Beitr Algebra Geom (2022) 63:287–320
https://doi.org/10.1007/s13366-021-00576-1

ORIGINAL PAPER

Multi-twisted additive codes over finite fields

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Received: 8 June 2020 / Accepted: 15 March 2021 / Published online: 31 March 2021

Abstract
In this paper, we introduce a new class of additive codes over finite fields, *viz.* multi-twisted (MT) additive codes, which are generalizations of constacyclic additive codes. We study their algebraic structures by writing a canonical form decomposition and provide an enumeration formula for these codes. By placing ordinary, Hermitian and $*$ trace bilinear forms, we further study their dual codes and derive necessary and sufficient conditions under which a MT additive code is self-dual and self-orthogonal. We also derive a necessary and sufficient condition for the existence of a self-dual MT additive code over a finite field, and provide enumeration formulae for all self-dual and self-orthogonal MT additive codes over finite fields with respect to the aforementioned trace bilinear forms. We also obtain several good codes within the family of MT additive codes over finite fields.

Keywords Witt decomposition · Witt index · Totally isotropic subspaces

Mathematics Subject Classification 94B15

1 Introduction

Aydin and Haliović (2017) introduced and studied multi-twisted (MT) codes over finite fields, which are generalizations of several well-known classes of linear codes, such as constacyclic codes (Berlekamp 1968) and quasi-cyclic (QC) codes (Townsend and Weldon 1967; Ling and Solé 2001), having rich algebraic structures and containing record-breaker codes. In the same work, they studied 1-generator MT codes over finite fields. They also presented several methods to construct these codes and also provided

Sandeep Sharma: Research support by UGC, India, is gratefully acknowledged.
Anuradha Sharma: Research support by DST-SERB, India, under Grant No. MTR/2017/000358 is gratefully acknowledged.

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bounds on their minimum distances. They also obtained MT codes with best-known parameters [33, 12, 12] over $\mathbb{F}_3$, [53, 18, 21] over $\mathbb{F}_5$, [23, 7, 13] over $\mathbb{F}_7$ and optimal parameters [54, 4, 44] over $\mathbb{F}_7$. Apart from this, they proved that the code parameters [53, 18, 21] over $\mathbb{F}_5$ and [33, 12, 12] over $\mathbb{F}_3$ can not be attained by constacyclic and quasi-cyclic codes, which suggests that this larger class of MT codes is more promising to find codes with better parameters than the current best known linear codes. Later, Sharma et al. (2018) thoroughly investigated algebraic structures of MT codes over finite fields and their dual codes with respect to the Euclidean inner product, and enumerated all MT codes over finite fields. They also derived a necessary and sufficient condition for the existence of a self-dual MT code and provided enumeration formulae for all self-orthogonal and self-dual MT codes over finite fields with respect to the Euclidean inner product. They also showed that MT codes over finite fields can be viewed as direct sums of certain concatenated codes, which gives rise to a method to construct these codes. They also derived a lower bound on their minimum Hamming distances using their multilevel concatenated structures. They also developed a generator theory for these codes, obtained a BCH type lower bound on their minimum Hamming distances, and derived some sufficient conditions under which a MT code is Euclidean self-dual, self-orthogonal and linear with complementary-dual (LCD). In a recent work, Sharma and Chauhan (2019) studied dual codes of MT codes over finite fields with respect to the Hermitian inner product. They also derived a necessary and sufficient condition for the existence of a self-dual MT code and enumerated all self-orthogonal, self-dual and LCD MT codes over finite fields with respect to the Hermitian inner product. Besides this, they counted all LCD MT codes with respect to the Euclidean inner product. In another related work, Sharma and Kaur (2018b) provided explicit enumeration formulae for all self-dual, self-orthogonal and LCD QC codes over finite fields with respect to the Euclidean inner product.

In another direction, additive codes over the finite field $\mathbb{F}_4$ were introduced and studied by Calderbank et al. (1998) as a natural generalization of linear codes. They investigated dual codes of additive codes over $\mathbb{F}_4$ with respect to the trace inner product. They also constructed quantum error-correcting codes from self-orthogonal additive codes over $\mathbb{F}_4$. Later, Bierbrauer and Edel (2000) and Rains (1999) generalized and studied additive codes over arbitrary finite fields. Huffman (2007) introduced and studied cyclic additive codes of odd lengths over $\mathbb{F}_4$ by writing a canonical form decomposition for these codes, and enumerated all such codes. Besides this, he provided explicit enumeration formulae for all self-dual and self-orthogonal cyclic additive codes over $\mathbb{F}_4$ with respect to the trace inner product. Huffman (2008) extended this work for cyclic additive codes of even lengths over $\mathbb{F}_4$. In another work, Huffman (2010) generalized this work for cyclic additive codes of length n over the finite field $\mathbb{F}_{q^t}$, where $t \geq 2$ is an integer, q is a prime power, and n is a positive integer with $\gcd(n, q) = 1$. By placing ordinary and Hermitian trace bilinear forms on $\mathbb{F}_{q^t}^n$, he studied their dual codes, derived necessary and sufficient conditions for the existence of a self-dual cyclic additive code and provided enumeration formulae for all self-orthogonal and self-dual cyclic additive codes over $\mathbb{F}_{q^t}$. When $t = 2$, he explicitly determined all self-dual and self-orthogonal cyclic additive codes of length n over $\mathbb{F}_{q^2}$ with respect to both ordinary and Hermitian trace bilinear forms on $\mathbb{F}_{q^2}^n$. Later, for any integer $t \geq 2$ satisfying $t \not\equiv 1 \pmod{p}$, Sharma and Kaur (2017) introduced a new trace bilinear form on $\mathbb{F}_{q^t}^n$, *viz.* $*$ trace bilinear form. They studied dual codes of cyclic additive codes of length n over $\mathbb{F}_{q^t}$ and provided enumeration formulae for all self-dual and self-orthogonal cyclic additive codes with respect to $*$ trace bilinear form. In another work, Sharma and Kaur (2018a) studied LCD cyclic additive codes of length n over $\mathbb{F}_{q^t}$ and provided enumeration formulae for all LCD cyclic additive codes with respect to ordinary, Hermitian and $*$ trace bilinear forms. Cao et al. (2015) further generalized cyclic additive codes to constacyclic additive codes of length n over $\mathbb{F}_{q^t}$ when t is a prime number and $\gcd(n, q) = 1$. They also studied their dual codes with respect to the ordinary trace bilinear form on $\mathbb{F}_{q^t}^n$. In the same work, they investigated the existence of a self-orthogonal and a self-dual negacyclic additive code of length n over $\mathbb{F}_{q^t}$. They also derived necessary and sufficient conditions for a negacyclic additive code of length n over $\mathbb{F}_{q^2}$ to be self-dual or self-orthogonal. Further, for any integer $t \geq 2$, Kaur and Sharma (2017) developed the theory of constacyclic additive codes over $\mathbb{F}_{q^t}$ by writing a canonical form decomposition for these codes. They also studied their dual codes and provided explicit enumeration formulae for all self-dual, self-orthogonal and LCD cyclic additive codes with respect to ordinary, Hermitian and $*$ trace bilinear forms on $\mathbb{F}_{q^t}^n$.

The main goal of this paper is to introduce and study MT additive codes over finite fields and their dual codes with respect to ordinary, Hermitian and $*$ trace bilinear forms. We shall also derive necessary and sufficient conditions for the existence of a self-dual MT additive code and provide explicit enumeration formulae for all self-dual and self-orthogonal MT additive codes with respect to each of the three above-mentioned trace bilinear forms. For this, throughout this paper, let $\mathbb{F}_q$ be the finite field of order q and characteristic p, and let $t \geq 2$ be an integer. Let $m_1, m_2, \ldots, m_\ell$ be positive integers coprime to q, and let $n = m_1 + m_2 + \cdots + m_\ell$. Let $\Omega = (\omega_1, \omega_2, \ldots, \omega_\ell)$, where $\omega_1, \omega_2, \cdots, \omega_\ell$ are non-zero elements of $\mathbb{F}_q$. This paper is structured as follows: In Sect. 2, we introduce and study Ω-MT additive codes of length n over the finite field $\mathbb{F}_{q^t}$ and their dual codes with respect to ordinary, Hermitian and $*$ trace bilinear forms by writing a canonical form decomposition for these codes. In Sect. 3, we derive necessary and sufficient conditions under which an Ω-MT additive code of length n over $\mathbb{F}_{q^t}$ is self-dual or self-orthogonal with respect to the aforementioned trace bilinear forms (Theorem 3.1). We also enumerate all self-orthogonal Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ (Theorem 3.2). We also derive necessary and sufficient conditions for the existence of a self-dual Ω-MT additive code of length n over $\mathbb{F}_{q^t}$, and provide enumeration formulae for all self-dual Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ with respect to ordinary, Hermitian and $*$ trace bilinear forms (Theorem 3.3). We also obtain several good codes within the family of MT additive codes over finite fields (Table 1).

2 Multi-twisted additive codes over finite fields and their dual codes

In this section, we shall define and study multi-twisted (MT) additive codes over finite fields and their dual codes with respect to three different trace bilinear forms. To do this, we assume, throughout this paper, that $\mathbb{F}_q$ is the finite field of order q

and characteristic p. Let $n = m_1 + m_2 + \cdots + m_\ell$, where $m_1, m_2, \ldots, m_\ell$ are positive integers coprime to q. Let $\omega_1, \omega_2, \ldots, \omega_\ell$ be non-zero elements of $\mathbb{F}_q$, and let $\Omega = (\omega_1, \omega_2, \ldots, \omega_\ell)$ and $\Omega' = (\omega_1^{-1}, \omega_2^{-1}, \ldots, \omega_\ell^{-1})$. Let us define $\mathcal{W} = \prod_{i=1}^{\ell} \mathcal{W}_i$, where $\mathcal{W}_i = \mathbb{F}_q[x]/\langle x^{m_i} - \omega_i \rangle$ for $1 \leq i \leq \ell$. The set $\mathcal{W}$ can be viewed as an $\mathbb{F}_q[x]$-module under componentwise addition and scalar multiplication, and is called the Ω-multi-twisted (MT) module. Now an Ω-multi-twisted (MT) code $\mathcal{C}$ of length n over $\mathbb{F}_q$ is defined as an $\mathbb{F}_q[x]$-submodule of $\mathcal{W}$ (see Aydin and Haliović 2017; Sharma et al. 2018).

Next let $t \geq 2$ be an integer, and let us define $\mathcal{V} = \prod_{i=1}^{\ell} \mathcal{V}_i$, where $\mathcal{V}_i = \mathbb{F}_{q^t}[x]/\langle x^{m_i} - \omega_i \rangle$ for $1 \leq i \leq \ell$. Note that the set $\mathcal{V}$ can be viewed as an $\mathbb{F}_q[x]$-module under componentwise addition and scalar multiplication. Now an Ω-MT additive code of length n over $\mathbb{F}_{q^t}$ is defined as an $\mathbb{F}_q[x]$-submodule of $\mathcal{V}$. Equivalently, an Ω-MT additive code of length n over $\mathbb{F}_{q^t}$ is defined as an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^t}^n$ satisfying the following property: $(\alpha_{1,0}, \alpha_{1,1}, \ldots, \alpha_{1,m_1-1}; \alpha_{2,0}, \alpha_{2,1}, \ldots, \alpha_{2,m_2-1}; \cdots; \alpha_{\ell,0}, \alpha_{\ell,1}, \ldots, \alpha_{\ell,m_\ell-1}) \in \mathcal{C}$ implies that $(\omega_1\alpha_{1,m_1-1}, \alpha_{1,0}, \ldots, \alpha_{1,m_1-2}; \omega_2\alpha_{2,m_2-1}, \alpha_{2,0}, \ldots, \alpha_{2,m_2-2}; \cdots; \omega_\ell\alpha_{\ell,m_\ell-1}, \alpha_{\ell,0}, \ldots, \alpha_{\ell,m_\ell-2}) \in \mathcal{C}$. In particular, Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ coincide with cyclic additive codes of length $n = m_1$ over $\mathbb{F}_{q^t}$ when $\ell = 1$ and $\omega_1 = 1$ (Huffman 2010), while Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ coincide with ω_1-constacyclic additive codes of length $n = m_1$ over $\mathbb{F}_{q^t}$ when $\ell = 1$ (Cao et al. 2015; Kaur and Sharma 2017).

From now on, we shall represent elements of the quotient ring $\mathbb{F}_{\mathfrak{Q}}[x]/\langle F(x) \rangle$ by their representatives in $\mathbb{F}_{\mathfrak{Q}}[x]$ of degree strictly less than the degree of $F(x)$ and we shall perform their addition and multiplication modulo $F(x)$, where $\mathbb{F}_{\mathfrak{Q}}$ is the finite field of order $\mathfrak{Q}$ and $F(x)$ is a non-zero, non-constant polynomial in $\mathbb{F}_{\mathfrak{Q}}[x]$. Moreover, we shall represent the vector $\alpha \in \mathbb{F}_{q^t}^n$ as $(\alpha_{1,0}, \alpha_{1,1}, \ldots, \alpha_{1,m_1-1}; \cdots; \alpha_{\ell,0}, \alpha_{\ell,1}, \ldots, \alpha_{\ell,m_\ell-1})$ and further identify the vector $\alpha \in \mathbb{F}_{q^t}^n$ with the element $\alpha(x) = (\alpha_1(x), \alpha_2(x), \ldots, \alpha_\ell(x)) \in \mathcal{V}$, where $\alpha_i(x) = \alpha_{i,0} + \alpha_{i,1}x + \cdots + \alpha_{i,m_i-1}x^{m_i-1} \in \mathcal{V}_i$ for $1 \leq i \leq \ell$.

In order to study the algebraic structure of Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$, let $g_1(x), g_2(x), \ldots, g_r(x)$ be all the distinct irreducible factors of the polynomials $x^{m_1} - \omega_1, x^{m_2} - \omega_2, \ldots, x^{m_\ell} - \omega_\ell$ in $\mathbb{F}_q[x]$. For $1 \leq u \leq r$ and $1 \leq i \leq \ell$, let us define

$$\epsilon_{u,i} = \begin{cases} 1 & \text{if } g_u(x) \mid x^{m_i} - \omega_i \text{ in } \mathbb{F}_q[x]; \\ 0 & \text{otherwise.} \end{cases}$$

Then for $1 \leq i \leq \ell$, we see that $x^{m_i} - \omega_i = \prod_{u=1}^{r} g_u(x)^{\epsilon_{u,i}}$ is the irreducible factorization of the polynomial $x^{m_i} - \omega_i$ in $\mathbb{F}_q[x]$. Now by applying Chinese Remainder Theorem, we see that $\mathcal{W}_i \simeq \bigoplus_{u=1}^{r} \epsilon_{u,i}\mathcal{F}_u$ for all i, where $\mathcal{F}_u = \dfrac{\mathbb{F}_q[x]}{\langle g_u(x) \rangle}$ for $1 \leq u \leq r$. Next for $1 \leq u \leq r$, if $d_u = \deg g_u(x)$, then $\mathcal{F}_u$ is the finite field of order q^{d_u}. Further, for $1 \leq u \leq r$, by Lemma 1 of Huffman (2010), we note that the polynomial $g_u(x)$ can be factorized into irreducible polynomials over $\mathbb{F}_{q^t}$ of the same degree. For each u, let $g_u(x) = g_{u,0}(x)g_{u,1}(x) \cdots g_{u,a_u-1}(x)$ be the irreducible factorization of the polynomial $g_u(x)$ in $\mathbb{F}_{q^t}[x]$, where $a_u = \gcd(t, d_u)$ and $\deg g_{u,j}(x) = d_u/a_u = D_u$ for $0 \le j \le a_u - 1$. From this, it follows that for $1 \le i \le \ell$, the irreducible factorization of the polynomial $x^{m_i} - \omega_i$ over $\mathbb{F}_{q^t}$ is given by $x^{m_i} - \omega_i = \prod_{u=1}^{r} \prod_{j=0}^{a_u-1} g_{u,j}(x)^{\epsilon_{u,i}}$. Further, by applying Chinese Remainder Theorem again, we get $\mathcal{V}_i \simeq \bigoplus_{u=1}^{r} \bigoplus_{j=0}^{a_u-1} \epsilon_{u,i}\mathcal{F}_{u,j}$ for each i, where $\mathcal{F}_{u,j} = \dfrac{\mathbb{F}_{q^t}[x]}{\langle g_{u,j}(x)\rangle}$ is the finite field of order q^{tD_u} for $1 \le u \le r$ and $0 \le j \le a_u - 1$. More precisely, the ring isomorphism from $\mathcal{V}_i$ onto $\bigoplus_{u=1}^{r} \bigoplus_{j=0}^{a_u-1} \epsilon_{u,i}\mathcal{F}_{u,j}$ is given by

$$\alpha_i(x) \mapsto \sum_{u=1}^{r} \sum_{j=0}^{a_u-1} \Big(\epsilon_{u,i}\alpha_i(x) + \langle g_{u,j}(x)\rangle\Big) \text{ for each } \alpha_i(x) \in \mathcal{V}_i.$$

In view of this, the ring isomorphism from $\mathcal{V}$ onto $\bigoplus_{u=1}^{r} \bigoplus_{j=0}^{a_u-1} \Big(\underbrace{\epsilon_{u,1}\mathcal{F}_{u,j}, \epsilon_{u,2}\mathcal{F}_{u,j}, \ldots, \epsilon_{u,\ell}\mathcal{F}_{u,j}}_{\mathcal{G}_{u,j}}\Big)$ is given by

$$\begin{aligned}(\alpha_1(x), \alpha_2(x), \ldots, \alpha_\ell(x)) \mapsto & \sum_{u=1}^{r} \sum_{j=0}^{a_u-1} (\epsilon_{u,1}\alpha_1(x) + \langle g_{u,j}(x)\rangle, \epsilon_{u,2}\alpha_2(x) \\ & + \langle g_{u,j}(x)\rangle, \ldots, \epsilon_{u,\ell}\alpha_\ell(x) + \langle g_{u,j}(x)\rangle)\end{aligned}$$

for each $(\alpha_1(x), \alpha_2(x), \ldots, \alpha_\ell(x)) \in \mathcal{V}$. For $1 \le u \le r$, let us define $\epsilon_u = \sum_{i=1}^{\ell} \epsilon_{u,i}$, and let $\mathcal{G}_u = \bigoplus_{j=0}^{a_u-1} \mathcal{G}_{u,j}$. We note that the set $\mathcal{G}_u$ can be viewed as a vector space over $\mathcal{F}_u$ under componentwise addition and scalar multiplication, and we observe the following:

Theorem 2.1 *For $1 \le u \le r$, $\mathcal{G}_u = \mathcal{G}_{u,0} \oplus \mathcal{G}_{u,1} \oplus \cdots \oplus \mathcal{G}_{u,a_u-1}$ is a vector space having dimension $\epsilon_u t$ over $\mathcal{F}_u$.*

Since $\mathcal{V} \simeq \mathcal{G} = \bigoplus_{u=1}^{r} \mathcal{G}_u$, from now on, we shall identify each element $(\alpha_1(x), \alpha_2(x), \ldots, \alpha_\ell(x)) \in \mathcal{V}$ with the element $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r) \in \mathcal{G}$, where $\mathcal{A}_u = (\mathcal{A}_{u,0}, \mathcal{A}_{u,1}, \ldots, \mathcal{A}_{u,a_u-1}) \in \mathcal{G}_u$ for each u. Here for $1 \le u \le r$ and $0 \le j \le a_u - 1$, the element $\mathcal{A}_{u,j} \in \mathcal{G}_{u,j}$ is given by $\mathcal{A}_{u,j} = (\mathcal{A}_{u,j}^{(1)}, \mathcal{A}_{u,j}^{(2)}, \ldots, \mathcal{A}_{u,j}^{(\ell)})$, where $\mathcal{A}_{u,j}^{(i)} := \epsilon_{u,i}\alpha_i(x) + \langle g_{u,j}(x)\rangle \in \epsilon_{u,i}\mathcal{F}_{u,j}$ for each i. More precisely, if Ψ denotes the isomorphism from $\mathcal{V}$ onto $\mathcal{G}$, then we shall write $\Psi(\alpha_1(x), \alpha_2(x), \ldots, \alpha_\ell(x)) = \mathcal{A}$. We further observe that the set $\mathcal{G}$ can be viewed as an $\mathbb{F}_q[x]$-module under componentwise addition and scalar multiplication.

In view of the above discussion, we have the following canonical form decomposition of each Ω-MT additive code of length n over $\mathbb{F}_{q^t}$.

Theorem 2.2 (a) *Let $\mathcal{C}(\subseteq \mathcal{V})$ be an Ω-MT additive code of length n over $\mathbb{F}_{q^t}$. For $1 \le u \le r$, let us define $\mathcal{C}_u = \mathcal{C} \cap \mathcal{G}_u$. Then for each u, $\mathcal{C}_u$ is an $\mathcal{F}_u$-linear* *subspace of* $\mathcal{G}_u$*. Furthermore, the code* $\mathcal{C}$ *has a unique direct sum decomposition* $\mathcal{C} = \bigoplus_{u=1}^{r} \mathcal{C}_u$*. (The subspaces* $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ *are called constituents of the code* $\mathcal{C}$*.)*

(b) *Conversely, if* $\mathcal{D}_u$ *is an* $\mathcal{F}_u$*-linear subspace of* $\mathcal{G}_u$ *for* $1 \leq u \leq r$*, and if* $\mathcal{D} = \sum_{u=1}^{r} \mathcal{D}_u$*, then we have* $\mathcal{D} = \bigoplus_{u=1}^{r} \mathcal{D}_u$ *and* $\mathcal{D}$ *is an* Ω*-MT additive code of length* n *over* $\mathbb{F}_{q^t}$*.*

Proof The proof is straightforward. □

The above theorem shows that $\mathcal{F}_u$-linear subspaces of $\mathcal{G}_u$, $1 \leq u \leq r$, are building blocks of all Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$. Now to count all Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$, we recall the following well-known result.

Lemma 2.1 *For any prime power* $\mathfrak{Q}$ *and positive integers* B, K *satisfying* $B \leq K$*, the number of distinct* B*-dimensional subspaces of a* K*-dimensional vector space over* $\mathbb{F}_{\mathfrak{Q}}$ *is given by the* $\mathfrak{Q}$*-binomial coefficient* $\begin{bmatrix} K \\ B \end{bmatrix}_{\mathfrak{Q}} = \prod_{b=0}^{B-1} \frac{(\mathfrak{Q}^{K-b}-1)}{(\mathfrak{Q}^{b+1}-1)}$*, (recall that the* $\mathfrak{Q}$*-binomial coefficient* $\begin{bmatrix} K \\ 0 \end{bmatrix}_{\mathfrak{Q}}$ *is assigned the value* 1*). As a consequence, the total number of distinct subspaces of a* K*-dimensional vector space over* $\mathbb{F}_{\mathfrak{Q}}$ *is* $N(K, \mathfrak{Q}) = \sum_{B=0}^{K} \begin{bmatrix} K \\ B \end{bmatrix}_{\mathfrak{Q}} = 1 + \sum_{B=1}^{K} \begin{bmatrix} K \\ B \end{bmatrix}_{\mathfrak{Q}}$*.*

In the next theorem, we enumerate all the distinct Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$.

