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ORIGINAL PAPER

Cyclic order: a geometric analysis

Rolf Struve[1]

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Abstract

Concepts of order play an important role in many branches of mathematics. We start the article with an analysis of the notion of cyclic order in algebraic structures, which includes a characterization of cyclically ordered groups by cyclic cones and the introduction of the notion of a cyclically ordered field. We then study the role of cyclic order in the foundations of geometry. In Euclidean and in absolute geometry, order structures are introduced by linear orders (see Hilbert in Grundlagen der Geometrie, Teubner, Stuttgart, 1972; Coxeter in Introduction to geometry, Wiley, New York, 1961; Sperner in Beziehungen zwischen geometrischer und algebraischer Anordnung. Sitzungsberichte der Heidelberger Akademie der Wissenschaften, 1949; Bachmann in Aufbau der Geometrie aus dem Spiegelungsbegriff, Springer, Heidelberg, 1973; H Struve and R Struve in J Geom 105:419–447, 2014; R Struve in J Geom 106:551–570, 2015). This excludes elliptic geometry. We show that the notion of cyclic order (on pencils of lines) allows the introduction of order structures in a unified way (including the elliptic case) and corresponds on the algebraic side to a linear order of the associated coordinate field. In addition we prove that the three classical geometries (Euclidean, hyperbolic, and elliptic) over fields K of characteristic $\neq 2$ are orderable if and only if a separation relation on rows of collinear points and a separation relation on pencils of concurrent lines can be defined which are 'compatible'. The article closes with a geometric interpretation of cyclically ordered fields as Gaußian coordinate fields of Euclidean Hilbert planes.

Keywords Cyclic order · Cyclically ordered group · Cyclically ordered field · Cyclic cone · Ordered geometry · Absolute geometry · Ordered Bachmann groups

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✉ Rolf Struve
rolf.struve@arcor.de

1 Auf der Panne 36, 44805 Bochum, Germany

1 Introduction

"*A discussion of order ... has become essential to any understanding of the foundations of mathematics*" observed Russell in *The principles of Mathematics* (Russell 1903, p. 199), since they allow an explication of infinity, continuity and other fundamental concepts of Arithmetic, Analysis and Geometry.

According to Russell (1903, p. 200) there are two types of order: a *linear* order which can be described by a ternary betweenness relation, and a *cyclic* order, which can be described by a quaternary separation relation. A betweenness relation corresponds to a pair $(<, >)$ of dual binary relations (which are asymmetric, transitive and total) and a separation relation corresponds to a pair of dual ternary relations $(\lessdot, \lessdot^*)$, which are cyclic, asymmetric, transitive and total.[1]

From a geometric point of view, linear and cyclic order are fundamental ideas of our intuition of space (in Euclidean geometry the points on a line are linearly ordered and the lines through a point are cyclically ordered)[2] but not easily captured by rigorous mathematical concepts, or, as Coxeter puts it: "*The intuitive idea of the two opposite directions along a line, or a round circle, is so familiar that we are apt to overlook the niceties of its theoretical basis*" (see Coxeter 1947, p. 31).

In the foundations of geometry cyclic order is studied in projective and elliptic geometry (see Lenz 1965; Prieß-Crampe 1983) but can hardly be found outside of this context.[3] In Hilbert's *Grundlagen der Geometrie* of 1899 and in the *Foundations of Geometry* of Borsuk and Szmielew (1960), for example, the notion of cyclic order is not introduced at all. One reason may be the widely held view that a separation relation can be "*reduced*" to a betweenesss relation so that the former notion is "*somewhat superficial*" (in Russell's words).

The aim of this article is to analyze the notions of linear and cyclic order in algebraic and geometric structures in detail, to discuss their relationship and to show that the notion of cyclic order is in no way superficial, but plays a significant role in the foundations of geometry.

In Sect. 2, we introduce the terminology of the theory of order relations, which is used in this article, and recall some basic results. We prove that the theory of linear orders and the theory of cyclic orders on a set $\mathcal{M}$ are not definitionally equivalent. Instead, a cyclic order on a set $\mathcal{M}$ corresponds to a pair of linear orders on complementary subsets of $\mathcal{M}$ (see Theorem 2.8).

In Sect. 3, we study ordered groups. Ordered groups can be characterized in purely group-theoretic terms. A group G is *linearly orderable* if there exists a (linear) cone of G, that is, a subset of G which satisfies three simple group-theoretic properties (see Blyth 2005).

[1] The first axiomatization of the notion of cyclic order and a detailed study of the relationship between different types of order relations are provided by Huntington (1916, 1924, 1935).

[2] We refer for aspects of this kind to the phenomenological discussion of Freudenthal (Bachmann and Behnke 1974, Chap. 1).

[3] For this observation we refer to Pambuccian's review (2011) of the axiomatics of ordered geometry and to the compendium of Karzel and Kroll (1988) about the history of geometry since Hilbert.

For cyclically ordered groups[4] we focus on *abelian groups G which contain an involution*. The motivating examples are, in the algebraic context, the multiplicative group of a field of characteristic $\neq 2$ and, in the geometric context, the group of rotations around a point O of a Euclidean plane. We introduce the notion of a *cyclic cone* and prove that (a) G is cyclically orderable if and only if there exists a cyclic cone and (b) the associated first-order theories are definitionally equivalent (see Theorem 3.12).

As an algebraic structure with two operations we study in Sect. 4 ordered fields. In the literature fields are called 'ordered' if they are endowed with a *linear* order relation (which is compatible with the field operations). The field $\mathbb{C}$ of complex numbers has a rich order structure but is not an ordered field. To capture order structures of this kind we introduce the notion of a *cyclically ordered field* (see Definition 4.3). They have an intuitive geometric interpretation: a field C is cyclically orderable if *generalized polar coordinates* can be introduced, i.e., if $(C, \cdot)$ is the direct product of a cyclically orderable subgroup $(U, \cdot)$ and the positive cone $(R^+, \cdot)$ of a subfield R of C.[5] The degree of the field extension $[C : R]$ is called the 'degree' of the cyclic order. Every linearly orderable field is cyclically orderable, but the converse statement does not hold (see Theorem 4.6).

In Sect. 5, we study ordered geometric structures. In the literature the order structures of the three classical geometries (Euclidean, hyperbolic, and elliptic) are introduced in different ways, either by a linear order (in the Euclidean and hyperbolic case) or by a cyclic order (in the elliptic case; see Hilbert 1971; Karzel and Kroll 1988). We show that the notion of cyclic order allows to introduce these order structures in a unified way, namely by a separation relation on the rows of collinear points and a separation relation on the pencils of concurrent lines which are 'compatible' (see Theorem 5.3).

Correspondingly in plane absolute geometry (the common substratum of Euclidean, hyperbolic and elliptic geometry) there is no unified notion of an order structure.[6] We close this gap and show in Theorem 5.7 that (a) an order structure can be introduced by a separation relation on the pencils of lines which is compatible with the orthogonality relation and invariant under perspectivities and (b) this geometric order structure corresponds to a linear order of the associated coordinate field (these results extend the correspondence between geometric and algebraic order structures, which is well-known from affine and projective geometry).[7]

We close this article with a geometric interpretation of cyclically ordered fields: a field F with $-1 \in F^2$ is cyclically orderable (by a cyclic order of degree 2) if and only if F is the Gaußian coordinate field of a Euclidean Hilbert plane (see Theorem 5.8).

[4] The notion of a cyclically ordered group is a generalization of the notion of a linearly ordered group: every linearly ordered group can be cyclically ordered, but the converse does not hold (e.g., the complex numbers of absolute value one, equipped with the natural cyclic order, cannot be linearly ordered).

[5] For an example let C be the field of complex numbers, R the field of real numbers and U the multiplicative group of complex numbers of value 1.

[6] Ordered absolute geometry is restricted to the non-elliptic case (see Pejas 1961; Ewald 2013; Kunze 1981, H Struve and R Struve 2014, R Struve 2015).

[7] See (Karzel and Kroll 1988; Pambuccian 2011).

2 Order relations

In this section, we introduce the terminology of the theory of order relations, which we use, and recall some basic results. As a reference we refer to Novak (1982, 1984).

An *order relation* $<$ on a set $\mathcal{M}$ is a binary relation which is irreflexive and transitive. If any two distinct elements $x, y \in \mathcal{M}$ are comparable then $<$ is a *total linear order* and we call $(\mathcal{M}, <)$ a *linearly ordered* set. The order relation of a linearly ordered set $\mathcal{M}$ is *non-empty* if $\mathcal{M}$ contains at least two elements. The restriction of $<$ to a subset $\mathcal{N}$ of $\mathcal{M}$ is an order on $\mathcal{N}$. The dual relation $<^*$ of an order $<$ on $\mathcal{M}$ is also an order on $\mathcal{M}$.

If $(\mathcal{M}, <_{\mathcal{M}})$ and $(\mathcal{N}, <_{\mathcal{N}})$ are linearly ordered sets with $\mathcal{M} \cap \mathcal{N} = \varnothing$ then the set $\mathcal{M} \cup \mathcal{N}$ with the binary relation $<$ defined by $x < y$ if either $x \in \mathcal{M}$ and $y \in \mathcal{N}$ or $x <_{\mathcal{M}} y$ or $x <_{\mathcal{N}} y$ is a linearly ordered set, which is called the *linear sum*[8] of $\mathcal{M}$ and $\mathcal{N}$ (denoted by $\mathcal{M} \oplus \mathcal{N}$).

