-(K\alpha^2+c_1)^m\right\}^{\frac{2m-1}{2m}}}{(K\alpha^2+c_1)^{\frac{2m-1}{2}}},$$

and we get the following.

Theorem 5.1 *A rotational surface in* $(\mathbb{R}^3, \|\cdot\|)$ *given by*

$$\bar{f}(\alpha, v) = (\alpha\cos v, \alpha\sin v, u(\alpha))$$

where $\alpha > 0$ *has non-zero constant Minkowski Gaussian curvature* K *if and only if*

$$u(\alpha) = \pm\int\frac{(K\alpha^2+c_1)^{\frac{2m-1}{2}}}{\left\{1-(K\alpha^2+c_1)^m\right\}^{\frac{2m-1}{2m}}}d\alpha$$

for a constant c_1.

Now, let us set

$$u_\pm(\alpha) := \pm\int\frac{(K\alpha^2+c_1)^{\frac{2m-1}{2}}}{\left\{1-(K\alpha^2+c_1)^m\right\}^{\frac{2m-1}{2m}}}d\alpha$$

for a constant c_1, and we discuss the behavior of the graph of $u_\pm(\alpha)$. It suffices to consider the case where $K = 1$ or $K = -1$.

(i) The case $K = 1$. We have $c_1 < 1$ and

$$u_\pm(\alpha) = \pm \int \frac{(\alpha^2 + c_1)^{\frac{2m-1}{2}}}{\left\{1 - (\alpha^2 + c_1)^m\right\}^{\frac{2m-1}{2m}}} d\alpha.$$

(i-1) The case $c_1 = 0$. In this case we have

$$u_\pm(\alpha) = \pm \int \frac{\alpha^{2m-1}}{\left(1 - \alpha^{2m}\right)^{\frac{2m-1}{2m}}} d\alpha = \mp(1 - \alpha^{2m})^{\frac{1}{2m}} + c_2$$

for a constant c_2. It satisfies

$$\alpha^{2m} + (u_\pm(\alpha) - c_2)^{2m} = 1.$$

So the resulting surface can be smoothly extended to a Minkowski sphere, which is a parallel translation of the unit sphere S.

(i-2) The case $0 < c_1 < 1$. In this case, we have $0 < \alpha < \sqrt{1 - c_1}$ and we can write

$$u_\pm(\alpha) = \pm \int_{\sqrt{1-c_1}}^{\alpha} \frac{(\rho^2 + c_1)^{\frac{2m-1}{2}}}{\left\{1 - (\rho^2 + c_1)^m\right\}^{\frac{2m-1}{2m}}} \, d\rho + c_3$$

for a constant c_3. Since

$$0 < \frac{2m-1}{2m} < 1,$$

the above integral converges. Set

$$d_1 := -\lim_{\alpha \to 0} \int_{\sqrt{1-c_1}}^{\alpha} \frac{(\rho^2 + c_1)^{\frac{2m-1}{2}}}{\left\{1 - (\rho^2 + c_1)^m\right\}^{\frac{2m-1}{2m}}} \, d\rho \ \ (> 0).$$

Then

$$\lim_{\alpha \to 0} u_+(\alpha) = c_3 - d_1, \quad \lim_{\alpha \to 0} u_-(\alpha) = c_3 + d_1, \quad \lim_{\alpha \to \sqrt{1-c_1}} u_\pm(\alpha) = c_3.$$

The function $u_+(\alpha)$ is an increasing function and

$$\lim_{\alpha \to \sqrt{1-c_1}} u'_+(\alpha) = \infty.$$

The function $u_-(\alpha)$ is a decreasing function and

$$\lim_{\alpha\to\sqrt{1-c_1}} u'_-(\alpha) = -\infty.$$

Let $\alpha_+(u)$ be the inverse function of $u_+(\alpha)$. It is an increasing function on (c_3-d_1, c_3) and

$$\lim_{u\to c_3-d_1} \alpha_+(u) = 0, \quad \lim_{u\to c_3} \alpha_+(u) = \sqrt{1-c_1}, \quad \lim_{u\to c_3} \alpha'_+(u) = 0.$$