Theorem 2.3 *The total number of distinct* Ω*-MT additive codes of length* n *over* $\mathbb{F}_{q^t}$ *is given by* $N_\Omega = \prod_{u=1}^{r} \left(\sum_{b=0}^{\epsilon_u t} \begin{bmatrix} \epsilon_u t \\ b \end{bmatrix}_{q^{d_u}} \right)$*.*

Proof It follows immediately from Theorems 2.1 and 2.2, and by Lemma 2.1. □

Remark 2.1 There are Ω-MT additive codes over $\mathbb{F}_{q^t}$ that can also be viewed as Λ-MT additive codes, where $\Omega \neq \Lambda$. For example, let $q = 5$, $t = 2$, $m_1 = 4$, $m_2 = 2$, and let a be a primitive element of $\mathbb{F}_{25}$. Let $\mathcal{C}$ be the $\mathbb{F}_5[x]$-submodule of $\frac{\mathbb{F}_{25}[x]}{\langle x^4-2 \rangle} \times \frac{\mathbb{F}_{25}[x]}{\langle x^2-3 \rangle}$ with the generating set $\{(a^5 + a^5 x + a^{15} x^2 + a^6 x^3, 1 + x)\}$. One can easily observe that the code $\mathcal{C}$ can also be viewed as an $\mathbb{F}_5[x]$-submodule of $\frac{\mathbb{F}_{25}[x]}{\langle x^4-2 \rangle} \times \frac{\mathbb{F}_{25}[x]}{\langle x^2-2 \rangle}$. That is, the code $\mathcal{C}$ is a (2, 3)-MT additive code as well as a (2, 2)-MT additive code of length 6 over $\mathbb{F}_{25}$. Thus the total number of distinct MT additive codes of length n over $\mathbb{F}_{q^t}$ is not equal to $(q-1)^\ell N_\Omega$.

Huffman (2010) and Sharma and Kaur (2017); Kaur and Sharma (2017) studied dual codes of constacyclic additive codes of length n over $\mathbb{F}_{q^t}$ with respect to ordinary, Hermitian and $*$ trace bilinear forms on $\mathbb{F}_{q^t}^n$, which are as defined below:

The ordinary trace bilinear form on $\mathbb{F}_{q^t}^n$ is a map $\langle \cdot, \cdot \rangle_0 : \mathbb{F}_{q^t}^n \times \mathbb{F}_{q^t}^n \to \mathbb{F}_q$, defined as

$$\langle \alpha, \beta \rangle_0 = \sum_{k=1}^{n} \mathrm{Tr}_{q^t,q}(\alpha_k \beta_k) = \sum_{i=1}^{\ell} \sum_{h=0}^{m_i-1} \mathrm{Tr}_{q^t,q}(\alpha_{i,h} \beta_{i,h})$$

for all $\alpha, \beta \in \mathbb{F}_{q^t}^n$, where $\text{Tr}_{q^t,q}$ denotes the trace map from $\mathbb{F}_{q^t}$ onto $\mathbb{F}_q$. By Lemma 5 of Huffman (2010), we see that the ordinary trace bilinear form $\langle\cdot,\cdot\rangle_0$ is a non-degenerate and symmetric bilinear form on $\mathbb{F}_{q^t}^n$.

To define the Hermitian trace bilinear form on $\mathbb{F}_{q^t}^n$, we assume that the integer t is even. Let us write $t = 2^a U = 2AU$ with $A = 2^{a-1}$, where $a \geq 1$ and the integer U is odd. One can easily observe that there exists a non-zero element $\gamma \in \mathbb{F}_{q^{2A}}$ satisfying $\gamma + \gamma^{q^A} = 0$. Note that the element γ lies in $\mathbb{F}_{q^t}$ and satisfies $\gamma^{q^{Ag}} = (-1)^g\gamma$ for each integer $g \geq 1$. The Hermitian trace bilinear form on $\mathbb{F}_{q^t}^n$ is a map $\langle\cdot,\cdot\rangle_\gamma : \mathbb{F}_{q^t}^n \times \mathbb{F}_{q^t}^n \to \mathbb{F}_q$, defined as

$$\langle\alpha,\beta\rangle_\gamma = \sum_{k=1}^{n} \text{Tr}_{q^t,q}(\gamma\alpha_k\beta_k^{q^{t/2}}) = \sum_{i=1}^{\ell}\sum_{h=0}^{m_i-1} \text{Tr}_{q^t,q}(\gamma\alpha_{i,h}\beta_{i,h}^{q^{t/2}})$$

for all $\alpha, \beta \in \mathbb{F}_{q^t}^n$. By Lemma 5 of Huffman Huffman (2010), we note that the Hermitian trace bilinear form $\langle\cdot,\cdot\rangle_\gamma$ is a non-degenerate, reflexive and an alternating bilinear form on $\mathbb{F}_{q^t}^n$.

To define the $*$ trace bilinear form on $\mathbb{F}_{q^t}^n$, let q be a power of the prime p, and let $t \not\equiv 1 \pmod p$. Then the map $\phi : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t}$, defined as $\phi(a) = \sum_{\lambda=1}^{t-1} a^{q^\lambda} = \text{Tr}_{q^t,q}(a) - a$ for each $a \in \mathbb{F}_{q^t}$, is an $\mathbb{F}_q$-linear vector space isomorphism. Now the $*$ trace bilinear form on $\mathbb{F}_{q^t}^n$ is a map $\langle\cdot,\cdot\rangle_* : \mathbb{F}_{q^t}^n \times \mathbb{F}_{q^t}^n \to \mathbb{F}_q$, defined as

$$\langle\alpha,\beta\rangle_* = \sum_{k=1}^{n} \text{Tr}_{q^t,q}(\alpha_k\phi(\beta_k)) = \sum_{i=1}^{\ell}\sum_{h=0}^{m_i-1} \text{Tr}_{q^t,q}(\alpha_{i,h}\phi(\beta_{i,h}))$$

for all $\alpha, \beta \in \mathbb{F}_{q^t}^n$. By Lemma 3.2 of Sharma and Kaur (2017), we see that the $*$ trace bilinear form $\langle\cdot,\cdot\rangle_*$ is a non-degenerate and symmetric bilinear form on $\mathbb{F}_{q^t}^n$, and is alternating in the case when q is even.

Now we shall study dual codes of Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ with respect to these three trace bilinear forms on $\mathbb{F}_{q^t}^n$. For this, let $\delta \in \{0, *, \gamma\}$, and let $\mathbb{T}_\delta$ be the set of (i) all integers $t \geq 2$ when $\delta = 0$, (ii) all integers $t \geq 2$ satisfying $t \not\equiv 1 \pmod p$ when $\delta = *$, and (iii) all even integers $t \geq 2$ when $\delta = \gamma$. From now on, let $t \in \mathbb{T}_\delta$ be fixed. If $\mathcal{C}$ is an Ω-MT additive code of length n over $\mathbb{F}_{q^t}$, then its δ-dual code $\mathcal{C}^{\perp_\delta}$ is defined as $\mathcal{C}^{\perp_\delta} = \{a \in \mathbb{F}_{q^t}^n : \langle a, c\rangle_\delta = 0 \text{ for all } c \in \mathcal{C}\}$. One can easily observe that the δ-dual code $\mathcal{C}^{\perp_\delta}$ is an Ω'-MT additive code of length n over $\mathbb{F}_{q^t}$. Equivalently, the δ-dual code $\mathcal{C}^{\perp_\delta}$ is an $\mathbb{F}_q[x]$-submodule of the Ω'-MT module $\mathcal{V}' = \prod_{i=1}^{\ell} \mathcal{V}_i'$, where $\mathcal{V}_i' = \mathbb{F}_{q^t}[x]/\langle x^{m_i} - \omega_i^{-1}\rangle$ for $1 \leq i \leq \ell$.

Let m be the least positive integer such that the polynomial $\text{lcm}\left[x^{m_1} - \omega_1, x^{m_2} - \omega_2, \ldots, x^{m_\ell} - \omega_\ell\right]$ divides $x^m - 1$ in $\mathbb{F}_q[x]$. We observe that $m = \text{lcm}\,[m_1 O(\omega_1), m_2 O(\omega_2), \ldots, m_\ell O(\omega_\ell)]$ and that $T_\Omega^m = T_{\Omega'}^m = I$, where I is the identity operator on $\mathbb{F}_{q^t}^n$ and $O(\omega_i)$ denotes the multiplicative order of ω_i for each i. Next let $\mathfrak{Q} = q^e$ with $e \geq 1$, $\pi \in \{1, -1\}$, and let θ be an integer satisfying $0 \leq \theta \leq e - 1$. Now for each monic divisor $F(x) = \sum_{h=0}^{d-1} a_h x^h + x^d$ of $x^m - 1$

in $\mathbb{F}_{\mathfrak{Q}}[x]$, let $F^{\dagger}(x) = a_0^{-1}\sum_{h=0}^{d-1} a_h x^{d-h} + a_0^{-1}$ denote the reciprocal polynomial of $F(x)$, and let us define

$$\widehat{F}(x) = \begin{cases} \sum\limits_{h=0}^{d-1} a_h^{q^\theta} x^h + x^d & \text{if } \pi = 1; \\ a_0^{-q^\theta} \sum\limits_{h=0}^{d-1} a_h^{q^\theta} x^{d-h} + a_0^{-q^\theta} & \text{if } \pi = -1. \end{cases}$$

Now the map $\tau_{q^\theta,\pi} : \frac{\mathbb{F}_{\mathfrak{Q}}[x]}{\langle F(x)\rangle} \to \frac{\mathbb{F}_{\mathfrak{Q}}[x]}{\langle \widehat{F}(x)\rangle}$, defined as $\tau_{q^\theta,\pi}\left(\sum_{h=0}^{d-1} a_h x^h\right) = \sum_{h=0}^{d-1} a_h^{q^\theta} x^{\pi h}$ for each $\sum_{h=0}^{d-1} a_h x^h \in \frac{\mathbb{F}_{\mathfrak{Q}}[x]}{\langle F(x)\rangle}$, is a ring isomorphism, (here $x^{-1} = x^{m-1}$ when $\pi = -1$). Moreover, the ring isomorphism $\tau_{q^{e-\theta},\pi}$ is the inverse of $\tau_{q^\theta,\pi}$. In particular, when $F(x) = x^{m_i} - \omega_i^{-1} \in \mathbb{F}_{q^t}[x]$, we see that $\widehat{F}(x) = x^{m_i} - \omega_i$, where $1 \le i \le \ell$. Further, for $1 \le i \le \ell$, the ring isomorphism $\tau_{1,-1} : \mathcal{V}_i' \to \mathcal{V}_i$ is defined as $\tau_{1,-1}(\beta_i(x)) = \beta_i(x^{-1})$ for each $\beta_i(x) \in \mathcal{V}_i'$, where $x^{-1} = \omega_i^{-1} x^{m_i-1} \in \mathcal{V}_i$. The map $\tau_{1,-1}$ can be further extended to the map $\tau_{1,-1} : \mathcal{V}' \to \mathcal{V}$ as $\tau_{1,-1}(\beta(x)) = \left(\tau_{1,-1}(\beta_1(x)), \tau_{1,-1}(\beta_2(x)), \ldots, \tau_{1,-1}(\beta_\ell(x))\right)$ for each $\beta(x) = (\beta_1(x), \beta_2(x), \ldots, \beta_\ell(x)) \in \mathcal{V}'$. On the other hand, when $F(x) = x^m - 1$, we see that $\widehat{F}(x) = x^m - 1$, and hence the map $\tau_{1,-1} : \mathbb{F}_q[x]/\langle x^m - 1\rangle \to \mathbb{F}_q[x]/\langle x^m - 1\rangle$ is defined as $\tau_{1,-1}(\sum_{h=0}^{m-1} a_h x^h) = \sum_{h=0}^{m-1} a_h x^{-h}$ for each $\sum_{h=0}^{m-1} a_h x^h \in \mathbb{F}_q[x]/\langle x^m - 1\rangle$, where $x^{-1} = x^{m-1}$ in $\mathbb{F}_q[x]/\langle x^m - 1\rangle$.

In order to study algebraic structures of δ-dual codes of Ω-MT additive codes over $\mathbb{F}_{q^t}$ for $\delta \in \{0, *, \gamma\}$, we define the map $(\cdot,\cdot)_\delta : \mathcal{V} \times \mathcal{V}' \to \mathbb{F}_q[x]/\langle x^m - 1\rangle$ as follows:

For $\alpha(x) \in \mathcal{V}$ and $\beta(x) \in \mathcal{V}'$, let us define

$$(\alpha(x), \beta(x))_\delta = \begin{cases} \sum\limits_{i=1}^{\ell}\sum\limits_{\mu=0}^{t-1} \omega_i \left(\frac{x^m-1}{x^{m_i}-\omega_i}\right) \tau_{q^\mu,1}\Big(\alpha_i(x)\tau_{1,-1}(\beta_i(x))\Big) & \text{when } \delta = 0; \\ \sum\limits_{i=1}^{\ell}\sum\limits_{\mu=0}^{t-1} \omega_i \left(\frac{x^m-1}{x^{m_i}-\omega_i}\right) \tau_{q^\mu,1}\Big(\alpha_i(x)\sum\limits_{\lambda=1}^{t-1} \tau_{q^\lambda,-1}(\beta_i(x))\Big) & \text{when } \delta = *; \\ \sum\limits_{i=1}^{\ell}\sum\limits_{\mu=0}^{t-1} \omega_i \left(\frac{x^m-1}{x^{m_i}-\omega_i}\right) \tau_{q^\mu,1}\Big(\gamma\alpha_i(x)\tau_{q^{t/2},-1}(\beta_i(x))\Big) & \text{when } \delta = \gamma. \end{cases}$$

Here the quotient ring $\frac{\mathbb{F}_q[x]}{\langle x^m-1\rangle}$ is viewed as an $\mathbb{F}_q[x]$-module.

In the following lemma, we relate the map $(\cdot,\cdot)_\delta$ with the bilinear form $\langle\cdot,\cdot\rangle_\delta$ on $\mathbb{F}_{q^t}^n$ and study its basic properties for each $\delta \in \{0, *, \gamma\}$.

Lemma 2.2 *Let $\alpha(x), \alpha_1(x) \in \mathcal{V}$, $\beta(x), \beta_1(x) \in \mathcal{V}'$ and $f(x), g(x) \in \mathbb{F}_q[x]/\langle x^m - 1\rangle$. Then for $\delta \in \{0, *, \gamma\}$, the following hold.*

- (a) $(\alpha(x), \beta(x))_\delta = \sum_{k=0}^{m-1} \langle \alpha, T_{\Omega'}^k(\beta)\rangle_\delta x^k$, *where $T_{\Omega'}^k(\beta)$ denotes the k^{th} Ω'-MT shift of $\beta \in \mathbb{F}_{q^t}^n$.*
- (b) $(\alpha(x), \beta(x)+\beta_1(x))_\delta = (\alpha(x), \beta(x))_\delta + (\alpha(x), \beta_1(x))_\delta$ *and* $(\alpha(x)+\alpha_1(x), \beta(x))_\delta = (\alpha(x), \beta(x))_\delta + (\alpha_1(x), \beta(x))_\delta$.

(c) $(f(x)\alpha(x), \beta(x))_\delta = f(x)(\alpha(x), \beta(x))_\delta$ *and* $(\alpha(x), g(x)\beta(x))_\delta = \tau_{1,-1}(g(x))$ $(\alpha(x), \beta(x))_\delta$.
(d) $(\alpha(x), \beta(x))_\delta = \tau_{1,-1}(\beta(x), \alpha(x))_\delta$ *for* $\delta \in \{0, *\}$, *while* $(\alpha(x), \beta(x))_\gamma = -\tau_{1,-1}((\beta(x), \alpha(x))_\gamma)$
and $(\alpha(x), \tau_{1,-1}(\alpha(x)))_\gamma = 0$.
(e) $(\cdot, \cdot)_\delta$ *is non-degenerate.*

Proof Working in a similar manner as in Lemma 6 of Huffman (2010) when $\delta \in \{0, \gamma\}$ and as in Lemma 3.3 of Sharma et al.Sharma and Kaur (2017) when $\delta = *$, the desired result follows. □

From the above discussion, we deduce the following:

Theorem 2.4 *Let $\mathcal{C}(\subseteq \mathcal{V})$ be an Ω-MT additive code of length n over $\mathbb{F}_{q^t}$. Then for $\delta \in \{0, *, \gamma\}$, the δ-dual code $\mathcal{C}^{\perp_\delta}(\subseteq \mathcal{V}')$ of the code $\mathcal{C}$ is an $\mathbb{F}_q[x]$-submodule of $\mathcal{V}'$ and is given by $\mathcal{C}^{\perp_\delta} = \{\beta(x) \in \mathcal{V}' : (\alpha(x), \beta(x))_\delta = 0$ for all $\alpha(x) \in \mathcal{C}\}$.*

Now we shall further study duality properties of Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$. To do this, working as above, we see that $\mathcal{V}' \simeq \mathcal{G}' = \bigoplus_{u=1}^{r} \mathcal{G}'_u$, where $\mathcal{G}'_u = \bigoplus_{j=0}^{a_u-1} \mathcal{G}'_{u,j}$ for $1 \leq u \leq r$. Here we have $\mathcal{G}'_{u,j} = \left(\epsilon_{u,1}\mathcal{F}^\dagger_{u,j}, \epsilon_{u,2}\mathcal{F}^\dagger_{u,j}, \ldots, \epsilon_{u,\ell}\mathcal{F}^\dagger_{u,j}\right)$, where $\mathcal{F}^\dagger_{u,j} = \frac{\mathbb{F}_{q^t}[x]}{\langle g^\dagger_{u,j}(x)\rangle}$ for $1 \leq u \leq r$ and $0 \leq j \leq a_u - 1$. In view of this, from now on, we shall identify each element $(\beta_1(x), \beta_2(x), \ldots, \beta_\ell(x)) \in \mathcal{V}'$ with the element $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r) \in \mathcal{G}'$, where $\mathcal{B}_u = (\mathcal{B}_{u,0}, \mathcal{B}_{u,1}, \ldots, \mathcal{B}_{u,a_u-1}) \in \mathcal{G}'_u$ for each u. Moreover, for $1 \leq u \leq r$ and $0 \leq j \leq a_u - 1$, the element $\mathcal{B}_{u,j} \in \mathcal{G}'_{u,j}$ is given by $\mathcal{B}_{u,j} = (\mathcal{B}^{(1)}_{u,j}, \mathcal{B}^{(2)}_{u,j}, \ldots, \mathcal{B}^{(\ell)}_{u,j})$, where $\mathcal{B}^{(i)}_{u,j} := \epsilon_{u,i}\beta_i(x) + \langle g^\dagger_{u,j}(x)\rangle \in \epsilon_{u,i}\mathcal{F}^\dagger_{u,j}$ for each i. More precisely, if Ψ' denotes the isomorphism from $\mathcal{V}'$ onto $\mathcal{G}'$, then we shall write $\Psi'(\beta_1(x), \beta_2(x), \ldots, \beta_\ell(x)) = \mathcal{B}$.

Now for $\delta \in \{0, *, \gamma\}$, we shall further relate the sesquilinear form $(\cdot, \cdot)_\delta$ on $\mathcal{V} \times \mathcal{V}'$ with the corresponding map $[\cdot, \cdot]_\delta$ on $\mathcal{G} \times \mathcal{G}'$ and study its properties. Towards this, we see that if $\epsilon_{u,i} = 1$ for some u and i, then $g_u(x)$ divides $x^{m_i} - \omega_i$ in $\mathbb{F}_q[x]$, which implies that $x^{m_i} = \omega_i$ in $\mathcal{F}_u$. This further implies that $\omega_i\left(\frac{x^m-1}{x^{m_i}-\omega_i}\right) = \sum_{h=0}^{\frac{m}{m_i}-1} \omega_i^{-h} x^{hm_i} = \frac{m}{m_i}$ in $\mathcal{F}_u$. Moreover, for $F(x) = g_{u,j}(x) \in \mathbb{F}_{q^t}[x]$, we have $\widehat{F}(x) = g_{u,j+\theta}(x)$ if $\pi = 1$, while $\widehat{F}(x) = g^\dagger_{u,j+\theta}(x)$ if $\pi = -1$, (here the subscript $j + \theta$ is considered modulo a_u). Therefore the ring isomorphism $\tau_{q^\theta,\pi} : \frac{\mathbb{F}_{q^t}[x]}{\langle F(x)\rangle} \to \frac{\mathbb{F}_{q^t}[x]}{\langle \widehat{F}(x)\rangle}$ is defined as $\tau_{q^\theta,\pi}\left(\sum_{h=0}^{D_u-1} a_h x^h\right) = \sum_{h=0}^{D_u-1} a_h^{q^\theta} x^{h\pi}$ for each $\sum_{h=0}^{D_u-1} a_h x^h \in \frac{\mathbb{F}_{q^t}[x]}{\langle F(x)\rangle}$. Furthermore, for $F(x) = g^\dagger_u(x) \in \mathbb{F}_q[x]$, we have $\widehat{F}(x) = g_u(x)$, and hence the ring isomorphism $\tau_{1,-1} : \mathcal{F}^\dagger_u \to \mathcal{F}_u$ is defined as $\tau_{1,-1}\left(\sum_{h=0}^{d_u-1} a_h x^h\right) = \sum_{h=0}^{d_u-1} a_h x^{-h}$ for all $\sum_{h=0}^{d_u-1} a_h x^h \in \mathcal{F}^\dagger_u$.

In view of the above discussion, we observe that for $\delta \in \{0, *, \gamma\}$, the sesquilinear form $(\cdot, \cdot)_\delta$ corresponds to the map $[\cdot, \cdot]_\delta : \mathcal{G} \times \mathcal{G}' \to \bigoplus_{u=1}^{r} \mathcal{F}_u$, which is defined as follows:

For $\delta = 0$, the map $[\cdot,\cdot]_0$ is defined as

$$[\mathcal{A},\mathcal{B}]_0 = \Bigg(\sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{1,i}\sum_{j=0}^{a_1-1}\sum_{\mu=0}^{(t/a_1)-1}\tau_{q^{\mu a_1+j},1}\Big(\mathcal{A}^{(i)}_{1,a_1-j}\tau_{1,-1}(\mathcal{B}^{(i)}_{1,a_1-j})\Big),$$
$$\sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{2,i}\sum_{j=0}^{a_2-1}\sum_{\mu=0}^{(t/a_2)-1}\tau_{q^{\mu a_2+j},1}\Big(\mathcal{A}^{(i)}_{2,a_2-j}\tau_{1,-1}(\mathcal{B}^{(i)}_{2,a_2-j})\Big),\ldots,$$
$$\sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{r,i}\sum_{j=0}^{a_r-1}\sum_{\mu=0}^{(t/a_r)-1}\tau_{q^{\mu a_r+j},1}\Big(\mathcal{A}^{(i)}_{r,a_r-j}\tau_{1,-1}(\mathcal{B}^{(i)}_{r,a_r-j})\Big)\Bigg). \tag{2.1}$$

For $\delta = *$, the map $[\cdot,\cdot]_*$ is defined as

$$[\mathcal{A},\mathcal{B}]_* = -[\mathcal{A},\mathcal{B}]_0 + \Bigg(\sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{1,i}\Bigg(\Big(\sum_{j=0}^{a_1-1}\sum_{\mu=0}^{(t/a_1)-1}\tau_{q^{\mu a_1+j},1}(\mathcal{A}^{(i)}_{1,a_1-j})\Big)$$
$$\Big(\sum_{j=0}^{a_1-1}\sum_{\sigma=0}^{(t/a_1)-1}\tau_{q^{\sigma a_1+j},1}(\tau_{1,-1}(\mathcal{B}^{(i)}_{1,a_1-j}))\Big)\Bigg),$$
$$\sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{2,i}\Bigg(\Big(\sum_{j=0}^{a_2-1}\sum_{\mu=0}^{(t/a_2)-1}\tau_{q^{\mu a_2+j},1}(\mathcal{A}^{(i)}_{2,a_2-j})\Big)$$
$$\Big(\sum_{j=0}^{a_2-1}\sum_{\sigma=0}^{(t/a_2)-1}\tau_{q^{\sigma a_2+j},1}(\tau_{1,-1}(\mathcal{B}^{(i)}_{2,a_2-j}))\Big)\Bigg),\ldots,$$
$$\sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{r,i}\Bigg(\Big(\sum_{j=0}^{a_r-1}\sum_{\mu=0}^{(t/a_r)-1}\tau_{q^{\mu a_r+j},1}(\mathcal{A}^{(i)}_{r,a_r-j})\Big)$$
$$\Big(\sum_{j=0}^{a_r-1}\sum_{\sigma=0}^{(t/a_r)-1}\tau_{q^{\sigma a_r+j},1}(\tau_{1,-1}(\mathcal{B}^{(i)}_{r,a_r-j}))\Big)\Bigg)\Bigg). \tag{2.2}$$

Further, for $g = (g_1, g_2, \ldots, g_r) \in \bigoplus_{u=1}^{r}\mathcal{F}_u^{\dagger}$, let us define $\tau_{1,-1}(g) = (\tau_{1,-1}(g_1), \tau_{1,-1}(g_2), \ldots, \tau_{1,-1}(g_r)) \in \bigoplus_{u=1}^{r}\mathcal{F}_u$.