Definition 2.1 A subset $\mathcal{I}$ of a linearly ordered set $(\mathcal{M}, <)$ is an *interval* of $(\mathcal{M}, <)$ if $x < z < y$ with $x, y \in \mathcal{I}$ implies $z \in \mathcal{I}$ (for all $z \in \mathcal{M}$). A non-empty subset $\langle a, b\rangle_< := \{x \in \mathcal{M} : a < x < b\}$ is called an *open interval* of $(\mathcal{M}, <)$. The associated *closed* and *half-closed* intervals are denoted by $[a, b]_< := \{a, b\} \cup \langle a, b\rangle_<$ and $[a, b\rangle_< := \langle a, b\rangle_< \cup \{a\}$ and $\langle a, b]_< := \langle a, b\rangle_< \cup \{b\}$.

Theorem 2.2 *Let $(\mathcal{M}, <)$ be a linearly ordered set. A subset $\mathcal{I} \subseteq \mathcal{M}$ is an interval of $\mathcal{M}$ if and only if there exist subsets $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M}$ with $\mathcal{M} = \mathcal{N}_1 \oplus \mathcal{I} \oplus \mathcal{N}_2$.*

Proof See Novak (1984, Theorem 1.3). □

A *cyclic order relation* $\lessdot$ on a set $\mathcal{M}$ is a ternary relation which is

(1) *cyclic*: If $\lessdot(x, y, z)$ then $\lessdot(y, z, x)$.
(2) *asymmetric*: If $\lessdot(x, y, z)$ then not $\lessdot(z, y, x)$.
(3) *transitive*: If $\lessdot(x, y, z)$ and $\lessdot(y, u, z)$ then $\lessdot(x, y, u)$.

If for any distinct elements x, y, z either $\lessdot(x, y, z)$ or $\lessdot(z, y, x)$, then $\lessdot$ is a *total* cyclic order and $(\mathcal{M}, \lessdot)$ is called a *cyclically ordered* set. The order relation of a cyclically ordered set $\mathcal{M}$ is *non-empty* if $\mathcal{M}$ contains at least three elements.

A cyclic order relation is *irreflexive*, i.e., if $\lessdot(x, y, z)$ then $x \neq y \neq z \neq x$ (see Novak 1982, Lemma 1.4).

The restriction of $\lessdot$ to a subset $\mathcal{N}$ of $\mathcal{M}$ is a cyclic order on $\mathcal{N}$. The *dual relation* $\lessdot^*$ of a cyclic order relation $\lessdot$, which is defined by $\lessdot^*(x, y, z)$ *if and only if* $\lessdot(z, y, x)$, is also a cyclic order on $\mathcal{M}$ (see Novak 1982, Remark 1.6).

We use the convention that $\lessdot(a, b, c, d)$ is an abbreviation for the conjunction $[\lessdot(a, b, c)$ and $\lessdot(b, c, d)]$.

Definition 2.3 Let $(\mathcal{M}, \lessdot)$ be a cyclically ordered set. A non-empty subset $\langle a, b\rangle_\lessdot := \{x \in \mathcal{M} : \lessdot(a, x, b)\}$ is called an *open cyclic interval* of $(\mathcal{M}, \lessdot)$. The associated *closed* and *half-closed* intervals are denoted by $[a, b]_\lessdot := \langle a, b\rangle_\lessdot \cup \{a, b\}$ and $[a, b\rangle_\lessdot := \langle a, b\rangle_\lessdot \cup \{a\}$ and $\langle a, b]_\lessdot := \langle a, b\rangle_\lessdot \cup \{b\}$.

[8] or *ordinal sum* (see Novak 1984)

There are close relationships between linear and cyclic orders on a set $\mathcal{M}$.
An order relation $<$ on $\mathcal{M}$ induces a ternary relation $\lessdot$ on $\mathcal{M}$ by

$$\lessdot(x, y, z) \quad :\Leftrightarrow \quad [x < y < z] \text{ or } [y < z < x] \text{ or } [z < x < y] \tag{2.1}$$

Theorem 2.4 *Let $<$ be an order relation on a set $\mathcal{M}$ and $\lessdot$ the associated ternary relation (2.1). If $(\mathcal{M}, <)$ is linearly ordered then $(\mathcal{M}, \lessdot)$ is cyclically ordered. The relation $\lessdot$ is the unique cyclic order on $\mathcal{M}$ such that $x < y < z$ implies $\lessdot(x, y, z)$ (the* natural cyclic order *which is associated to $<$).*

Proof See Novak (1982, Theorem 3.5, Lemma 3.7; 1984, Theorem 2.4). □

Conversely, a cyclic order relation $\lessdot$ on a set $\mathcal{M}$ induces for every element $e \in \mathcal{M}$ a binary relation $<_e$ on $\mathcal{M}$ by

$$x <_e y \quad :\Leftrightarrow \quad [e = x \neq y] \text{ or } [\lessdot(e, x, y)] \tag{2.2}$$

Theorem 2.5 *Let $\lessdot$ be a non-empty cyclic order relation on a set $\mathcal{M}$ and $e \in \mathcal{M}$ and $<_e$ the associated binary relation (2.2). If $(\mathcal{M}, \lessdot)$ is cyclically ordered then $(\mathcal{M}, <_e)$ is linearly ordered with the least element e.*

Proof See Novak (1982, Theorem 3.1 and Lemma 3.4). □

The theory of linear orders on a set $\mathcal{M}$ and the theory of cyclic orders on $\mathcal{M}$ are closely related by the mappings (2.1) and (2.2) which induce functors between the associated categories of models.

Theorem 2.6 *Let $(\mathcal{M}, \lessdot)$ be a cyclically ordered set with a non-empty order relation and $e \in \mathcal{M}$. If $(\mathcal{M}, <_e)$ is the linearly ordered set, which is associated to e by Theorem 2.5, and $(\mathcal{M}, \lessdot_e)$ the cyclically ordered set, which is associated to $(\mathcal{M}, <_e)$ by Theorem 2.4, then $(\mathcal{M}, \lessdot) = (\mathcal{M}, \lessdot_e)$.*

Proof See Novak (1982, Lemma 3.11). □

However, the theories of linear and of cyclic order on a set $\mathcal{M}$ are *not definitionally equivalent* (according to Padoa's method since every cyclically ordered set allows, by Theorem 2.6, the definition of linear orders with distinct least elements).

Definition 2.7 Let $(\mathcal{M}, <_{\mathcal{M}})$ and $(\mathcal{N}, <_{\mathcal{N}})$ be disjoint linearly ordered sets and $\lessdot$ the natural cyclic order (2.1), which is associated to the linear sum of $\mathcal{M}$ and $\mathcal{N}$. Then $(\mathcal{M} \cup \mathcal{N}, \lessdot)$ is a cyclically ordered set which we call the *cyclic sum* of $(\mathcal{M}, <_{\mathcal{M}})$ and $(\mathcal{N}, <_{\mathcal{N}})$, in symbols $\mathcal{M} \odot \mathcal{N}$.

Obviously, the cyclic sum of two linearly ordered sets $\mathcal{M}$ and $\mathcal{N}$ is a cyclically ordered set and $\mathcal{M} \odot \mathcal{N} = \mathcal{N} \odot \mathcal{M}$. According to the next theorem every cyclically ordered set can be represented as the cyclic sum of two linearly ordered sets.

Theorem 2.8 *Let $(\mathcal{M}, \lessdot)$ be a cyclically ordered set and $<_a$ and $<_b$ the linear orders on $\mathcal{M}$, which are induced by two distinct elements $a, b \in \mathcal{M}$ according to (2.2). Then $[a, b\rangle_{<_a}$ and $[b, a\rangle_{<_b}$ are linearly ordered sets with the cyclic sum $(\mathcal{M}, \lessdot)$.*

Proof The theorem is an immediate consequence of Theorem 2.5 and Definition 2.7. □

3 Ordered groups

In this section, we introduce the terminology of ordered groups and summarize some basic results (see Fuchs 1963; Blyth 2005; Świerczkowski 1959; Jakubíc and Pringerová 1988). Please note that in this article ordered groups will be defined as *totally* ordered groups.

$(G, \cdot, <)$ is a *linearly ordered group* if $(G, \cdot)$ is a group and $(G, <)$ a linearly ordered set with a compatible linear order, i.e., if $\alpha, \beta, \gamma \in G$ and $\alpha < \beta$ then $\gamma\alpha < \gamma\beta$ and $\alpha\gamma < \beta\gamma$.

If $(G, \cdot, <)$ is a linearly ordered group then $G^+ = \{\alpha \in G : 1 < \alpha\}$ is called the *positive cone* and $G^- := (G^+)^{-1} = \{\alpha \in G : \alpha^{-1} \in G^+\} = \{\alpha \in G : \alpha < 1\}$ the *negative cone*. According to Fuchs (1963) and Blyth (2005) the following holds:

(1) $G = G^+ \cup G^- \cup \{1\}$
(2) $G^+ \cap G^- = \varnothing$
(3) If $\alpha, \beta \in G^+$ then $\alpha\beta \in G^+$.
(4) If $\alpha \in G^+$ and $\beta \in G$ then $\beta^{-1}\alpha\beta \in G^+$.