Let $\alpha_-(u)$ be the inverse function of $u_-(\alpha)$. It is a decreasing function on (c_3, c_3+d_1) and

$$\lim_{u\to c_3+d_1} \alpha_-(u) = 0, \quad \lim_{u\to c_3} \alpha_-(u) = \sqrt{1-c_1}, \quad \lim_{u\to c_3} \alpha'_-(u) = 0.$$

We define a function $\hat{\alpha}(u)$ on (c_3-d_1, c_3+d_1) by

$$\hat{\alpha}(u) = \begin{cases} \alpha_+(u), & c_3-d_1 < u < c_3 \\ \alpha_-(u), & c_3 < u < c_3+d_1 \\ \sqrt{1-c_1}, & u = c_3 \end{cases}.$$

Then $\hat{\alpha}(u)$ is a C^1-function on (c_3-d_1, c_3+d_1) such that

$$\hat{\alpha}'(u) = \begin{cases} \alpha'_+(u), & c_3-d_1 < u < c_3 \\ \alpha'_-(u), & c_3 < u < c_3+d_1 \\ 0, & u = c_3 \end{cases}.$$

For $u \in (c_3-d_1, c_3) \cup (c_3, c_3+d_1)$, $\hat{\alpha}(u)$ satisfies the Eq. (5.1) for $K = 1$. Then, noting that $m \geq 2$, we can see that

$$\lim_{u\to c_3} (\hat{\alpha}'(u))^{-\frac{2m-2}{2m-1}} \hat{\alpha}''(u) = -(2m-1)\sqrt{1-c_1}$$

and

$$\lim_{u\to c_3} \hat{\alpha}''(u) = 0.$$

Thus the function $\hat{\alpha}(u)$ is a C^2-function on (c_3-d_1, c_3+d_1). Also by (3.5) and (3.6), we find that the Birkhoff-Gauss map can be C^1-extended for $u \in (c_3-d_1, c_3+d_1)$.

On the other hand, we have

$$\lim_{u\to c_3-d_1} \hat{\alpha}(u) = 0, \quad \lim_{u\to c_3+d_1} \hat{\alpha}(u) = 0$$

and

$$\lim_{u\to c_3-d_1}\hat{\alpha}'(u)=\frac{(1-c_1^m)^{\frac{2m-1}{2m}}}{c_1^{\frac{2m-1}{2}}},\qquad \lim_{u\to c_3+d_1}\hat{\alpha}'(u)=-\frac{(1-c_1^m)^{\frac{2m-1}{2m}}}{c_1^{\frac{2m-1}{2}}}.$$

So the surface has singularities at $(0,0,c_3-d_1)$ and $(0,0,c_3+d_1)$.

(i-3) The case $c_1<0$. In this case, we have $\sqrt{-c_1}<\alpha<\sqrt{1-c_1}$ and

$$u_\pm(\alpha)=\pm\int_{\sqrt{1-c_1}}^{\alpha}\frac{(\rho^2+c_1)^{\frac{2m-1}{2}}}{\left\{1-(\rho^2+c_1)^m\right\}^{\frac{2m-1}{2m}}}\,d\rho+c_4$$

for a constant c_4. As in the case (i-2), we can see that the graphs of $u_+(\alpha)$ and $u_-(\alpha)$ are connected smoothly at $\alpha=\sqrt{1-c_1}$. But the surface has singularities at points where $\alpha=\sqrt{-c_1}$.

(ii) The case $K=-1$. We have $c_1>0$ and

$$u_\pm(\alpha)=\pm\int\frac{(c_1-\alpha^2)^{\frac{2m-1}{2}}}{\left\{1-(c_1-\alpha^2)^m\right\}^{\frac{2m-1}{2m}}}d\alpha.$$

(ii-1) The case $0<c_1\leq 1$. In this case, we have $0<\alpha<\sqrt{c_1}$ and

$$u_\pm(\alpha)=\pm\int_{\sqrt{c_1}}^{\alpha}\frac{(c_1-\rho^2)^{\frac{2m-1}{2}}}{\left\{1-(c_1-\rho^2)^m\right\}^{\frac{2m-1}{2m}}}d\rho+c_5$$

for a constant c_5. Then

$$\lim_{\alpha\to\sqrt{c_1}}u_\pm(\alpha)=c_5,\qquad \lim_{\alpha\to\sqrt{c_1}}u'_\pm(\alpha)=0.$$

(ii-1-1) When $c_1=1$, since

$$1<\frac{2m-1}{m}<2,$$

we have

$$\lim_{\alpha\to 0}u_\pm(\alpha)=\mp\infty.$$

The corresponding surface has singularities at points where $\alpha=1$, and it can be seen as a generalization of the pseudo-sphere in the Euclidean 3-space.

