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ORIGINAL PAPER

On power series Armendariz modules

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Abstract
All rings are commutative with identity, and all modules are unital. Let R be a ring, an R-module M is called Power Series Armendariz Module if $f(x)m(x) = 0$ implies that $a_i m_j = 0$, for all i and j, where $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$.The purpose of this paper is to investigate power series Armendariz property for some class of Modules, homomorphic image and direct product of modules. The article includes a brief discussion of the scope and precision of our results.

Keywords Power series Armendariz modules · Power series Armendariz ring · Von Neumann regular ring · Reduced module · Prime submodule · Maximal submodule

Mathematics Subject Classification Primary: 13C13 · 13C99

1 Introduction

Throughout this paper, all rings are commutative with identity, and all modules are unital. Let R be a ring, M an R-module and N a submodule of M. We denote by $(N :_R M)$ the set of all $r \in R$ such that $rM \subset N$. The annihilator of M denoted by $ann_R(M)$ is $(0 :_R M)$. An R-module M is called faithful if $ann_R(M) = 0$. $R[x]$, $M[x]$, $R[[x]]$, $M[[x]]$ denotes respectively, the polynomial ring over the ring R, the polynomial module over the module M, the formal power series ring over the ring R, the formal power series module over the module M. An R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal of R. A submodule N of M is idempotent if $(N :_R M)N = N$. Also, N is prime whenever $rm \in N$, for some $r \in R$ and $m \in M$ implies that $m \in N$ or $r \in (N :_R M)$. In this case $P = (N :_R M)$ is a prime ideal of R and N is called a it P-prime submodule of M.

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We recall that an ideal I of R is nilpotent if $I^k = 0$ for some positive integer k, and we denote by $Nil(R)$ the set of all nilpotent element of R. According to Ali (2008), a submodule N of M is called nilpotent if $(N :_R M)^k N = 0$ for some positive integer k. We say that $m \in M$ is nilpotent if Rm is a nilpotent submodule of M. $Nil(M)$ denotes the set of all nilpotent element of M. If M is a faithful multiplication R-module, then $Nil(M) = Nil(R)M = \cap P$, where P runs over all prime submodules of M by Ali (2008, Theorem 6).

Following (Kasch (1982), p. 105), an R-module M is called a Von Neumann regular module if and only if every cyclic submodule M is a direct summand in M. Ali (2008) introduced and studied the concept of faithful multiplication Von Neumann regular module which generalizes Von Neumann regular rings. A ring R is Von Neumann regular ring if every element is Von Neumann regular element, that is, for all $r \in R$, there exists $a \in R$ such that $r = ar^2$.

Armendariz (1974) proved that $a_i b_j = 0$ for all i, j whenever polynomials $f = \sum_{i=0}^{i=n} a_i x^i$ and $g = \sum_{i=0}^{i=m} b_i x^i$ over a reduced ring satisfy $fg = 0$. Rege and Chhawchharia (1997) called such a ring (not necessarily reduced) Armendariz. Armendariz rings are thus a generalization of reduced rings. It is easy to see that subrings of Armendariz rings are Armendariz. Also, Anderson and Camillo (1998) show that a commutative ring R is Gaussian if and only if every homomorphic image of R is Armendariz. See for instance (Anderson and Camillo 1998; Armendariz 1974; Lee and Wong 2003; Rege and Chhawchharia 1997).

Kim et al. (2006) define power serieswise Armendariz rings as rings such that for every $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{i=0}^{\infty} b_i x^i \in R[[x]]$ such that $fg = 0$, then $a_i b_j = 0$ for every i and j. Power serieswise Armendariz rings are clearly Armendariz rings, but the converse is false by Antoine (2010, Example 2). Recall that a reduced ring is power serieswise Armendariz. It is easy to see that a subring of power serieswise Armendariz is also power serieswise Armendariz. See for instance (Antoine 2008, 2010; Hizem 2010; Kim et al. 2006)

An R-module M is called Power Series Armendariz Module if $f(x)m(x) = 0$ implies that $a_i m_j = 0$, for all i and j, where $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$. It is clear that a submodule of power series Armendariz module is also power series Armendariz.