We call a subset G^+ of a group G a *linear cone* if the conditions (1)–(4) are satisfied. If G^+ is a linear cone of G then G^- is also a linear cone of G which we call the associated *inverse cone*.

According to (2), (3) and (4), a linear cone is an invariant subset of G which contains with two elements α, β their product $\alpha\beta$ but no elements δ, ε with $\delta\varepsilon = 1$. If G is an abelian group then G^+ is a cone if (1), (2) and (3) are satisfied. Condition (2) implies that no element of a linearly ordered group is an involution.

Ordered groups can be characterized in purely group-theoretic terms: a group $(G, \cdot)$ is orderable if and only if there exists a linear cone $G^+ \subseteq G$. In this case $\alpha < \beta$ is defined by $\alpha^{-1}\beta \in G^+$ (for all $\alpha, \beta \in G$).

Now let $(G, \cdot)$ be a group and $\lessdot$ a cyclic order on G. We call $\lessdot$ *compatible* with $(G, \cdot)$, if $\alpha, \beta, \gamma, \delta \in G$ and $\lessdot(\alpha, \beta, \gamma)$ imply $\lessdot(\delta\alpha, \delta\beta, \delta\gamma)$ and $\lessdot(\alpha\delta, \beta\delta, \gamma\delta)$.

Definition 3.1 $(G, \cdot, \lessdot)$ is a *cyclically ordered group* if $(G, \cdot)$ is a group and $\lessdot$ a compatible total cyclic order on G.

Remark 3.2 The cyclic groups of order 1 and 2 are cyclically ordered groups and the only ones with an empty order relation.

The notion of a cyclically ordered group is a generalization of the notion of a linearly ordered group according to the next theorem.

Theorem 3.3 *Let $(G, \cdot, <)$ be a linearly ordered group and $\lessdot$ the cyclic order which is induced on G by (2.1). Then $(G, \cdot, \lessdot)$ is a cyclically ordered group and if $<_{\lessdot}$ is the order relation on G which is given by*

$$1 <_{\lessdot} \alpha \quad :\Leftrightarrow \quad \lessdot(\alpha^{-1}, 1, \alpha) \tag{3.1}$$

then $(G, \cdot, <) = (G, \cdot, <_{\lessdot})$.

Proof See (Jakubíc and Pringerová 1988, Lemma 3.1). □

Every linearly ordered group can be cyclically ordered. The converse does not hold: the multiplicative group $\mathbb{C}^1$ of complex numbers of absolute value one, equipped with the natural cyclic order, is a cyclically ordered group which *cannot* be linearly ordered. More generally, every cyclically ordered group with an involutory element cannot be linearly ordered (since linearly ordered groups have no involution).

However, to every cyclically ordered group $(G, \cdot, \lessdot)$ there exists a linearly ordered group H, such that $(G, \cdot, \lessdot)$ is isomorphic to some subgroup of the direct product $\mathbb{C}^1 \times H$, equipped with the natural induced (lexicographic) cyclic order (a representation theorem known as Świerszkowski's Theorem; see Świerczkowski (1959) or Jakubíc and Pringerová (1988, Lemma 2.6)).

Theorem 3.4 *Let $(G, \cdot, \lessdot)$ be a cyclically ordered group with identity 1. Then the following holds for all $\alpha, \beta \in G$ with $\alpha \neq \beta$:*

(1) *If $\alpha^2 = \beta^2 = 1$ then $\alpha = 1$ or $\beta = 1$ (there is at most one involution)*
(2) *If $\lessdot(1, \alpha, \beta)$ then $\lessdot(1, \beta^{-1}, \alpha^{-1})$.*

Proof For a proof of (1) let $\alpha, \beta \in G$ with $\alpha^2 = \beta^2 = 1$ and $\lessdot(1, \alpha, \beta)$. Then $\lessdot(1, \alpha, \beta)$ implies $\lessdot(\alpha, 1, \alpha\beta)$ and $\lessdot(\beta, \alpha\beta, 1)$. According to the transitivity law $\lessdot(\alpha, 1, \alpha\beta)$ and $\lessdot(1, \beta, \alpha\beta)$ imply $\lessdot(\alpha, 1, \beta)$. which is a contradiction to the assumption $\lessdot(1, \alpha, \beta)$.

(2) holds according to Jakubíc and Pringerová (1988, Lemma 2.6). □

According to Theorem 3.4 a cyclically ordered group has either no involution or exactly one involution. We now focus our investigations on the *geometric context* and study abelian groups which contain an involution. The motivating examples are the groups of rotations around a point O of the classical Euclidean and non-Euclidean geometries, which contain the reflection in O as an involutory element.

Definition 3.5 Let $(G, \cdot)$ be an abelian group with an involution O. If $G^+ \subseteq G$ and $G^- := \{\alpha^{-1} : \alpha \in G^+\}$ and $\{O\}G^+ := \{O\alpha \in G : \alpha \in G^+\}$) then G^+ is called a *cyclic cone* if the following properties hold:

(1) $G = G^+ \cup \{O\}G^+ \cup \{1, O\}$
(2) $G^+ \cap G^- = \varnothing$
(3) If $\alpha, \beta, \gamma, \alpha^{-1}\beta, \beta^{-1}\gamma \in G^+$ then $\alpha^{-1}\gamma \in G^+$.

Remark 3.6 The conditions (1) and (2) of the definition of a cyclic cone correspond to the conditions (1) and (2) of the definition of a linear cone, especially since $\{O\}G^+ = G^-$ (according to Theorem 3.7).Condition (3) is well-known from the theory of linear cones: if G^+ is a linear cone of an abelian group G and if the associated linear order relation is defined, as usual, by $\alpha < \beta$ if $\alpha^{-1}\beta \in G^+$ (for all $\alpha, \beta \in G$) then condition (3) states the transitivity of the $<$-relation: if $\alpha, \beta, \gamma \in G^+$ and $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$.

The following theorem lists basic properties of a cyclic cone which are consequences of the conditions (1) and (2) of Definition 3.5.

Theorem 3.7 *Let $(G, \cdot)$ be an abelian group with an involution O and $G^+ \subseteq G$ a cyclic cone and $G^- := \{\alpha^{-1} : \alpha \in G^+\}$. Then the following hold:*

(1) $G = G^+ \cup G^- \cup \{1, O\}$ *(disjoint union)*
(2) $G^- = \{O\}G^+$

Proof We prove the theorem in several steps.
(a) $G^+ \cap G^- = \varnothing$
This holds by Definition 3.5, (2).
(b) $1, O \notin G^+$ and $1, O \notin \{O\}G^+$
It is $1, O \notin G^+$, according to Definition 3.5, (2), and this implies $1, O \notin \{O\}G^+$.
(c) $G^- \subseteq \{O\}G^+$
This is a consequence of Definition 3.5, (2) and (1).
(d) $G^- \cap \{O\}G^+ = \varnothing$
For a proof of (d) suppose $\alpha, \beta \in G^+$ and $\alpha = O\beta$. Then $\alpha^{-1} = O\beta^{-1}$ and according to (c) it is $\beta^{-1}, O\beta^{-1} \in \{O\}G^+$. Hence $\beta, \beta^{-1} \in G^+$ which is contradiction to 3.5, (2).
(e) $\{O\}G^+ \subseteq G^-$
This holds according to (d) and Definition 3.5, (1) and (2).
Statement (2) of the theorem is a consequence of (c) and (e). Statement (1) of the theorem is implied by (2) and Definition 3.5, (1) and (a) and (b). □

In analogy to linear cones the following theorem holds for cyclic cones.

Theorem 3.8 *Let G be an abelian group with an involution O. If G^+ is a cyclic cone of G then $G^- := (G^+)^{-1}$ is also a cyclic cone.*

Proof We show that the conditions (1), (2) and (3) of Definition 3.5 of a cyclic cone hold. According to our assumptions G^+ satisfies (1), (2) and (3) and by Theorem 3.7 it is $G^- = \{O\}G^+$.

Condition (1) holds since $G = G^+ \cup \{O\}G^+ \cup \{1, O\}$ implies $G = \{O\}G^+ \cup \{O\} \cdot \{O\}G^+ \cup \{1, O\}$. Condition (2) holds since $G^+ \cap G^- = \varnothing$ implies $G^- \cap G^+ = \varnothing$. For a proof of condition (3) let $\alpha, \beta, \gamma, \alpha^{-1}\beta, \beta^{-1}\gamma \in G^-$. Then $\alpha^{-1}, \beta^{-1}, \gamma^{-1}, \alpha\beta^{-1}, \beta\gamma^{-1} \in G^+$ and since (3) holds with respect to G^+ it is $\alpha\gamma^{-1} \in G^+$ and $\alpha^{-1}\gamma \in G^-$. □

Theorem 3.9 *Let $(G, \cdot)$ be an abelian group with an involution O and $G^+ \subseteq G$ a cyclic cone. If for $\alpha, \beta \in G$ the binary relation $\alpha < \beta$ is defined by $\alpha^{-1}\beta \in G^+$ and if $[1, O) := \{\alpha \in G : 1 < \alpha$ and $\alpha < O\} \cup \{1\}$ and $[O, 1) := \{\alpha \in G : O < \alpha$ and $\alpha < 1\} \cup \{O\}$ then the following holds:*

(a) $[1, O) = G^+ \cup \{1\}$
(b) $[O, 1) = \{O\}G^+ \cup \{O\}$
(c) $([1, O), <)$ *and* $([O, 1), <)$ *are linearly ordered sets.*
(d) $[1, O) \cup [O, 1)$ *is a (set-theoretic) partition of* G.
(e) $(G, \cdot, \lessdot)$ *is a cyclically ordered group if* $\lessdot$ *denotes the order relation of the cyclic sum of* $([1, O), <)$ *and* $([O, 1), <)$.
(f) $\alpha \in G^+$ *if and only if* $\lessdot(1, \alpha, O)$.
(g) $\alpha \in G^-$ *if and only if* $\lessdot(\mathrm{O}, \alpha, 1)$.