(ii-1-2) When $0<c_1<1$, we have

$$\lim_{\alpha\to 0}u_\pm(\alpha)=c_5\mp d_2$$

where

$$d_2 := -\lim_{\alpha\to 0}\int_{\sqrt{c_1}}^{\alpha}\frac{(c_1-\rho^2)^{\frac{2m-1}{2}}}{\left\{1-(c_1-\rho^2)^m\right\}^{\frac{2m-1}{2m}}}d\rho\ \ (>0),$$

and

$$\lim_{\alpha\to 0}u'_{\pm}(\alpha)=\pm\frac{c_1^{\frac{2m-1}{2}}}{(1-c_1^m)^{\frac{2m-1}{2m}}}.$$

So the surface has singularities at points where $\alpha=\sqrt{c_1}$ and $\alpha=0$.

(ii-2) The case $c_1>1$. In this case, we have $\sqrt{c_1-1}<\alpha<\sqrt{c_1}$ and

$$u_{\pm}(\alpha)=\pm\int_{\sqrt{c_1-1}}^{\alpha}\frac{(c_1-\rho^2)^{\frac{2m-1}{2}}}{\left\{1-(c_1-\rho^2)^m\right\}^{\frac{2m-1}{2m}}}d\rho+c_6$$

for a constant c_6. By the discussion as before, the graphs of u_+ and u_- can be C^2-connected at $\alpha=\sqrt{c_1-1}$. But the surface has singularities at points where $\alpha=\sqrt{c_1}$.

6 Non-zero constant mean curvature

Let $(\mathbb{R}^3,\|\cdot\|)$ be the 3-dimensional normed space as in Sect. 3. Let M be a rotational surface in $(\mathbb{R}^3,\|\cdot\|)$ parametrized by

$$f(u,v)=(\alpha(u)\cos v,\alpha(u)\sin v,u)$$

where $\alpha>0$ and $\alpha'\neq 0$.

By (3.8), if H is a non-zero constant, then

$$\begin{aligned}&\frac{1}{2m-1}\alpha\left(\left(\alpha'\right)^{\frac{2m}{2m-1}}+1\right)^{-\frac{2m+1}{2m}}(\alpha')^{-\frac{2m-2}{2m-1}}\alpha''-\left(\left(\alpha'\right)^{\frac{2m}{2m-1}}+1\right)^{-\frac{1}{2m}}\\&=2H\alpha.\end{aligned}\tag{6.1}$$

Multiplying by $-\alpha'$ we have

$$-\frac{1}{2m-1}\alpha\left(\left(\alpha'\right)^{\frac{2m}{2m-1}}+1\right)^{-\frac{2m+1}{2m}}(\alpha')^{\frac{1}{2m-1}}\alpha''+\alpha'\left(\left(\alpha'\right)^{\frac{2m}{2m-1}}+1\right)^{-\frac{1}{2m}}=-H(\alpha^2)'.$$

Integrating it we can get

$$\alpha\left(\left(\alpha'\right)^{\frac{2m}{2m-1}}+1\right)^{-\frac{1}{2m}}=c_1-H\alpha^2\ \ (>0)$$

for a constant c_1. Then

$$\frac{d\alpha}{du} = \pm \frac{\left\{\alpha^{2m} - (c_1 - H\alpha^2)^{2m}\right\}^{\frac{2m-1}{2m}}}{(c_1 - H\alpha^2)^{2m-1}},$$

and we get the following.