In the following lemma, we show that the map $[\cdot,\cdot]_\delta : \mathcal{G}\times\mathcal{G}' \to \bigoplus_{u=1}^{r}\mathcal{F}_u$ is a reflexive and a non-degenerate $\tau_{1,-1}$-sesquilinear form for $\delta \in \{0,*\}$.

Lemma 2.3 *Let $\mathcal{A},\mathcal{A}' \in \mathcal{G}$, $\mathcal{B},\mathcal{B}' \in \mathcal{G}'$, $f \in \bigoplus_{u=1}^{r}\mathcal{F}_u$, and let $g \in \bigoplus_{u=1}^{r}\mathcal{F}_u^{\dagger}$. Then for $\delta \in \{0,*\}$, the following hold.*

(a) $[\mathcal{A}+\mathcal{A}',\mathcal{B}]_\delta = [\mathcal{A},\mathcal{B}]_\delta + [\mathcal{A}',\mathcal{B}]_\delta$ *and* $[\mathcal{A},\mathcal{B}+\mathcal{B}']_\delta = [\mathcal{A},\mathcal{B}]_\delta + [\mathcal{A},\mathcal{B}']_\delta$.
(b) $[f\mathcal{A},\mathcal{B}]_\delta = f[\mathcal{A},\mathcal{B}]_\delta$ *and* $[\mathcal{A},g\mathcal{B}]_\delta = \tau_{1,-1}(g)[\mathcal{A},\mathcal{B}]_\delta$.
(c) $[\mathcal{A},\mathcal{B}]_\delta = \tau_{1,-1}([\mathcal{B},\mathcal{A}]_\delta)$.
(d) $[\cdot,\cdot]_\delta$ *is non-degenerate.*

Proof Proofs of parts (a)–(c) are trivial. To prove (d), let $\mathcal{A} \in \mathcal{G}$ be such that $[\mathcal{A}, \mathcal{B}]_\delta = 0$ for all $\mathcal{B} \in \mathcal{G}'$. We assert that $\mathcal{A} = 0$. Suppose, on the contrary, that $\mathcal{A}$ is non-zero. As Ψ is the isomorphism from $\mathcal{V}$ onto $\mathcal{G}$ and Ψ' is the isomorphism from $\mathcal{V}'$ onto $\mathcal{G}'$, there exist unique elements $(\alpha_1(x), \alpha_2(x), \ldots, \alpha_\ell(x)) \in \mathcal{V}$ and $(\beta_1(x), \beta_2(x), \ldots, \beta_\ell(x)) \in \mathcal{V}'$ satisfying $\Psi(\alpha_1(x), \alpha_2(x), \ldots, \alpha_\ell(x)) = \mathcal{A}$ and $\Psi'(\beta_1(x), \beta_2(x), \ldots, \beta_\ell(x)) = \mathcal{B}$. Now since $\mathcal{A}$ is non-zero, we note that $\mathcal{A}_u = (\mathcal{A}_{u,0}, \mathcal{A}_{u,1}, \ldots, \mathcal{A}_{u,a_u-1}) \neq 0$ for some u, $1 \leq u \leq r$. From this, we see that $\epsilon_{u,h} = 1$ and $\alpha_h(x) \neq 0$ for some h, $1 \leq h \leq \ell$. Let us write $\alpha_h(x) = \sum_{k=0}^{D_u-1} \alpha_{h,k} x^k$. As $\alpha_h(x) \neq 0$, we note that $\alpha_{h,k} \neq 0$ for some k. Since $\mathrm{Tr}_{q^t,q}$ is a non-zero and an onto map, there exist $c, d \in \mathbb{F}_{q^t}$ satisfying $\mathrm{Tr}_{q^t,q}(\alpha_{h,k} c) \neq 0$ and $\mathrm{Tr}_{q^t,q}(d) \neq 0$. Now let us take $\beta_i(x) = 0$ for $1 \leq i (\neq h) \leq \ell$, and let us take $\beta_h(x) = c$ when $\delta = 0$, while let us take $\beta_h(x) = \phi^{-1}(d\alpha_{h,k}^{-1})$ when $\delta = *$.

Now when $\delta = 0$, we see that $\tau_{1,-1}(\beta_h(x) = c$. This, by (2.1), implies that $[\mathcal{A}, \mathcal{B}]_0 = \dfrac{m}{m_h} \sum_{k=0}^{D_u-1} \mathrm{Tr}_{q^t,q}(\alpha_{h,k} c) x^k \neq 0$, which is a contradiction. This shows that the map $[\cdot, \cdot]_0$ is non-degenerate.

When $\delta = *$, we see that $\tau_{1,-1}(\beta_h(x)) = \phi^{-1}(d\alpha_{h,k}^{-1})$. By (2.2), we obtain $[\mathcal{A}, \mathcal{B}]_* = \dfrac{m}{m_h} \sum_{k=0}^{D_u-1} \mathrm{Tr}_{q^t,q}(\alpha_{h,k}\phi(\phi^{-1}(d\alpha_{h,k}^{-1}))x^k = \dfrac{m}{m_h} \sum_{k=0}^{D_u-1} \mathrm{Tr}_{q^t,q}(d) x^k \neq 0$, which is a contradiction. This shows that the map $[\cdot, \cdot]_*$ is non-degenerate. □

Finally, for $\delta = \gamma$, the map $[\cdot, \cdot]_\gamma : \mathcal{G} \times \mathcal{G}' \to \bigoplus_{u=1}^r \mathcal{F}_u$ is defined as

$$[\mathcal{A}, \mathcal{B}]_\gamma = \Bigg(\sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{1,i} \sum_{j=0}^{a_1-1} \sum_{\mu=0}^{(t/a_1)-1} \tau_{q^{\mu a_1 + j},1} \Big(\gamma \mathcal{A}_{1,a_1-j}^{(i)} \tau_{q^{\frac{t}{2}},-1} \Big(\mathcal{B}_{1,\frac{t}{2}-j}^{(i)} \Big) \Big),$$
$$\sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{2,i} \sum_{j=0}^{a_2-1} \sum_{\mu=0}^{(t/a_2)-1} \tau_{q^{\mu a_2 + j},1} \Big(\gamma \mathcal{A}_{2,a_2-j}^{(i)} \tau_{q^{\frac{t}{2}},-1} \Big(\mathcal{B}_{2,\frac{t}{2}-j}^{(i)} \Big) \Big), \ldots,$$
$$\sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{r,i} \sum_{j=0}^{a_r-1} \sum_{\mu=0}^{(t/a_r)-1} \tau_{q^{\mu a_r + j},1} \Big(\gamma \mathcal{A}_{r,a_r-j}^{(i)} \tau_{q^{\frac{t}{2}},-1} \Big(\mathcal{B}_{r,\frac{t}{2}-j}^{(i)} \Big) \Big) \Bigg). \quad (2.3)$$

In the following lemma, we show that the map $[\cdot, \cdot]_\gamma : \mathcal{G} \times \mathcal{G}' \to \bigoplus_{u=1}^r \mathcal{F}_u$ is a reflexive and a non-degenerate $\tau_{1,-1}$-sesquilinear form.

Lemma 2.4 *Let $\mathcal{A}, \mathcal{A}' \in \mathcal{G}$, $\mathcal{B}, \mathcal{B}' \in \mathcal{G}'$, $f \in \bigoplus_{u=1}^r \mathcal{F}_u$, and let $g \in \bigoplus_{u=1}^r \mathcal{F}_u^\dagger$. Then the following hold.*

(a) $[\mathcal{A} + \mathcal{A}', \mathcal{B}]_\gamma = [\mathcal{A}, \mathcal{B}]_\gamma + [\mathcal{A}', \mathcal{B}]_\gamma$ *and* $[\mathcal{A}, \mathcal{B} + \mathcal{B}']_\gamma = [\mathcal{A}, \mathcal{B}]_\gamma + [\mathcal{A}, \mathcal{B}']_\gamma$.
(b) $[f\mathcal{A}, \mathcal{B}]_\gamma = f[\mathcal{A}, \mathcal{B}]_\gamma$ *and* $[\mathcal{A}, g\mathcal{B}]_\gamma = \tau_{1,-1}(g)[\mathcal{A}, \mathcal{B}]_\gamma$.
(c) $[\mathcal{A}, \mathcal{B}]_\gamma = -\tau_{1,-1}([\mathcal{B}, \mathcal{A}]_\gamma)$.
(d) $[\cdot, \cdot]_\gamma$ *is non-degenerate.*

Proof Proofs of parts (a) and (b) are trivial. To prove (c), we first recall that $t = 2AU$, where U is odd. One can easily observe that $\gamma^{q^{\frac{t}{2}}} = \gamma^{AU} = (-1)^U \gamma = -\gamma$. From this and by (2.3), part (c) follows immediately.

Now it remains to show that the map $[\cdot,\cdot]_\gamma$ is non-degenerate. For this, let $\mathcal{A} \in \mathcal{G}$ be such that $[\mathcal{A},\mathcal{B}]_\gamma = 0$ for all $\mathcal{B} \in \mathcal{G}'$. We assert that $\mathcal{A} = 0$.

Suppose, on the contrary, that $\mathcal{A}$ is non-zero. Since $\mathcal{A}$ is non-zero, we note that $\mathcal{A}_u = (\mathcal{A}_{u,0}, \mathcal{A}_{u,1}, \ldots, \mathcal{A}_{u,a_u-1}) \neq 0$ for some u, $1 \le u \le r$. From this, we see that $\epsilon_{u,h} = 1$ and $\alpha_h(x) \neq 0$ for some h, $1 \le h \le \ell$. Let us write $\alpha_h(x) = \sum_{k=0}^{D_u-1} \alpha_{h,k} x^k$. As $\alpha_h(x) \neq 0$, we note that $\alpha_{h,k} \neq 0$ for some k. Since $\mathrm{Tr}_{q^t,q}$ is a non-zero and an onto map, there exists $\xi \in \mathbb{F}_{q^t}$ satisfying $\mathrm{Tr}_{q^t,q}(\gamma \alpha_{h,k} \xi^{q^{\frac{t}{2}}}) \neq 0$. Now let us take $\beta_i(x) = 0$ for $1 \le i (\neq h) \le \ell$, and let us take $\beta_h(x) = \xi$. Then we see, by (2.3), that $[\mathcal{A},\mathcal{B}]_\gamma = \frac{m}{m_h} \sum_{k=0}^{D_u-1} \mathrm{Tr}_{q^t,q}(\gamma \alpha_{h,k} \xi^{q^{\frac{t}{2}}}) x^k \neq 0$, which is a contradiction.

This shows that the $\tau_{1,-1}$-sesquilinear form $[\cdot,\cdot]_\gamma$ is non-degenerate. □

The following proposition is useful in investigating the algebraic structures of δ-dual codes of Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ for each $\delta \in \{0, *, \gamma\}$.

Proposition 2.1 *Let $\alpha(x) \in \mathcal{V}$ and $\beta(x) \in \mathcal{V}'$ be identified with $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_r) \in \mathcal{G}$ and $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r) \in \mathcal{G}'$, respectively. Then for $\delta \in \{0, *, \gamma\}$, we have $[\mathcal{A},\mathcal{B}]_\delta = 0$ if and only if $[\mathcal{A}_u,\mathcal{B}_u]_\delta = 0$ for $1 \le u \le r$.*

In the following theorem, we relate the constituents of an Ω-MT additive code with that of its δ-dual code, where $\delta \in \{0, *, \gamma\}$.

Theorem 2.5 *Let $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots \oplus \mathcal{C}_r$ and $\mathcal{C}^{\perp_\delta} = \mathcal{C}_1^{(\delta)} \oplus \mathcal{C}_2^{(\delta)} \oplus \cdots \oplus \mathcal{C}_r^{(\delta)}$, where $\mathcal{C}_u = \mathcal{C} \cap \mathcal{G}_u$ and $\mathcal{C}_u^{(\delta)} = \mathcal{C}^{\perp_\delta} \cap \mathcal{G}'_u$ for $1 \le u \le r$. Then for $1 \le u \le r$, we have $\mathcal{C}_u^{(\delta)} = \{\mathcal{B}_u \in \mathcal{G}'_u : [\mathcal{A}_u,\mathcal{B}_u]_\delta = 0$ for all $\mathcal{A}_u \in \mathcal{C}_u\} = \mathcal{C}_u^{\perp_\delta}$ and $\dim_{\mathcal{F}_u^\dagger} \mathcal{C}_u^{(\delta)} = \dim_{\mathcal{F}_u^\dagger} \mathcal{C}_u^{\perp_\delta} = \epsilon_u t - \dim_{\mathcal{F}_u} \mathcal{C}_u$ for $1 \le u \le r$. (Throughout this paper, $\dim_F V$ denotes the dimension of a vector space V over the field F.)*

Proof To prove the result, let $1 \le u \le r$ be fixed. Now we first observe, by Proposition 2.1, that $\mathcal{C}_u^{(\delta)} = \{\mathcal{B}_u \in \mathcal{G}'_u : [\mathcal{A}_u,\mathcal{B}_u]_\delta = 0 \text{ for all } \mathcal{A}_u \in \mathcal{C}_u\} = \mathcal{C}_u^{\perp_\delta}$.

To prove the second part, let $k_u = \dim_{\mathcal{F}_u} \mathcal{C}_u$. Here we assert that $\dim_{\mathcal{F}_u^\dagger} \mathcal{C}_u^{\perp_\delta} = \epsilon_u t - k_u$.

To prove the assertion, for each non-zero $\mathcal{A} \in \mathcal{G}_u$, let us define a map $\varphi_{\mathcal{A}} : \mathcal{G}'_u \to \mathcal{F}_u^\dagger$ as $\varphi_{\mathcal{A}}(\mathcal{B}) = [\mathcal{B},\mathcal{A}]_\delta$ for each $\mathcal{B} \in \mathcal{G}'_u$. One can easily observe that $\varphi_{\mathcal{A}}$ is an $\mathcal{F}_u^\dagger$-linear transformation. Further, working as in Theorem 2.1, we note that $\mathcal{G}'_u$ is an $\epsilon_u t$-dimensional vector space over $\mathcal{F}_u^\dagger$ for each u. So for $1 \le u \le r$, by the rank-nullity theorem, we see that $\mathrm{Nullity}(\varphi_{\mathcal{A}}) = \dim(\ker(\varphi_{\mathcal{A}}))$ is either $\epsilon_u t$ or $\epsilon_u t - 1$, where $\ker(\varphi_{\mathcal{A}})$ denotes the kernel (or the null-space) of $\varphi_{\mathcal{A}}$. Further, if $\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{k_u}\}$ is an $\mathcal{F}_u$-basis of $\mathcal{C}_u$, then one can easily observe that $\ker(\varphi_{\mathcal{A}_1}) \cap \ker(\varphi_{\mathcal{A}_2}) \cap \cdots \cap \ker(\varphi_{\mathcal{A}_{k_u}}) = \mathcal{C}_u^{\perp_\delta}$ and that $\dim_{\mathcal{F}_u^\dagger}(\mathcal{C}_u^{\perp_\delta}) = \dim_{\mathcal{F}_u^\dagger}(\ker(\varphi_{\mathcal{A}_1}) \cap \ker(\varphi_{\mathcal{A}_2}) \cap \cdots \cap \ker(\varphi_{\mathcal{A}_{k_u}})) \ge \epsilon_u t - k_u$ for each u. As $\mathcal{F}_u^\dagger \simeq \mathbb{F}_{q^{d_u}}$, we get $\dim_{\mathbb{F}_q}(\mathcal{C}_u^{\perp_\delta}) \ge (\epsilon_u t - k_u) d_u$. Next it is easy to see that $\dim_{\mathbb{F}_q}(\mathcal{C}) = \sum_{u=1}^r k_u d_u$. Further, by Theorem 4.2.4(ii) of Ling and Xing (2004), we get $\dim_{\mathbb{F}_q}(\mathcal{C}^{\perp_\delta}) = nt - \dim_{\mathbb{F}_q}(\mathcal{C}) = t \sum_{u=1}^r \epsilon_u d_u - \sum_{u=1}^r k_u d_u = \sum_{u=1}^r (\epsilon_u t - k_u) d_u$. Now as $\dim_{\mathbb{F}_q}(\mathcal{C}_u^{\perp_\delta}) \ge (\epsilon_u t - k_u) d_u$, we must have $\dim_{\mathbb{F}_q}(\mathcal{C}_u^{\perp_\delta}) = (\epsilon_u t - k_u) d_u$, which gives $\dim_{\mathcal{F}_u^\dagger}(\mathcal{C}_u^{\perp_\delta}) = \epsilon_u t - k_u = \epsilon_u t - \dim_{\mathcal{F}_u} \mathcal{C}_u$ for each u. □

In the following section, we shall study algebraic structures of self-orthogonal and self-dual Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ with respect to the aforementioned $\tau_{1,-1}$-sesquilinear forms $[\cdot,\cdot]_0$, $[\cdot,\cdot]_*$ and $[\cdot,\cdot]_\gamma$ on $\mathcal{G}\times\mathcal{G}'$.

3 Self-dual and self-orthogonal Ω-MT additive codes over finite fields

Let $\mathcal{C}$ be an Ω-MT additive code of length n over $\mathbb{F}_{q^t}$. For $\delta\in\{0,*,\gamma\}$, the code $\mathcal{C}$ is said to be (i) δ-self-dual if it satisfies $\mathcal{C}=\mathcal{C}^{\perp_\delta}$ and (ii) δ-self-orthogonal if it satisfies $\mathcal{C}\subseteq\mathcal{C}^{\perp_\delta}$. Now we shall study all δ-self-dual and δ-self-orthogonal Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$. To do this, we proceed as follows:

Let $f(x)$ be a non-zero polynomial in $\mathbb{F}_{\mathfrak{Q}}[x]$, and let $f^\dagger(x)$ be its reciprocal polynomial. The polynomial $f(x)$ is said to be self-reciprocal if it satisfies $f(x)=f^\dagger(x)$. We say that two non-zero coprime polynomials $f(x),g(x)\in\mathbb{F}_{\mathfrak{Q}}[x]$ form a reciprocal pair if they satisfy $f^\dagger(x)=g(x)$. Now we recall that $g_1(x),g_2(x),\ldots,g_r(x)$ are all the distinct irreducible factors of the polynomials $x^{m_1}-\omega_1,\ x^{m_2}-\omega_2,\ldots,x^{m_\ell}-\omega_\ell$ in $\mathbb{F}_q[x]$, and that $g_u(x)=\prod_{j=0}^{a_u-1}g_{u,j}(x)$ is the irreducible factorization of the polynomial $g_u(x)$ in $\mathbb{F}_{q^t}[x]$ for $1\le u\le r$. That is, the polynomials $g_{1,0}(x),\ g_{1,1}(x),\ldots,g_{1,a_1-1}(x),\ g_{2,0}(x),\ g_{2,1}(x),\ldots,g_{2,a_2-1}(x),\ldots\cdots,g_{r,0}(x),g_{r,1}(x),\ldots,g_{r,a_r-1}(x)$ are all the distinct irreducible factors of the polynomials $x^{m_1}-\omega_1,x^{m_2}-\omega_2,\ldots,x^{m_\ell}-\omega_\ell$ in $\mathbb{F}_{q^t}[x]$. Further, by Lemma 2 of Huffman (2010), we observe that if for some u and j, the polynomial $g_{u,j}(x)$ is self-reciprocal, then the polynomial $g_{u,j+1}(x)$ is also self-reciprocal, where the subscript $j+1$ is considered modulo a_u. As a consequence, if for a given u, the polynomial $g_{u,j}(x)$ is self-reciprocal for some j $(0\le j\le a_u-1)$, then the polynomial $g_u(x)$ is also self-reciprocal, (but the converse is not true in general). Moreover, for $1\le u,u'\le r$, if there exist integers h,h' satisfying $0\le h,h'\le a_u-1$ and $g_{u,h}(x)=g^\dagger_{u',h'}(x)$, then we have $g_u(x)=g^\dagger_{u'}(x)$. In view of this and by reordering $g_{u,j}(x)'s$ (if required), we assume that

- $g_{1,0}(x),g_{1,1}(x),\ldots,g_{1,a_1-1}(x),\ldots\cdots,g_{e_1,0}(x),g_{e_1,1}(x),\ldots,g_{e_1,a_{e_1}-1}(x)$ are all the distinct self-reciprocal irreducible factors,
- $g_{e_1+1,0}(x),g^\dagger_{e_1+1,0}(x),\ldots,g_{e_1+1,a_{e_1+1}-1}(x),g^\dagger_{e_1+1,a_{e_1+1}-1}(x),\ldots\cdots,g_{e_2,0}(x),g^\dagger_{e_2,0}(x),\ldots,g_{e_2,a_{e_2}-1}(x),g^\dagger_{e_2,a_{e_2}-1}(x)$ are the irreducible factors forming reciprocal pairs, and
- $g_{e_2+1,0}(x),g_{e_2+1,1}(x),\ldots,g_{e_2+1,a_{e_2+1}-1}(x),\ldots\ldots,g_{e_3,0}(x),g_{e_3,1}(x),\ldots,g_{e_3,a_{e_3}-1}(x)$ are the remaining irreducible factors (i.e., neither they are self-reciprocal nor do they form reciprocal pairs)

of the polynomials $x^{m_1}-\omega_1,x^{m_2}-\omega_2,\ldots,x^{m_\ell}-\omega_\ell$ in $\mathbb{F}_{q^t}[x]$. Note that $r=e_2+e_3-e_1$.