Proof (a) If $\alpha \neq 1$ then $\alpha \in [1, O)$ if $1 < \alpha$ and $\alpha < O$. But $1 < \alpha$ is equivalent with $\alpha \in G^+$ and $\alpha < O$ is equivalent with $\alpha^{-1}O \in G^+$ and $\alpha^{-1} \in \{O\}G^+$ and $\alpha^{-1} \in G^-$ (according to Theorem 3.7, (2)) and $\alpha \in G^+$.

(b) If $\alpha \neq O$ then $\alpha \in [O, 1)$ if $O < \alpha$ and $\alpha < 1$. But $O < \alpha$ is equivalent with $O\alpha \in G^+$ and $\alpha \in \{O\}G^+$ and the statement $\alpha < 1$ is equivalent with $\alpha^{-1} \in G^+$ and $\alpha \in G^-$ and $\alpha \in \{O\}G^+$ (according to Theorem 3.7, (2)).

(c) The relation $<$ on $[1, O)$ is irreflexive according to the definition of $<$, transitive according to Definition 3.5, (3) and total by Definition 3.5, (1).

(d) is an immediate consequence of (a) and (b) and Definition 3.5, (1).

(e) $(G, \cdot, \lessdot)$ is a cyclically ordered set according to Theorem 2.8. The cyclic order is compatible with the group operation since the relation $<$ is compatible with the group operation: if $\alpha < \beta$ then $\alpha^{-1}\beta \in G^+$ and $(\alpha\delta)^{-1} \cdot \beta\delta = \alpha^{-1}\beta \in G^+$ and hence $\alpha\delta < \beta\delta$ for all $\delta \in G$. Since G is an abelian group it is $\delta\alpha < \delta\beta$.

(f), (g) are immediate consequences of the definitions of $[1, O)$ and $[O, 1)$. □

Theorem 3.9 induces a mapping which associates to every cyclic cone G^+ of G a cyclic order on G, in other words, a functor F from the class of models of $(G, \cdot, G^+)$ into the class of models of $(G, \cdot, \lessdot)$. Conversely, there exists a functor G from the class of models of $(G, \cdot, \lessdot)$ into the class of models of $(G, \cdot, G^+)$, according to the next theorem.

Theorem 3.10 *Let* $(G, \cdot, \lessdot)$ *be a cyclically ordered abelian group with an involution* O. *Then* $G^+ = \{\alpha \in G : \lessdot(1, \alpha, O)\}$ *is a cyclic cone which, in turn, induces a cyclic order* $\lessdot^*$ *on G (see Theorem* 3.9*, (e)) with* $(G, \cdot, \lessdot) = (G, \cdot, \lessdot^*)$.

Proof Let $(G, \cdot, \lessdot)$ be a cyclically ordered abelian group with an involution O and $G^+ = \{\alpha \in G : \lessdot(1, \alpha, O)\}$ and $G^- := (G^+)^{-1}$. It is $G^- = \{\beta : \beta^{-1} \in G^+\} = \{\beta : \lessdot(1, \beta^{-1}, O\} = \{\beta : \lessdot(1, O, \beta\} = \{\beta : \lessdot(O, \beta, 1)\}$ according to Theorem 3.4, (2).

For a proof, that G^+ is a cyclic cone, we show in a first step $G^- = \{O\}G^+$. If $\alpha \in G^-$ then $\lessdot(1, O, \alpha)$ and (since $\lessdot$ is a compatible order relation) $\lessdot(O, 1, O\alpha)$ and (since $\lessdot$ is cyclic) $\lessdot(1, O\alpha, O)$. Hence $O\alpha \in G^+$ and since $\alpha = O \cdot O\alpha$ it is $G^- \subseteq \{O\}G^+$.

If, conversely, $\alpha \in G^+$ then $\lessdot(1, \alpha, O)$ and $\lessdot(1, O, \alpha^{-1})$ (according to Theorem 3.4, (2)) and, since $\lessdot$ is a compatible order relation, $\lessdot(O, 1, O\alpha^{-1})$. Since $\lessdot$ is cyclic this implies $\lessdot(1, O\alpha^{-1}, O)$ and $O\alpha^{-1} \in G^+$ and $O\alpha \in G^-$ (since $(O\alpha)^{-1} = O\alpha^{-1} \in G^+$). Hence $\{O\}G^+ \subseteq G^-$ and $G^- = \{O\}G^+$.

It remains to show that the defining properties (1), (2) and (3) of a cyclic cone are satisfied (see Definition 3.5). It is $\{1, O\} \nsubseteq G^+ \cup G^-$. Hence $G^+ \cap G^- = \varnothing$ and condition (2) holds. Since $\lessdot$ is a total cyclic order on G it is $G = G^+ \cup G^- \cup \{1, O\}$ and condition (1) holds.

For a proof of condition (3) we first show that $\lessdot(1, \alpha, \beta)$ if and only if $\lessdot(1, \alpha^{-1}\beta, O)$. If $\lessdot(1, \alpha, \beta)$ then $\lessdot(1, \beta^{-1}, \alpha^{-1})$ (according to Theorem 3.4, (2)) and $\lessdot(\beta, 1, \alpha^{-1}\beta)$ (since $\lessdot$ is a compatible order relation) and hence $\lessdot(1, \alpha^{-1}\beta, \beta)$. Since the relation $\lessdot$ is transitive, $\lessdot(1, \alpha^{-1}\beta, \beta)$ and $\lessdot(1, \beta, O)$ imply $\lessdot(1, \alpha^{-1}\beta, O)$.

If, conversely, $\lessdot(1, \alpha^{-1}\beta, O)$ then $\lessdot(\alpha, \beta, O\alpha)$ (since $\lessdot$ is a compatible order relation). Now, $\alpha \in G^+$ implies $\lessdot(1, \alpha, 0)$ and $\lessdot(O, O\alpha, 1)$ and $\lessdot(1, O, O\alpha)$. Since, in addition, $\lessdot(1, \alpha, O)$ it is $\lessdot(1, \alpha, O\alpha)$ (by the transitivity of $\lessdot$). $\lessdot(1, \alpha, O\alpha)$ and $\lessdot(\alpha, \beta, O\alpha)$ imply $\lessdot(1, \alpha, \beta)$ (by the law of transitivity).

Let $\lessdot^*$ be the cyclic order on G which is associated to G^+ according to Theorem 3.9. By Theorem 3.9, (f) and (g) it is $(G, \cdot, \lessdot) = (G, \cdot, \lessdot^*)$. □

Theorem 3.11 *Let $(G, \cdot)$ be an abelian group with an involution O and $G^+ \subseteq G$ a cyclic cone. G^+ induces a compatible total cyclic order $\lessdot$ on G (see Theorem 3.9). If $G^+_\lessdot$ denotes the associated cyclic cone (see Theorem 3.10) then $G^+ = G^+_\lessdot$.*

Proof Let $(G, \cdot)$ be an abelian group with an involution O and $G^+ \subseteq G$ a cyclic cone, $\lessdot$ the associated cyclic order on G (see Theorem 3.9) and $G^+_\lessdot = \{\alpha \in G : \lessdot(1, \alpha, O)\}$ the cone which, in turn, is associated to $\lessdot$ (see Theorem 3.10). Then $\alpha \in G^+$ if and only if $\lessdot(1, \alpha, O)$ (by Theorem 3.9, (f)) and hence $G^+ = G^+_\lessdot$ (by Theorem 3.10). □

The results of the last two theorems can be summarized in terms of the functors F and G, which were introduced above. If $\mathcal{M}$ is a model of the theory of cyclically ordered abelian groups with an involution O and if $\mathcal{N}$ is a model of the theory of cyclic cones of abelian groups with an involution O then $\mathcal{M}^{\mathrm{GF}} = \mathcal{M}$ and $\mathcal{N}^{\mathrm{FG}} = \mathcal{N}$. Hence the following theorem holds.

Theorem 3.12 *The first-order theory of cyclically ordered abelian groups (which contain an involution) and the first-order theory of cyclic cones of abelian groups (which contain an involution) are definitionally equivalent.*[9]

4 Cyclically ordered fields

An *ordered field* is a field $(K, +, \cdot)$ with a compatible linear order, i.e., if $x, y, z \in K$ and $x < y$ then $x + z < y + z$ and if $0 < x$ and $0 < y$ then $0 < xy$. Every ordered field is of characteristic 0.

If K is an ordered field then $(K, +, <)$ is a linearly ordered abelian group with the positive cone $P = \{x \in K : 0 < x\}$ and $(P, \cdot)$ is a linearly ordered subgroup of index 2 of the multiplicative group $\dot{K}$ of K.[10] A subset P of a field K with these properties is called a *positive cone of* K (see Karzel and Kroll 1988). A field K is orderable if and only if there exists a positive cone of K. An alternative characterization is given in the following theorem.

