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ORIGINAL PAPER

On morphic modules over commutative rings

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Abstract
A commutative ring R is said to be a *morphic ring* if for each $a \in R$ there exists $b \in R$ such that $ann(a) = Rb$ and $ann(b) = Ra$. In this paper, we extend the notion of morphic rings to modules and we study the introduced concept by comparing it with some related notions.

Keywords Comorphic ring · Comorphic module

Mathematics Subject Classification 13A15

1 Introduction

Throughout, all rings considered are commutative with nonzero identity and all modules are unital. A ring R is said to be a von Neumann regular if, for every $x \in R$, there is $y \in R$ such that $x = x^2y$, or equivalently, if every principal (resp. finitely generated) ideal of R is generated by an idempotent (Wang and Kim 2016, Theorem 3.6.13). Recently, the notion of von Neumann regular rings was extended to modules

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in Jayaram and Tekir (2018). An R-module M is said to be von Neumann regular if, for each $m \in M$, there exists $a \in R$ such that $Rm = aM = a^2M$. The concepts of regular von Neumann rings have been widely studied and several generalizations of these rings have been given and studied in recent years (see for example, Alkan et al. 2018; Nicholson and Sánchez Campos 2004; Kist 1963). In Nicholson and Sánchez Campos (2004), Nicholson and Sánchez Campos introduced the concept of morphic rings (over associative rings). A commutative ring R is called a morphic ring if, for every element $a \in R$, $R/Ra \cong \text{ann}_R(a)$ (isomorphism of R-modules). It is proved that R is morphic if and only if, for every $a \in R$, there exists $b \in R$ such that $Ra = \text{ann}_R(b)$ and $\text{ann}_R(a) = Rb$ (Nicholson and Sánchez Campos 2004, Lemma1). Morphic (associative) rings have been widely studied and many variants of this notion have been proposed and studied in recent papers. Among these variants, the concepts of quasi-morphic rings (Camillo and Nicholson 2007) and comorphic rings (Alkan et al. 2018) which are equivalent to the concept of morphic rings over commutative rings.

In 2005, Nicholson and Sánchez Campos extended the notion of morphic rings to modules (Nicholson and Sánchez Campos 2005). An endomorphism α of a module M is called morphic if $M/\alpha(M) \cong \ker(\alpha)$. The module M is called a morphic module if every endomorphism of M is morphic. Note that the endomorphisms of the R-module R are only the multiplication morphisms. So, clearly, a ring R is morphic if and only if R is a morphic R-module.

In this article, we extend, in another way, the notion of morphic rings to modules. An R-module M is said to be a morphic module if, for each $m \in M$, there exists $a \in R$ such that $Rm = \text{ann}_M(a)$ and $\text{ann}_R(m) = Ra + \text{ann}_R(M)$. We investigate many properties of this notion. Among other results, we show that a ring R is morphic if and only if it a morphic R-module (Example 1). Some other classes of examples of morphic modules are given in Examples 2, 3, and 4. It is also proved that the intersection of two (finite number of) cyclic submodules of a morphic module is again cyclic (Proposition1 and Corollary1). Theorem 1 gives a characterization of morphic modules. Let M be an R-module. A submodule N of M is called a Baer submodule if $m \in N$ implies that $\text{ann}_M(\text{ann}_R(m)) \subseteq N$ (see Jayaram et al. 2021). Proposition 4 shows that every submodule of a morphic module is a Baer submodule. Recall that a module is called a Bézout module if its finitely generated submodules are all cyclic. If M is a non-torsion morphic module, then M is a Bézout module (Corollary 3). Theorem 3 shows that every finitely generated von Neumann regular module is also a morphic module with equivalence if M is a torsion-free module (Proposition 6). We end this paper with some results about the stability of the notion of morphic modules under homomorphisms, direct products of modules, and localization of modules.

2 Main results

We begin by recalling the key definition of this paper.

Definition 1 An R-module M is said to be a morphic module if, for each $m \in M$, there exists $a \in R$ such that $Rm = \text{ann}_M(a)$ and $\text{ann}_R(m) = (a) + \text{ann}_R(M)$

The first definition of morphic rings was in the associative context. The definition above deals with a generalization of morphic rings but only in the commutative case. In this case, all modules (resp. ideals) are bi-modules (resp. bi-ideals).

Example 1 A commutative ring R is a morphic ring if and only if it is a morphic R-module.

Example 2 Every simple module is morphic. In particular, for each prime integer p, $\mathbb{Z}_p$ is a morphic $\mathbb{Z}$-module.