This paper aims at studying the property of power serieswise Armendariz modules. It contains in addition to the introduction three sections. The first one deals with the study of power series Armendariz property for some class of modules. The second and third sections investigate the transfer of the pre-mentioned property to the homomorphic image and direct product.

It is worth mentioning that some parts of proofs are very similar.

2 Power series Armendariz property in some class of modules

We start this section by the following definitions and a lemma which characterizes when a module is reduced.

Definition 2.1 Let R be a ring. An R-module M is called Power Series Armendariz Module if $f(x)m(x) = 0$ implies that $a_i m_j = 0$, for all i and j, where $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$.

Definition 2.2 *(*Lee and Zhou 2004, *Definition 1.1)* A module M is called α-reduced if, for any $m \in M$ and any $a \in R$

1 $ma = 0$ implies $mR \cap Ma = 0$;
2 $ma = 0$ if and only if $m\alpha(a) = 0$. The module M is called reduced if M is 1-reduced.

Lemma 2.3 (Lee and Zhou 2004, Lemma 1.2) *The following are equivalent for a module M_R:*

1 *M_R is α-reduced.*
2 *The following three conditions: For any $m \in M$ and $a \in R$,*

  (a) $ma = 0$ *implies* $mRa = mR\alpha(a) = 0$.
  (b) $ma\alpha(a) = 0$ *implies* $ma = 0$.
  (c) $ma^2 = 0$ *implies* $ma = 0$.

Now, we study the Power Series Armendariz property for some class of modules, in particular reduced Modules.

Theorem 2.4 *Let R be a ring. Then every reduced R-module is Power Series Armendariz Module.*

Proof Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ such that $f(x)m(x) = 0$. Then $\sum_{i+j=k} a_i b_j = 0$ for all k. Thus

$$\begin{cases} a_0 m_0 = 0 & (1) \\ a_1 m_0 + a_0 m_1 = 0 & (2) \\ \vdots & \\ a_k m_0 + \cdots + a_0 m_k = 0 & (k+1) \\ \vdots & \end{cases}$$

We will show that $a_i m_j = 0$ by induction on $i + j$.

If $i + j = 0$, then $a_0 m_0 = 0$.

Now suppose that k is a positive integer such that $a_i m_j = 0$ when $i + j < k$. We will show that $a_i m_j = 0$ when $i + j = k$.

Multiplying the equations (2), (3), (4), ... above by $a_0, a_0^2, a_0^3, \ldots$ (respectively), we get $a_0 m_0 = 0$, $a_0^2 m_1 = 0, \ldots, a_0^{k+1} m_k = 0 \ldots$. Which implies that $a_0 m_0 = 0$,

$a_0m_1 = 0, \ldots, a_0m_k = 0$ Since M is reduced 2.3. Thus we have

$$\begin{cases} a_1m_0 = 0 \\ a_2m_0 + a_1m_1 = 0 \\ \vdots \\ a_km_0 + \cdots + a_1m_{k-1} = 0 \\ \vdots \end{cases}$$

Similarly, we show that $a_1m_0 = 0, a_1m_1 = 0, \cdots, a_1m_{k-1} = 0$

Continuing this procedure, we show that $a_im_j = 0$ when $i + j = k$. Hence $a_im_j = 0$ for all i and j. Consequently M is power series Armendariz module. □

Example 2.5 1 Every reduced ring is power series Armendariz ring.

2 Let p be a prime number and $n > 1$. Then $p^{n-1}\mathbb{Z}_{p^n}$ is power series Armendariz since it is reduced.

Corollary 2.6 *Let R be a ring and M a finitely generated faithful multiplication R-module. Then $M/nil(M)$ is power series Armendariz as $R/nil(R)$ module.*

Proof $M/nil(M)$ is reduced $R/nil(R)$-module. □

Theorem 2.7 *Let R be a ring, and M an R-module such that for all $m \in M$, $A/Ann(m)$ is reduced ring. Then M is power series Armendariz module.*

Proof Let $f(x) = \sum_{i=0}^{\infty} a_ix^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_jx^j \in M[[x]]$ such that $f(x)m(x) = 0$. Then $\sum_{i+j=k} a_ib_j = 0$ for all k. Thus

$$\begin{cases} a_0m_0 = 0 & (1) \\ a_1m_0 + a_0m_1 = 0 & (2) \\ \vdots \\ a_km_0 + \cdots + a_0m_k = 0 & (k+1) \\ \vdots \end{cases}$$

We will show that $a_im_j = 0$ by induction on $i + j$.