[9] For the terminology we refer to H Struve and R Struve (2019a, 2019b).

[10] $(\dot{K}, \cdot)$ is not linearly orderable, since $-1 \in \dot{K}$ is an involution, but cyclically orderable with the cyclic cone $\{x \in K : -1 < x < 0\} \cup \{y \in K : 1 < y\}$.

Theorem 4.1 *A commutative field* $(K, +, \cdot)$ *of characteristic* $\neq 2$ *is orderable if and only if there exists a homomorphism* φ *from* $(\dot{K}, \cdot)$ *onto the cyclic group* $(\{1, -1\}, \cdot)$ *of order* 2 *whose kernel is a linear cone of* $(K, +)$.

Proof Let $(K, +, \cdot)$ be a commutative field of characteristic $\neq 2$ and P the kernel of a homomorphism φ from $(\dot{K}, \cdot)$ onto a cyclic group of order 2. If P is a linear cone of $(K, +)$ then $(K, +)$ is a linearly orderable group. Since the kernel of φ is a group it is $P \cdot P \subseteq P$ and hence K an orderable field. The inverse statement is an immediate consequence of the definition of an ordered field. □

The field $\mathbb{C}$ of complex numbers cannot be equipped with an order relation. However, $\mathbb{C}$ can be endowed with an order structure in the following way. The multiplicative group $(\dot{\mathbb{C}}, \cdot)$ of complex numbers is the direct product of the group $(\mathbb{C}^1, \cdot)$ of complex numbers of absolute value 1 and the group $(\mathbb{R}^+, \cdot)$ of positive real numbers (in other words: complex numbers allow the introduction of *polar coordinates*). Since $(\mathbb{C}^1, \cdot)$ is a cyclically ordered group (if endowed with the natural cyclic order relation) and $(\mathbb{R}^+, \cdot)$ a linearly ordered group (if endowed with the natural linear order of real numbers) their direct product $(\dot{\mathbb{C}}, \cdot)$ is a cyclically ordered group (according to the representation theorem of Świerczkowski 1959). This is the natural order structure on the *multiplicative* group of complex numbers.

Since $\mathbb{C}$ is a vector space of dimension 2 over $\mathbb{R}$, the linear order on $(\mathbb{R}, +)$ induces a linear order on $(\mathbb{C}, +)$[11] which, in turn, induces a cyclic order on $(\mathbb{C}, +)$ (see Theorem 3.3). This is the natural order structure on the *additive* group of complex numbers.

The order structures of $(\dot{\mathbb{C}}, \cdot)$ and $(\mathbb{C}, +)$ are 'compatible' in the following sense: the kernel $\mathbb{R}^+$ of the homomorphism from $(\dot{\mathbb{C}}, \cdot)$ onto $(\mathbb{C}^1, \cdot)$ is a linear cone of the additive group $(\mathbb{R}, +)$ of the subfield $\mathbb{R}$ of $\mathbb{C}$.

The compatability condition can be expressed in a more general form if the notion of a *partially ordered group* is introduced. $(G, \cdot, <)$ is a partially linearly ordered group if $<$ is a partial linear order on G, which is compatible with the group operation. This is equivalent with the existence of a non-empty invariant subset P of G with $P \cdot P \subseteq P$ and $P \cap P^{-1} = \varnothing$ (with $P^{-1} := \{\alpha^{-1} : \alpha \in P\}$). The subset P is called a *cone of the partial order* $<$ of G.

If G is an abelian group then every partial order of G is a total order of a subgroup of G, as the next theorem shows.

Theorem 4.2 *Let* P *be a cone of a partial linear order* $<$ *of an abelian group* $(G, \cdot)$. *Then* $(P \cup P^{-1} \cup \{1\}, \cdot, <)$ *is a totally ordered subgroup of* G.

Proof If P is a cone of a partial linear order $<$ of G then $P \cdot P \subseteq P$ and $P \cap P^{-1} = \varnothing$. Since G is an abelian group this implies that $(P \cup P^{-1} \cup \{1\}, \cdot)$ is a group. Obviously P is a linear cone of this group and $<$ a total order. □

The field of complex numbers is the motivating example for our notion of a cyclically ordered field.

[11] The lexicographical order with the linear cone $\{x + iy \in \mathbb{C} : y > 0 \text{ or } (y = 0 \text{ and } x > 0)\}$, or, in polar coordinates, the set of complex numbers with an angular coordinate α satisfying $0 \leq \alpha < \pi$.

Definition 4.3 A commutative field $(C, +, \cdot)$ of characteristic $\neq 2$ is *cyclically orderable* if there exists a homomorphism φ from $(\dot{C}, \cdot)$ onto a cyclically orderable subgroup of $(\dot{C}, \cdot)$ whose kernel is the cone of a partial linear order of $(C, +)$.

Remark 4.4 A cyclically ordered field is a pair (C, φ) of a field C and a homomorphism φ which satisfies the conditions of Definition 4.3. According to Sperner (1949) a *half-ordered field* is a pair (C, φ) with a field C and a homomorphism φ from $(\dot{C}, \cdot)$ onto the cyclic subgroup $(\{1, -1\}, \cdot)$ of $\dot{C}$ of order 2. However, the notion of a cyclically ordered field is not a generalization of half-ordered fields, since every finite field of characteristic $\neq 2$ can be endowed with a half-order, but not with a cyclic order (since cyclically ordered fields are of characteristic 0; see Theorem 4.5).

For the sake of a precise terminology we now call 'ordered fields' (which satisfy the definition at the beginning of this section) *linearly ordered fields*.

Theorem 4.5 *Let $(C, +, \cdot)$ be a cyclically orderable field and P the kernel of the associated homomorphism from $(\dot{C}, \cdot)$ onto a cyclically orderable subgroup of $(\dot{C}, \cdot)$. Then P is the positive cone of a linearly orderable field $(R, +, \cdot)$ and $(\dot{C}, \cdot)$ is a cyclically orderable group. The fields R and C are of characteristic* 0.

Proof Let $(C, +, \cdot)$ be a field and φ a homomorphism from $(\dot{C}, \cdot)$ onto a cyclically orderable subgroup C^* of $(\dot{C}, \cdot)$ whose kernel P is a cone of a partial linear order of $(C, +)$. Theorem 4.2 implies that P is the cone of a totally ordered subgroup $(R, +)$ of $(C, +)$ with $R := P \cup P^{-1} \cup \{1\}$. Since P is a subgroup of $(\dot{C}, \cdot)$, it is $P \cdot P \subseteq P$ and hence $R \cdot R \subseteq R$. Thus $(R, +, \cdot)$ is a linearly orderable field with positive cone P. This implies that $(P, \cdot)$ is a linearly orderable subgroup of index 2 of $(R, \cdot)$ and that R and C are fields of characteristic 0.

The mapping φ is an epimorphism from the abelian group $\dot{C}$ onto the subgroup C^* of $\dot{C}$. Hence $(\dot{C}, \cdot)$ is the direct product of the kernel of φ and the image of φ, that is, of $(P, \cdot)$ and $(C^*, \cdot)$. Since $(C^*, \cdot)$ is cyclically and $(P, \cdot)$ linearly orderable, their direct product $(\dot{C}, \cdot)$ is a cyclically orderable group (according to the representation theorem of Świerczkowski (1959)). □

According to Theorem 4.5, to every cyclically ordered field C there is associated a subfield R which is linearly orderable. The degree $[C : R]$ of the field extension is the dimension of the vector space C over R and the index of the abelian subgroup $(R, +)$ of $(C, +)$ (cp. Theorem 4.2). We call this value the *degree of a cyclic order* on a field C.

The field of complex numbers can be endowed with a cyclic order of degree 2 with the field of real numbers as associated subfield. This situation will be of particular interest in plane geometry (see Sect. 5.3). In the general algebraic context we make no assumptions about the degree of a cyclic order.

Theorem 4.6 *Every linearly ordered field is cyclically orderable. The converse statement does not hold.*

Proof Let $(K, +, \cdot)$ be a linearly ordered field. According to Theorem 4.1 there exists a homomorphism φ from $(\dot{K}, \cdot)$ onto the cyclic group $(\{1, -1\}, \cdot)$ of order 2 whose kernel is a linear cone of $(K, +)$. Since the cyclic group of order 2 is cyclically orderable (see Remark 3.2) this shows that K is a cyclically orderable field (according to Definition 4.3). The field of complex numbers is cyclically but not linearly orderable. □

We close this section with an intuitive geometric interpretations of cyclically ordered fields: a field C is cyclically orderable if *generalized polar coordinates* can be introduced, i.e., if $(C, \cdot)$ is the direct product of a cyclically orderable subgroup $(U, \cdot)$ and the positive cone $(R^+, \cdot)$ of a subfield R of C.[12]

5 Ordered geometric structures

The study of the concept of order differs in various aspects between algebra and geometry. We note that geometric structures are provided with an order structure by a *betweenness relation* (which corresponds to a pair of linear order relations $(\leq, \geq)$) or by a *separation relation* (which corresponds to a pair of cyclic order relations $(\lessdot, \lessdot^*)$; see Pambuccian 2011). Thus from a geometric point of view it seems preferable to define a cyclic order structure on a group G by a separation relation, but this is neither definitionally equivalent nor bi-interpretable with the algebraic notion of a cyclically ordered group (see Definition 3.1) since a separation relation cannot distinguish between the two associated cyclic orders. However, the following theorem holds.