Example 3 Let $n \geq 2$ be an integer. Consider the $\mathbb{Z}$-module $\mathbb{Z}_n$. Let $m \in \mathbb{Z}$. If $\gcd(m, n) = 1$ then $\mathbb{Z}\overline{m} = \mathbb{Z}_n = \mathrm{ann}_{\mathbb{Z}_n}(0)$ and $\mathrm{ann}_{\mathbb{Z}}(\overline{m}) = \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}_n) = (0) + \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}_n)$. Assume now that $\gcd(m, n) = d > 1$ and write $n = dn'$ and $m = dm'$ for some $n', m' \in \mathbb{Z}$. We have $\mathbb{Z}\overline{m} = \mathbb{Z}\overline{d} = \mathrm{ann}_{\mathbb{Z}_n}(n')$, and furthermore $\mathrm{ann}_{\mathbb{Z}}(\overline{m}) = n'\mathbb{Z} = (n') + \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}_n)$. Accordingly, $\mathbb{Z}_n$ is a morphic $\mathbb{Z}$-module. More generally, every cyclic group of finite order is a morphic $\mathbb{Z}$-module.

Example 4 Let G be a finite abelian group. By the Structure Theorem of Finite Abelian Groups, G is isomorphic, as a $\mathbb{Z}$-module, to $M = \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_r^{\alpha_r}}$ where p_i's are prime integers (which may not be distinct) and $\alpha_i \geq 1$ for each $i = 1, 2, \ldots, n$. Assume that G is a morphic $\mathbb{Z}$-module. Then, so is M. We claim that all p_i's are distinct, that is G is cyclic. Suppose that $p_i = p_j$ for some $i \neq j$. For example, suppose that $p_1 = p_2$. Firstly, suppose that $\alpha_1 = \alpha_2$. Set $m = (\overline{1}, \overline{0}, \overline{0}, \ldots, \overline{0}) \in M$. There exists $a \in \mathbb{Z}$ such that $\mathbb{Z}m = \mathrm{ann}_M(a)$ and $\mathrm{ann}_{\mathbb{Z}}(m) = (a) + \mathrm{ann}_{\mathbb{Z}}(M)$. Thus, $(a) \subseteq \mathrm{ann}_{\mathbb{Z}}(m) = (p_1^{\alpha_1})$. Hence, $\mathrm{ann}_M(p_1^{\alpha_1}) \subseteq \mathrm{ann}_M(a) = \mathbb{Z}m$. However, $(\overline{0}, \overline{1}, \overline{0}, \ldots, \overline{0}) \in \mathrm{ann}_M(p_1^{\alpha_1}) \backslash \mathbb{Z}m$, a contradiction. So, we may assume for example that $\alpha_1 < \alpha_2$. Set $m = (\overline{0}, \overline{1}, \overline{0}, \ldots, \overline{0}) \in M$. There exists $a \in \mathbb{Z}$ such that $\mathbb{Z}m = \mathrm{ann}_M(a)$ and $\mathrm{ann}_{\mathbb{Z}}(m) = (a) + \mathrm{ann}_{\mathbb{Z}}(M)$. Thus, $(a) \subseteq \mathrm{ann}_{\mathbb{Z}}(m) = (p_1^{\alpha_2})$. Hence, $\mathrm{ann}_M(p_1^{\alpha_2}) \subseteq \mathrm{ann}_M(a) = \mathbb{Z}m$. However, $(\overline{1}, \overline{0}, \overline{0}, \ldots, \overline{0}) \in \mathrm{ann}_M(p_1^{\alpha_2}) \backslash \mathbb{Z}m$, a contradiction. Consequently, $p_1 \neq p_2$, and so by the same way we get that all p_i's are distinct. Keeping in mind Example 3, we conclude that a finite abelian group G is a morphic $\mathbb{Z}$-module if and only if G is a cyclic group.