For $e_1+1\le w\le e_2$ and $1\le i\le\ell$, let us define

$$\epsilon^\dagger_{w,i}=\begin{cases}1 & \text{if } g^\dagger_{w,j}(x)\mid x^{m_i}-\omega_i \text{ in } \mathbb{F}_{q^t}[x] \text{ for some } j;\\ 0 & \text{otherwise.}\end{cases}$$

We also note that for $e_1+1 \le w \le e_2$ and $1 \le i \le \ell$, if $g^{\dagger}_{w,j}(x) \mid x^{m_i} - \omega_i$ for some j, then $g^{\dagger}_{w,j+1}(x) \mid x^{m_i} - \omega_i$. From this, we get

$$x^{m_i} - \omega_i = \prod_{v=1}^{e_1} \prod_{j=0}^{a_v-1} g_{v,j}(x)^{\epsilon_{v,i}} \prod_{w=e_1+1}^{e_2} \prod_{j=0}^{a_w-1} g_{w,j}(x)^{\epsilon_{w,i}} g^{\dagger}_{w,j}(x)^{\epsilon^{\dagger}_{w,i}} \prod_{s=e_2+1}^{e_3} \prod_{j=0}^{a_s-1} g_{s,j}(x)^{\epsilon_{s,i}}$$

and

$$x^{m_i} - \omega_i^{-1} = \prod_{v=1}^{e_1} \prod_{j=0}^{a_v-1} g_{v,j}(x)^{\epsilon_{v,i}} \prod_{w=e_1+1}^{e_2} \prod_{j=0}^{a_w-1} g^{\dagger}_{w,j}(x)^{\epsilon_{w,i}} g_{w,j}(x)^{\epsilon^{\dagger}_{w,i}} \prod_{s=e_2+1}^{e_3} \prod_{j=0}^{a_s-1} g^{\dagger}_{s,j}(x)^{\epsilon_{s,i}}.$$

For each relevant v, w, s and j, we note that

$$\begin{aligned}
\mathcal{F}_{v,j} = \frac{\mathbb{F}_{q^t}[x]}{\langle g_{v,j}(x)\rangle} \simeq \mathbb{F}_{q^{tD_v}}, \quad \mathcal{F}_{w,j} = \frac{\mathbb{F}_{q^t}[x]}{\langle g_{w,j}(x)\rangle} \simeq \mathbb{F}_{q^{tD_w}}, \\
\mathcal{F}^{\dagger}_{w,j} = \frac{\mathbb{F}_{q^t}[x]}{\langle g^{\dagger}_{w,j}(x)\rangle} \simeq \mathbb{F}_{q^{tD_w}}, \quad \mathcal{F}_{s,j} = \frac{\mathbb{F}_{q^t}[x]}{\langle g_{s,j}(x)\rangle} \simeq \mathbb{F}_{q^{tD_s}}, \\
\mathcal{F}^{\dagger}_{s,j} = \frac{\mathbb{F}_{q^t}[x]}{\langle g^{\dagger}_{s,j}(x)\rangle} \simeq \mathbb{F}_{q^{tD_s}}, \quad \mathcal{F}_{v} = \frac{\mathbb{F}_{q}[x]}{\langle g_{v}(x)\rangle} \simeq \mathbb{F}_{q^{d_v}}, \\
\mathcal{F}_{w} = \frac{\mathbb{F}_{q}[x]}{\langle g_{w}(x)\rangle} \simeq \mathbb{F}_{q^{d_w}}, \quad \mathcal{F}^{\dagger}_{w} = \frac{\mathbb{F}_{q}[x]}{\langle g^{\dagger}_{w}(x)\rangle} \simeq \mathbb{F}_{q^{d_w}}, \\
\mathcal{F}_{s} = \frac{\mathbb{F}_{q}[x]}{\langle g_{s}(x)\rangle} \simeq \mathbb{F}_{q^{d_s}}, \quad \mathcal{F}^{\dagger}_{s} = \frac{\mathbb{F}_{q}[x]}{\langle g^{\dagger}_{s}(x)\rangle} \simeq \mathbb{F}_{q^{d_s}}.
\end{aligned}$$

.

Now by applying the Chinese Remainder Theorem, we see that

$$\begin{aligned}
\mathcal{V} \simeq \mathcal{G} = \Bigg(\bigoplus_{v=1}^{e_1} \bigoplus_{j=0}^{a_v-1} \Big(\underbrace{\epsilon_{v,1}\mathcal{F}_{v,j}, \epsilon_{v,2}\mathcal{F}_{v,j}, \ldots, \epsilon_{v,\ell}\mathcal{F}_{v,j}}_{\mathcal{G}_{v,j}} \Big)\Bigg) \\
\oplus \Bigg(\bigoplus_{w=e_1+1}^{e_2} \bigoplus_{j=0}^{a_w-1} \Big\{ \Big(\underbrace{\epsilon_{w,1}\mathcal{F}_{w,j}, \epsilon_{w,2}\mathcal{F}_{w,j}, \ldots, \epsilon_{w,\ell}\mathcal{F}_{w,j}}_{\mathcal{G}_{w,j}} \Big) \\
\oplus \Big(\underbrace{\epsilon^{\dagger}_{w,1}\mathcal{F}^{\dagger}_{w,j}, \epsilon^{\dagger}_{w,2}\mathcal{F}^{\dagger}_{w,j}, \ldots, \epsilon^{\dagger}_{w,\ell}\mathcal{F}^{\dagger}_{w,j}}_{\mathcal{G}^{\dagger}_{w,j}} \Big) \Big\} \Bigg) \\
\oplus \Bigg(\bigoplus_{s=e_2+1}^{e_3} \bigoplus_{j=0}^{a_s-1} \Big(\underbrace{\epsilon_{s,1}\mathcal{F}_{s,j}, \epsilon_{s,2}\mathcal{F}_{s,j}, \ldots, \epsilon_{s,\ell}\mathcal{F}_{s,j}}_{\mathcal{G}_{s,j}} \Big)\Bigg)
\end{aligned}$$

$$= \left(\bigoplus_{v=1}^{e_1}\bigoplus_{j=0}^{a_v-1}\mathcal{G}_{v,j}\right)\oplus\left(\bigoplus_{w=e_1+1}^{e_2}\bigoplus_{j=0}^{a_w-1}\left(\mathcal{G}_{w,j}\oplus\mathcal{G}^{\dagger}_{w,j}\right)\right)\oplus\left(\bigoplus_{s=e_2+1}^{e_3}\bigoplus_{j=0}^{a_s-1}\mathcal{G}_{s,j}\right)$$

$$= \left(\bigoplus_{v=1}^{e_1}\mathcal{G}_{v}\right)\oplus\left(\bigoplus_{w=e_1+1}^{e_2}\left(\mathcal{G}_{w}\oplus\mathcal{G}^{\dagger}_{w}\right)\right)\oplus\left(\bigoplus_{s=e_2+1}^{e_3}\mathcal{G}_{s}\right), \tag{3.1}$$

where $\mathcal{G}_v$ (resp. $\mathcal{G}_w$, $\mathcal{G}_w^{\dagger}$ and $\mathcal{G}_s$) is a vector space over $\mathcal{F}_v$ (resp. $\mathcal{F}_w$, $\mathcal{F}_w^{\dagger}$ and $\mathcal{F}_s$) for each v (resp. w and s).

In an analogous manner, we have

$$\mathcal{V}' \simeq \mathcal{G}' = \left(\bigoplus_{v=1}^{e_1}\bigoplus_{j=0}^{a_v-1}\Big(\underbrace{\epsilon_{v,1}\mathcal{F}_{v,j},\epsilon_{v,2}\mathcal{F}_{v,j},\ldots,\epsilon_{v,\ell}\mathcal{F}_{v,j}}_{\mathcal{G}_{v,j}}\Big)\right)$$

$$\oplus\left(\bigoplus_{w=e_1+1}^{e_2}\bigoplus_{j=0}^{a_w-1}\left\{\Big(\underbrace{\epsilon^{\dagger}_{w,1}\mathcal{F}_{w,j},\epsilon^{\dagger}_{w,2}\mathcal{F}_{w,j},\ldots,\epsilon^{\dagger}_{w,\ell}\mathcal{F}_{w,j}}_{\mathcal{H}_{w,j}}\Big)\right.\right.$$

$$\left.\left.\oplus\Big(\underbrace{\epsilon_{w,1}\mathcal{F}^{\dagger}_{w,j},\epsilon_{w,2}\mathcal{F}^{\dagger}_{w,j},\ldots,\epsilon_{w,\ell}F^{\dagger}_{w,j}}_{\mathcal{H}^{\dagger}_{w,j}}\Big)\right\}\right)$$

$$\oplus\left(\bigoplus_{s=e_2+1}^{e_3}\bigoplus_{j=0}^{a_s-1}\Big(\underbrace{\epsilon_{s,1}\mathcal{F}^{\dagger}_{s,j},\epsilon_{s,2}\mathcal{F}^{\dagger}_{s,j},\ldots,\epsilon_{s,\ell}\mathcal{F}^{\dagger}_{s,j}}_{\mathcal{G}^{\dagger}_{s,j}}\Big)\right)$$

$$= \left(\bigoplus_{v=1}^{e_1}\bigoplus_{j=0}^{a_v-1}\mathcal{G}_{v,j}\right)\oplus\left(\bigoplus_{w=e_1+1}^{e_2}\bigoplus_{j=0}^{a_w-1}\left(\mathcal{H}_{w,j}\oplus\mathcal{H}^{\dagger}_{w,j}\right)\right)\oplus\left(\bigoplus_{s=e_2+1}^{e_3}\bigoplus_{j=0}^{a_s-1}\mathcal{G}^{\dagger}_{s,j}\right)$$

$$= \left(\bigoplus_{v=1}^{e_1}\mathcal{G}_{v}\right)\oplus\left(\bigoplus_{w=e_1+1}^{e_2}\left(\mathcal{H}_{w}\oplus\mathcal{H}^{\dagger}_{w}\right)\right)\oplus\left(\bigoplus_{s=e_2+1}^{e_3}\mathcal{G}^{\dagger}_{s}\right), \tag{3.2}$$

where $\mathcal{G}_v$ (resp. $\mathcal{H}_w$, $\mathcal{H}_w^{\dagger}$ and $\mathcal{G}_s^{\dagger}$) is a vector space over $\mathcal{F}_v$ (resp. $\mathcal{F}_w$, $\mathcal{F}_w^{\dagger}$ and $\mathcal{F}_s^{\dagger}$) for each v (resp. w and s). We also recall that for $1 \le u \le r$, $\epsilon_u = \sum_{i=1}^{\ell}\epsilon_{u,i}$ and $\dim_{\mathcal{F}_u}\mathcal{G}_u = \epsilon_u t$. Further, for $e_1+1 \le w \le e_2$, we see that if $\epsilon_w^{\dagger} = \sum_{i=1}^{\ell}\epsilon^{\dagger}_{w,i}$, then $\dim_{\mathcal{F}_w^{\dagger}}\mathcal{G}_w^{\dagger} = \dim_{\mathcal{F}_w}\mathcal{H}_w = \epsilon_w^{\dagger}t$.

Hereafter, we shall identify each element $\alpha(x) = (\alpha_1(x),\alpha_2(x),\ldots,\alpha_\ell(x)) \in \mathcal{V}$ with the element $\mathcal{A} = \big(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_{e_1},\mathcal{A}_{e_1+1},\mathcal{A}^{\dagger}_{e_1+1},\ldots,\mathcal{A}_{e_2},\mathcal{A}^{\dagger}_{e_2},\mathcal{A}_{e_2+1},\mathcal{A}_{e_2+2},\cdots,\mathcal{A}_{e_3}) \in \mathcal{G}$, where $\mathcal{A}_v = (\mathcal{A}_{v,0},\mathcal{A}_{v,1},\ldots,\mathcal{A}_{v,a_v-1}) \in \mathcal{G}_v$, $\mathcal{A}_w = (\mathcal{A}_{w,0},\mathcal{A}_{w,1},\cdots,\mathcal{A}_{w,a_w-1}) \in \mathcal{G}_w$, $\mathcal{A}_w^{\dagger} = (\mathcal{A}^{\dagger}_{w,0},\mathcal{A}^{\dagger}_{w,1},\ldots,\mathcal{A}^{\dagger}_{w,a_w-1}) \in \mathcal{G}_w^{\dagger}$ and $\mathcal{A}_s = (\mathcal{A}_{s,0},\mathcal{A}_{s,1},\ldots,\mathcal{A}_{s,a_s-1}) \in \mathcal{G}_s$ for $1 \le v \le e_1$, $e_1+1 \le w \le e_2$ and $e_2+1 \le s \le e_3$. Here for each relevant v, w, s and j, we have $\mathcal{A}_{v,j} = (\mathcal{A}^{(1)}_{v,j},\mathcal{A}^{(2)}_{v,j},\ldots,\mathcal{A}^{(\ell)}_{v,j}) \in \mathcal{G}_{v,j}$, $\mathcal{A}_{w,j} = (\mathcal{A}^{(1)}_{w,j},\mathcal{A}^{(2)}_{w,j},\ldots,\mathcal{A}^{(\ell)}_{w,j}) \in \mathcal{G}_{w,j}$, $\mathcal{A}^{\dagger}_{w,j} = (\mathcal{A}^{\dagger(1)}_{w,j},\mathcal{A}^{\dagger(2)}_{w,j},\ldots,\mathcal{A}^{\dagger(\ell)}_{w,j}) \in \mathcal{G}^{\dagger}_{w,j}$

and $\mathcal{A}_{s,j} = (\mathcal{A}_{s,j}^{(1)}, \mathcal{A}_{s,j}^{(2)}, \ldots, \mathcal{A}_{s,j}^{(\ell)}) \in \mathcal{G}_{s,j}$ with $\mathcal{A}_{v,j}^{(i)} := \epsilon_{v,i}\alpha_i(x) + \langle g_{v,j}(x)\rangle \in \epsilon_{v,i}\mathcal{F}_{v,j}$, $\mathcal{A}_{w,j}^{(i)} := \epsilon_{w,i}\alpha_i(x) + \langle g_{w,j}(x)\rangle \in \epsilon_{w,i}\mathcal{F}_{w,j}$, $\mathcal{A}_{w,j}^{\dagger(i)} := \epsilon_{w,i}^{\dagger}\alpha_i(x) + \langle g_{w,j}^{\dagger}(x)\rangle \in \epsilon_{w,i}^{\dagger}\mathcal{F}_{w,j}^{\dagger}$ and $\mathcal{A}_{s,j}^{(i)} := \epsilon_{s,i}\alpha_i(x) + \langle g_{s,j}(x)\rangle \in \epsilon_{s,i}\mathcal{F}_{s,j}$ for $1 \le i \le \ell$.

Similarly, each element $\beta(x) = (\beta_1(x), \beta_2(x), \ldots, \beta_\ell(x)) \in \mathcal{V}'$ is identified with $\mathcal{B} = \big(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{e_1}, \mathcal{B}_{e_1+1}, \mathcal{B}_{e_1+1}^{\dagger}, \ldots, \mathcal{B}_{e_2}, \mathcal{B}_{e_2}^{\dagger}, \mathcal{B}_{e_2+1}, \mathcal{B}_{e_2+2}, \ldots, \mathcal{B}_{e_3}) \in \mathcal{G}'$, where $\mathcal{B}_v = (\mathcal{B}_{v,0}, \mathcal{B}_{v,1}, \ldots, \mathcal{B}_{v,a_v-1}) \in \mathcal{G}_v$, $\mathcal{B}_w = (\mathcal{B}_{w,0}, \mathcal{B}_{w,1}, \ldots, \mathcal{B}_{w,a_w-1}) \in \mathcal{H}_w$, $\mathcal{B}_w^{\dagger} = (\mathcal{B}_{w,0}^{\dagger}, \mathcal{B}_{w,1}^{\dagger}, \ldots, \mathcal{B}_{w,a_w-1}^{\dagger}) \in \mathcal{H}_w^{\dagger}$ and $\mathcal{B}_s = (\mathcal{B}_{s,0}, \mathcal{B}_{s,1}, \ldots, \mathcal{B}_{s,a_s-1}) \in \mathcal{G}_s^{\dagger}$ for $1 \le v \le e_1$, $e_1+1 \le w \le e_2$ and $e_2+1 \le s \le e_3$. Here also, for each relevant v, w, s and j, we have $\mathcal{B}_{v,j} = (\mathcal{B}_{v,j}^{(1)}, \mathcal{B}_{v,j}^{(2)}, \ldots, \mathcal{B}_{v,j}^{(\ell)}) \in \mathcal{G}_{v,j}$, $\mathcal{B}_{w,j} = (\mathcal{B}_{w,j}^{(1)}, \mathcal{B}_{w,j}^{(2)}, \ldots, \mathcal{B}_{w,j}^{(\ell)}) \in \mathcal{H}_{w,j}$, $\mathcal{B}_{w,j}^{\dagger} = (\mathcal{B}_{w,j}^{\dagger(1)}, \mathcal{B}_{w,j}^{\dagger(2)}, \ldots, \mathcal{B}_{w,j}^{\dagger(\ell)}) \in \mathcal{H}_{w,j}^{\dagger}$ and $\mathcal{B}_{s,j} = (\mathcal{B}_{s,j}^{(1)}, \mathcal{B}_{s,j}^{(2)}, \ldots, \mathcal{B}_{s,j}^{(\ell)}) \in \mathcal{G}_{s,j}^{\dagger}$ with $\mathcal{B}_{v,j}^{(i)} := \epsilon_{v,i}\beta_i(x) + \langle g_{v,j}(x)\rangle \in \epsilon_{v,i}\mathcal{F}_{v,j}$, $\mathcal{B}_{w,j}^{(i)} := \epsilon_{w,i}^{\dagger}\beta_i(x) + \langle g_{w,j}(x)\rangle \in \epsilon_{w,i}^{\dagger}\mathcal{F}_{w,j}$, $\mathcal{B}_{w,j}^{\dagger(i)} := \epsilon_{w,i}\beta_i(x) + \langle g_{w,j}^{\dagger}(x)\rangle \in \epsilon_{w,i}\mathcal{F}_{w,j}^{\dagger}$ and $\mathcal{B}_{s,j}^{(i)} := \epsilon_{s,i}\beta_i(x) + \langle g_{s,j}^{\dagger}(x)\rangle \in \epsilon_{s,i}\mathcal{F}_{s,j}^{\dagger}$ for $1 \le i \le \ell$.
Further, by Theorem 2.2 and by (3.1), we see that each Ω-MT additive code $\mathcal{C} \subseteq \mathcal{G}$ of length n over $\mathbb{F}_{q^t}$ has a unique representation of the form

$$\mathcal{C} = \Big(\bigoplus_{v=1}^{e_1}\mathcal{C}_v\Big) \oplus \Big(\bigoplus_{w=e_1+1}^{e_2}\big(\mathcal{C}_w \oplus \mathcal{C}_w^{\dagger}\big)\Big) \oplus \Big(\bigoplus_{s=e_2+1}^{e_3}\mathcal{C}_s\Big), \tag{3.3}$$

where $\mathcal{C}_v$ (resp. $\mathcal{C}_w$, $\mathcal{C}_w^{\dagger}$ and $\mathcal{C}_s$) is a subspace of $\mathcal{G}_v$ (resp. $\mathcal{G}_w$, $\mathcal{G}_w^{\dagger}$ and $\mathcal{G}_s$) over $\mathcal{F}_v$ (resp. $\mathcal{F}_w$, $\mathcal{F}_w^{\dagger}$ and $\mathcal{F}_s$) for each v (resp. w and s). Furthermore, for $\delta \in \{0, *, \gamma\}$ and for all $\mathcal{A} \in \mathcal{G}$ and $\mathcal{B} \in \mathcal{G}'$, the sesquilinear forms $[\cdot,\cdot]_\delta$, defined by (2.1)-(2.3), can be rewritten as

$$[\mathcal{A},\mathcal{B}]_0 = (\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_{e_1}, \mathcal{R}_{e_1+1}, \mathcal{R}_{e_1+1}^{\dagger}, \ldots, \mathcal{R}_{e_2}, \mathcal{R}_{e_2}^{\dagger}, \mathcal{R}_{e_2+1}, \mathcal{R}_{e_2+2}, \ldots, \mathcal{R}_{e_3}), \tag{3.4}$$

$$\begin{aligned}[\mathcal{A},\mathcal{B}]_* &= (\mathcal{S}_1 - \mathcal{R}_1, \ldots, \mathcal{S}_{e_1} - \mathcal{R}_{e_1}, \mathcal{S}_{e_1+1} - \mathcal{R}_{e_1+1}, \mathcal{S}_{e_1+1}^{\dagger} \\ &\quad - \mathcal{R}_{e_1+1}^{\dagger}, \ldots, \mathcal{S}_{e_2} - \mathcal{R}_{e_2}, \mathcal{S}_{e_2}^{\dagger} - \mathcal{R}_{e_2}^{\dagger}, \mathcal{S}_{e_2+1} - \mathcal{R}_{e_2+1}, \ldots, \mathcal{S}_{e_3} - \mathcal{R}_{e_3}), \\ &= (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{e_1}, \mathcal{S}_{e_1+1}, \mathcal{S}_{e_1+1}^{\dagger}, \ldots, \mathcal{S}_{e_2}, \mathcal{S}_{e_2}^{\dagger}, \mathcal{S}_{e_2+1}, \mathcal{S}_{e_2+2}, \ldots, \mathcal{S}_{e_3}) - [\mathcal{A},\mathcal{B}]_0,\end{aligned} \tag{3.5}$$

and

$$[\mathcal{A},\mathcal{B}]_\gamma = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{e_1}, \mathcal{T}_{e_1+1}, \mathcal{T}_{e_1+1}^{\dagger}, \ldots, \mathcal{T}_{e_2}, \mathcal{T}_{e_2}^{\dagger}, \mathcal{T}_{e_2+1}, \mathcal{T}_{e_2+2}, \ldots, \mathcal{T}_{e_3}), \tag{3.6}$$

where for $1 \le v \le e_1$, $e_1+1 \le w \le e_2$ and $e_2+1 \le s \le e_3$,

$$\mathcal{R}_v = \sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{v,i}\Bigg(\sum_{j=0}^{a_v-1}\sum_{\mu=0}^{(t/a_v)-1}\tau_{q^{\mu a_v+j},1}\Big(\mathcal{A}_{v,a_v-j}^{(i)}\tau_{1,-1}(\mathcal{B}_{v,a_v-j}^{(i)})\Big)\Bigg),$$

$$\mathcal{R}_w = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{w,i} \left(\sum_{j=0}^{a_w-1} \sum_{\mu=0}^{(t/a_w)-1} \tau_{q^{\mu a_w + j},1} \left(\mathcal{A}^{(i)}_{w,a_w-j} \tau_{1,-1} (\mathcal{B}^{\dagger(i)}_{w,a_w-j}) \right) \right),$$

$$\mathcal{R}^{\dagger}_w = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon^{\dagger}_{w,i} \left(\sum_{j=0}^{a_w-1} \sum_{\mu=0}^{(t/a_w)-1} \tau_{q^{\mu a_w + j},1} \left(\mathcal{A}^{\dagger(i)}_{w,a_w-j} \tau_{1,-1} (\mathcal{B}^{(i)}_{w,a_w-j}) \right) \right),$$