If $i + j = 0$, then $a_0m_0 = 0$.

Now suppose that k is a positive integer such that $a_im_j = 0$ when $i + j < k$. We will show that $a_im_j = 0$ when $i + j = k$.

Multiplying the equations (2), (3), (4), ... above by $a_0, a_0^2, a_0^3, \ldots$(respectively), we get $a_0m_0 = 0, a_0^2m_1 = 0, \ldots, a_0^{k+1}m_k = 0 \ldots$. Which implies that $a_0^{k+1} \in Ann(m_k)$, hence $\overline{a_0^{k+1}} = \overline{a_0}^{k+1} = \overline{0}$ in $A/Ann(m_k)$, which implies that $\overline{a_0} = \overline{0}$ in $A/Ann(m_k)$ since $A/Ann(m_k)$ is reduced. Hence $a_0 \in Ann(m_k)$ and thus $a_0m_k = 0$. Also $a_0 \in$

$Ann(m_t)$ for all $t \in \{0, \ldots, k-1\}$, thus we have

$$\begin{cases} a_1 m_0 = 0 \\ a_2 m_0 + a_1 m_1 = 0 \\ \vdots \\ a_k m_0 + \cdots + a_1 m_{k-1} = 0 \\ \vdots \end{cases}$$

Similarly, we show that $a_1 m_{k-1} = 0$. Continuing this procedure, we show that $a_i m_j = 0$ when $i + j = k$. Hence $a_i m_j = 0$ for all i and j. Consequently M is power series Armendariz module. □

Theorem 2.8 *Let R be a ring and M a torsion-free R-module, then M is power series module.*

Proof Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ such that $f(x)m(x) = 0$. Then $\sum_{i+j=k} a_i b_j = 0$ for all k. Thus

$$\begin{cases} a_0 m_0 = 0 \\ a_1 m_0 + a_0 m_1 = 0 \\ \vdots \\ a_k m_0 + \cdots + a_0 m_k = 0 \\ \vdots \end{cases}$$

Suppose that $a_0 \neq 0$, if not we set $f(x) = g(x)x^l$ and $g(x)m(x) = 0$. From the equation (1) we get $m_0 = 0$ since M is torsion free. The same fact implies that $m_1 = 0$, $m_2 = 0$,.... Thus $m_j = 0$ for all j. Hence $a_i m_j = 0$ for all i and j. Consequently M is power series Armendariz module. □

Example 2.9 1 Every flat module is power series Armendariz.
2 Every projective module is power series Armendariz.
3 Every vector space is power series Armendariz.

Proof Flat, projective and vector space are torsion free modules. □

Theorem 2.10 *Let R be a Von Neumann regular ring and M an R-module, then M is power series Armendariz module.*

Proof Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ such that $f(x)m(x) = 0$. Then $\sum_{i+j=k} a_i b_j = 0$ for all k. Thus

$$\begin{cases} a_0 m_0 = 0 & (1) \\ a_1 m_0 + a_0 m_1 = 0 & (2) \\ \vdots & \\ a_k m_0 + \cdots + a_0 m_k = 0 & (k+1) \\ \vdots & \end{cases}$$

We will show that $a_i m_j = 0$ by induction on $i + j$.

If $i + j = 0$, then $a_0 m_0 = 0$.

Now suppose that k is a positive integer such that $a_i m_j = 0$ when $i + j < k$. We will show that $a_i m_j = 0$ when $i + j = k$.