Proposition 1 *Let R be a ring. If M is a morphic R-module then, the intersection of two cyclic submodules of M is also cyclic.*

Proof Let m, $m' \in M$. Since M is a morphic R-module, then $Rm = \mathrm{ann}_M(a)$, $\mathrm{ann}_R(m) = (a) + \mathrm{ann}_R(M)$, $Rm' = \mathrm{ann}_M(a')$ and $\mathrm{ann}_R(m') = (a') + \mathrm{ann}_R(M)$ for some a, $a' \in R$. We have also $Ram' = \mathrm{ann}_M(c)$ and $\mathrm{ann}_R(am') = (c) + \mathrm{ann}_R(M)$ for some $c \in R$. We will show that $Rm \cap Rm' = Rcm'$. Let $x \in Rm \cap Rm'$, then $x = rm = r'm'$ for some $r, r' \in R$. It follows that $r'm' \in Rm = \mathrm{ann}_M(a)$. Hence, $r'(am') = ar'm' = 0$. Then, $r' \in \mathrm{ann}_R(am') = (c) + \mathrm{ann}_R(M)$. So, $r' = \alpha c + b$ for some $\alpha \in R$ and some $b \in \mathrm{ann}_R(M)$. Then, $x = r'm' = (\alpha c + b)m' = \alpha cm'$, and so we conclude that $x \in Rcm'$. Thus, $Rm \cap Rm' \subseteq Rcm'$. For the other inclusion, it suffices to show that $Rcm' \subseteq Rm$. We have $cam' = 0$. Then, $cm' \in \mathrm{ann}_M(a) = Rm$. Thus, $Rcm' \subseteq Rm$, as desired. □

By induction, we obtain the following result.

Corollary 1 *Let R be a ring. If M is a morphic R-module then, the intersection of finite number of cyclic submodules of M is also cyclic.*

Corollary 2 *If R is a morphic ring, the intersection of finite number of principal ideals of R is again principal.*

Proposition 2 *Let M be a morphic R-module and $m, m' \in M$. If $Rm \cap Rm' = (0)$, then $Rm \oplus Rm' = R(m + m')$.*

Proof Since M is a morphic R-module, there exists $a \in R$ such that $R(m + m') = \text{ann}_M(a)$. Thus, $a(m + m') = 0$ and so $am = -am' \in Rm \cap Rm' = (0)$. Hence, m, $m' \in \text{ann}_M(a) = R(m + m')$, and then $Rm + Rm' \subseteq R(m + m')$. The other inclusion is trivial. So, since $Rm \cap Rm' = (0)$, we get $Rm \oplus Rm' = Rm + Rm' = R(m + m')$. □

Theorem 1 *Let M be an R-module. The following are equivalent:*

(1) M is a morphic R-module.
(2) For every finitely generated submodule N of M, there exists $a \in R$ such that $N = \text{ann}_M(a)$ and $\text{ann}_R(N) = (a) + \text{ann}_R(M)$.

Proof The implication (2) $\Rightarrow$ (1) is clear.
(1) $\Rightarrow$ (2) Let N be a submodule of M and let n be the minimal number of generators of N, and then proceed by induction on n. The induction start is just the definition of morphic modules. If $n \geq 1$, set $N = \sum_{i=1}^{n} Rm_i$ and $J = \sum_{i=1}^{n} Rm_i$. It is clear that $\{m_i\}_{i=1}^{n-1}$ is a minimal set of generators of J. The induction hypothesis yields that $J = \text{ann}_M(a_1)$ and $\text{ann}_R(J) = (a_1) + \text{ann}_R(M)$ for some $a_1 \in R$. Since M is a morphic R-module, we get that $Rm_n = \text{ann}_M(a_2)$, $\text{ann}_R(m_n) = (a_2) + \text{ann}_R(M)$, $Ra_1m_n = \text{ann}_M(a_3)$, and $\text{ann}_R(a_1m_n) = (a_3) + \text{ann}_R(M)$ for some $a_2, a_3 \in R$. Set $a = a_1a_3$. We claim that $N = \text{ann}_M(a)$. Let $x \in \text{ann}_M(a)$. Then, $a_1a_3x = 0$. Therefore, $a_1x \in \text{ann}_M(a_3) = Ra_1m_n$. Write $a_1x = \alpha a_1m_n$ for some $\alpha \in R$. We have $a_1(x - \alpha m_n) = 0$. Then, $x - \alpha m_n \in \text{ann}_M(a_1) = J$. So, $x - \alpha m_n = j$ for some $j \in J$. Then, $x = j + \alpha m_n$, and so $x \in N$. Thus, $\text{ann}_M(a) \subseteq N$. Now, let $x \in N$. Then, $x = j + \beta m_n$ for some $j \in J$ and $\beta \in R$. So, $ax = a_1a_3j + a_1a_3\alpha m_n = 0$, and then $x \in \text{ann}_M(a)$. Thus, $N \subseteq \text{ann}_M(a)$. Consequently, $N = \text{ann}_M(a)$.
We claim that $\text{ann}_R(N) = (a) + \text{ann}_R(M)$. First, it is clear that $(a) + \text{ann}_R(M) = (a_1a_3) + \text{ann}_R(M) \subseteq \text{ann}_R(J) \cap \text{ann}_R(m_n) = \text{ann}_R(N)$. Let $t \in \text{ann}_R(N) = \text{ann}_R(J) \cap \text{ann}_R(m_n)$. Then, $t \in \text{ann}_R(J)$ implies that $t = sa_1 + u$ for some $s \in R$ and $u \in \text{ann}_R(M)$. Since $tm_n = 0$ we have $0 = (sa_1 + u)m_n = sa_1m_n$, and so $s \in \text{ann}_R(a_1m_n) = (a_3) + \text{ann}_R(M)$. Then, there exist $v \in R$ and $w \in \text{ann}_R(M)$ such that $s = a_3v + w$ and so $t = v(a_1a_3) + wa_1 + u \in (a_1a_3) + \text{ann}_R(M)$. Thus, $\text{ann}_R(N) \subseteq (a) + \text{ann}_R(M)$. Consequently, $\text{ann}_R(N) = (a) + \text{ann}_R(M)$. So, we have the desired result. □