$$\mathcal{R}_s = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{s,i} \left(\sum_{j=0}^{a_s-1} \sum_{\mu=0}^{(t/a_s)-1} \tau_{q^{\mu a_s + j},1} \left(\mathcal{A}^{(i)}_{s,a_s-j} \tau_{1,-1} (\mathcal{B}^{(i)}_{s,a_s-j}) \right) \right),$$

$$\begin{aligned} \mathcal{S}_v = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{v,i} \Bigg(\left(\sum_{j=0}^{a_v-1} \sum_{\mu=0}^{(t/a_v)-1} \tau_{q^{\mu a_v + j},1} (\mathcal{A}^{(i)}_{v,a_v-j}) \right) \\ \times \left(\sum_{j=0}^{a_v-1} \sum_{\sigma=0}^{(t/a_v)-1} \tau_{q^{\sigma a_v + j},1} (\tau_{1,-1} (\mathcal{B}^{(i)}_{v,a_v-j})) \right) \Bigg), \end{aligned}$$

$$\begin{aligned} \mathcal{S}_w = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{w,i} \Bigg(\left(\sum_{j=0}^{a_w-1} \sum_{\mu=0}^{(t/a_w)-1} \tau_{q^{\mu a_w + j},1} (\mathcal{A}^{(i)}_{w,a_w-j}) \right) \\ \left(\sum_{j=0}^{a_w-1} \sum_{\sigma=0}^{(t/a_w)-1} \tau_{q^{\sigma a_w + j},1} (\tau_{1,-1} (\mathcal{B}^{\dagger(i)}_{w,a_w-j})) \right) \Bigg), \end{aligned}$$

$$\begin{aligned} \mathcal{S}^{\dagger}_w = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon^{\dagger}_{w,i} \Bigg(\left(\sum_{j=0}^{a_w-1} \sum_{\mu=0}^{(t/a_w)-1} \tau_{q^{\mu a_w + j},1} (\mathcal{A}^{\dagger(i)}_{w,a_w-j}) \right) \\ \left(\sum_{j=0}^{a_w-1} \sum_{\sigma=0}^{(t/a_w)-1} \tau_{q^{\sigma a_w + j},1} (\tau_{1,-1} (\mathcal{B}^{(i)}_{w,a_w-j})) \right) \Bigg), \end{aligned}$$

$$\begin{aligned} \mathcal{S}_s = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{s,i} \Bigg(\left(\sum_{j=0}^{a_s-1} \sum_{\mu=0}^{(t/a_s)-1} \tau_{q^{\mu a_s + j},1} (\mathcal{A}^{(i)}_{s,a_s-j}) \right) \\ \times \left(\sum_{j=0}^{a_s-1} \sum_{\sigma=0}^{(t/a_s)-1} \tau_{q^{\sigma a_s + j},1} (\tau_{1,-1} (\mathcal{B}^{(i)}_{s,a_s-j})) \right) \Bigg), \end{aligned}$$

$$\mathcal{T}_v = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{v,i} \left(\sum_{j=0}^{a_v-1} \sum_{\mu=0}^{(t/a_v)-1} \tau_{q^{\mu a_v + j},1} \left(\gamma \mathcal{A}^{(i)}_{v,a_v-j} \tau_{q^{t/2},-1} (\mathcal{B}^{(i)}_{v,\frac{t}{2}-j}) \right) \right),$$

$$\mathcal{T}_w = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{w,i} \left(\sum_{j=0}^{a_w-1} \sum_{\mu=0}^{(t/a_w)-1} \tau_{q^{\mu a_w + j},1} \left(\gamma \mathcal{A}^{(i)}_{w,a_w-j} \tau_{q^{t/2},-1} (\mathcal{B}^{\dagger(i)}_{w,\frac{t}{2}-j}) \right) \right),$$

$$\mathcal{T}^{\dagger}_w = \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon^{\dagger}_{w,i} \left(\sum_{j=0}^{a_w-1} \sum_{\mu=0}^{(t/a_w)-1} \tau_{q^{\mu a_w + j},1} \left(\gamma \mathcal{A}^{\dagger(i)}_{w,a_w-j} \tau_{q^{t/2},-1} (\mathcal{B}^{(i)}_{w,\frac{t}{2}-j}) \right) \right),$$

$$\mathcal{T}_s = \sum_{i=1}^{\ell} \frac{m}{m_i}\epsilon_{s,i}\Bigg(\sum_{j=0}^{a_s-1}\sum_{\mu=0}^{(t/a_s)-1} \tau_{q^{\mu a_s+j},1}\Big(\gamma \mathcal{A}^{(i)}_{s,a_s-j}\tau_{q^{t/2},-1}(\mathcal{B}^{(i)}_{s,\frac{t}{2}-j})\Big)\Bigg).$$

Moreover, by (3.4)-(3.6) and by applying Theorem 2.5, we see that the δ-dual code $\mathcal{C}^{\perp_\delta}(\subseteq \mathcal{G}')$ of the code $\mathcal{C}$ is given by

$$\mathcal{C}^{\perp_\delta} = \Big(\bigoplus_{v=1}^{e_1}\mathcal{C}_v^{\perp_\delta}\Big) \oplus \Big(\bigoplus_{w=e_1+1}^{e_2}(\mathcal{C}_w^{\dagger\perp_\delta}\oplus \mathcal{C}_w^{\perp_\delta})\Big) \oplus \Big(\bigoplus_{s=e_2+1}^{e_3}\mathcal{C}_s^{\perp_\delta}\Big), \tag{3.7}$$

where $\mathcal{C}_v^{\perp_\delta} \subseteq \mathcal{G}_v$ is the orthogonal complement of $\mathcal{C}_v$ with respect to $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ for $1 \leq v \leq e_1$, $\mathcal{C}_w^{\perp_\delta} \subseteq \mathcal{H}_w^\dagger$ is the orthogonal complement of $\mathcal{C}_w$ with respect to $[\cdot,\cdot]_{\delta\restriction_{\mathcal{H}_w^\dagger\times\mathcal{G}_w}}$, $\mathcal{C}_w^{\dagger\perp_\delta} \subseteq \mathcal{H}_w$ is the orthogonal complement of $\mathcal{C}_w^\dagger$ with respect to $[\cdot,\cdot]_{\delta\restriction_{\mathcal{H}_w\times\mathcal{G}_w^\dagger}}$ for $e_1+1 \leq w \leq e_2$, and $\mathcal{C}_s^{\perp_\delta} \subseteq \mathcal{G}_s^\dagger$ is the orthogonal complement of $\mathcal{C}_s$ with respect to $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_s^\dagger\times\mathcal{G}_s}}$ for $e_2+1 \leq s \leq e_3$. Here $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ (resp. $[\cdot,\cdot]_{\delta\restriction_{\mathcal{H}_w^\dagger\times\mathcal{G}_w}}$, $[\cdot,\cdot]_{\delta\restriction_{\mathcal{H}_w\times\mathcal{G}_w^\dagger}}$ and $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_s^\dagger\times\mathcal{G}_s}}$) denotes the restriction of the sesquilinear form $[\cdot,\cdot]_\delta$ to $\mathcal{G}_v\times\mathcal{G}_v$ (resp. $\mathcal{H}_w^\dagger\times\mathcal{G}_w$, $\mathcal{H}_w\times\mathcal{G}_w^\dagger$ and $\mathcal{G}_s^\dagger\times\mathcal{G}_s$) for $1 \leq v \leq e_1$ (resp. for $e_1+1 \leq w \leq e_2$ and $e_2+1 \leq s \leq e_3$).

Now for $e_1+1 \leq w \leq e_2$ and $0 \leq j \leq a_w-1$, let us define $\mathcal{K}_{w,j} = \mathcal{G}_{w,j}\cap\mathcal{H}_{w,j} = \big(\epsilon_{w,1}\epsilon_{w,1}^\dagger\mathcal{F}_{w,j},\ldots,\epsilon_{w,\ell}\epsilon_{w,\ell}^\dagger\mathcal{F}_{w,j}\big)$ and $\mathcal{K}_{w,j}^\dagger = \mathcal{G}_{w,j}^\dagger\cap\mathcal{H}_{w,j}^\dagger = \big(\epsilon_{w,1}\epsilon_{w,1}^\dagger\mathcal{F}_{w,j}^\dagger,\ldots,\epsilon_{w,\ell}\epsilon_{w,\ell}^\dagger\mathcal{F}_{w,j}^\dagger\big)$. Next let $\mathcal{K}_w = \mathcal{G}_w\cap\mathcal{H}_w = \bigoplus_{j=0}^{a_w-1}\mathcal{K}_{w,j}$, $\mathcal{K}_w^\dagger = \mathcal{G}_w^\dagger\cap\mathcal{H}_w^\dagger = \bigoplus_{j=0}^{a_w-1}\mathcal{K}_{w,j}^\dagger$, $\kappa_w = \sum_{i=1}^{\ell}\epsilon_{w,i}\epsilon_{w,i}^\dagger$, and let $\eta_w = \kappa_w t$ for $e_1+1 \leq w \leq e_2$. One can easily observe that $\mathcal{K}_w$ (resp. $\mathcal{K}_w^\dagger$) is an η_w-dimensional vector space over $\mathcal{F}_w$ (resp. $\mathcal{F}_w^\dagger$) for $e_1+1 \leq w \leq e_2$. It is easy to observe that the restriction $[\cdot,\cdot]_{\delta\restriction_{\mathcal{K}_w\times\mathcal{K}_w^\dagger}}$ of the form $[\cdot,\cdot]_\delta$ to $\mathcal{K}_w\times\mathcal{K}_w^\dagger$ is a reflexive and a non-degenerate $\tau_{1,-1}$-sesquilinear form.

Next one can easily see that $x+1$ and $x-1$ are the only irreducible self-reciprocal polynomials over finite fields of odd degree, and that all other irreducible self-reciprocal polynomials over finite fields are of even degrees. More precisely, for $1 \leq v \leq e_1$ and $0 \leq j \leq a_v-1$, if $\deg g_{v,j}(x) = D_v$ is odd, then we must have $D_v = 1$ and $g_{v,j}(x)$ is either $x+1$ or $x-1$. Further, it is easy to observe that $d_v = 1$ if and only if $D_v = 1$.

Moreover, for $1 \leq v \leq e_1$ and $0 \leq j \leq a_v-1$, one can easily observe that the map $\tau_{1,-1} : \mathcal{F}_{v,j} \to \mathcal{F}_{v,j}$ is an automorphism of $\mathcal{F}_{v,j}$ satisfying $\tau_{1,-1}^2 = \tau_{1,1}$, the identity map. That is, the map $\tau_{1,-1}$ is either the identity automorphism or the automorphism of $\mathcal{F}_{v,j}$ of order 2. In view of this and using the fact that $\mathcal{F}_v = \bigoplus_{j=0}^{a_v-1}\mathcal{F}_{v,j}$, we see that the map $\tau_{1,-1}$ is either the identity automorphism or the automorphism of $\mathcal{F}_v$ of order 2, where $1 \leq v \leq e_1$.

Next for $1 \leq v \leq e_1$, let us define $\mathcal{J}_1 = \{v : 1 \leq v \leq e_1, d_v = 1\}$ and $\mathcal{J}_2 = \{v : 1 \leq v \leq e_1, d_v \text{ is even}\}$. Note that $\{1,2,\ldots,e_1\} = \mathcal{J}_1\cup\mathcal{J}_2$. Now the following three lemmas are useful in studying and counting all δ-self-dual and δ-self-orthogonal Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ for each $\delta \in \{0, *, \gamma\}$.

Lemma 3.1 *For $1 \leq v \leq e_1$ and $0 \leq j \leq a_v - 1$, the following hold.*

(a) *The map $\tau_{1,-1} : \mathcal{F}_{v,j} \to \mathcal{F}_{v,j}$ is the identity automorphism if and only if $v \in \mathcal{J}_1$.*
(b) *The map $\tau_{1,-1}$ is the automorphism of $\mathcal{F}_{v,j}$ of order 2 if and only if $v \in \mathcal{J}_2$.*

Proof (a) If $v \in \mathcal{J}_1$, then we have $d_v = D_v = 1$, from which it follows that $\tau_{1,-1}$ is the identity map on $\mathcal{F}_{v,j} \simeq \mathbb{F}_{q^t}$. To prove the forward part, let us suppose that $\tau_{1,-1}$ is the identity automorphism of $\mathcal{F}_{v,j}$. Here we assert that $v \in \mathcal{J}_1$. Since $\tau_{1,-1}$ is the identity map on $\mathcal{F}_{v,j}$, we have $x + \langle g_{v,j}(x)\rangle = \tau_{1,-1}(x + \langle g_{v,j}(x)\rangle) = x^{-1} + \langle g_{v,j}(x)\rangle$. This gives $x - x^{-1} \in \langle g_{v,j}(x)\rangle$. Note that there exists an integer i $(1 \leq i \leq \ell)$ such that $g_{v,j}(x)$ divides $x^{m_i} - \omega_i$, which implies that $x^{m_i} \equiv \omega_i \pmod{g_{v,j}(x)}$, and hence $x^{-1} + \langle g_{v,j}(x)\rangle = \omega_i^{-1} x^{m_i - 1} + \langle g_{v,j}(x)\rangle$. From this, it follows that $g_{v,j}(x)$ divides $\omega_i x^2 - x^{m_i}$. This implies that $g_{v,j}(x)$ divides $\omega_i x^2 - x^{m_i} + x^{m_i} - \omega_i = \omega_i (x^2 - 1)$. As $g_{v,j}(x)$ is a monic irreducible polynomial over $\mathbb{F}_{q^t}$, we see that $g_{v,j}(x)$ is either $x + 1$ or $x - 1$, which gives $D_v = 1$. This shows that $v \in \mathcal{J}_1$.

(b) It follows immediately from part (a) and the fact that the map $\tau_{1,-1}$ is either the identity automorphism of $\mathcal{F}_{v,j}$ or the automorphism of $\mathcal{F}_{v,j}$ having order 2.

□

Lemma 3.2 *For $1 \leq v \leq e_1$, the following hold.*

(a) *The map $\tau_{1,-1}$ is the identity automorphism of $\mathcal{F}_v$ if and only if $v \in \mathcal{J}_1$.*
(b) *The map $\tau_{1,-1}$ is the automorphism of $\mathcal{F}_v$ of order 2 if and only if $v \in \mathcal{J}_2$.*

Proof Working in a similar manner as in Lemma 3.1, the desired result follows. □

Lemma 3.3 *Let $t \geq 2$ be an even integer. For $1 \leq v \leq e_1$, the following hold.*

(a) *The integer $\frac{t}{a_v} = \frac{t}{\gcd(t, d_v)}$ is even if and only if $v \in \mathcal{J}_1$.*
(b) *If $v \in \mathcal{J}_2$, then we have $j + \frac{t}{2} \equiv j + \frac{a_v}{2} \pmod{a_v}$ for all integers j.*

Proof (a) We know that $D_v = 1$ if and only if $d_v = 1$.
Now if $v \in \mathcal{J}_1$, then we have $d_v = 1$, which gives $a_v = \gcd(t, d_v) = \gcd(t, 1) = 1$. This implies that the integer $\frac{t}{a_v} = t$ is even.
To prove the forward part, suppose that the integer $\frac{t}{a_v}$ is even. Since $a_v = \gcd(t, d_v)$, we have $\gcd(\frac{t}{a_v}, \frac{d_v}{a_v}) = \gcd(\frac{t}{a_v}, D_v) = 1$, which implies that the integer D_v must be odd. This further implies that $d_v = D_v = 1$, from which it follows that $v \in \mathcal{J}_1$.

(b) Now if $v \in \mathcal{J}_2$, then the integer d_v must be even, and hence $a_v = \gcd(t, d_v)$ is an even integer. Further, by part (a), we see that the integer $\frac{t}{a_v}$ is odd. In view of this, we have $\frac{t}{2} - \frac{a_v}{2} = a_v \times \frac{1}{2}\big(\frac{t}{a_v} - 1\big) \equiv 0 \pmod{a_v}$. From this, it follows that $j + \frac{t}{2} \equiv j + \frac{a_v}{2} \pmod{a_v}$ for all integers j.

□

In the following lemma, we study some basic properties of the restriction map $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ for each $\delta\in\{0,*,\gamma\}$ and $1\le v\le e_1$.

Lemma 3.4 *Let $1\le v\le e_1$ be fixed.*

(a) *For $v\in\mathcal{J}_1$ and $\delta\in\{0,*\}$, the map $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ is a symmetric and a non-degenerate bilinear form on $\mathcal{G}_v$, i.e., $(\mathcal{G}_v,[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}})$ is an orthogonal space of dimension $\epsilon_v t$ over $\mathcal{F}_v\simeq\mathbb{F}_q$.*

(b) *For $v\in\mathcal{J}_2$ and $\delta\in\{0,*\}$, the map $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ is a non-degenerate and a Hermitian $\tau_{1,-1}$-sesquilinear form on $\mathcal{G}_v$, i.e., $(\mathcal{G}_v,[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}})$ is a unitary space of dimension $\epsilon_v t$ over $\mathcal{F}_v\simeq\mathbb{F}_{q^{d_v}}$.*

(c) *For $v\in\mathcal{J}_1$, the map $[\cdot,\cdot]_{\gamma\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ is an alternating, reflexive and a non-degenerate bilinear form on $\mathcal{G}_v$, i.e., $(\mathcal{G}_v,[\cdot,\cdot]_{\gamma\restriction_{\mathcal{G}_v\times\mathcal{G}_v}})$ is a symplectic space of dimension $\epsilon_v t$ over $\mathcal{F}_v\simeq\mathbb{F}_q$.*

(d) *For $v\in\mathcal{J}_2$, the map $[\cdot,\cdot]_{\gamma\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ is a non-degenerate, reflexive and a skew–Hermitian $\tau_{1,-1}$-sesquilinear form on $\mathcal{G}_v$.*

Proof One can easily verify that the map $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ is a $\tau_{1,-1}$-sesquilinear form on $\mathcal{G}_v$ for each v. Now to show that the map $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ is non-degenerate, let us suppose that $[\mathcal{A}_v,\mathcal{B}_v]_\delta=0$ for all $\mathcal{B}_v\in\mathcal{G}_v$. Here we assert that $\mathcal{A}_v=0$. Let $\mathcal{C}'=(\mathcal{C}_1,\ldots,\mathcal{C}_v,\ldots,\mathcal{C}_{e_3})$ be an arbitrary element of $\mathcal{G}'$ with $\mathcal{C}_v=\mathcal{B}_v$. Note that as $\mathcal{B}_v$ runs over $\mathcal{G}_v$, the element $\mathcal{C}'$ runs over $\mathcal{G}'$. Also let $\mathcal{A}=(0,\ldots,0,\underbrace{\mathcal{A}_v}_{v^{\text{th}}},0,\ldots,0)\in\mathcal{G}$.

Then we see that $[\mathcal{A},\mathcal{C}']_\delta=[\mathcal{A}_v,\mathcal{B}_v]_\delta=0$ for all $\mathcal{C}\in\mathcal{G}'$. By Lemmas 2.3–2.4, we see that the sesquilinear form $[\cdot,\cdot]_\delta$ is non-degenerate on $\mathcal{G}\times\mathcal{G}'$. From this, it follows that $\mathcal{A}=0$, which gives $\mathcal{A}_v=0$.

Further, when $\delta\in\{0,*\}$, one can easily observe that $[\mathcal{A}_v,\mathcal{B}_v]_\delta=\tau_{1,-1}\big([\mathcal{B}_v,\mathcal{A}_v]_\delta\big)$ for all $\mathcal{A}_v,\mathcal{B}_v\in\mathcal{G}_v$. From this and by applying Lemma 3.1, parts (a) and (b) follows immediately.

Now to prove (c), let $v\in\mathcal{J}_1$. Here we have $d_v=D_v=a_v=1$ and $\mathcal{F}_{v,0}\simeq\mathbb{F}_{q^t}$. We also see, by Lemma 3.2, that the map $\tau_{1,-1}$ is the identity automorphism of $\mathcal{F}_v$. Further, we observe that $[\mathcal{A}_v,\mathcal{B}_v]_\gamma=-[\mathcal{B}_v,\mathcal{A}_v]_\gamma$ for all $\mathcal{A}_v,\mathcal{B}_v\in\mathcal{G}_v$. In particular, we have $[\mathcal{A}_v,\mathcal{A}_v]_\gamma=-[\mathcal{A}_v,\mathcal{A}_v]_\gamma$, which gives $2[\mathcal{A}_v,\mathcal{A}_v]_\gamma=0$ for all $\mathcal{A}_v\in\mathcal{G}_v$. From this, it follows that $[\mathcal{A}_v,\mathcal{A}_v]_\gamma=0$ when q is odd. On the other hand, when q is even, we see that $\gamma^{q^{\frac{t}{2}}}=\gamma$, which implies that

$$
\begin{aligned}
[\mathcal{A}_v,\mathcal{A}_v]_\gamma &= \sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{v,i}\Bigg(\sum_{\mu=0}^{t-1}\tau_{q^\mu,1}(\gamma\mathcal{A}_{v,0}^{(i)}\tau_{q^{t/2},-1}(\mathcal{A}_{v,0}^{(i)}))\Bigg)\\
&= \sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{v,i}\Bigg(\sum_{\mu=0}^{t-1}(\gamma\alpha_{i,0}\alpha_{i,0}^{q^{t/2}})^{q^\mu}+\langle g_{v,0}(x)\rangle\Bigg)\\
&= \sum_{i=1}^{\ell}\frac{m}{m_i}\epsilon_{v,i}\mathrm{Tr}_{q^t,q}(\gamma\alpha_{i,0}\alpha_{i,0}^{q^{t/2}})+\langle g_{v,0}(x)\rangle
\end{aligned}
$$

$$= \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{v,i} \mathrm{Tr}_{q^{t/2},q}(\mathrm{Tr}_{q^t,q^{t/2}}(\gamma \alpha_{i,0} \alpha_{i,0}^{q^{t/2}})) + \langle g_{v,0}(x) \rangle$$
$$= \sum_{i=1}^{\ell} \frac{m}{m_i} \epsilon_{v,i} \mathrm{Tr}_{q^{t/2},q}(2\gamma \alpha_{i,0} \alpha_{i,0}^{q^{t/2}}) + \langle g_{v,0}(x) \rangle$$
$$= 0 + \langle g_{v,0}(x) \rangle \text{ for all } \mathcal{A}_v \in \mathcal{G}_v.$$

This shows that the $\tau_{1,-1}$-sesquilinear form $[\cdot,\cdot]_{\gamma}\restriction_{\mathcal{G}_v \times \mathcal{G}_v}$ is alternating when $v \in \mathcal{J}_1$.

Finally, to prove (d), let $v \in \mathcal{J}_2$. Here by Lemma 3.2, the map $\tau_{1,-1}$ is the automorphism of $\mathcal{F}_v$ of order 2. Further, it is easy to observe that $[\mathcal{A}_v, \mathcal{B}_v]_{\gamma} = -\tau_{1,-1}([\mathcal{B}_v, \mathcal{A}_v]_{\gamma})$ for all $\mathcal{A}_v, \mathcal{B}_v \in \mathcal{G}_v$. This shows that the $\tau_{1,-1}$-sesquilinear form $[\cdot,\cdot]_{\gamma}\restriction_{\mathcal{G}_v \times \mathcal{G}_v}$ is a skew-Hermitian form on $\mathcal{G}_v$.