Multiplying the equations (2), (3), (4), $\ldots$, $(k + 1)$ above by $a_0, a_0^2, a_0^3, \ldots$ (respectively), we get $a_0 m_0 = 0$, $a_0^2 m_1 = 0, \ldots, a_0^{k+1} m_k = 0$. Let $I = (a_0)$ and $N = (m_0, \ldots, m_k)$. R is Von Neumann regular ring implies that $a_0 = \alpha a_0^2$ for some $\alpha \in R$. It is easy to show that I is an idempotent ideal of R, that is, $I^2 = I$, hence $I^{k+1} = I$. Thus $IN = I^{k+1}N = 0$ hence $a_0 m_0 = 0$, $a_0 m_1 = 0$,...,$a_0 m_k = 0$. Thus

$$\begin{cases} a_1 m_0 = 0 \\ a_2 m_0 + a_1 m_1 = 0 \\ \vdots \\ a_k m_0 + \cdots + a_1 m_{k-1} = \\ \vdots \end{cases}$$

Similarly, we show that $a_1 m_0 = 0, a_1 m_1 = 0, \ldots, a_1 m_{k-1} = 0$ by taking $I = (a_1)$ and $N = (m_0, \ldots, m_{k-1})$.

Continuing this procedure, we show $a_i m_j = 0$ for all i, j such that $i + j = k$. By induction we have $a_i m_j = 0$ for all i, j which implies that M is power series Armendariz.

3 Power series Armendariz property and homomorphic image of modules

This section deals with the transfer of the power series Armendariz property into homomorphic image of modules.

Theorem 3.1 *Let R be a ring, M an R-module and N is a submodule of M such that $Nil(M) \cap N = (0)$. If M/N is power series Armendariz, then so is M.*

Proof Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ such that $f(x)m(x) = 0$. Then $\sum_{i+j=k} a_i m_j = 0$ for all k. Thus

$$\begin{cases} a_0m_0 & = 0 \quad (1) \\ a_1m_0 + a_0m_1 & = 0 \quad (2) \\ \vdots & \\ a_km_0 + \cdots + a_0m_k & = 0 \ (k+1) \\ \vdots & \end{cases}$$

$f(x)m(x) = 0$ implies that $f(x)\overline{m}(x) = \overline{0}$, where $\overline{m}(x) \in M/N[[x]]$. Which implies that $a_i\overline{m_j} = \overline{a_imj} = \overline{0}$ since M/N is power series Armendariz, hence $a_im_j \in N$ for all i and j. We will show that $a_im_j = 0$ by induction on $i + j$.

$i + j = 0$, then $a_0m_0 = 0$.

Now suppose that k is a positive integer such that $a_im_j = 0$ when $i + j < k$. We will show that $a_im_j = 0$ when $i + j = k$.

Multiplying the equations (2), (3), (4), ... above by $a_0, a_0^2, a_0^3, \ldots$(respectively), we get $a_0m_0 = 0, a_0^2m_1 = 0, \ldots, a_0^{k+1}m_k = 0 \ldots$.

For all $t \in \{1, \ldots, k\}$, $a_0^{t+1}m_t = 0$ implies that $[Ra_0m_t : M]^t Ra_0m_t = 0$. Thus, a_0m_t is nilpotent in M. Hence $a_0m_t \in Nil(M) \cap N = (0)$ for all $t \in \{1, \ldots, k\}$. Thus we have

$$\begin{cases} a_1m_0 & = 0 \\ a_2m_0 + a_1m_1 & = 0 \\ \vdots & \\ a_km_0 + \cdots + a_1m_{k-1} & = 0 \\ \vdots & \end{cases}$$

Similarly, by multiplying the equations by $a_1, a_1^2, a_1^3, \ldots$(respectively) and the same fact as above, we show that $a_1m_0 = 0, a_1m_1 = 0, \ldots, a_1m_{k-1} = 0$.
Continuing this procedure, yields that $a_im_j = 0$ when $i + j = k$. Hence $a_im_j = 0$ for all i and j. Consequently M is power series Armendariz module. □

In the case of rings, $Nil(R) \cap I = (0)$ it is equivalent to I is a reduced ideal, where I is an ideal of a ring R.