Let M be an R-module. The set of all torsion elements of M is denoted by $T(M) = \{m \in M : \text{ann}_R(m) \neq 0\}$. M is said to be a torsion-free module if $T(M) = \{0\}$, and M is said to be a torsion module if $T(M) = M$. Hence, M is said to be a non-torsion module if $M \neq T(M)$.

Proposition 3 *Let M be a non-torsion R-module. If M is a morphic R-module, then R is a morphic ring.*

Proof Suppose that M is a non-torsion morphic module. Choose $m \in M - T(M)$ and $a \in R$. Then note that $\text{ann}_R(am) = \text{ann}_R(a)$ and $\text{ann}_R(M) = 0$. Since M is morphic, there exists $b \in R$ such that $\text{ann}_R(am) = (b) + \text{ann}_R(M) = (b)$ and $\text{ann}_M(b) = Ram$. So we have $\text{ann}_R(a) = (b)$ and thus $(a) \subseteq \text{ann}_R(b)$. Let $r \in \text{ann}_R(b)$. Then we have $rb = 0$ and so $rm \in \text{ann}_M(b) = Ram$. Then there exists $s \in R$ such that $rm = sam$ which implies that $(r - sa)m = 0$. As $\text{ann}_R(m) = 0$, we get $r \in (a)$ and so $\text{ann}_R(b) \subseteq (a)$ which completes the proof. □

Let M be an R-module. A submodule N of M is called a Baer submodule if $m \in N$ implies that $\text{ann}_M(\text{ann}_R(m)) \subseteq N$ (Jayaram et al. 2021). Recall from Jayaram et al. (2021) that a submodule N of M is said to be an annihilator submodule if $N = \text{ann}_M(\text{ann}_R(N))$. Note that a cyclic submodule Rm is a Baer submodule if and only if it is an annihilator submodule. For various informations of Baer submodules and annihilator submodules, the reader may consult (Jayaram et al. 2021; Jayaram et al. 2021; Jayaram et al. 2022).

Proposition 4 *Let M be a morphic R-module. Then,*

(1) Every finitely generated submodule N of M is an annihilator submodule.
(2) Every submodule N of M is a Baer submodule.

Proof (2) Suppose that N is a finitely generated submodule of M. By Theorem 1, $N = \text{ann}_M(a)$ and $\text{ann}_R(N) = (a) + \text{ann}_R(M)$. Thus, $(a) \subseteq \text{ann}_R(N)$ implies that $\text{ann}_M(\text{ann}_R(N)) \subseteq \text{ann}_M(a) = N$. Moreover, $N \subseteq \text{ann}_M(\text{ann}_R(N))$. Thus, $\text{ann}_M(\text{ann}_R(N)) = N$. So N is an annihilator submodule. of M.
(2) Let $m \in N$. Since Rm is a finitely generated submodule of M, by (1), we get that $\text{ann}_M(\text{ann}_R(m)) = Rm \subseteq N$. Hence, N is a Baer submodule of M. □

Recall that a module is called a Bézout module if its finitely generated submodules are all cyclic. A ring R is called a Bézout ring if R is Bézout R-module. That is every finitely generated ideal is principal.