This completes the proof of the lemma. □

In the following theorem, we derive necessary and sufficient conditions for an Ω-MT additive code of length n over $\mathbb{F}_{q^t}$ to be δ-self-orthogonal and δ-self-dual for each $\delta \in \{0, *, \gamma\}$.

Theorem 3.1 *Let* $\Omega = (\omega_1, \omega_2, \ldots, \omega_\ell)$ *be fixed. Let* $\mathcal{C} = \left(\bigoplus_{v=1}^{e_1} \mathcal{C}_v\right) \oplus \left(\bigoplus_{w=e_1+1}^{e_2} \left(\mathcal{C}_w \oplus \mathcal{C}_w^{\dagger}\right)\right) \oplus \left(\bigoplus_{s=e_2+1}^{e_3} \mathcal{C}_s\right)$ *be an* Ω*-MT additive code of length* n *over* $\mathbb{F}_{q^t}$*, where* $\mathcal{C}_v$ *(resp.* $\mathcal{C}_w$*,* $\mathcal{C}_w^{\dagger}$ *and* $\mathcal{C}_s$*) is a subspace of* $\mathcal{G}_v$ *(resp.* $\mathcal{G}_w$*,* $\mathcal{G}_w^{\dagger}$ *and* $\mathcal{G}_s$*) over* $\mathcal{F}_v$ *(resp.* $\mathcal{F}_w$*,* $\mathcal{F}_w^{\dagger}$ *and* $\mathcal{F}_s$*) for all* v *(resp.* w *and* s*). Then for* $\delta \in \{0, *, \gamma\}$*, the following hold.*

(a) *The code* $\mathcal{C}$ *is* δ*-self-dual if and only if the following conditions are satisfied:*

- *Irreducible factors of the polynomials* $x^{m_1} - \omega_1, x^{m_2} - \omega_2, \ldots, x^{m_\ell} - \omega_\ell$ *in* $\mathbb{F}_{q^t}[x]$ *are either self-reciprocal or form reciprocal pairs.*
- *For* $1 \leq v \leq e_1$*,* $\dim_{\mathcal{F}_v} \mathcal{G}_v = \epsilon_v t$ *is even and* $\mathcal{C}_v$ *is a* δ*-self-dual* $\mathcal{F}_v$*-subspace of* $\mathcal{G}_v$*.*
- *For* $e_1 + 1 \leq w \leq e_2$*,* $\mathcal{C}_w$ *(resp.* $\mathcal{C}_w^{\dagger}$*) is a subspace of* $\mathcal{K}_w$ *(resp.* $\mathcal{K}_w^{\dagger}$*) over* $\mathcal{F}_w$ *(resp.* $\mathcal{F}_w^{\dagger}$*) satisfying* $\mathcal{C}_w^{\dagger} = \mathcal{C}_w^{\perp_\delta} \cap \mathcal{K}_w^{\dagger}$*.*

As a consequence, the total number of distinct δ*-self-dual* Ω*-MT additive codes of length* n *over* $\mathbb{F}_{q^t}$ *is given by*

$$\mathfrak{N} = \prod_{v=1}^{e_1} \mathfrak{N}_v \prod_{w=e_1+1}^{e_2} \mathfrak{N}_w, \tag{3.8}$$

where $\mathfrak{N}_v$ *denotes the number of distinct* δ*-self-dual* $\mathcal{F}_v$*-subspaces of* $\mathcal{G}_v$ *for* $1 \leq v \leq e_1$ *and* $\mathfrak{N}_w$ *denotes the number of distinct* $\mathcal{F}_w$*-subspaces of* $\mathcal{K}_w$ *for* $e_1 + 1 \leq w \leq e_2$*.*

(b) *The code* $\mathcal{C}$ *is* δ*-self-orthogonal if and only if the following conditions are satisfied:*

- *For* $1 \leq v \leq e_1$*,* $\mathcal{C}_v$ *is a* δ*-self-orthogonal* $\mathcal{F}_v$*-subspace of* $\mathcal{G}_v$*.*

- *For $e_1+1\le w\le e_2$, $\mathcal{C}_w$ (resp. $\mathcal{C}_w^{\dagger}$) is a subspace of $\mathcal{K}_w$ (resp. $\mathcal{K}_w^{\dagger}$) over $\mathcal{F}_w$ (resp. $\mathcal{F}_w^{\dagger}$) satisfying $\mathcal{C}_w^{\dagger}\subseteq\mathcal{C}_w^{\perp_\delta}\cap\mathcal{K}_w^{\dagger}$.*
- *$\mathcal{C}_s=\{0\}$ for $e_2+1\le s\le e_3$.*

As a consequence, the total number of distinct δ-self-orthogonal Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ is given by

$$\mathfrak{M}=\prod_{v=1}^{e_1}\mathfrak{M}_v\prod_{w=e_1+1}^{e_2}\mathfrak{M}_w, \tag{3.9}$$

where $\mathfrak{M}_v$ denotes the number of distinct δ-self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{G}_v$ for $1\le v\le e_1$ and $\mathfrak{M}_w$ denotes the number of distinct pairs $(\mathcal{C}_w,\mathcal{C}_w^{\dagger})$, where $\mathcal{C}_w$ (resp. $\mathcal{C}_w^{\dagger}$) is a subspace of $\mathcal{K}_w$ (resp. $\mathcal{K}_w^{\dagger}$) over $\mathcal{F}_w$ (resp. $\mathcal{F}_w^{\dagger}$) satisfying $\mathcal{C}_w^{\dagger}\subseteq\mathcal{C}_w^{\perp_\delta}\cap\mathcal{K}_w^{\dagger}$ for $e_1+1\le w\le e_2$.

Proof (a) By (3.3) and (3.7), we see that $\mathcal{C}=\mathcal{C}^{\perp_\delta}$ if and only if all the irreducible factors of the polynomials $x^{m_1}-\omega_1, x^{m_2}-\omega_2,\ldots,x^{m_\ell}-\omega_\ell$ in $\mathbb{F}_{q^t}[x]$ are either self-reciprocal or form reciprocal pairs, $\mathcal{C}_v=\mathcal{C}_v^{\perp_\delta}$ for $1\le v\le e_1$, $\mathcal{C}_w=\mathcal{C}_w^{\dagger\perp_\delta}$ and $\mathcal{C}_w^{\dagger}=\mathcal{C}_w^{\perp_\delta}$ for $e_1+1\le w\le e_2$.

For $1\le v\le e_1$, we further note, by Lemma 3.4, that the $\tau_{1,-1}$-sesquilinear form $[\cdot,\cdot]_{\delta\restriction_{\mathcal{G}_v\times\mathcal{G}_v}}$ is non-degenerate. We also recall that $\dim_{\mathcal{F}_v}\mathcal{G}_v=\epsilon_v t$. Now if $\mathcal{C}_v$ is an $\mathcal{F}_v$-subspace of $\mathcal{G}_v$ satisfying $\mathcal{C}_v=\mathcal{C}_v^{\perp_\delta}$, then by Theorem 2.5, we see that $\dim_{\mathcal{F}_v}\mathcal{C}_v=\dim_{\mathcal{F}_v}\mathcal{C}_v^{\perp_\delta}=\dim_{\mathcal{F}_v}\mathcal{G}_v-\dim_{\mathcal{F}_v}\mathcal{C}_v=\epsilon_v t-\dim_{\mathcal{F}_v}\mathcal{C}_v$, which implies that $\epsilon_v t=2\dim_{\mathcal{F}_v}\mathcal{C}_v$ is an even integer.

For $e_1+1\le w\le e_2$, we see that $\mathcal{C}_w=\mathcal{C}_w^{\dagger\perp_\delta}$ and $\mathcal{C}_w^{\dagger}=\mathcal{C}_w^{\perp_\delta}$ holds if and only if $\mathcal{C}_w$ (resp. $\mathcal{C}_w^{\dagger}$) is a subspace of $\mathcal{K}_w$ (resp. $\mathcal{K}_w^{\dagger}$) over $\mathcal{F}_w$ (resp. $\mathcal{F}_w^{\dagger}$) satisfying $\mathcal{C}_w^{\dagger}=\mathcal{C}_w^{\perp_\delta}\cap\mathcal{K}_w^{\dagger}$.

From this and by applying Theorem 2.2, part (a) follows immediately.

(b) By (3.3) and (3.7), we see that $\mathcal{C}\subseteq\mathcal{C}^{\perp_\delta}$ if and only if $\mathcal{C}_v\subseteq\mathcal{C}_v^{\perp_\delta}$ for $1\le v\le e_1$, $\mathcal{C}_w$ (resp. $\mathcal{C}_w^{\dagger}$) is a subspace of $\mathcal{K}_w$ (resp. $\mathcal{K}_w^{\dagger}$) over $\mathcal{F}_w$ (resp. $\mathcal{F}_w^{\dagger}$) satisfying $\mathcal{C}_w\subseteq\mathcal{C}_w^{\dagger\perp_\delta}\cap\mathcal{K}_w$ and $\mathcal{C}_w^{\dagger}\subseteq\mathcal{C}_w^{\perp_\delta}\cap\mathcal{K}_w^{\dagger}$ for $e_1+1\le w\le e_2$, and $\mathcal{C}_s\subseteq\{0\}$ and $\{0\}\subseteq\mathcal{C}_s^{\perp_\delta}$ for $e_2+1\le s\le e_3$. Further, for $e_1+1\le w\le e_2$, we observe that $\mathcal{C}_w\subseteq\mathcal{C}_w^{\dagger\perp_\delta}\cap\mathcal{K}_w$ and $\mathcal{C}_w^{\dagger}\subseteq\mathcal{C}_w^{\perp_\delta}\cap\mathcal{K}_w^{\dagger}$ hold if and only if $\mathcal{C}_w$ (resp. $\mathcal{C}_w^{\dagger}$) is a subspace of $\mathcal{K}_w$ (resp. $\mathcal{K}_w^{\dagger}$) over $\mathcal{F}_w$ (resp. $\mathcal{F}_w^{\dagger}$) satisfying $\mathcal{C}_w^{\dagger}\subseteq\mathcal{C}_w^{\perp_\delta}\cap\mathcal{K}_w^{\dagger}$. Moreover, for $e_2+1\le s\le e_3$, we see that $\mathcal{C}_s\subseteq\{0\}$ and $\{0\}\subseteq\mathcal{C}_s^{\perp_\delta}$ hold if and only if $\mathcal{C}_s=\{0\}$. From this, part (b) follows immediately.

□

Now we shall apply Theorem 3.1 and Witt decomposition theory to count all δ-self-orthogonal and δ-self-dual Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ for each $\delta\in\{0,*,\gamma\}$.

3.1 Enumeration formulae for δ-self-orthogonal Ω-multi-twisted additive codes

In the following theorem, we provide enumeration formulae for all δ-self-orthogonal Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$, where $\delta \in \{0, *, \gamma\}$.

Theorem 3.2 *Let $\Omega = (\omega_1, \omega_2, \ldots, \omega_\ell)$ be fixed. For $\delta \in \{0, *, \gamma\}$, the total number $\mathfrak{M}$ of distinct δ-self-orthogonal Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ is given by*

$$\mathfrak{M} = \prod_{v=1}^{e_1} \mathfrak{M}_v \prod_{w=e_1+1}^{e_2} \left(\sum_{k_1=0}^{\eta_w} \begin{bmatrix} \eta_w \\ k_1 \end{bmatrix}_{q^{d_w}} \sum_{k_2=0}^{\eta_w - k_1} \begin{bmatrix} \eta_w - k_1 \\ k_2 \end{bmatrix}_{q^{d_w}} \right),$$

where for $1 \le v \le e_1$, the number $\mathfrak{M}_v$ equals

- $\displaystyle\sum_{k=0}^{\epsilon_v t/2} \begin{bmatrix} \epsilon_v t/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d - 2}{2}} + 1 \right)$ *when $v \in \mathcal{J}_1$ and $\delta \in \{0, *\}$ with either $\epsilon_v t$ is even and $q \equiv 1 \pmod 4$ or $\epsilon_v t \equiv 0 \pmod 4$ and $q \equiv 3 \pmod 4$.*
- $\displaystyle\sum_{k=0}^{(\epsilon_v t-2)/2} \begin{bmatrix} (\epsilon_v t - 2)/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d}{2}} + 1 \right)$ *when $v \in \mathcal{J}_1$, $\delta \in \{0, *\}$, $q \equiv 3 \pmod 4$ and $\epsilon_v t \equiv 2 \pmod 4$.*
- $\displaystyle\sum_{k=0}^{(\epsilon_v t-2)/2} \begin{bmatrix} (\epsilon_v t - 2)/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d - 2}{2}} + 1 \right) + \sum_{k'=1}^{\epsilon_v t/2} q^{\epsilon_v t - 2k'} \begin{bmatrix} (\epsilon_v t - 2)/2 \\ k' - 1 \end{bmatrix}_q \prod_{d'=0}^{k'-2} \left(q^{\frac{\epsilon_v t - 2d' - 2}{2}} + 1 \right)$ *when $v \in \mathcal{J}_1$, $\delta = 0$ and both $\epsilon_v t, q$ are even.*
- $\displaystyle\sum_{k=0}^{(\epsilon_v t-1)/2} \begin{bmatrix} (\epsilon_v t - 1)/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} (q^{\frac{\epsilon_v t - 2d - 1}{2}} + 1)$ *when $v \in \mathcal{J}_1$ with either $\delta = *$ and both $\epsilon_v t$, q are odd or $\delta = 0$ and $\epsilon_v t$ is odd.*
- $\displaystyle\sum_{k=0}^{\epsilon_v t/2} \begin{bmatrix} \epsilon_v t/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d}{2}} + 1 \right)$ *when $v \in \mathcal{J}_1$ with either $\delta = \gamma$ or $\delta = *$ and both $\epsilon_v t, q$ are even.*
- $\displaystyle\sum_{k=0}^{\epsilon_v t/2} \begin{bmatrix} \epsilon_v t/2 \\ k \end{bmatrix}_{q^{d_v}} \prod_{b=0}^{k-1} \left(q^{\frac{d_v(\epsilon_v t - 2b - 1)}{2}} + 1 \right)$ *when $v \in \mathcal{J}_2$ and $\epsilon_v t$ is even.*
- $\displaystyle\sum_{k=0}^{(\epsilon_v t-1)/2} \begin{bmatrix} (\epsilon_v t - 1)/2 \\ k \end{bmatrix}_{q^{d_v}} \prod_{b=0}^{k-1} \left(q^{\frac{d_v(\epsilon_v t - 2b)}{2}} + 1 \right)$ *when $v \in \mathcal{J}_2$ and $\epsilon_v t$ is odd.*

To prove the above theorem, we see, by (3.9), that it is enough to determine the number $\mathfrak{M}_v$ of distinct δ-self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{G}_v$ for $1 \le v \le e_1$, and to determine the number $\mathfrak{M}_w$ of distinct pairs $(\mathcal{C}_w, \mathcal{C}_w^\dagger)$, where $\mathcal{C}_w$ is an $\mathcal{F}_w$-subspace of $\mathcal{K}_w$ and $\mathcal{C}_w^\dagger$ is an $\mathcal{F}_w^\dagger$-subspace of $\mathcal{K}_w^\dagger$ satisfying $\mathcal{C}_w^\dagger \subseteq \mathcal{C}_w^{\perp_\delta} \cap \mathcal{K}_w^\dagger$ for $e_1 + 1 \le w \le e_2$.

In the following lemma, we determine the number $\mathfrak{M}_v$ for $1 \le v \le e_1$.

Lemma 3.5 *For $\delta \in \{0, *, \gamma\}$, the following hold.*

(a) *When $v \in \mathcal{J}_1$, we have*

$$\mathfrak{M}_v = \begin{cases} \sum_{k=0}^{\epsilon_v t/2} \begin{bmatrix} \epsilon_v t/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d - 2}{2}} + 1\right) & \text{if } \delta \in \{0, *\} \text{ with either } \epsilon_v t \text{ is even and } q \equiv 1 \ (\mathrm{mod}\ 4) \\ & \text{or } \epsilon_v t \equiv 0 \ (\mathrm{mod}\ 4) \text{ and } q \equiv 3 \ (\mathrm{mod}\ 4); \\ \sum_{k=0}^{(\epsilon_v t-2)/2} \begin{bmatrix} (\epsilon_v t-2)/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d}{2}} + 1\right) & \text{if } \delta \in \{0, *\}, \epsilon_v t \equiv 2 \ (\mathrm{mod}\ 4) \text{ and } q \equiv 3 \ (\mathrm{mod}\ 4); \\ \sum_{k=0}^{(\epsilon_v t-1)/2} \begin{bmatrix} (\epsilon_v t-1)/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d - 1}{2}} + 1\right) & \text{if } \delta \in \{0, *\} \text{ and both } \epsilon_v t, q \text{ are odd}; \\ \sum_{k=0}^{(\epsilon_v t-1)/2} \begin{bmatrix} (\epsilon_v t-1)/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d - 1}{2}} + 1\right) & \text{if } \delta = 0, \ q \text{ is even and } \epsilon_v t \text{ is odd}; \\ \sum_{k=0}^{(\epsilon_v t-2)/2} \begin{bmatrix} (\epsilon_v t-2)/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d - 2}{2}} + 1\right) & \\ + \sum_{k'=1}^{\epsilon_v t/2} q^{\epsilon_v t - 2k'} \begin{bmatrix} (\epsilon_v t-2)/2 \\ k'-1 \end{bmatrix}_q \prod_{d'=0}^{k'-2} \left(q^{\frac{\epsilon_v t - 2d' - 2}{2}} + 1\right) & \text{if } \delta = 0 \text{ and both } \epsilon_v t, q \text{ are even}; \\ \sum_{k=0}^{\epsilon_v t/2} \begin{bmatrix} \epsilon_v t/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d}{2}} + 1\right) & \text{if } \delta = * \text{ and both } \epsilon_v t, q \text{ are even}; \\ \sum_{k=0}^{\epsilon_v t/2} \begin{bmatrix} \epsilon_v t/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d}{2}} + 1\right) & \text{if } \delta = \gamma \text{ and } \epsilon_v t \text{ is even}. \end{cases}$$

(b) *When $v \in \mathcal{J}_2$, we have*

$$\mathfrak{M}_v = \begin{cases} \sum_{k=0}^{\epsilon_v t/2} \begin{bmatrix} \epsilon_v t/2 \\ k \end{bmatrix}_{q^{d_v}} \prod_{d=0}^{k-1} \left(q^{\frac{d_v(\epsilon_v t - 2d - 1)}{2}} + 1\right) & \text{if } \epsilon_v t \text{ is even}; \\ \sum_{k=0}^{(\epsilon_v t-1)/2} \begin{bmatrix} (\epsilon_v t-1)/2 \\ k \end{bmatrix}_{q^{d_v}} \prod_{d=0}^{k-1} \left(q^{\frac{d_v(\epsilon_v t - 2d)}{2}} + 1\right) & \text{if } \epsilon_v t \text{ is odd}. \end{cases}$$

Proof (a) Let $v \in \mathcal{J}_1$. Here we have $d_v = D_v = 1$, and hence $\mathcal{F}_v \simeq \mathbb{F}_q$. Now to determine the number $\mathfrak{M}_v$, we proceed as follows:
When $\delta = \gamma$, we see, by Lemma 3.4(c), that $(\mathcal{G}_v, [\cdot,\cdot]_{\gamma \restriction \mathcal{G}_v \times \mathcal{G}_v})$ is a symplectic space of dimension $\epsilon_v t$ over $\mathcal{F}_v \simeq \mathbb{F}_q$. Further, by (Taylor 1992, p.69), we note that the Witt index of $\mathcal{G}_v$ (i.e., the dimension of a maximal self-orthogonal $\mathcal{F}_v$-subspace of $\mathcal{G}_v$) is $\epsilon_v t/2$. Using this and by Exercise 8.1(ii) of Taylor (1992), we get

$$\mathfrak{M}_v = \sum_{k=0}^{\epsilon_v t/2} \left(\begin{bmatrix} \epsilon_v t/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1} \left(q^{\frac{\epsilon_v t - 2d}{2}} + 1\right) \right).$$

From this point on, let $\delta \in \{0, *\}$. Here by Lemma 3.4(a), we see that $(\mathcal{G}_v, [\cdot,\cdot]_{\delta \restriction \mathcal{G}_v \times \mathcal{G}_v})$ is an orthogonal space of dimension $\epsilon_v t$ over $\mathcal{F}_v$. Now we shall distinguish the following three cases: **I.** q is odd, **II.** q is even and $\delta = 0$ and **III.** q is even and $\delta = *$.

I. When q is odd, we see that the orthogonal space $(\mathcal{G}_v, [\cdot,\cdot]_{\delta \restriction \mathcal{G}_v \times \mathcal{G}_v})$ can also be viewed as a non-degenerate quadratic space $(\mathcal{G}_v, \mathcal{Q}_v)$ over $\mathcal{F}_v$, where $\mathcal{Q}_v : \mathcal{G}_v \to \mathcal{F}_v$ is the quadratic map, defined as $\mathcal{Q}_v(\mathcal{A}_v) = \frac{1}{2}[\mathcal{A}_v, \mathcal{A}_v]_\delta$ for all $\mathcal{A}_v \in \mathcal{G}_v$. Further, by Theorem 1 of Pless (1968), we note that the Witt index

m_v of $\mathcal{G}_v$ is given by

$$m_v = \begin{cases} (\epsilon_v t - 1)/2 & \text{if } \epsilon_v t \text{ is odd;} \\ \epsilon_v t/2 & \text{if either } \epsilon_v t \text{ is even and } q \equiv 1 \pmod 4 \\ & \text{or } \epsilon_v t \equiv 0 \pmod 4 \text{ and } q \equiv 3 \pmod 4; \\ (\epsilon_v t - 2)/2 & \text{if } \epsilon_v t \equiv 2 \pmod 4 \text{ and } q \equiv 3 \pmod 4. \end{cases} \tag{3.10}$$

Therefore when $\delta \in \{0, *\}$ and q is odd, by Exercise 11.3 of Taylor (1992), we get $\mathfrak{M}_v = \sum_{k=0}^{m_v} \begin{bmatrix} m_v \\ k \end{bmatrix}_q \prod_{d=0}^{k-1}(q^{m_v-\varrho-d}+1)$, where $\varrho = 2m_v - \epsilon_v t + 1$. Now on substituting the value of m_v from (3.10), we get the desired value of $\mathfrak{M}_v$.