Theorem 5.1 *For a group $(G, \cdot)$ with at least four elements are equivalent:*

(a) *G can be endowed with a separation relation //, which is compatible: if $\alpha\beta // \gamma\delta$ then $\alpha\varepsilon, \beta\varepsilon // \gamma\varepsilon, \delta\varepsilon$ and $\varepsilon\alpha, \varepsilon\beta // \varepsilon\gamma, \varepsilon\delta$ for all $\alpha, \beta, \gamma, \delta, \varepsilon \in G$.*
(b) *G can be endowed with a cyclic order $\lessdot$, which is compatible: if $\lessdot(\alpha, \beta, \gamma)$ then $\lessdot(\delta\alpha, \delta\beta, \delta\gamma)$ and $\lessdot(\alpha\delta, \beta\delta, \gamma\delta)$ for all $\alpha, \beta, \gamma, \delta \in G$.*

Proof (b) ⇒ (a): Let $(G, \cdot)$ be a group with at least four elements which can be endowed with a cyclic order $\lessdot$. Then a separation relation can be defined by $\alpha\gamma // \beta\delta :\Leftrightarrow \lessdot(\alpha, \beta, \gamma, \delta)$ or $\lessdot(\alpha, \delta, \gamma, \beta)$ (see Huntington 1935, Sect. 1.5). If $\lessdot$ is compatible with the group operation of G then // is also compatible.

(a) ⇒ (b): Let $(G, \cdot)$ be a group with at least four elements which can be endowed with a compatible separation relation //. Then // determines a pair $(\lessdot, \lessdot^*)$ of cyclic order relations which can be defined in terms of // as in Huntington (1935, Sect. 4.2). If, in addition, // is compatible with the group operation of G then $\lessdot$ and $\lessdot^*$ are also compatible. □

According Theorem 5.1, a cyclically ordered group can be given by a cyclic order or by a separation relation. We will use this equivalence without further ado.

5.1 The order structure of the classical plane geometries

There are three classical plane geometries, namely the Euclidean, hyperbolic, and elliptic planes over fields of characteristic $\neq 2$ (see Bachmann 1973 for an axiomatic

[12] For the classical example let C be the field of complex numbers, R the field of real numbers and U the multiplicative group of complex numbers of value 1.

foundation of these geometries). Their order structures are introduced in the literature in different ways, either by a linear order (a betweenness relation) or—in the elliptic case—by a cyclic order (a separation relation; see Hilbert 1971; Karzel and Kroll 1988; Pambuccian 2011).

We show that the notion of cyclic order allows the introduction of order structures in a unified way.

Definition 5.2 A Euclidean, hyperbolic or elliptic plane is called *orderable* if the following holds:

(1) A separation relation // can be defined on every row[13] of collinear points.
(2) A separation relation //* can be defined on every pencil of concurrent lines.
(3) The relations // and //* are *compatible*, i.e., if A, B, C, D are collinear points and a, b, c, d concurrent lines and $A|a$ and $B|b$ and $C|c$ and $D|d$ then $AB//CD$ if and only if $ab//^*cd$.

Theorem 5.3 *For a Euclidean, hyperbolic or elliptic plane $\mathcal{E}$ the following two conditions are equivalent:*

(a) *The plane $\mathcal{E}$ is orderable in the sense of Definition 5.2.*
(b) *The coordinate field of $\mathcal{E}$ is orderable.*

Proof *Case 1:* Let $\mathcal{E}$ be an elliptic plane (see Bachmann 1973, Chap. VI). The incidence structure of $\mathcal{E}$ is a projective plane with a field K of coordinates which is commutative and of characteristic $\neq 2$ (see Bachmann 1973, Sect. 16.3). Hence, according to Prieß-Crampe (Prieß-Crampe 1983, Chap. V, Sects. 1, 2), the conditions (a) and (b) of Theorem 5.3 are equivalent.

Case 2: Let $\mathcal{E}$ be a Euclidean plane (see Bachmann 1973, Chap. IV). The incidence structure of $\mathcal{E}$ is an affine plane with a field K of coordinates which is commutative and of characteristic $\neq 2$. Hence, according to R Struve (2018, Theorem 4.5), the conditions (a) and (b) are equivalent.

Case 3: Let $\mathcal{E}$ be a hyperbolic plane (see Bachmann 1973, Chap. V). Then $\mathcal{E}$ has an orderable field K of coordinates which is commutative and of characteristic $\neq 2$ (see Bachmann 1973, Sect. 15.1) and there exist separation relations // and //* which satisfy the conditions (1), (2) and (3) of Definition 5.2 (see R Struve 2012 for a reflection geometric definition of these relations). Hence for any hyperbolic plane the statements (a) and (b) of Theorem 5.3 are (trivially) equivalent. □

5.2 Ordered absolute geometry

The term ‘absolute geometry’ was coined by Bolyai to characterize the part of Euclidean geometry that does not depend on the parallel postulate. This implies that a theorem is not only valid in Euclidean geometry but also in hyperbolic geometry (the non-Euclidean geometry of Bolyai, Gauß and Lobatschewsky) and in elliptic geometry.

[13] The set of points on a line a is called a *row of collinear points*. The set of lines through a point A is called a *pencil of concurrent lines*.

Bachmann (1973) defines a common substratum of Euclidean, hyperbolic, and elliptic geometry by a group-theoretic axiom system whose models are called *Bachmann groups* or—in the geometric interpretation—*planes of absolute geometry*. No assumptions are made about order or continuity.

An order structure is introduced in an additional step either by a linear order relation or—in the elliptic case—by a cyclic order relation (see Pejas 1961; Kunze 1981, H Struve and R Struve 2014; R Struve 2015).

In this section we introduce an order structure for all planes of absolute geometry in a unified way, based on the notion of cyclic order. We follow the group-theoretical approach of Bachmann.

Basic assumption. Let G be a group which is generated by an invariant set S of involutory elements.

Notation: The elements of S are called *lines* and will be denoted by lower case latin letters. The involutory elements of S^2 are called *points* and will be denoted by upper case letters $A, B, \ldots$. Let P be the set of points. The 'stroke relation' $\alpha \mid \beta$ is an abbreviation for the statement that α, β and $\alpha\beta$ are involutory elements. The statement $\alpha \mid \delta$ and $\beta \mid \delta$ is abbreviated by $\alpha, \beta \mid \delta$. A point A and a line b are *incident* if $A \mid b$. Lines $a, b \in S$ are *orthogonal* if $a \mid b$. Points $A, B \in P$ are *polar* if $A \mid B$.

The mapping $a \to a^\alpha$, $A \to A^\alpha$ of S onto S and P onto P is called the *motion* induced by $\alpha \in G$ (we write β^α instead of $\alpha^{-1}\beta\alpha$). A point M is a *midpoint* of A and B if $A^M = B$. Dually, a line m is a *midline* of a and b if $a^m = b$. If a and b have a common point then a midline is called an *angle bisector* of a and b.

B1. For A, B there exists c with $A, B \mid c$.
B2. If $A, B \mid c, d$ then $A = B$ or $c = d$.
B3. If $a, b, c \mid D$ then $abc \in S$.
B4. If $a, b, c \mid d$ then $abc \in S$.
B5. There exists a, b, c with $a \mid b$ and $c \nmid a$ and $c \nmid b$ and $c \nmid ba$.

The axioms make the following statements: according to **B1** and **B2** any two points have a unique joining line and according to **B3** and **B4** the *theorem of three reflections* holds: if three lines have a common point or a common perpendicular, then the product of the reflections in these lines is a line reflection. According to **B5** there exist two orthogonal lines a and b and a point C which is not incident with a or b.

If (G, S, P) satisfies the Basic Assumption and the axioms **B1**–**B5** then (G, S, P) is a *Bachmann group* and the associated geometric structure a *plane of absolute geometry*. If there exist three mutual orthogonal lines then (G, S, P) is an *elliptic* Bachmann group respectively an elliptic plane of absolute geometry (see Bachmann 1973). If any two intersecting lines have an angle bisector and any two points have a midpoint then (G, S, P) is a Bachmann group with *free mobility*.

Now we will introduce an order structure on pencil of lines and on rows of points. Let $S(O)$ denote the pencil of lines through a point O. By **B3** it is $S(O)^3 \subseteq S(O)$ and $S(O) \cup S(O)^2$ is a generalized dihedral group with the abelian subgroup $D(O) := S(O)^2$ of rotations around O. The next theorem shows that a cyclic order on $D(O)$ induces a cyclic order on $S(O)$ and vice versa.

Theorem 5.4 *For a point O of a Bachmann group (G, S, P) are equivalent:*

(1) *The group $D(O)$ of rotations around O is cyclically orderable.*
(2) *The pencil $S(O)$ of lines through O is orderable by a cyclic order relation $\lessdot$ such that $\lessdot(a, b, c)$ implies $\lessdot(auv, buv, cuv)$ for all lines a, b, c, u, v through O.*
(3) *The pencil $S(O)$ of lines through O is orderable by a separation relation // such that $ab//cd$ implies $(auv)(buv)//(cuv)(duv)$ for all lines a, b, c, d, u, v through O.*

Proof Let (G, S, P) be a Bachmann group and $O \in P$.