Corollary 3 *Let M be a non-torsion R-module. If M is a morphic R-module, then M is a Bézout module.*

Proof Let N be a finitely generated submodule of M and $m \in M - T(M)$. Since $\text{ann}_R(M) = 0$, by Theorem 1, there exists $a \in R$ such that $\text{ann}_R(N) = (a)$ and $N = \text{ann}_M(a)$. On the other hand, by Proposition 3, R is a morphic ring and so there exists $b \in R$ such that $(a) = \text{ann}_R(b)$ and $(b) = \text{ann}_R(a)$. Thus, $\text{ann}_R(N) = (a) = \text{ann}_R(b) = \text{ann}_R(bm)$. So, $N = \text{ann}_M(\text{ann}_R(N)) = \text{ann}_M(\text{ann}_R(bm))$. By Proposition 4, $\text{ann}_M(\text{ann}_R(bm)) = Rbm$. Consequently, $N = Rbm$. □

Since a ring is always a non-torsion R-module, we obtain the commutative version of (Camillo and Nicholson 2007, Theorem 15).

Corollary 4 *Every morphic ring is a Bézout ring.*

Theorem 2 *Let M be a morphic module. Consider the following conditions:*

(1) M satisfies the DCC on cyclic submodules.
(2) R satisfies the ACC on $\{\text{ann}_R(m) : m \in M\}$.
(3) M satisfies the DCC on finitely generated submodules.
(4) R satisfies the ACC on $\{\text{ann}_R(N) : N$ is finitely generated submodule of $M\}$.

Then, $(1) \Leftrightarrow (2) \Leftarrow (3) \Leftrightarrow (4)$. *Furthermore, if M is a non-torsion module, then* $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.

Proof $(1) \Rightarrow (2)$ Consider the following ascending chain of ideals

$$\text{ann}_R(m_1) \subseteq \text{ann}_R(m_2) \subseteq \cdots \subseteq \text{ann}_R(m_n) \subseteq \cdots$$

where $m_i \in M$. By Proposition 4, $Rm_i = \text{ann}_M(\text{ann}_R(m_i))$ for each i. Thus, we obtain the following descending chain of cyclic submodules of M:

$$Rm_1 \supseteq Rm_2 \supseteq \cdots \supseteq Rm_n \supseteq \cdots$$

Thus, by hypothesis, there exists an integer $k \geq 1$ such that $Rm_k = Rm_{n+k}$ for each $n \geq 0$. This gives $\text{ann}_R(m_k) = \text{ann}_R(m_{k+n})$, as needed.
$(2) \Rightarrow (1)$ is similar to $(1) \Rightarrow (2)$.
$(3) \Leftrightarrow (4)$ Similar to $(1) \Leftrightarrow (2)$ (by using Proposition 4).
$(3) \Rightarrow (1)$ Trivial.
If M is a non-torsion morphic module, then the desired result follows from Corollary 3. □

Let M be an R-module. M is said to be a wq-regular module if for each $m \in M$, there is an $a \in R$ such that $\text{ann}_M(\text{ann}_R(m)) = \text{ann}_M(a)$ (see Jayaram et al. 2021). The ring R is called wq-regular ring if R is wq-regular R-module.

Proposition 5 *Every morphic module is a wq-regular module.*

Proof Let $m \in M$. Since M is a morphic module, then there exists $a \in R$ such that $Rm = \text{ann}_M(a)$ and $\text{ann}_R(m) = (a) + \text{ann}_R(M)$. Hence, $\text{ann}_M(\text{ann}_R(m)) = Rm = \text{ann}_M(a)$, and so M is wq-regular module. □

Corollary 5 *Every morphic ring R is wq-regular ring.*

The converse part of the two previous results is not always true as witnessed by the ring $\mathbb{Z}$ which is clearly a wq-regular but not a morphic ring (since it is a non-field domain).

Recall that a commutative ring R is said to be a von Neumann regular ring if for each $a \in R$ there exists $x \in R$ such that $a = a^2x$. In this case $e = ax$ is an idempotent element. It is easy to see that a ring R is a von Neumann regular ring if each principal ideal of R is generated by an idempotent element. Jayaram and Tekir extended the notion of idempotent elements in commutative rings to modules and they studied von Neumann regular modules in terms of this extension. Let M be an R-module. Recall from Jayaram and Tekir (2018) that an element $a \in R$ is said to be M-vn-regular (resp. weak idempotent relatively to M) if $aM = a^2M$ (resp. $am = a^2m$ for each $m \in M$). Note that for a given R-module M and for every $a \in R$, we have the following implications:

$$a \text{ is an idempotent } \Rightarrow a \text{ is a weak idempotent } \Rightarrow a \text{ is an } M\text{-vn-regular element}$$

Note that if M is a faithful R-module (i.e. $\mathrm{ann}_R(M) = 0$) then the right arrow is reversible. However, these three classes of elements of R are not equal in general.