Theorem 6.1 *A rotational surface in* $(\mathbb{R}^3, \|\cdot\|)$ *given by*

$$\bar{f}(\alpha, v) = (\alpha \cos v, \alpha \sin v, u(\alpha))$$

where $\alpha > 0$ *has non-zero constant Minkowski mean curvature* H *if and only if*

$$u(\alpha) = \pm \int \frac{(c_1 - H\alpha^2)^{2m-1}}{\left\{\alpha^{2m} - (c_1 - H\alpha^2)^{2m}\right\}^{\frac{2m-1}{2m}}} d\alpha$$

for a constant c_1.

Set

$$u_{\pm}(\alpha) := \pm \int \frac{(c_1 - H\alpha^2)^{2m-1}}{\left\{\alpha^{2m} - (c_1 - H\alpha^2)^{2m}\right\}^{\frac{2m-1}{2m}}} d\alpha.$$

To study the behavior of the graph of $u_{\pm}(\alpha)$, it suffices to consider the case where $H = \pm 1$. The signature of H changes if the orientation of the parametrization changes. So we treat the both cases $H = 1$ and $H = -1$.

(i) The case $H = 1$. In this case, we have $c_1 > 0$,

$$b_1 := \frac{\sqrt{1 + 4c_1} - 1}{2} < \alpha < \sqrt{c_1}$$

and

$$u_{\pm}(\alpha) = \pm \int_{\sqrt{c_1}}^{\alpha} \frac{(c_1 - \rho^2)^{2m-1}}{\left\{\rho^{2m} - (c_1 - \rho^2)^{2m}\right\}^{\frac{2m-1}{2m}}} d\rho + c_2^{\pm}$$

for a constant $c_2^{\pm}$. This integral converges as α tends to $\sqrt{c_1}$, and since $0 < (2m - 1)/2m < 1$, it converges also as α tends to b_1. Set

$$d_1 := -\lim_{\alpha \to b_1} \int_{\sqrt{c_1}}^{\alpha} \frac{(c_1 - \rho^2)^{2m-1}}{\left\{\rho^{2m} - (c_1 - \rho^2)^{2m}\right\}^{\frac{2m-1}{2m}}} d\rho \ \ (> 0).$$

Then

$$\lim_{\alpha \to b_1} u_{+}(\alpha) = c_2^{+} - d_1, \quad \lim_{\alpha \to b_1} u_{-}(\alpha) = c_2^{-} + d_1, \quad \lim_{\alpha \to \sqrt{c_1}} u_{\pm}(\alpha) = c_2^{\pm}.$$

The function $u_+(\alpha)$ is an increasing function and

$$\lim_{\alpha\to b_1} u'_+(\alpha) = \infty, \qquad \lim_{\alpha\to\sqrt{c_1}} u'_+(\alpha) = 0.$$

The function $u_-(\alpha)$ is a decreasing function and

$$\lim_{\alpha\to b_1} u'_-(\alpha) = -\infty, \qquad \lim_{\alpha\to\sqrt{c_1}} u'_-(\alpha) = 0.$$

(ii) The case $H = -1$. We have

$$u_\pm(\alpha) = \pm\int \frac{(c_3+\alpha^2)^{2m-1}}{\left\{\alpha^{2m} - (c_3+\alpha^2)^{2m}\right\}^{\frac{2m-1}{2m}}} d\alpha$$

for a constant c_3. Here we use c_3 instead of c_1 because we will later choose c_3 different from c_1.

(ii-1) The case $c_3 = 0$. Then

$$u_\pm(\alpha) = \pm\int \frac{\alpha^{2m-1}}{(1-\alpha^{2m})^{\frac{2m-1}{2m}}} d\alpha = \mp(1-\alpha^{2m})^{\frac{1}{2m}} + c_4$$

for a constant c_4. It satisfies

$$\alpha^{2m} + (u_\pm(\alpha) - c_4)^{2m} = 1.$$

So the surface can be smoothly extended to a Minkowski sphere, which is a parallel translation of the unit sphere S.