Corollary 3.2 (El Ouarrachi and Mahdou 2017, Theorem 2.10) *Let I be a reduced ideal of a ring R such that R/I is power series Armendariz. Then R is power series Armendariz.*

Theorem 3.3 *Let R be a ring, M an R-module and N is a torsion free submodule of M. If M/N is power series Armendariz, then so is M.*

Proof Let $f(x) = \sum_{i=0}^{\infty} a_ix^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_jx^j \in M[[x]]$ such that $f(x)m(x) = 0$. Then $\sum_{i+j=k} a_im_j = 0$ for all k. Thus

$$\begin{cases} a_0m_0 & = 0 \quad (1) \\ a_1m_0 + a_0m_1 & = 0 \quad (2) \\ \vdots & \\ a_km_0 + \cdots + a_0m_k & = 0 \ (k+1) \\ \vdots & \end{cases}$$

Suppose that $a_0 \neq 0$, if not we set $f(x) = g(x)x^l$ and $g(x)m(x) = 0$.
$f(x)m(x) = 0$ implies that $f(x)\overline{m}(x) = \overline{0}$, where $\overline{m}(x) \in M/N[[x]]$. Which implies that $a_i\overline{m_j} = \overline{a_i mj} = \overline{0}$ since M/N is power series Armendariz, hence $a_i m_j \in N$ for all i and j. We will show that $a_i m_j = 0$ by induction on $i + j$.
If $i + j = 0$, then $a_0 m_0 = 0$.
Now suppose that k is a positive integer such that $a_i m_j = 0$ when $i + j < k$. We will show that $a_i m_j = 0$ when $i + j = k$.
***Multiplying the equations (2), (3), (4), ... above by $a_0, a_0^2, a_0^3, \ldots$(respectively), we get $a_0 m_0 = 0$, $a_0^2 m_1 = 0, \ldots, a_0^{k+1} m_k = 0 \ldots$. Which implies that $a_0 m_0 = 0$, $a_0 m_1 = 0, \ldots, a_0 m_k = 0 \ldots$ Since N is torsion free and $a_0 \neq 0$. Thus we have

$$\begin{cases} a_1 m_0 & = 0 \\ a_2 m_0 + a_1 m_1 & = 0 \\ \vdots & \\ a_k m_0 + \cdots + a_1 m_{k-1} & = 0 \\ \vdots & \end{cases}$$

If $a_1 \neq 0$. Similarly as above, we show that $a_1 m_0 = 0, a_1 m_1 = 0, \ldots, a_1 m_{k-1} = 0$.
If $a_1 = 0$, it is obvious that $a_1 m_0 = 0, a_1 m_1 = 0, \ldots, a_1 m_{k-1} = 0$.
Continuing this procedure, yields that $a_i m_j = 0$ when $i + j = k$. Hence $a_i m_j = 0$ for all i and j. Consequently M is power series Armendariz module. □

Theorem 3.4 *Let R be a ring, M an R-module and N is a submodule of M. If N is prime submodule, then M/N is power series Armendariz.*

Proof Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $\overline{m}(x) = \sum_{j=0}^{\infty} \overline{m_j} x^j \in M/N[[x]]$ such that $f(x)\overline{m}(x) = \overline{0}$. Then $\sum_{i+j=k} a_i m_j \in N$ for all k.
Thus

$$\begin{cases} a_0 m_0 & \in N \\ a_1 m_0 + a_0 m_1 & \in N \\ \vdots & \\ a_k m_0 + \cdots + a_0 m_k & \in N \\ \vdots & \end{cases}$$

$a_0 m_0 \in N$ and N prime submodule implies that $a_0 \in [N : M]$ or $m_0 \in N$.

1) $a_0 \in [N : M]$ and $m_0 \notin N$
   $a_0 \in [N : M]$ implies that $a_0 m_1 \in N$. But $a_1 m_0 + a_0 m_1 \in N$, then $a_1 m_0 \in N$ which implies that $a_1 \in [N : M]$ since $m_0 \notin N$.
   $a_1 \in [N : M]$ implies that $a_1 m_1 \in N$. We have $a_0 m_2 \in N$, then $a_2 m_0 \in N$. Hence $a_2 \in [N : M]$.
   Continuing this procedure, we get $a_i \in [N : M]$ for all i. Consequently, $a_i m_j \in N$ for all i, j and $a_i \overline{m_j} = \overline{0}$.
2) $a_0 \notin [N : M]$ and $m_0 \in N$
   $m_0 \in N$ implies that $a_1 m_0 \in N$. But $a_1 m_0 + a_0 m_1 \in N$, then $a_0 m_1 \in N$ which implies that $m_1 \in N$ since $a_0 \notin [N : M]$.