(1) ⇒ (2): Let $(D(O), \lessdot)$ be a cyclically ordered group and g a fixed line through O. The cyclic order on $D(O)$ induces a cyclic order $\lessdot^*$ on $S(O)$ by $\lessdot^*(a, b, c)$ if $\lessdot(ga, gb, gc)$ for $a, b, c \mid O$.

$\lessdot(ga, gb, gc)$ implies $\lessdot(ga \cdot uv, gb \cdot uv, gc \cdot uv)$ for $u, v \mid O$ since $\lessdot$ is compatible with the group operation of $D(O)$. Hence $\lessdot^*(a, b, c)$ implies $\lessdot^*(auv, buv, cuv)$ for all lines a, b, c, u, v through O.

(2) ⇒ (1): Let $S(O)$ be a pencil of lines through O which is orderable by a cyclic order $\lessdot^*$ such that $\lessdot^*(a, b, c)$ implies $\lessdot^*(auv, buv, cuv)$ for all lines a, b, c, u, v through O.

The cyclic order on $S(O)$ induces a cyclic order $\lessdot$ on $D(O) = \{ga : a \mid O\}$ by $\lessdot(ga, gb, gc)$ if $\lessdot^*(a, b, c)$. It remains to show that the relation $\lessdot$ is compatible with the group operation of $D(O)$, that is, if $\lessdot(ga, gb, gc)$ then $\lessdot(ga \cdot uv, gb \cdot uv, gc \cdot uv)$ for all $uv \in D(O)$. This holds since $\lessdot(ga, gb, gc)$ implies $\lessdot^*(a, b, c)$ and—by (2)—$\lessdot(auv, buv, cuv)$ and $\lessdot(ga \cdot uv, gb \cdot uv, gc \cdot uv)$.

(2) ⇔ (3): This is an immediate consequence of the mutual definability of a cyclic order relation $\lessdot$ on a group G and a separation relation // on G (see Theorem 5.1 and Huntington (1935, Sects. 1.5, 4.2)). □

Hence on a pencil of lines the notion of cyclic order can be introduced in purely group-theoretical terms.

Definition 5.5 A pencil of lines through a point O is called *cyclically orderable* if the group of rotations around O is cyclically orderable (in other words, if the group of rotations around O contains a cyclic cone).

We now show that the notion of cyclic order allows the introduction of an order structure in absolute geometry in a unified way (including the elliptic case).

Definition 5.6 A plane of absolute geometry is called *orderable* if the following holds:

- **O1.** A separation relation // can be defined on every pencil of lines.
- **O2.** If a, b, c, d and a', b', c', d' are concurrent lines, which are perspectively related (i.e., there exist distinct collinear points A, B, C, D with $A \mid a, a'$ and $B \mid b, b'$ and $C \mid c, c'$ and $D \mid d, d'$) then $ab//cd$ if and only if $a'b'//c'd'$.
- **O3.** The relation // is *compatible with the orthogonality relation*: if a, b, c, d are concurrent lines and a', b', c', d' the perpendiculars from a point O to a, b, c, d, respectively, then $ab//cd$ if and only if $a'b'//c'd'$.

Theorem 5.7 *For a plane $\mathcal{E}$ of absolute geometry are equivalent:*

(a) *The plane $\mathcal{E}$ is orderable in the sense of Definition 5.6.*

(b) *The coordinate field of $\mathcal{E}$ is linearly orderable.*

Proof (a) ⇒ (b): Let $\mathcal{E}$ be a plane of absolute geometry which is orderable in the sense of Definition 5.6.

If $\mathcal{E}$ is an elliptic plane then the incidence structure of $\mathcal{E}$ is a projective plane and the separation relation // on the pencils of lines induces a separation relation $//^*$ on the row of points: if A, B, C, D are collinear points of $\mathcal{E}$ then AB separate CD (in symbols $AB\ //^*\ CD$) if there exist concurrent lines a, b, c, d of $\mathcal{E}$ with $A \mid a$ and $B \mid b$ and $C \mid c$ and $D \mid d$ and $ab//cd$. According to condition **O2** of Definition 5.6 the relation $//^*$ is well-defined and compatible with //, i.e., if A, B, C, D are collinear points and a, b, c, d concurrent lines of $\mathcal{E}$ and $A \mid a$ and $B \mid b$ and $C \mid c$ and $D \mid d$ then $AB\ //^*\ CD$ if and only if $ab\ //\ cd$. Hence the conditions of Definition 5.2 are satisfied and the field of coordinates of $\mathcal{E}$ is orderable by Theorem 5.3.

Now let $\mathcal{E}$ be a non-elliptic plane. According to the main theorem of Bachmann (1973, Sects. 6, 11) $\mathcal{E}$ can be extended to a Pappian Fanoian projective plane (the *projective ideal plane*) by introducing ideal points and ideal lines. The *ideal points* are the pencils of lines $S(ab) = \{c : abc \in S\}$ with $a \neq b$. The set of lines through a point E is called a *proper pencil* (or a proper ideal point). The proper pencils correspond in a one-to-one way to the points of $\mathcal{E}$.

Ideal lines are defined by means of *contractions* with center O (see Bachmann 1973, p. 307). A contraction is a mapping from S into S which is induced by the product of two semi-rotations χ_{uv} and χ_{vu} around O (with lines $u, v \mid O$ and $u \nmid v$) and maps a line a onto the axis of the glide reflections auv resp. avu. The line $a\chi_{uv}$ can constructively be obtained by dropping perpendiculars and constructing fourth reflection lines: a) drop the perpendicular g from O to the line a with foot F; b) construct the fourth reflection line guv; c) drop the perpendicular from F to guv (this line is $a\chi_{uv}$). A contraction maps a line, which is orthogonal to a line e through O, onto a line which is orthogonal to e (according to Bachmann 1973, p. 307).

For contractions the following holds (see Bachmann 1973, p. 307):

(†) A contraction maps a proper pencil into a proper pencil.

(‡) For any improper pencil which is not a pencil of perpendiculars of a line through O there exists a contraction with center O which takes the improper pencil into a proper one.

Ideal lines are sets of ideal points. A set of pencils that can be transformed by a contraction with center O into the set of pencils which have a common line g is called an *ideal line*. An ideal line whose pencils have a common line a is a *proper (ideal) line*.

The set of pencils, whose lines have a common perpendicular through O, is called the *line at infinity* of the projective ideal plane. Let $\mathcal{A}$ denote the affine specialization with respect to this line at infinity. $\mathcal{E}$ can be represented as a subplane of $\mathcal{A}$ which contains with a point of $\mathcal{A}$ all lines of $\mathcal{A}$ which are incident with this point.

A contraction of $\mathcal{E}$ with center O induces a dilatation of $\mathcal{A}$ with center O, i.e., a collineation with fixed point O which maps every line of $\mathcal{A}$ onto a parallel line (see Bachmann 1973, p. 307; Bachmann and Behnke 1974, p. 79). We denote the group of dilatations of $\mathcal{A}$ which is generated by the set of contractions of $\mathcal{E}$ with center O by

$\mathcal{D}(O)$. The group $\mathcal{D}(O)$ is a subgroup of the full group of dilatations of $\mathcal{A}$ with center O. According to (†) and (‡) any finite set of collinear points of $\mathcal{A}$ can be mapped by an element of $\mathcal{D}(O)$ onto a set of collinear points of $\mathcal{E}$ and any finite set of concurrent lines of $\mathcal{A}$ can be mapped by an element of $\mathcal{D}(O)$ onto a set of concurrent lines of $\mathcal{E}$.

A dilatation κ of $\mathcal{A}$ with center O preserves the separation relation // on the pencils of lines of $\mathcal{E}$: let $a, b, c, d \in S$ and $E \in P$ and $a, b, c, d \mid E$ and $E \neq O$ and $ab//cd$. A contraction is the product of two semi-rotations around O. A semi-rotation χ around O maps the proper pencil of lines through E onto a proper pencil of lines through a point $E\chi$ and operates on a, b, c, d by a) dropping perpendiculars from O to a, b, c, d (which preserves the separation relation // by **O3**); b) the construction of fourth reflection lines of lines through O (which preserves the separation relation // by Definition 5.5, Theorem 5.4); c) by dropping perpendiculars from $E\chi$ to the lines through O, which are constructed in b) (which preserves the separation relation // by **O3**).

We now show that the ideal affine plane $\mathcal{A}$ is orderable and this implies—as is well-known—that the field of coordinates is orderable.

We start with an extension of the separation relation // of $\mathcal{E}$ to $\mathcal{A}$ and define for lines a, b, c, d of $\mathcal{A}$ that a, b separate c, d (in symbols $ab \parallel cd$) if a, b, c, d are concurrent lines and if there exists a dilatation of $\mathcal{D}(O)$ which maps a, b, c, d onto lines a', b', c', d' of $\mathcal{E}$ with $a'b'//c'd'$.