(ii-2) The case $c_3 > 0$. In this case, we have $0 < c_3 < 1/4$,

$$b_2 := \frac{1-\sqrt{1-4c_3}}{2} < \alpha < \frac{1+\sqrt{1-4c_3}}{2} =: b_3$$

and

$$u_\pm(\alpha) = \pm\int_{b_2}^{\alpha} \frac{(c_3+\rho^2)^{2m-1}}{\left\{\rho^{2m} - (c_3+\rho^2)^{2m}\right\}^{\frac{2m-1}{2m}}} d\rho + c_5$$

for a constant c_5. This integral converges as α tends to b_2 and b_3. Set

$$d_2 := \lim_{\alpha\to b_3}\int_{b_2}^{\alpha} \frac{(c_3+\rho^2)^{2m-1}}{\left\{\rho^{2m} - (c_3+\rho^2)^{2m}\right\}^{\frac{2m-1}{2m}}} d\rho.$$

Then

$$\lim_{\alpha\to b_2} u_\pm(\alpha) = c_5, \quad \lim_{\alpha\to b_3} u_+(\alpha) = c_5 + d_2, \quad \lim_{\alpha\to b_3} u_-(\alpha) = c_5 - d_2.$$

The function $u_+(\alpha)$ is an increasing function and

$$\lim_{\alpha\to b_2} u'_+(\alpha) = \infty, \quad \lim_{\alpha\to b_3} u'_+(\alpha) = \infty.$$

The function $u_-(\alpha)$ is a decreasing function and

$$\lim_{\alpha\to b_2} u'_-(\alpha) = -\infty, \quad \lim_{\alpha\to b_3} u'_-(\alpha) = -\infty.$$

Let $\alpha_+(u)$ be the inverse function of $u_+(\alpha)$. It is increasing on (c_5, c_5+d_2) and

$$\lim_{u\to c_5} \alpha_+(u) = b_2, \quad \lim_{u\to c_5+d_2} \alpha_+(u) = b_3, \quad \lim_{u\to c_5} \alpha'_+(u) = \lim_{u\to c_5+d_2} \alpha'_+(u) = 0.$$

Let $\alpha_-(u)$ be the inverse function of $u_-(\alpha)$. It is decreasing on (c_5-d_2, c_5) and

$$\lim_{u\to c_5} \alpha_-(u) = b_2, \quad \lim_{u\to c_5-d_2} \alpha_-(u) = b_3, \quad \lim_{u\to c_5} \alpha'_-(u) = \lim_{u\to c_5-d_2} \alpha'_-(u) = 0.$$

We define a function $\hat{\alpha}(u)$ on $[c_5-d_2, c_5+d_2]$ by

$$\hat{\alpha}(u) = \begin{cases} \alpha_+(u), & c_5 < u < c_5+d_2 \\ \alpha_-(u), & c_5-d_2 < u < c_5 \\ b_2, & u = c_5 \\ b_3, & u = c_5 \pm d_2 \end{cases}.$$

Then it is a C^1-function on $[c_5-d_2, c_5+d_2]$ such that

$$\hat{\alpha}'(u) = \begin{cases} \alpha'_+(u), & c_5 < u < c_5+d_2 \\ \alpha'_-(u), & c_5-d_2 < u < c_5 \\ 0, & u = c_5,\ c_5 \pm d_2 \end{cases}.$$

For $u \in (c_5-d_2, c_5) \cup (c_5, c_5+d_2)$, $\hat{\alpha}(u)$ satisfies the Eq. (6.1) for $H = -1$. Then, noting that $m \geq 2$, we can see that

$$\lim_{u\to c_5} (\hat{\alpha}'(u))^{-\frac{2m-2}{2m-1}} \hat{\alpha}''(u) = \frac{(2m-1)(1-2b_2)}{b_2},$$
$$\lim_{u\to c_5\pm d_2} (\hat{\alpha}'(u))^{-\frac{2m-2}{2m-1}} \hat{\alpha}''(u) = \frac{(2m-1)(1-2b_3)}{b_3}$$

and

$$\lim_{u\to c_5} \hat{\alpha}''(u) = \lim_{u\to c_5\pm d_2} \hat{\alpha}''(u) = 0.$$

So the function $\hat{\alpha}(u)$ is a C^2-function on $[c_5 - d_2, c_5 + d_2]$. By (3.5) and (3.6), the Birkhoff-Gauss map can be C^1-extended for $u \in [c_5 - d_2, c_5 + d_2]$.