Example 5 For the $\mathbb{Z}$-module $\mathbb{Z}_6$,

(1) 4 is a weak idempotent element which is not idempotent.
(2) 2 is a $\mathbb{Z}_6$-vn-regular which is not a weak idempotent element.

Following (Jayaram and Tekir 2018), an R-module M is said to be a von Neumann regular module if, for each $m \in M$, there exists an M-vn-regular element $a \in R$ such that $Rm = aM$.

Theorem 3 *Every finitely generated von Neumann regular module is also a morphic module.*

Proof Let M be a finitely generated von Neumann regular R-module and $m \in M$. By [Jayaram and Tekir, 2018, Lemma 5], $Rm = eM$ for some weak idempotent element $e \in R$. Hence, $(1-e)Rm = (1-e)eM = 0$, and so $(1-e) \in \mathrm{ann}_R(m)$. It follows that $(1-e) + \mathrm{ann}_R(M) \subseteq \mathrm{ann}_R(m)$. Now, let $t \in \mathrm{ann}_R(m)$. Then, $tm = 0$ and so $teM = 0$. Thus, $te \in \mathrm{ann}_R(M)$, and then $t = te + t(1-e) \in (1-e) + \mathrm{ann}_R(M)$. Hence, $\mathrm{ann}_R(m) = (1-e) + \mathrm{ann}_R(M)$. Since $(1-e)Rm = 0$, we also have $Rm \subseteq \mathrm{ann}_M(1-e)$. Now, let $y \in \mathrm{ann}_M(1-e)$, that is $(1-e)y = 0$. This gives $y = ey \in eM = Rm$, and thus $Rm = \mathrm{ann}_M(1-e)$. Hence, M is a morphic module. □

Corollary 6 *Every von Neumann regular ring is a morphic ring.*

The converse of previous Theorem is not always true as shown by the following example.

Example 6 (1) Let p be a prime integer and consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^2}$. Suppose that $\mathbb{Z}_{p^2}$ is a von Neumann regular module. Hence, $p\mathbb{Z}_{p^2} = \mathbb{Z}\overline{p} = a\mathbb{Z}_{p^2} = a^2\mathbb{Z}_{p^2}$ for some integer a. Write $\overline{a} = \overline{pm}$ for some $m \in \mathbb{Z}$. Then, $\overline{a^2} = \overline{0}$. Hence, $p\mathbb{Z}_{p^2} = (\overline{0})$, a contradiction. Thus, $\mathbb{Z}_{p^2}$ is not a von Neumann regular module. However, by Example 3, it is a morphic module.
(2) The ring $\mathbb{Z}_4$ is a morphic ring but not a von Neumann regular ring.

Proposition 6 *Let M be a torsion-free R-module. Then the following statements are equivalent:*

(1) M is a morphic module.
(2) M is a simple module.
(3) M is a von Neumann regular module

Proof (1) $\Rightarrow$ (2) Let $0 \neq m \in M$. Since M is a morphic module, then $Rm = \text{ann}_M(a)$ and $\text{ann}_R(m) = (a) + \text{ann}_R(M)$ for some $a \in R$. As M is a torsion-free R-module, we have $\text{ann}_R(m) = (0)$ and so $a = 0$. Hence, $Rm = \text{ann}_M(0) = M$. Consequently, M is a simple module.
(2) $\Rightarrow$ (1) Follows from Example 2.
(2) $\Leftrightarrow$ (3) Follows from (Jayaram and Tekir 2018, Proposition 1). □

Recall that an R-module M is said to be a reduced module if, whenever $a \in R$ and $m \in M$ such that $am = 0$, we have $aM \cap Rm = 0$ (Lee and Zhou 2004).

Proposition 7 *Let M be a finitely generated R-module. Then, M is a reduced multiplication morphic module if and only if M is a von Neumann regular module.*

Proof The "only if" part follows from Theorem 3 and (Jayaram et al. 2021, Theorem 3.10). Suppose that M is a reduced multiplication morphic module and let $m, m' \in M$ such that $\text{ann}_R(m) = \text{ann}_R(m')$. Since M is a morphic module, then $Rm = \text{ann}_M(a)$ and $\text{ann}_R(m) = (a) + \text{ann}_R(M)$ for some $a \in R$. Thus, $a \in \text{ann}_R(m) = \text{ann}_R(m')$. Hence, $am' = 0$. So, $m' \in \text{ann}_M(a) = Rm$. Similarly, $m \in Rm'$. Hence, $Rm = Rm'$. Using (Jayaram et al. 2021, Theorem 3.10), we conclude that M is a von Neumann regular module. □

Proposition 8 *Let $\{R_i\}_{i\in\Delta}$ be a family of rings and M_i be an R_i module for each $i \in \Delta$. Set $R = \prod_{i\in\Delta} R_i$ and $M = \prod_{i\in\Delta} M_i$. Then, the following statements are equivalent:*

(1) M is a morphic R-module.
(2) M_i is a morphic R_i-module for each $i \in \Delta$.