II. Next let $\delta = 0$ and q be even. Here all m_i's are odd integers, which implies that the integer m is odd, and hence $\frac{m}{m_i} = 1$ in $\mathcal{F}_v \simeq \mathbb{F}_q$. Since $d_v = 1$, we have $a_v = \gcd(t, d_v) = 1$. So each element $\mathcal{A}_v \in \mathcal{G}_v$ can be expressed as $\mathcal{A}_v = \mathcal{A}_{v,0} = (\mathcal{A}_{v,0}^{(1)}, \mathcal{A}_{v,0}^{(2)}, \ldots, \mathcal{A}_{v,0}^{(\ell)})$, where $\mathcal{A}_{v,0}^{(i)} \in \epsilon_{v,i}\mathcal{F}_{v,0}$ for $1 \le i \le \ell$. Now let us define the set $\mathcal{M}_v = \{\mathcal{A}_v \in \mathcal{G}_v :$ $\sum_{i=1}^{\ell} \epsilon_{v,i}\left(\mathcal{A}_{v,0}^{(i)} + \tau_{q,1}(\mathcal{A}_{v,0}^{(i)}) + \cdots + \tau_{q^{t-1},1}(\mathcal{A}_{v,0}^{(i)})\right) = 0\}$. One can easily observe that $\mathcal{M}_v$ is an $\mathcal{F}_v$-subspace of $\mathcal{G}_v$ having dimension $\epsilon_v t - 1$. Next let $\Theta_v = (\epsilon_{v,1}, \epsilon_{v,2}, \ldots, \epsilon_{v,\ell}) \in \mathcal{G}_v$. Then it is easy to see that $\Theta_v \in \mathcal{M}_v$ if and only if $\sum_{i=1}^{\ell} \epsilon_{v,i}(1 + \tau_{q,1}(1) + \cdots + \tau_{q^{t-1},1}(1)) = \epsilon_v t = 0$ if and only if $\epsilon_v t$ is even. Now the following two cases arise:

(i) When $\epsilon_v t$ is odd, we note that $\Theta_v \notin \mathcal{M}_v$. Further, we see that $[\mathcal{A}_v, \Theta_v]_0 = 0$ for all $\mathcal{A}_v \in \mathcal{M}_v$. This implies that $\mathcal{G}_v = \mathcal{M}_v \perp \langle\Theta_v\rangle$, an orthogonal direct sum of the $\mathcal{F}_v$-subspaces $\mathcal{M}_v$ and $\langle\Theta_v\rangle$ of $\mathcal{G}_v$. Further, one can easily observe that any self-orthogonal $\mathcal{F}_v$-subspace of $\mathcal{G}_v$ is contained in $\mathcal{M}_v$.

Next we assert that the restriction $[\cdot,\cdot]_{0\restriction_{\mathcal{M}_v\times\mathcal{M}_v}}$ of the $\tau_{1,-1}$-sesquilinear form $[\cdot,\cdot]_0$ to $\mathcal{M}_v \times \mathcal{M}_v$ is non-degenerate. For this, suppose that there exists $\mathcal{A}_v \in \mathcal{M}_v$ satisfying $[\mathcal{A}_v, \mathcal{B}_v]_0 = 0$ for all $\mathcal{B}_v \in \mathcal{M}_v$. Here we shall show that $\mathcal{A}_v = 0$.

Let $\mathcal{Y}_v = \mathcal{B}_v + a_v\Theta_v \in \mathcal{G}_v$, where $a_v \in \mathcal{F}_v$. As a_v runs over $\mathcal{F}_v$ and $\mathcal{B}_v$ runs over $\mathcal{M}_v$, we see that $\mathcal{Y}_v$ runs over $\mathcal{G}_v$. Now let us consider $[\mathcal{A}_v, \mathcal{Y}_v]_0 = [\mathcal{A}_v, \mathcal{B}_v]_0 + a_v[\mathcal{A}_v, \Theta_v]_0 = 0$ for all $\mathcal{Y}_v \in \mathcal{G}_v$. As the $\tau_{1,-1}$-sesquilinear form $[\cdot,\cdot]_0$ is non-degenerate on $\mathcal{G}_v$, we get $\mathcal{A}_v = 0$. This proves the assertion.

Next for each $\mathcal{A}_v \in \mathcal{M}_v$, we observe that $[\mathcal{A}_v, \mathcal{A}_v]_0 = \sum_{i=1}^{\ell} \epsilon_{v,i}((\mathcal{A}_{v,0}^{(i)})^2 + \tau_{q,1}(\mathcal{A}_{v,0}^{(i)})^2 + \cdots + \tau_{q^{t-1},1}(\mathcal{A}_{v,0}^{(i)})^2) = \left(\sum_{i=1}^{\ell} \epsilon_{v,i}(\mathcal{A}_{v,0}^{(i)} + \tau_{q,1}(\mathcal{A}_{v,0}^{(i)}) + \cdots + \tau_{q^{t-1},1}(\mathcal{A}_{v,0}^{(i)}))\right)^2 = 0$. This shows that $(\mathcal{M}_v, [\cdot,\cdot]_{0\restriction_{\mathcal{M}_v\times\mathcal{M}_v}})$ is a symplectic space over $\mathcal{F}_v$, whose dimension is $\epsilon_v t - 1$ and Witt index is $\frac{\epsilon_v t-1}{2}$. Now by Exercise 8.1 of Taylor (1992), for $0 \le k \le \frac{\epsilon_v t-1}{2}$, we see that the total number of k-dimensional self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{M}_v$ (and hence of $\mathcal{G}_v$) is given by $\begin{bmatrix} (\epsilon_v t-1)/2 \\ k \end{bmatrix}_q \prod_{d=0}^{k-1}\left(q^{\frac{\epsilon_v t-2d-1}{2}} + 1\right)$. From this and using the fact that the dimension of a maximal self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{M}_v$ is $\frac{\epsilon_v t-1}{2}$, we obtain $\mathfrak{M}_v = \sum_{k=0}^{(\epsilon_v t-1)/2}\left(\left[{(\epsilon_v t-1)/2 \atop k}\right]_q \prod_{d=0}^{k-1}\left(q^{\frac{\epsilon_v t-2d-1}{2}}+1\right)\right)$.

(ii) When $\epsilon_v t$ is even, we see that $\Theta_v \in \mathcal{M}_v \cap \mathcal{M}_v^{\perp_0}$. In this case, let $\widehat{\mathcal{M}}_v$ be an $(\epsilon_v t-2)$-dimensional $\mathcal{F}_v$-subspace of $\mathcal{M}_v$ satisfying $\Theta_v \notin \widehat{\mathcal{M}}_v$. Then we have $\mathcal{M}_v = \widehat{\mathcal{M}}_v \oplus \langle\Theta_v\rangle$. Further, it is easy to observe that there exists an element $y_v \in \widehat{\mathcal{M}}_v^{\perp_0} \setminus \mathcal{M}_v$. This implies that $\mathcal{G}_v = \widehat{\mathcal{M}}_v \oplus \langle\Theta_v\rangle \oplus \langle y_v\rangle$. We also observe that any self-orthogonal $\mathcal{F}_v$-subspace of $\mathcal{G}_v$ is either (i) contained in $\widehat{\mathcal{M}}_v$, or (ii) contained in $\widehat{\mathcal{M}}_v \oplus \langle\Theta_v\rangle$ but not in $\widehat{\mathcal{M}}_v$.
To determine the number $\mathfrak{M}_v$, we shall first count the number of self-orthogonal $\mathcal{F}_v$-subspaces of $\widehat{\mathcal{M}}_v$. Towards this, we note that $[\mathcal{A}_v, \mathcal{A}_v]_0 = 0$ for all $\mathcal{A}_v \in \widehat{\mathcal{M}}_v$. Now we assert that the $\tau_{1,-1}$-sesquilinear form $[\cdot,\cdot]_0\restriction_{\widehat{\mathcal{M}}_v\times\widehat{\mathcal{M}}_v}$ is non-degenerate. For this, suppose that there exists $\mathcal{A}_v \in \widehat{\mathcal{M}}_v$ satisfying $[\mathcal{A}_v, \mathcal{B}_v]_0 = 0$ for all $\mathcal{B}_v \in \widehat{\mathcal{M}}_v$. Here it suffices to show that $\mathcal{A}_v = 0$. Let $\mathcal{X}_v = \mathcal{B}_v + a_v y_v + b_v\Theta_v \in \mathcal{G}_v$, where $a_v, b_v \in \mathcal{F}_v$. We note that as $\mathcal{A}_v$ runs over the set $\widehat{\mathcal{M}}_v$ and elements a_v, b_v run over $\mathcal{F}_v$, the element $\mathcal{X}_v$ runs over $\mathcal{G}_v$. Then we see that $[\mathcal{A}_{v,}, \mathcal{X}_v]_0 = [\mathcal{A}_v, \mathcal{B}_v]_0 + a_v[\mathcal{A}_v, y_v]_0 + b_v[\mathcal{A}_v, \Theta_v]_0 = 0$ for all $\mathcal{X}_v \in \mathcal{G}_v$. Since the map $[\cdot,\cdot]_0$ is non-degenerate on $\mathcal{G}_v$, we get $\mathcal{A}_v = 0$. From this, it follows that $(\widehat{\mathcal{M}}_v, [\cdot,\cdot]_0\restriction_{\widehat{\mathcal{M}}_v\times\widehat{\mathcal{M}}_v})$ is a symplectic space over $\mathcal{F}_v$, whose dimension is $\epsilon_v t-2$ and Witt index is $\frac{\epsilon_v t-2}{2}$. Now by Exercise 8.1 of Taylor (1992), we see that the total number of distinct self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{G}_v$ that are contained in $\widehat{\mathcal{M}}_v$ is given by $\mathfrak{T}_v = \sum_{k=0}^{(\epsilon_v t-2)/2}\left[{(\epsilon_v t-2)/2 \atop k}\right]_q \prod_{d=0}^{k-1}\left(q^{\frac{\epsilon_v t-2d-2}{2}}+1\right)$.
Now we proceed to count all distinct self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{G}_v$ that are contained in $\widehat{\mathcal{M}}_v \oplus \langle\Theta_v\rangle$ but not in $\widehat{\mathcal{M}}_v$. For this, let X be a k-dimensional self-orthogonal $\mathcal{F}_v$-subspace of $\mathcal{G}_v$, which is contained in $\widehat{\mathcal{M}}_v \oplus \langle\Theta_v\rangle$, but not in $\widehat{\mathcal{M}}_v$. Then we see that $1 \le k \le \epsilon_v t/2$ and that any such k-dimensional $\mathcal{F}_v$-subspace of $\mathcal{G}_v$ has a basis set of the form $\{z_1, z_2, \ldots, z_{k-1}, z_k + \Theta_v\}$, where $z_\nu \in \widehat{\mathcal{M}}_v \setminus \{0\}$ for $1 \le \nu \le k-1$ and $z_k \in \widehat{\mathcal{M}}_v$. Now it is easy to observe that $\langle z_1, z_2, \ldots, z_{k-1}, z_k + \Theta_v\rangle \subseteq \langle z_1, z_2, \ldots, z_{k-1}, z_k + \Theta_v\rangle^{\perp_0}$ if and only if $\langle z_1, z_2, \ldots, z_{k-1}\rangle \subseteq \langle z_1, z_2, \ldots, z_{k-1}\rangle^{\perp_0}$ in $\widehat{\mathcal{M}}_v$ and $z_k \in \langle z_1, z_2, \ldots, z_{k-1}\rangle^{\perp_0}$. Since $(\widehat{\mathcal{M}}_v, [\cdot,\cdot]_0\restriction_{\widehat{\mathcal{M}}_v\times\widehat{\mathcal{M}}_v})$ is a symplectic space of dimension $\epsilon_v t-2$ and Witt index $\frac{\epsilon_v t-2}{2}$, by Exercise 8.1 of Taylor (1992), we see that for $1 \le k \le \epsilon_v t/2$, the number of distinct $(k-1)$-dimensional self-orthogonal $\mathcal{F}_v$-subspaces of $\widehat{\mathcal{M}}_v$ is given by $\left[{(\epsilon_v t-2)/2 \atop k-1}\right]_q \prod_{d=0}^{k-2}\left(q^{\frac{\epsilon_v t-2d-2}{2}}+1\right)$. Further, for a given $(k-1)$-dimensional self-orthogonal $\mathcal{F}_v$-subspace $\langle z_1, z_2, \ldots, z_{k-1}\rangle$ of $\widehat{\mathcal{M}}_v$, we observe that $\langle z_1, z_2, \ldots, z_{k-1}, z_k + \Theta_v\rangle = \langle z_1, z_2, \ldots, z_{k-1}, z_k' + \Theta_v\rangle$ for some $z_k, z_k' \in \langle z_1, z_2, \ldots, z_{k-1}\rangle^{\perp_0}$ if and only if $z_k - z_k' \in \langle z_1, z_2, \ldots, z_{k-1}\rangle$. This implies that all z_k's lying in the distinct cosets of the quotient space $\langle z_1, z_2, \ldots, z_{k-1}\rangle^{\perp_0}/\langle z_1, z_2, \ldots, z_{k-1}\rangle$ give rise to distinct self-orthogonal $\mathcal{F}_v$-subspaces of the form $\langle z_1, z_2, \ldots, z_{k-1}, z_k+\Theta_v\rangle$, and vice versa. Fur-

ther, we see that the $\mathcal{F}_v$-dimension of $\langle z_1, z_2, \ldots, z_{k-1}\rangle^{\perp_0}/\langle z_1, z_2, \ldots, z_{k-1}\rangle$ is $\epsilon_v t-2k$, which implies that the element z_k has $q^{\epsilon_v t-2k}$ relevant choices. Therefore for $1 \leq k \leq \epsilon_v t/2$, we see that the number of distinct k-dimensional self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{G}_v$ that are contained in $\widehat{\mathcal{M}}_v \oplus \langle\Theta_v\rangle$ but not in $\widehat{\mathcal{M}}_v$ is given by $q^{\epsilon_v t-2k}\left[{(\epsilon_v t-2)/2 \atop k-1}\right]_q \prod_{d=0}^{k-2}\left(q^{\frac{\epsilon_v t-2d-2}{2}}+1\right)$. This shows that the total number of distinct self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{G}_v$ that are contained in $\widehat{\mathcal{M}}_v \oplus \langle\Theta_v\rangle$ but not in $\widehat{\mathcal{M}}_v$, is given by $\mathfrak{U}_v = \sum_{k=1}^{\epsilon_v t/2} q^{\epsilon_v t-2k}\left[{(\epsilon_v t-2)/2 \atop k-1}\right]_q \prod_{d=0}^{k-2}\left(q^{\frac{\epsilon_v t-2d-2}{2}}+1\right)$. On combining both the cases, we obtain

$$\begin{aligned}\mathfrak{M}_v &= \mathfrak{T}_v + \mathfrak{U}_v = \sum_{k=0}^{(\epsilon_v t-2)/2} \begin{bmatrix}(\epsilon_v t-2)/2 \\ k\end{bmatrix}_q \prod_{d=0}^{k-1}\left(q^{\frac{\epsilon_v t-2d-2}{2}}+1\right) \\ &\quad + \sum_{k'=1}^{\epsilon_v t/2} q^{\epsilon_v t-2k'}\left(\begin{bmatrix}(\epsilon_v t-2)/2 \\ k'-1\end{bmatrix}_q \prod_{d'=0}^{k'-2}\left(q^{\frac{\epsilon_v t-2d'-2}{2}}+1\right)\right).\end{aligned}$$

III. Finally, let $\delta = *$ and q be even. Since $t \not\equiv 1 \pmod{p}$, the integer t is even. Here all m_i's are odd integers, which implies that the integer m is odd, and hence $\dfrac{m}{m_i} = 1$ in $\mathcal{F}_v \simeq \mathbb{F}_q$. Since $d_v = 1$, we have $a_v = \gcd(t, d_v) = 1$. So each element $\mathcal{A}_v \in \mathcal{G}_v$ can be expressed as $\mathcal{A}_v = \mathcal{A}_{v,0} = (\mathcal{A}_{v,0}^{(1)}, \mathcal{A}_{v,0}^{(2)}, \ldots, \mathcal{A}_{v,0}^{(\ell)})$, where $\mathcal{A}_{v,0}^{(i)} \in \epsilon_{v,i}\mathcal{F}_{v,0}$ for $1 \leq i \leq \ell$. Now by (3.6), we see that $[\mathcal{A}_v, \mathcal{A}_v]_* = \sum_{i=1}^{\ell} \epsilon_{v,i}\left(\left(\sum_{\mu=0}^{t-1}\tau_{q^\mu,1}(\mathcal{A}_{v,0}^{(i)})\right)^2 - \left(\sum_{\mu=0}^{t-1}\tau_{q^\mu,1}({\mathcal{A}_{v,0}^{(i)}}^2)\right)\right) = 0$ for all $\mathcal{A}_v \in \mathcal{G}_v$. Further, by Lemma 3.4(a), we see that $[\cdot,\cdot]_{*\restriction \mathcal{G}_v\times\mathcal{G}_v}$ is a reflexive and non-degenerate form. Hence $(\mathcal{G}_v, [\cdot,\cdot]_{*\restriction \mathcal{G}_v\times\mathcal{G}_v})$ is a symplectic space over $\mathcal{F}_v$ with dimension $\epsilon_v t$ and Witt index $\epsilon_v t/2$. Now by Exercise 8.1 of Taylor (1992), we get $\mathfrak{M}_v = \sum_{k=0}^{\epsilon_v t/2}\left(\left[{\epsilon_v t/2 \atop k}\right]_q \prod_{d=0}^{k-1}\left(q^{\frac{\epsilon_v t-2d}{2}}+1\right)\right)$.

(b) Next let $v \in \mathcal{J}_2$. Here $d_v = \deg g_v(x)$ is even. To determine the number $\mathfrak{M}_v$, we first note, by Lemma 3.4(b), that $(\mathcal{G}_v, [\cdot,\cdot]_{\delta\restriction \mathcal{G}_v\times\mathcal{G}_v})$ is a unitary space of dimension $\epsilon_v t$ over $\mathcal{F}_v$ when $\delta \in \{0, *\}$.
When $\delta = \gamma$, by Lemma 3.4(d), we see that $[\cdot,\cdot]_{\gamma\restriction \mathcal{G}_v\times\mathcal{G}_v}$ is a non-degenerate, reflexive and skew-Hermitian $\tau_{1,-1}$-sesquilinear form. Now we shall associate an orthogonality preserving Hermitian $\tau_{1,-1}$-sesquilinear form with the skew-Hermitain form $[\cdot,\cdot]_{\gamma\restriction \mathcal{G}_v\times\mathcal{G}_v}$. For this, we see, by Lemma 3.2, that $\tau_{1,-1}$ is a non-identity map on $\mathcal{F}_v$, and hence there exists $\varsigma \in \mathcal{F}_v$ satisfying $\tau_{1,-1}(\varsigma) \neq \varsigma$. Now let us define $\xi = \varsigma - \tau_{1,-1}(\varsigma)(\neq 0) \in \mathcal{F}_v$, which satisfies $\tau_{1,-1}(\xi) = -\xi$. We further define a map $[\cdot,\cdot]'_\gamma : \mathcal{G}_v \times \mathcal{G}_v \to \mathcal{F}_v$ as $[\mathcal{A}_v, \mathcal{B}_v]'_\gamma = \xi[\mathcal{A}_v, \mathcal{B}_v]_\gamma$ for all $\mathcal{A}_v, \mathcal{B}_v \in \mathcal{G}_v$. It is easy to see that the map $[\cdot,\cdot]'_\gamma$ is a non-degenerate and Hermitian $\tau_{1,-1}$-sesquilinear form on $\mathcal{G}_v$. That is, $(\mathcal{G}_v, [\cdot,\cdot]'_{\gamma\restriction \mathcal{G}_v\times\mathcal{G}_v})$ is a unitary space of dimension $\epsilon_v t$ over $\mathcal{F}_v \simeq \mathbb{F}_{q^{d_v}}$. Furthermore, one can easily observe that any $\mathcal{F}_v$-subspace of $\mathcal{G}_v$ is self-orthogonal with respect to $[\cdot,\cdot]_\gamma$ if and only if it is

self-orthogonal with respect to $[\cdot,\cdot]'_\gamma$. Therefore the number $\mathfrak{M}_v$ equals the number of distinct self-orthogonal $\mathcal{F}_v$-subspaces of the unitary space $(\mathcal{G}_v, [\cdot,\cdot]'_{\gamma\restriction \mathcal{G}_v\times\mathcal{G}_v})$. Further, by (Taylor 1992, p.116), we see that the Witt index m_v of the unitary space $(\mathcal{G}_v, [\cdot,\cdot]'_{\gamma\restriction \mathcal{G}_v\times\mathcal{G}_v})$ is given by

$$m_v = \begin{cases} \epsilon_v t/2 & \text{if } \epsilon_v t \text{ is even;} \\ (\epsilon_v t-1)/2 & \text{if } \epsilon_v t \text{ is odd.} \end{cases} \tag{3.11}$$

Now for each $\delta \in \{0,*,\gamma\}$, by applying Exercise 10.4 of Taylor (1992), we get $\mathfrak{M}_v = \sum_{k=0}^{m_v} \begin{bmatrix} m_v \\ k \end{bmatrix}_{q^{d_v}} \prod_{d=0}^{k-1} \left(q^{d_v(m_v-\rho-d)}+1\right)$, where $\rho = \frac{1}{2}$ if $\epsilon_v t$ is even and $\rho = \frac{-1}{2}$ if $\epsilon_v t$ is odd. Further, on substituting the value of the Witt index m_v in (3.11), part (b) follows immediately.

This completes the proof of the lemma. □

In the following lemma, we determine the number $\mathfrak{M}_w$ for $e_1+1 \le w \le e_2$.

Lemma 3.6 *For $\delta \in \{0,*,\gamma\}$ and $e_1+1 \le w \le e_2$, we have* $\mathfrak{M}_w = \sum_{k_1=0}^{\eta_w} \begin{bmatrix} \eta_w \\ k_1 \end{bmatrix}_{q^{d_w}} \sum_{k_2=0}^{\eta_w-k_1} \begin{bmatrix} \eta_w-k_1 \\ k_2 \end{bmatrix}_{q^{d_w}}$.

Proof To prove the result, let $e_1+1 \le w \le e_2$ be fixed. Here we first observe that the restriction $[\cdot,\cdot]_{\delta\restriction \mathcal{K}_w\times\mathcal{K}_w^\dagger}$ of the $\tau_{1,-1}$-sesquilinear form $[\cdot,\cdot]_\delta$ to $\mathcal{K}_w \times \mathcal{K}_w^\dagger$ is non-degenerate. Now let $\mathcal{C}_w$ be an $\mathcal{F}_w$-subspace of $\mathcal{K}_w$ and $\mathcal{C}_w^\dagger$ be an $\mathcal{F}_w^\dagger$-subspace of $\mathcal{K}_w^\dagger$ satisfying $\mathcal{C}_w^\dagger \subseteq \mathcal{C}_w^{\perp_\delta} \cap \mathcal{K}_w^\dagger$. Working as in the proof of Theorem 2.5, we see that if the $\mathcal{F}_w$-dimension of $\mathcal{C}_w$ is k_1, then the $\mathcal{F}_w^\dagger$-dimension of $\mathcal{C}_w^{\perp_\delta} \cap \mathcal{K}_w^\dagger$ is $\eta_w - k_1$, where $0 \le k_1 \le \eta_w$. Since $\mathcal{C}_w^\dagger$ is an $\mathcal{F}_w^\dagger$-subspace of $\mathcal{C}_w^{\perp_\delta} \cap \mathcal{K}_w^\dagger$, by Lemma 2.1, there are precisely $\sum_{k_2=0}^{\eta_w-k_1} \begin{bmatrix} \eta_w-k_1 \\ k_2 \end{bmatrix}_{q^{d_w}}$ distinct choices of $\mathcal{C}_w^\dagger$ for a given choice of the k_1-dimensional $\mathcal{F}_w$-subspace $\mathcal{C}_w$ of $\mathcal{K}_w$. Further, as $\mathcal{C}_w$ is an $\mathcal{F}_w$-subspace of $\mathcal{K}_w$ having dimension k_1, by using Lemma 2.1 again, the number of choices for $\mathcal{C}_w$ is given by $\begin{bmatrix} \eta_w \\ k_1 \end{bmatrix}_{q^{d_w}}$ for $0 \le k_1 \le \eta_w$. Therefore, the number $\mathfrak{M}_w$ of distinct pairs $(\mathcal{C}_w, \mathcal{C}_w^\dagger)$ with $\mathcal{C}_w$ as an $\mathcal{F}_w$-subspace of $\mathcal{K}_w$ and $\mathcal{C}_w^\dagger$ as an $\mathcal{F}_w^\dagger$-subspace of $\mathcal{K}_w^\dagger$ satisfying $\mathcal{C}_w^\dagger \subseteq \mathcal{C}_w^{\perp_\delta} \cap \mathcal{K}_w^\dagger$, is given by $\mathfrak{M}_w = \sum_{k_1=0}^{\eta_w} \begin{bmatrix} \eta_w \\ k_1 \end{bmatrix}_{q^{d_w}} \sum_{k_2=0}^{\eta_w-k_1} \begin{bmatrix} \eta_w-k_1 \\ k_2 \end{bmatrix}_{q^{d_w}}$. This proves the lemma. □

Proof of Theorem 3.2. On substituting the values of the numbers $\mathfrak{M}_v$ for $1 \le v \le e_1$ and $\mathfrak{M}_w$ for $e_1+1 \le w \le e_2$ from Lemmas 3.5 and 3.6 in Eq. (3.9), the desired result follows immediately. □

3.2 Enumeration formulae for δ-self-dual Ω-multi-twisted additive codes

In the following theorem, we derive a necessary and sufficient condition for the existence of a δ-self-dual Ω-MT additive code of length n over $\mathbb{F}_{q^t}$, and we also provide enumeration formulae for all δ-self-dual Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ for each $\delta \in \{0,*,\gamma\}$.