$m_1 \in N$ implies that $a_1 m_1 \in N$. We have $a_2 m_0 \in N$, then $a_0 m_2 \in N$. Hence $m_2 \in N$.
Continuing this procedure, we get $m_j \in N$ for all j. Consequently, $a_i m_j \in N$ for all i, j and $a_i \overline{m_j} = \overline{0}$.

3) $a_0 \in [N : M]$ and $m_0 \in N$
It suffice to repeat the procedure above for $a_1 m_1 \in N$.
This complete the proof.

□

Example 3.5 Let $p\mathbb{Z}[x]$ be the submodule of the $\mathbb{Z}$-module $\mathbb{Z}[x]$, then $(\mathbb{Z}/p\mathbb{Z})[x]$ is power series Armendariz $\mathbb{Z}$-module.

Proof $p\mathbb{Z}[x]$ is prime ideal of $\mathbb{Z}[x]$, hence $p\mathbb{Z}[x]$ is prime submodule of $\mathbb{Z}[x]$. Consequently, $\mathbb{Z}[x]/p\mathbb{Z}[x] \simeq (\mathbb{Z}/p\mathbb{Z})[x]$ is power series Armendariz. □

Corollary 3.6 *Let R be a ring, M an R-module and N is a submodule of M. If N is a maximal submodule, then M/N is power series Armendariz.*

Proof Every maximal submodule is a prime submodule. □

Remark 3.7 1 Every Noetherian R-module M contains a maximal submodule N, then M/N is power series Armendariz.
2 Every finitely generated R-module M contains a maximal submodule N, then M/N is power series Armendariz.

□

4 Direct product of power series Armendariz modules

In this section, we investigate the transfer of the power series Armendariz property into direct product.

Theorem 4.1 *Let R be a ring, $I = \{1, 2, \ldots, n\}$ where n is a positive integer and $\{M_i\}_{i \in I}$ a family of R-modules. Then the direct sum $M = \bigoplus_{i \in I} M_i$ is power series Armendariz module if and only if M_i is, for all $i \in I$.*

Proof It suffice to show that $M = M_1 \bigoplus M_2$ is power series Armendariz if and only if M_1 and M_2 are both power series Armendariz modules.
Suppose that M_1 and M_2 are power series Armendariz modules . Let $f(x) = \sum_{k=0}^{\infty} a_k x^k \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ such that $f(x)m(x) = 0$. We can assume that $m_j = (m_{ji})_{i \in \{1,2\}} \in M = M_1 \bigoplus M_2$. Set $m^i(x) = \sum_{j=0}^{\infty} m_{ij} x^j$, $i \in \{1, 2\}$. It is clear that $f(x)m(x) = 0$ implies that $f(x)m^i(x) = 0$ for all $i \in \{1, 2\}$. Since each M_i is power series Armendariz, $a_k m_{ij} = 0$ for all k, j and all $i \in \{1, 2\}$, which implies that $a_k m_j = 0$ for all k, j. Hence M is power series Armendariz module.
Conversely, Let $f(x) = \sum_{k=0}^{\infty} a_k x^k \in R[[x]]$ and $m_1(x) = \sum_{j=0}^{\infty} m_{1j} x^j \in M_1[[x]]$ such that $f(x)m_1(x) = 0$. Set $m(x) = \sum_{j=0}^{\infty} (m_{1j}, 0)x^j \in M[[x]]$. Then we have

$f(x)m(x) = 0$. Since M is power series Armendariz, we have $a_k m_{1j} = 0$ for all j, k. Hence M_1 is power series Armendariz.
Similarly, we show that M_2 is power series Armendariz. and this complete the proof. □

Corollary 4.2 *Let R be a power series Armendariz ring and let M be a free module of finite rank. Then M is power series Armendariz module.*

Proof $M \simeq R^n$ for some positive integer n. □

Example 4.3 1 Every finitely generated free abelian group G is power series Armendariz.
2 Let $\mathcal{O}_d$ be the ring of integers of algebraic number field $\mathbb{Q}[\sqrt{d}]$, where $d \in \mathbb{N}$. Then $\mathcal{O}_d$ is power series Armendariz.