We note that $\hat{\alpha}(u)$ has the same derivatives at the end points $u = c_5 - d_2$ and $u = c_5 + d_2$. Thus we can extend $\hat{\alpha}(u)$ periodically as a C^2-function on $\mathbb{R}$ as follows:

$$\alpha^*(u + 2kd_2) := \hat{\alpha}(u), \quad u \in [c_5 - d_2, c_5 + d_2], \ \ k \in \mathbb{Z},$$

and we get the following.

Theorem 6.2 *Under the notation above, the rotational surface in* $(\mathbb{R}^3, \|\cdot\|)$ *parametrized by*

$$f^*(u, v) = (\alpha^*(u)\cos v, \alpha^*(u)\sin v, u), \quad (u, v) \in \mathbb{R} \times [0, 2\pi]$$

has constant Minkowski mean curvature -1.

Remark. The surface in Theorem 6.2 can be seen as a generalization of the unduloid (Delaunay 1841).

(ii-3) The case $c_3 < 0$. In this case we have

$$\sqrt{-c_3} < \alpha < \frac{1 + \sqrt{1 - 4c_3}}{2} =: b_4$$

and

$$u_{\pm}(\alpha) = \pm \int_{b_4}^{\alpha} \frac{(c_3 + \rho^2)^{2m-1}}{\left\{\rho^{2m} - (c_3 + \rho^2)^{2m}\right\}^{\frac{2m-1}{2m}}} d\rho + c_6$$

for a constant c_6. This integral converges as α tends to $\sqrt{-c_3}$ and b_4. Set

$$d_3 := -\lim_{\alpha \to \sqrt{-c_3}} \int_{b_4}^{\alpha} \frac{(c_3 + \rho^2)^{2m-1}}{\left\{\rho^{2m} - (c_3 + \rho^2)^{2m}\right\}^{\frac{2m-1}{2m}}} d\rho \ \ (> 0).$$

Then

$$\lim_{\alpha \to \sqrt{-c_3}} u_+(\alpha) = c_6 - d_3, \quad \lim_{\alpha \to \sqrt{-c_3}} u_-(\alpha) = c_6 + d_3, \quad \lim_{\alpha \to b_4} u_{\pm}(\alpha) = c_6.$$

The function $u_+(\alpha)$ is an increasing function and

$$\lim_{\alpha \to \sqrt{-c_3}} u_+'(\alpha) = 0, \quad \lim_{\alpha \to b_4} u_+'(\alpha) = \infty.$$

The function $u_-(\alpha)$ is a decreasing function and

$$\lim_{\alpha \to \sqrt{-c_3}} u_-'(\alpha) = 0, \quad \lim_{\alpha \to b_4} u_-'(\alpha) = -\infty.$$

In the following, we will connect the curves in the cases (i) and (ii-3). For distinguishment, we denote $u_{\pm}(\alpha)$ in the case (i) by $u_{1\pm}(\alpha)$, and $u_{\pm}(\alpha)$ in the case (ii-3) by $u_{2\pm}(\alpha)$.

We take the graph G_1 of $u_{1+}(\alpha)$ for $b_1 < \alpha < \sqrt{c_1}$. Next, choosing $c_3 := -c_1$ and $c_6 := c_2^+ - d_3$, we take the graph G_2 of $u_{2-}(\alpha)$ for

$$\sqrt{-c_3} = \sqrt{c_1} < \alpha < b_4 = \frac{1+\sqrt{1+4c_1}}{2}.$$

Then G_1 and G_2 are C^1-connected at $(\alpha, u) = (\sqrt{c_1}, c_2^+)$.

Next we take the graph G_3 of $u_{2+}(\alpha)$ for $\sqrt{c_1} < \alpha < b_4$. Then G_2 and G_3 are C^1-connected at $(\alpha, u) = (b_4, c_2^+ - d_3)$.