Proof (1) $\Rightarrow$ (2) Let i be a fixed element of Δ and let $x \in M_i$. Set $m = (m_j)_{j\in\Delta}$ such that $m_i = x$ and $m_j = 0$ for each $j \neq i$. There exists $a \in R$ such that $Rm = \text{ann}_M(a)$ and $\text{ann}_R(m) = Ra + \text{ann}_R(M)$. Write $a = (a_j)_{j\in\Delta}$. Since $am = 0$, we get $a_i x = 0$. Hence, $R_i x \subseteq \text{ann}_{M_i}(a_i)$ and $R_i a_i + \text{ann}_{R_i}(M_i) \subseteq \text{ann}_{R_i}(x)$. Let $y \in \text{ann}_{M_i}(a_i)$ and set $m' = (m'_j)_{j\in\Delta}$ such that $m'_i = y$ and $m'_j = 0$ for each $j \neq i$. We have $am' = 0$. Thus, $m' \in Rm$. Hence, $y \in R_i x$, and so $\text{ann}_{M_i}(a_i) \subseteq R_i x$. Consequently, $R_i x = \text{ann}_{M_i}(a_i)$. Now, let $\alpha \in \text{ann}_{R_i}(x)$ and set $b = (b_j)_{j\in\Delta}$ such that $b_i = \alpha$. Then, $bm = 0$. Hence, $b \in \text{ann}_R(m) = Ra + \text{ann}_R(M)$. Write $b = ra + c$ with $c = (c_j)_{j\in\Delta} \in \text{ann}_R(M)$ and $r = (r_j)_{j\in\Delta} \in R$. Hence, $\alpha = r_i a_i + c_i$ with $c_i \in \text{ann}_{R_i}(M_i)$. So, $\alpha \in R_i a_i + \text{ann}_{R_i}(M_i)$. Thus, $\text{ann}_{R_i}(x) \subseteq R_i a_i + \text{ann}_{R_i}(M_i)$. Hence, $\text{ann}_{R_i}(x) = R_i a_i + \text{ann}_{R_i}(M_i)$. Consequently, M_i is a morphic R_i-module.
(2) $\Rightarrow$ (1) Let $m = (m_j)_{j\in\Delta} \in M$. For each $j \in \Delta$, there exists $a_j \in R_j$ such that $R_j m_j = \text{ann}_{M_j}(a_j)$ and $\text{ann}_{R_j}(m_j) = R_j a_j + \text{ann}_{R_j}(M_j)$. Set $a = (a_j)_{j\in\Delta}$. Thus,

$$Rm = \prod_{j\in\Delta} R_j m_j = \prod_{j\in\Delta} \text{ann}_{M_j}(a_j) = \text{ann}_M(a),$$

and

$$\begin{aligned}\operatorname{ann}_R(m) &= \prod_{j\in\Delta} \operatorname{ann}_{R_j}(m_j)\\ &= \prod_{j\in\Delta} \left(R_j a_j + \operatorname{ann}_{R_j}(M_j)\right)\\ &= \prod_{j\in\Delta} R_j a_j + \prod_{i\in\Delta} \operatorname{ann}_{R_j}(M_j)\\ &= Ra + \operatorname{ann}_R(M).\end{aligned}$$

Accordingly, M is a morphic R-module. □

Proposition 9 *Let $f : M \to M'$ be a monomorphism of R-modules. If M' is a morphic R-module then so is M.*