Theorem 3.3 *Let $\Omega = (\omega_1, \omega_2, \ldots, \omega_\ell)$ be fixed. For $\delta \in \{0,*,\gamma\}$, there exists a δ-self-dual Ω-MT additive code of length n over $\mathbb{F}_{q^t}$ if and only if irreducible factors* *of the polynomials $x^{m_1} - \omega_1, x^{m_2} - \omega_2, \ldots, x^{m_\ell} - \omega_\ell$ in $\mathbb{F}_{q^t}[x]$ are either self-reciprocal or they form reciprocal pairs, $\epsilon_v t$ is an even integer for $1 \le v \le e_1$, and in the case when $\delta \in \{0, *\}$, $(-1)^{\epsilon_v t/2}$ is a square in $\mathbb{F}_q$ for each $v \in \mathcal{J}_1$. Under these conditions, the total number $\mathfrak{N}$ of distinct δ-self-dual Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ is given by*

$$\mathfrak{N} = \prod_{v \in \mathcal{J}_1} \mathfrak{N}_v \prod_{v \in \mathcal{J}_2} \left(\prod_{b=0}^{(\epsilon_v t/2)-1} (q^{\frac{d_u(\epsilon_v t - 2b-1)}{2}} + 1) \right) \prod_{w=e_1+1}^{e_2} \left(\sum_{d=0}^{\eta_w} \begin{bmatrix} \eta_w \\ d \end{bmatrix}_{q^{d_w}} \right),$$

where for each $v \in \mathcal{J}_1$, the number $\mathfrak{N}_v$ equals

- $\prod_{a=0}^{(\epsilon_v t/2)-1} \left(q^{\frac{\epsilon_v t - 2a-2}{2}} + 1 \right)$ *when $\delta \in \{0, *\}$ with either $q \equiv 3 \pmod 4$ and $\epsilon_v t \equiv 0 \pmod 4$ or $q \equiv 1 \pmod 4$ and $\epsilon_v t \equiv 0 \pmod 2$.*
- $\prod_{a=0}^{(\epsilon_v t/2)-2} \left(q^{\frac{\epsilon_v t - 2a-2}{2}} + 1 \right)$ *when $\delta = 0$ and q is even.*
- $\prod_{a=0}^{(\epsilon_v t/2)-1} \left(q^{\frac{\epsilon_v t - 2a}{2}} + 1 \right)$ *when either $\delta = \gamma$ or $\delta = *$ and q is even.*

In order to derive a necessary and sufficient condition for the existence of a δ-self-dual Ω-MT additive code of length n over $\mathbb{F}_{q^t}$, we first prove the following lemma.

Lemma 3.7 *Let $1 \le v \le e_1$ be fixed. There exists a δ-self-dual $\mathcal{F}_v$-subspace of $\mathcal{G}_v$ if and only if the following two conditions are satisfied: (i) $\epsilon_v t$ is an even integer, and (ii) the element $(-1)^{\epsilon_v t/2}$ is a square in $\mathbb{F}_q$ when $\delta \in \{0, *\}$ and $v \in \mathcal{J}_1$.*

Proof To prove the forward part, let $\mathcal{C}_v$ be a δ-self-dual $\mathcal{F}_v$-subspace of $\mathcal{G}_v$, i.e., $\mathcal{C}_v$ satisfies $\mathcal{C}_v = \mathcal{C}_v^{\perp_\delta}$. From this and by Theorem 2.5, we get $\dim_{\mathcal{F}_v} \mathcal{C}_v = \dim_{\mathcal{F}_v} \mathcal{C}_v^{\perp_\delta} = \dim_{\mathcal{F}_v} \mathcal{G}_v - \dim_{\mathcal{F}_v} \mathcal{C}_v = \epsilon_v t - \dim_{\mathcal{F}_v} \mathcal{C}_v$. This implies that $\epsilon_v t = \dim_{\mathcal{F}_v} \mathcal{G}_v = 2 \dim_{\mathcal{F}_v} \mathcal{C}_v$ is an even integer and $\dim_{\mathcal{F}_v} \mathcal{C}_v = \epsilon_v t/2$.

To prove the converse part, let $\epsilon_v t$ be even. Now when $v \in \mathcal{J}_2$, by Lemma 3.4 and (Taylor 1992, p.116), we see that $(\mathcal{G}_v, [\cdot, \cdot]_{\delta \restriction \mathcal{G}_v \times \mathcal{G}_v})$ is a unitary space of dimension $\epsilon_v t$ and Witt index $\epsilon_v t/2$ over $\mathcal{F}_v$. Hence there exists an $\mathcal{F}_v$-subspace $\mathcal{C}_v$ of $\mathcal{G}_v$ satisfying $\mathcal{C}_v = \mathcal{C}_v^{\perp_\delta}$.

When $v \in \mathcal{J}_1$ and $\delta \in \{0, *\}$, by Lemma 3.4(a), we note that $(\mathcal{G}_v, [\cdot, \cdot]_{\delta \restriction \mathcal{G}_v \times \mathcal{G}_v})$ is an orthogonal space over $\mathcal{F}_v$. Since $\epsilon_v t$ is even, we see, by Theorem 9.1.3 of Huffman and Pless (2003), that the Witt index of $(\mathcal{G}_v, [\cdot, \cdot]_{\delta \restriction \mathcal{G}_v \times \mathcal{G}_v})$ is $\epsilon_v t/2$ if and only if $(-1)^{\epsilon_v t/2}$ is a square in $\mathbb{F}_q$.

From this, the desired result follows. □

From this point on, throughout this section, we assume that the irreducible factors of the polynomials $x^{m_1} - \omega_1, x^{m_2} - \omega_2, \ldots, x^{m_\ell} - \omega_\ell$ in $\mathbb{F}_{q^t}[x]$ are either self-reciprocal or they form reciprocal pairs, $\epsilon_v t$ is an even integer for $1 \le v \le e_1$, and in the case when $\delta \in \{0, *\}$, $(-1)^{\frac{\epsilon_v t}{2}}$ is a square in $\mathbb{F}_q$ for each $v \in \mathcal{J}_1$. Further, we see, by (3.8), that to count all δ-self-dual Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$, it is enough to determine the numbers $\mathfrak{N}_v$ for $1 \le v \le e_1$ and the numbers $\mathfrak{N}_w$ for $e_1 + 1 \le w \le e_2$.

In the following lemma, we determine the number $\mathfrak{N}_v$ for $1 \le v \le e_1$.

Lemma 3.8 *Suppose that $\epsilon_v t$ is an even integer for $1 \le v \le e_1$, and that $(-1)^{\epsilon_v t/2}$ is a square in $\mathbb{F}_q$ when $\delta \in \{0, *\}$ and $v \in \mathcal{J}_1$. Then for $\delta \in \{0, *, \gamma\}$, the following hold.*

(a) *When $v \in \mathcal{J}_1$, we have*

$$\mathfrak{N}_v = \begin{cases} \prod_{a=0}^{(\epsilon_v t/2)-1} \left(q^{\frac{\epsilon_v t-2a-2}{2}}+1\right) & \text{if either } \epsilon_v t \text{ is even and } q \equiv 1 \ (mod\ 4) \\ & \text{or } \epsilon_v t \equiv 0 \ (mod\ 4) \text{ and } q \equiv 3 \ (mod\ 4); \\ \prod_{a=0}^{(\epsilon_v t/2)-2} \left(q^{\frac{\epsilon_v t-2a-2}{2}}+1\right) & \text{if } \delta = 0 \text{ and } q \text{ is even}; \\ \prod_{a=0}^{(\epsilon_v t/2)-1} \left(q^{\frac{\epsilon_v t-2a}{2}}+1\right) & \text{if either } \delta = \gamma \text{ or } \delta = * \text{ and } q \text{ is even.} \end{cases}$$

(b) *When $v \in \mathcal{J}_2$, we have*

$$\mathfrak{N}_v = \prod_{b=0}^{(\epsilon_v t/2)-1} \left(q^{\frac{d_v(\epsilon_v t-2b-1)}{2}}+1\right).$$

Proof (a) When $v \in \mathcal{J}_1$, we note that $d_v = D_v = 1$.

First of all, let $\delta = \gamma$. Here by Lemma 3.4(c), we see that $(\mathcal{G}_v, [\cdot,\cdot]_{\gamma\restriction \mathcal{G}_v \times \mathcal{G}_v})$ is a symplectic space of dimension $\epsilon_v t$ over $\mathcal{F}_v$. Therefore in this case, $\mathfrak{N}_v$ equals the number of distinct $\epsilon_v t/2$-dimensional self-orthogonal $\mathcal{F}_v$-subspaces of $(\mathcal{G}_v, [\cdot,\cdot]_{\gamma\restriction \mathcal{G}_v \times \mathcal{G}_v})$. Now by Exercise 8.1 of Taylor (1992), we get $\mathfrak{N}_v = \prod_{a=0}^{(\epsilon_v t/2)-1} (q^{\epsilon_v t-2a}-1)/(q^{a+1}-1) = \prod_{a=0}^{(\epsilon_v t/2)-1} (q^{\frac{\epsilon_v t-2a}{2}}+1)$.
From this point on, let $\delta \in \{0, *\}$. Here $(-1)^{\epsilon_v t/2}$ is a square in $\mathbb{F}_q$. Further, by Lemma 3.4(a), we see that $(\mathcal{G}_v, [\cdot,\cdot]_{\delta\restriction \mathcal{G}_v \times \mathcal{G}_v})$ is an orthogonal space of dimension $\epsilon_v t$ over $\mathcal{F}_v \simeq \mathbb{F}_q$. Now the following three cases arise: **I.** q is odd, **II.** q is even and $\delta = 0$ and **III.** q is even and $\delta = *$.

I. Let q be odd. Here working as in Lemma 3.5, we see that the orthogonal space $(\mathcal{G}_v, [\cdot,\cdot]_{0\restriction \mathcal{G}_v \times \mathcal{G}_v})$ can also be viewed as a non-degenerate quadratic space $(\mathcal{G}_v, \mathcal{Q}_v)$ over $\mathcal{F}_v$, where $\mathcal{Q}_v : \mathcal{G}_v \to \mathcal{F}_v$ is the quadratic map, defined as $\mathcal{Q}_v(\mathcal{A}_v) = \frac{1}{2}[\mathcal{A}_v, \mathcal{A}_v]_\delta$ for all $\mathcal{A}_v \in \mathcal{G}_v$. Further, by Theorem 1 of Pless (1968), we note that the Witt index of the corresponding quadratic space $(\mathcal{G}_v, \mathcal{Q}_v)$ is $\epsilon_v t/2$. Since $\mathcal{C}_v$ is a self-dual $\mathcal{F}_v$-subspace of $\mathcal{G}_v$, by Lemma 3.7, we see that the $\mathcal{F}_v$-dimension of $\mathcal{C}_v$ is $\epsilon_v t/2$. Now by Exercise 11.3 of Taylor (1992), we see that the number $\mathfrak{N}_v$ of distinct $\epsilon_v t/2$-dimensional self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{G}_v$ is given by $\mathfrak{N}_v = \prod_{a=0}^{(\epsilon_v t/2)-1} (q^{\frac{\epsilon_v t-2a-2}{2}}+1)$.

II. Next let $\delta = 0$ and q be even. If $\mathcal{M}_v = \{(\mathcal{A}_{v,0}^{(1)}, \mathcal{A}_{v,0}^{(2)}, \ldots, \mathcal{A}_{v,0}^{(\ell)}) \in \mathcal{G}_v : \sum_{i=1}^{\ell} \epsilon_{v,i}(\mathcal{A}_{v,0}^{(i)} + \tau_{q,1}(\mathcal{A}_{v,0}^{(i)}) + \cdots + \tau_{q^{t-1},1}(\mathcal{A}_{v,0}^{(i)})) = 0\}$, $\Theta_v = (\epsilon_{v,1}, \epsilon_{v,2}, \ldots, \epsilon_{v,\ell}) \in \mathcal{M}_v$ and $\widehat{\mathcal{M}_v}$ is an $(\epsilon_v t-2)$-dimensional $\mathcal{F}_v$-subspace of $\mathcal{M}_v$ such that $\Theta_v \notin \widehat{\mathcal{M}_v}$, then we see that there exists an element $y_v \in \widehat{\mathcal{M}_v}^{\perp_0} \setminus \mathcal{M}_v$

such that $\mathcal{G}_v = \widehat{\mathcal{M}_v} \oplus \langle \Theta_v \rangle \oplus \langle y_v \rangle$. Further, we observe that any $\epsilon_v t/2$-dimensional self-orthogonal $\mathcal{F}_v$-subspace of $\mathcal{G}_v$ is contained in $\widehat{\mathcal{M}_v} \oplus \langle \Theta_v \rangle$, but not in $\widehat{\mathcal{M}_v}$. Now working as in the proof of Case **II(ii)** of Lemma 3.5, we get $\mathfrak{N}_v = \prod_{a=0}^{(\epsilon_v t/2)-2} \left(q^{\frac{\epsilon_v t - 2a - 2}{2}} + 1 \right)$.

III. Let $\delta = *$ and q be even. Working as in the proof of Case **III** of Lemma 3.5, we see that $(\mathcal{G}_v, [\cdot, \cdot]_{*\restriction \mathcal{G}_v \times \mathcal{G}_v})$ is a symplectic space of dimension $\epsilon_v t$ over $\mathcal{F}_v$, whose Witt index is given by $\epsilon_v t/2$. Now by Exercise 8.1 of Taylor (1992), we see that the number $\mathfrak{N}_v$ of distinct $\epsilon_v t/2$-dimensional self-orthogonal $\mathcal{F}_v$-subspaces of $\mathcal{G}_v$ is given by $\mathfrak{N}_v = \prod_{a=0}^{(\epsilon_v t/2)-1} \left(q^{\frac{\epsilon_v t - 2a}{2}} + 1 \right)$.

(b) Let $v \in \mathcal{J}_2$. Here $d_v = \deg g_v(x)$ is an even integer. By Lemma 3.4(b), we see that $(\mathcal{G}_v, [\cdot, \cdot]_{\delta \restriction \mathcal{G}_v \times \mathcal{G}_v})$ is a unitary space of dimension $\epsilon_v t$ over $\mathcal{F}_v$ when $\delta \in \{0, *\}$. Further, by Lemma 3.4(d), we note that $[\cdot, \cdot]_{\gamma \restriction \mathcal{G}_v \times \mathcal{G}_v}$ is a reflexive, non-degenerate and a skew-Hermitian $\tau_{1,-1}$-sesquilinear form on $\mathcal{G}_v$. Here working in a similar manner as in the proof of Lemma 3.5, we can associate an orthogonality preserving non-degenerate and Hermitian $\tau_{1,-1}$-sesquilinear form $[\cdot, \cdot]'_{\gamma \restriction \mathcal{G}_v \times \mathcal{G}_v}$ with the form $[\cdot, \cdot]_{\gamma \restriction \mathcal{G}_v \times \mathcal{G}_v}$. That is, $(\mathcal{G}_v, [\cdot, \cdot]'_{\gamma \restriction \mathcal{G}_v \times \mathcal{G}_v})$ is a unitary space of dimension $\epsilon_v t$ over $\mathcal{F}_v$.

In view of this, $\mathfrak{N}_v$ equals the number of distinct $\epsilon_v t/2$-dimensional self-orthogonal $\mathcal{F}_v$-subspaces of an $\epsilon_v t$-dimensional unitary space $\mathcal{G}_v$ over $\mathcal{F}_v$. From this and by Exercise 10.4 of Taylor (1992), we get $\mathfrak{N}_v = \prod_{b=0}^{(\epsilon_v t/2)-1} \left(q^{\frac{d_v(\epsilon_v t - 2b - 1)}{2}} + 1 \right)$.

This proves the lemma. □

In the following lemma, we determine the number $\mathfrak{N}_w$ for $e_1 + 1 \leq w \leq e_2$.

Lemma 3.9 *For $\delta \in \{0, *, \gamma\}$ and $e_1 + 1 \leq w \leq e_2$, we have $\mathfrak{N}_w = \sum_{d=0}^{\eta_w} \begin{bmatrix} \eta_w \\ d \end{bmatrix}_{q^{d_w}}$.*

Proof By (3.8), we see that $\mathfrak{N}_w$ equals the number of distinct $\mathcal{F}_w$-subspaces of $\mathcal{K}_w$ for all $e_1 + 1 \leq w \leq e_2$. As $\dim_{\mathcal{F}_w} \mathcal{K}_w = \eta_w$ and $\mathcal{F}_w \simeq \mathbb{F}_{q^{d_w}}$, by Lemma 2.1, we obtain $\mathfrak{N}_w = \sum_{d=0}^{\eta_w} \begin{bmatrix} \eta_w \\ d \end{bmatrix}_{q^{d_w}}$. □

Proof of Theorem 3.3. The first part of the theorem follows immediately from Lemma 3.7 and Theorem 3.1(a). Further, on substituting the value of $\mathfrak{N}_v$ for $1 \leq v \leq e_1$ from Lemma 3.8 and the value of $\mathfrak{N}_w$ for $e_1 + 1 \leq w \leq e_2$ from Lemma 3.9 in Eq. (3.8), we get the desired enumeration formulae for all δ-self-dual Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$. □

Let $\mathcal{C}$ be an Ω-MT additive code of length n over $\mathbb{F}_{q^t}$. Then the dimension of the code $\mathcal{C}$ is defined as the rational number k satisfying $|\mathcal{C}| = q^{kt}$. Note that the dimension k of the code $\mathcal{C}$ need not be an integer, but kt is always an integer. Further, the code $\mathcal{C}$ is said to have k_g number of generators if there exist k_g number of codewords of $\mathcal{C}$ such that every codeword of $\mathcal{C}$ is an $\mathbb{F}_q$-linear combination of these k_g codewords and k_g is the smallest positive integer with this property. Note that the number k_g need not be equal to the dimension k of the code $\mathcal{C}$, and that $k_g \leq nt$. The Ω-MT additive code

Table 1 Here $\ell = 2$, a is a primitive element of $\mathbb{F}_{q^t}$ and the element $\alpha_i(x) = \alpha_{i,0} + \alpha_{i,1}x + \alpha_{i,2}x^2 + \cdots + \alpha_{i,m_i-1}x^{m_i-1} \in \mathcal{V}_i$ is identified with the sequence $\alpha_i = \alpha_{i,0}\alpha_{i,1}\alpha_{i,2}\cdots\alpha_{i,m_i-1}$ for $1 < i <= 2$

q	t	(m_1, m_2)	Ω	$(\alpha_1(x), \alpha_2(x))$	$[n, k : k_g, d]$
3	2	(2,14)	(2,2)	$(a^5 1, 0a^5 1a^2 1a^2 1a^5 a^2 a^2 011a^5)$	$[16, 7 : 14, 8]$
5	2	(16,2)	(1,2)	$(a^{14} a^3 3aa^2 aa^{22} a^5 a^8 a^{19} 2a^{22} a^{14} a^9 a^{23} a^{16}, a^5 a^{22})$	$[18, 8 : 16, 8]$
7	2	(4,16)	(3,4)	$(a^5 a^2 a^{13} a^{45}, a^{10} a^{21} 6a^{18} a^{19} a^{39} a^{42} 2a^{10} a^{33} 2a^{14} a^{22} a^{29} a^{43} a^{31})$	$[20, 8 : 16, 10]$
5	2	(18,3)	(4,2)	$(a^{23} a^3 a^7 a^7 a^{23} a^{14} a^9 a^7 a^9 a^8 a^{16} a^{20} a^3 4a^{15} 30a^{11}, a^2 a^{20} a^9)$	$[21, 9 : 18, 10]$
3	2	(20,2)	(1,2)	$(aa0a^2 a^6 1a^5 a^6 a112a^6 0aa^5 a^3 a^5 aa^6, a^6 a^5)$	$[22, 10 : 20, 8]$
3	2	(22,2)	(2,2)	$(a^2 1a^6 a^2 a^7 aaa^3 a^7 a^5 a^2 a^7 10a^5 1a^6 2a^3 0a^7 1, 2a)$	$[24, 11 : 22, 9]$

of length n, dimension k, Hamming distance d and having k_g generators is referred to as an $[n, k : k_g, d]$-additive code over $\mathbb{F}_{q^t}$.

In Table 1, we obtain some good Ω-MT additive codes of length $n = m_1 + m_2$ dimension k, distance d and having k_g generators over $\mathbb{F}_{q^t}$ generated by the element $(\alpha_1(x), \alpha_2(x)) \in \mathcal{V}$ as an $\mathbb{F}_q[x]$-submodule of $\mathcal{V}$, by carrying out computations in the Magma Computational Algebra System.

4 Conclusion

In this paper, a new class of additive codes over finite fields, *viz.* multi-twisted (MT) additive codes is introduced and studied. By placing ordinary, Hermitian and $*$ trace bilinear forms, the dual codes of all MT additive codes over finite fields are studied. More precisely, necessary and sufficient conditions for a MT additive code to be self-dual or self-orthogonal are also derived. Besides this, a necessary and sufficient condition for the existence of a self-dual MT additive code is derived. Explicit enumeration formulae for all self-orthogonal and self-dual Ω-MT additive codes of length n over $\mathbb{F}_{q^t}$ are also obtained. These enumeration formulae are useful in classifying these special classes of MT additive codes over finite fields up to equivalence. Some MT additive codes over finite fields with good parameters are also obtained.

It would be interesting to enumerate all LCD MT additive codes over finite fields with respect to the aforementioned trace bilinear forms. Another interesting problem is to classify self-dual, self-orthogonal and LCD MT additive codes over finite fields up to equivalence.

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