Proof 1 G is a free $\mathbb{Z}$-module of finite rank and $\mathbb{Z}$ is power series Armendariz ring.
2 $\mathcal{O}_d$ is a free $\mathbb{Z}$-module of rank 2 and $\mathbb{Z}$ is power series Armendariz ring. □

Theorem 4.4 *Let R be a ring containing an idempotent e, and M an R-module. Then M is power series Armendariz module if and only if so is eM and $(1-e)M$.*

Proof Assume that M is power series Armendariz. Since every submodule of a power series Armendariz module is power series Armendariz, then eM and $(1-e)M$ are power series Armendariz modules.
Conversely, suppose that eM and $(1-e)M$ are power series Armendariz modules. Note that $M = eM \bigoplus (1-e)M$.

Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ such that $f(x)m(x) = 0$. Set $m_1(x) = em(x) \in eM[[x]]$ and $m_2(x) = (1-e)m(x) \in (1-e)M[[x]]$.
$f(x)m(x) = 0$ implies that $f(x)m_1(x) = f(x)em(x) = 0$ and $f(x)m_2(x) = f(x)(1-e)m(x) = 0$. Since eM and $(1-e)M$ are power series Armendariz modules, $a_i em_j = a_i(1-e)m_j = 0$ for all i, j. Hence $a_i m_j = a_i em_j + a_i(1-e)m_j = 0$. Thus M is power series Armendariz. □

Corollary 4.5 *Let M be an R-module. Then the following statements are equivalent:*

1) *M is power series Armendariz;*
2) *eM and $(1-e)M$ are power series Armendariz R- modules for every idempotent element e of R;*
3) *eM and $(1-e)M$ are power series Armendariz R- modules for some idempotent element e of R;*

Proof (1) $\Rightarrow$ (2) is obvious since eM and $(1-e)M$ are submodules of M.
(2) $\Rightarrow$ (3) Straightforward.
(3) $\Rightarrow$ (1) Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$ and $m(x) = \sum_{j=0}^{\infty} m_j x^j \in M[[x]]$ such that $f(x)m(x) = 0$. For some $e^2 = e \in R$, set $m_1(x) = em(x) \in eM[[x]]$ and $m_2(x) = (1-e)m(x) \in (1-e)M[[x]]$.

$f(x)m(x) = 0$ implies that $f(x)m_1(x) = f(x)em(x) = 0$ and $f(x)m_2(x) = f(x)(1-e)m(x) = 0$. Since eM and $(1-e)M$ are power series Armendariz modules, then $a_i em_j = a_i(1-e)m_j = 0$ for all i, j. Hence $a_i m_j = a_i em_j + a_i(1-e)m_j = 0$. Thus M is power series Armendariz. □

References

Ali, M.: Idempotent and nilpotent submodules of multiplication modules. Commun. Algebra **36**, 4620–4642 (2008)

Anderson, D.D., Camillo, V.: Armendariz rings and Gaussian rings. Commun. Algebra **26**, 2265–2272 (1998)

Antoine, R.: Nilpotent elements and Armendariz rings. J. Algebra **319**, 3128–3140 (2008)

Antoine, R.: Examples of Armendariz rings. Commun. Algebra **38**(11), 4130–4143 (2010)

Armendariz, E.: A note on extensions of Baer and P. P. rings. J. Austral. Math. Soc. **18**, 470–473 (1974)

Hizem, S.: A note on nil power serieswise Armendariz rings. Rend. Circ. Mat. Palermo **59**, 87–99 (2010)

El Ouarrachi, M., Mahdou, N.: On power serieswise Armendariz rings. Palestine J. Math. **6**(1), 1–9 (2017)

Kasch, F.: Modules and rings. London Academic Press (1982)

Kim, N.K., Lee, K.H., Lee, Y.: Power series rings satisfying a zero divisor property. Commun. Algebra. **34**, 2205–2218 (2006)

Lee, T.K., Wong, T.L.: On Armendariz rings. Houston J. Math. **28**(3), 583–593 (2003)

Lee, T.K., Zhou, Y.: Reduced modules, in Rings, modules, algebras, and Abelian groups. In: Proceedings of the Algebra Conference-Venezia 2002, Venice, Italy, June 3–8, 2002. New York, NY: Marcel Dekker, 365–377 (2004)

Rege, M.B., Chhawchharia, S.: Armendariz rings. Proc. Jpn. Acad. Ser. A Math. Sci. **73**, 14–17 (1997)