Proof Let $m \in M$. For $m' = f(m) \in M'$, there exists $a \in R$ such that $Rm' = \operatorname{ann}_{M'}(a)$ and $\operatorname{ann}_R(m') = (a) + \operatorname{ann}_R(M')$. We have $0 = am' = f(am)$. Thus, $am = 0$ since f is a monomorphism. Hence, $Rm \subseteq \operatorname{ann}_M(a)$ and $Ra+\operatorname{ann}_R(M) \subseteq \operatorname{ann}_R(m)$. Let $m'' \in \operatorname{ann}_M(a)$. Then, $af(m'') = f(am'') = 0$. Thus, $f(m'') \in \operatorname{ann}_{M'}(a) = Rm'$. So, $f(m'') = rf(m) = f(rm)$ for some $r \in R$. Thus, $m'' = rm \in Rm$. Hence, $\operatorname{ann}_M(a) \subseteq Rm$. So, $\operatorname{ann}_M(a) = Rm$. Let $b \in \operatorname{ann}_R(m)$. Then, $bm' = f(bm) = 0$. Thus, $b \in \operatorname{ann}_R(m') = (a)+\operatorname{ann}_R(M') \subseteq (a)+\operatorname{ann}_R(M)$. So, $\operatorname{ann}_R(m) \subseteq (a) + \operatorname{ann}_R(M)$. Consequently, $\operatorname{ann}_R(m) = (a) + \operatorname{ann}_R(M)$. Thus, M is a morphic R-module. □

Corollary 7 *Every submodule of a morphic module is also morphic.*

Let R be a ring, M be an R-module, and N be a proper submodule of M. Recall that N is called a maximal submodule of M if it is maximal (with respect to set inclusion) among all proper submodules of M. Recall also that N is said to be a prime submodule of M if, for every $a \in R$, the induced homothety $h_a : M/N \to M/N$, $h_a(\overline{x}) = a.\overline{x}$, is either injective or zero. It is well know that every maximal submodule of a module M (if there exists) is also a prime submodule of M.

Theorem 4 *Let M be an R-module and N a submodule of M. Then the following are equivalent:*

(1) N is a maximal submodule of M.
(2) N is prime submodule and M/N is a simple R-module.
(3) N is prime submodule and M/N is a morphic R-module.

Proof (1) ⇒ (2) N is clearly a prime submodule of M. Let $m \in M$. Since N is a proper submodule of M (as a maximal submodule of M), we can consider $m \in M\backslash N$. We have $M = N + Rm$. Thus, $M/N = R\overline{m}$. Hence, M/N is a simple R-module.
(2) ⇒ (3) Clear.
(3) ⇒ (1) Let $m \in M\backslash N$. There exists $a \in R$ such that $R\overline{m} = \operatorname{ann}_{M/N}(a)$ and $\operatorname{ann}_R(\overline{m}) = (a)+\operatorname{ann}_R(M/N)$. Since N is prime, the homothety $h_a : M/N \to M/N$, $h_a(\overline{x}) = a.\overline{x}$, is either injective or zero. Note that $\ker(h_a) = \operatorname{ann}_{M/N}(a) = R\overline{m}$. Since $m \notin N$, we have $\ker(h_a) \neq (\overline{0})$. Thus, $R\overline{m} = \ker(h_a) = M/N$. Thus, $M = Rm + N$, and so N is a maximal submodule of M. □

Theorem 5 *Let M be a morphic R-module and S be a multiplicatively closed subset of R. Then $S^{-1}M$ is a morphic $S^{-1}R$-module.*

Proof Let $\frac{m}{s} \in S^{-1}M$ where $m \in M$ and $s \in S$. Since M is a morphic module, there exists $a \in R$ such that $Rm = \text{ann}_M(a)$ and $\text{ann}_R(m) = Ra + \text{ann}_R(M)$. Thus,

$$\left(S^{-1}R\right)\frac{m}{s} = \left(S^{-1}R\right)\frac{m}{1} = S^{-1}(Rm) = S^{-1}\left(\text{ann}_M(a)\right) = \text{ann}_{S^{-1}M}\left(a/1\right).$$

Moreover,

$$\begin{aligned} \text{ann}_{S^{-1}R}\left(\frac{m}{s}\right) &= \text{ann}_{S^{-1}R}\left(\frac{m}{1}\right) \\ &= S^{-1}\left(\text{ann}_R(m)\right) \\ &= S^{-1}\left(Ra + \text{ann}_R(M)\right) \\ &= S^{-1}\left(Ra\right) + S^{-1}\left(\text{ann}_R(M)\right) \\ &= (a/1) + \text{ann}_{S^{-1}R}(S^{-1}M). \end{aligned}$$

Therefore, $S^{-1}M$ is a morphic $S^{-1}R$-module. □

We end this paper with following **open question**.
Question: What is the relation between morphic modules introduced in this paper and morphic modules in the sense of Nicholson and Sánchez Campos (2005)?

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