Finally, letting $c_2^- := c_2^+ - 2d_3$, we take the graph G_4 of $u_{1-}(\alpha)$ for $b_1 < \alpha < \sqrt{c_1}$. Then G_3 and G_4 are C^1-connected at $(\alpha, u) = (\sqrt{c_1}, c_2^+ - 2d_3)$. Thus we get a C^1-curve Γ which is constructed by connecting G_1, G_2, G_3 and G_4.

With respect to the parameter u, $H = 1$ for the G_1 and G_4 parts, and $H = -1$ for the G_2 and G_3 parts. On the other hand, with respect to the parameter α, $H = 1$ for the G_1 and G_2 parts, and $H = -1$ for the G_3 and G_4 parts. Then, with respect to a parametrization of Γ in the order of G_1, G_2, G_3 and G_4, we have $H = 1$ for all parts.

The C^2-connectedness of G_2 and G_3 is shown by the discussion as before. Similarly we can see that G_1 and G_4 are C^2 at $\alpha = b_1$.

Let us prove the C^2-connectedness of G_1 and G_2. We define a function $\hat{u}(\alpha)$ on (b_1, b_4) by

$$\hat{u}(\alpha) = \begin{cases} u_{1+}(\alpha), \ b_1 < \alpha < \sqrt{c_1} \\ u_{2-}(\alpha), \ \sqrt{c_1} < \alpha < b_4 \, . \\ c_2^+, \qquad \alpha = \sqrt{c_1} \end{cases}$$

Then it is a C^1-function on (b_1, b_4) such that

$$\hat{u}'(\alpha) = \begin{cases} u_{1+}'(\alpha), \ b_1 < \alpha < \sqrt{c_1} \\ u_{2-}'(\alpha), \ \sqrt{c_1} < \alpha < b_4 \, . \\ 0, \qquad \alpha = \sqrt{c_1} \end{cases}$$

For $\alpha \in (b_1, \sqrt{c_1}) \cup (\sqrt{c_1}, b_4)$, $\hat{u}(\alpha)$ satisfies the Eq. (3.4) for "$H = 1$", where α is the parameter and $\beta = \hat{u}(\alpha)$. Then, noting that $m \geq 2$, we can see that

$$\lim_{\alpha \to \sqrt{c_1}} \left(\hat{u}'(\alpha)\right)^{-\frac{2m-2}{2m-1}} \hat{u}''(\alpha) = -2(2m-1)$$

and

$$\lim_{\alpha \to \sqrt{c_1}} \hat{u}''(\alpha) = 0.$$

So the function $\hat{u}(\alpha)$ is a C^2-function on (b_1, b_4). By (3.1) and (3.2), the Birkhoff-Gauss map can be C^1-extended for $\alpha \in (b_1, b_4)$. Thus the C^2-connectedness of G_1 and G_2 is proved. The C^2-connectedness of G_3 and G_4 is proved similarly.

Now we have obtained a C^2-curve Γ which is constructed by connecting G_1, G_2, G_3 and G_4. The curve Γ has the same derivatives at the end points. Then, as in the case (ii-2), we can extend it periodically as a C^2-curve Γ^*, which can be parametrized as $(\alpha^*(t), \beta^*(t))$ for $t \in \mathbb{R}$.

Theorem 6.3 *Under the notation above, the rotational surface in* $(\mathbb{R}^3, \|\cdot\|)$ *parametrized by*

$$f^*(t, v) = (\alpha^*(t)\cos v, \alpha^*(t)\sin v, \beta^*(t)), \qquad (t, v) \in \mathbb{R} \times [0, 2\pi]$$

has constant Minkowski mean curvature 1.

Remark. (i) The surface in Theorem 6.3 can be seen as a generalization of the nodoid (Delaunay 1841).

(ii) By "Mathematica" we know that: (a) When $c_1 = -c_3 = 2$ and $m = 2$, $d_1 = 0.34459...$ and $d_3 = 0.65540...$, (b) When $c_1 = -c_3 = 2$ and $m = 3$, $d_1 = 0.33886...$ and $d_3 = 0.66113...$, and (c) When $c_1 = -c_3 = 6$ and $m = 2$, $d_1 = 0.40710...$ and $d_3 = 0.59289...$. Thus the curve Γ is not closed in those cases.

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