We note that if a, b, c, d are concurrent lines of $\mathcal{A}$ and $\delta, \kappa \in \mathcal{D}(O)$ dilatations, which map a, b, c, d onto concurrent lines $a\delta, b\delta, c\delta, d\delta$ and $a\kappa, b\kappa, c\kappa, d\kappa$ of $\mathcal{E}$ then the dilatation $\delta^{-1}\kappa$ of $\mathcal{A}$ maps the lines $a\delta, b\delta, c\delta, d\delta$ onto $a\kappa, b\kappa, c\kappa, d\kappa$, respectively. Hence $a\delta, b\delta//c\delta, d\delta$ is equivalent with $a\kappa, b\kappa//c\kappa, d\kappa$ (since contractions preserve the relation //). This implies that the extension of the separation relation // to $\mathcal{A}$ is also a separation relation.

Hence the Pappian affine plane $\mathcal{A}$ can be endowed with a separation relation $\parallel$ on the pencils of lines which induces a separation relation $\parallel^*$ on the rows of points: if A, B, C, D are collinear points of $\mathcal{A}$ then AB separate CD (in symbols $AB \parallel^* CD$) if there exist concurrent lines a, b, c, d of $\mathcal{A}$ with $A \mid a$ and $B \mid b$ and $C \mid c$ and $D \mid d$ and $ab \parallel cd$. The relation is well-defined since $\parallel^*$ is defined by $\parallel$ which, in turn, is defined by the relation // of $\mathcal{E}$ which satisfies **O2**. According to **O2** the relations $\parallel$ and $\parallel^*$ are compatible, i.e., if A, B, C, D are collinear points and a, b, c, d concurrent lines of $\mathcal{A}$ and $A \mid a$ and $B \mid b$ and $C \mid c$ and $D \mid d$ then $AB \parallel^* CD$ if and only if $ab \parallel cd$. This implies according to R Struve (2018, Theorem 4.5) that the field of coordinates of $\mathcal{A}$ is orderable.

(b) $\Rightarrow$ (a): Let $\mathcal{E}$ be a plane of absolute geometry and $\mathcal{P}_{\mathcal{E}}$ the associated projective-metric ideal plane over a field K and a symmetric bilinear form f. A motion of $\mathcal{E}$ can be extended to a projective collineation of $\mathcal{P}_{\mathcal{E}}$. The group of motions of $\mathcal{E}$ can be represented as a subgroup of the orthogonal group $O_3(K, f)$ (see Bachmann (1973, Sects. 6.9, 6.10).

If K is a (linearly) orderable field then $\mathcal{P}_{\mathcal{E}}$ is an orderable projective plane, i.e., $\mathcal{P}_{\mathcal{E}}$ can be provided with an order structure by a separation relation on the rows of points which is invariant with respect to perspectivities (see Prieß-Crampe (1983, Chap. V, Sect. 1)). This implies that the dual projective plane is also orderable (see Prieß-Crampe 1983, p. 175). Hence $\mathcal{P}_{\mathcal{E}}$ can be endowed with a separation relation //

on the pencils of lines which is invariant with respect to perspectivities and satisfies **O1** and **O2**. The restriction of // to $\mathcal{E}$ is a separation relation on $\mathcal{E}$ which satisfies **O1** and **O2**. It remains to show that also **O3** is satisfied.

For a proof let a, b, c, d be lines through a point E and a', b', c', d' the perpendiculars from a point O to a, b, c, d, respectively. The permutation of the pencil of lines through E, which associates to every line the orthogonal line through E, is a product of perspectivities (see Bachmann 1973, Sect. 5.6) and hence preserves the separation relation // of $\mathcal{P}_{\mathcal{E}}$. Thus if a^*, b^*, c^*, d^* are the lines through E which are orthogonal to a, b, c, d, respectively, then $ab//cd$ if and only if $a^*b^*//c^*d^*$. The quadruples of concurrent lines a^*, b^*, c^*, d^* and a', b', c', d' are perspectively related: if A, B, C, D are the poles of a, b, c, d (i.e., the unique points which are incident with all lines which are orthogonal to a, b, c, d, respectively) then $A \mid a^*, a'$ and $B \mid b^*, b'$ and $C \mid c^*, c'$ and $D \mid d^*, d'$. By **O2** it is $a^*b^*//c^*d^*$ if and only if $a'b'//c'd'$. Hence $ab//cd$ is equivalent with $a'b'//c'd'$. The restriction of // to $\mathcal{E}$ is a separation relation on $\mathcal{E}$ which satisfies **O1**, **O2** and **O3**. □

5.3 Cartesian and Gaußian coordinate planes

Let $\mathcal{E}$ be a Euclidean plane of absolute geometry (see Bachmann (1973) or, equivalently, the axiomatic characterization of Euclidean planes without order or continuity by Schnabel (1981)).

In the analytic geometry of Descartes the points of a line of $\mathcal{E}$ (of a 'coordinate axis') are interpreted as the elements of a *Cartesian coordinate field*. In the *Cartesian coordinate plane* over a field K of characteristic $\neq 2$ points of the plane are pairs (x, y) of elements from K, lines are triples $[u, v, w]$, point-line incidence is given by $ux + vy + w = 0$, whereas the orthogonality of the lines $[u, v, w]$ and $[u', v', w']$ is given by kuu′ + vv′ = 0 (in terms of a constant k, with $-k$ not a square in K). The Cartesian interpretation of a field corresponds thus to the classical interpretation of real numbers as elements of a 'number line'.

In the associated *Gaußian coordinate plane* the set of all points of the plane is interpreted as a *Gaußian coordinate field* F (F is isomorphic to $K[x]/(x^2 + k)$; an element $z \in F$ can be represented in the form $z = x + y\varepsilon$ with $x, y \in K$ and $\varepsilon^2 = -k$; the mapping κ, which associates to $x + y\varepsilon$ the element $x - y\varepsilon$ is an involutory automorphism of F which leaves the elements of K fixed; $\|z\| := zz^\kappa$ is called the *norm* of z which satisfies $\|zz'\| = \|z\|\|z'\|$). Points of the Gaußian coordinate plane are the elements from F, lines are the sets $uK + v$ with $u, v \in F$ and $u \neq 0$, point-line incidence is given by the $\in$−relation and the orthogonality of the lines $uK + v$ and $u'K + v'$ is given by $uu'^\kappa + u'u^\kappa = 0$.[14]

A Euclidean plane is a *Hilbert plane* if the plane axioms of incidence, order and congruence of Hilbert's *Grundlagen der Geometrie* are satisfied.

Theorem 5.8 *For a field F with $-1 \in F^2$ are equivalent:*

(a) *F is cyclically orderable by a cyclic order of degree 2.*

[14] For this 'complex representation' of singular geometries (i.e., in any quadrangle with three right angles the fourth angle is a right one) see Schnabel (1981) and Bachmann (1989, Sect. 7).

(b) *F is the Gaußian coordinate field of a Euclidean Hilbert plane.*

Proof (a) $\Rightarrow$ (b): Let F be a field with $-1 \in F^2$ which is endowed with a cyclic order of degree 2. Then F is a quadratic field extension of a linearly orderable field K (see Theorem 4.5). Hence K is formally real and $-1 \notin K^2$ and $F = K(i)$ with $i = \sqrt{-1}$ (since $\{i, 1\}$ is a basis of the vector space F over K). We show that K is a Pythagorean field.

Let $K^+ = \{x \in K : x > 0\}$ and φ an epimorphism from $(\dot{F}, \cdot)$ onto a cyclically orderable subgroup $(U, \cdot)$ with kernel $ker(\varphi) = K^+$. Then $(\dot{F}, \cdot)$ is the direct product of the kernel of φ and the image of φ, that is, $\dot{F} = K^+ \otimes U$. If $z = au$ is the unique representation of an element $z \in \dot{F}$ with $a \in K^+$ and $u \in U$ then $z^\kappa = a^\kappa u^\kappa = au^\kappa$ is the unique representation of z^κ. Hence $u \in U$ implies $u^\kappa, u^{-1} \in U$ and since $\|u\|u^{-1} = u^\kappa$ it is $\|u\| \in K^+ \cap U$ and $\|u\| = 1$. Hence $\|z\| = \|au\| = \|a\|\|u\| = a^2\|u\| = a^2$ is a square and since $\|z\| = \|x + yi\| = x^2 + y^2$ with $x, y \in K$ this shows that K is a Pythagorean field.

This implies (according to Pejas (1961) and Bachmann (1973, Sect. 20.13)) that the Cartesian coordinate plane over K with the orthogonality constant $k = 1$ is a Hilbert plane and F the associated Gaußian coordinate field.

(b) $\Rightarrow$ (a): Let $\mathcal{E}$ be a Euclidean Hilbert plane with the Cartesian coordinate field K, the orthogonality constant k and the Gaußian coordinate field F. Since every right angle of $\mathcal{E}$ has an angle bisector, we can assume $k = 1$ and $F = K(i)$ with $i := \sqrt{-1}$. According to Pejas (1961) and Bachmann (1973, Sect. 20.13) K is a linearly orderable Pythagorean field.

Let O denote the origin of the Gaußian coordinate plane (i.e., the neutral element of the additive group of F), and let $U_F = \{x + yi \in F : x^2 + y^2 = 1\}$ be the unit circle with center O (the set of elements of $(F, \cdot)$ with norm 1).

Then it is well-known that the pencil of lines through O is cyclically orderable (in the sense of Definition 5.5), that every line through O has a common point with the unit circle U_F and that $(U_F, \cdot)$ is a cyclically orderable subgroup of $(F, \cdot)$ (see Hessenberg and Diller 1967; Bachmann 1973; Lenz 1967, Sect. 2). As for the field $\mathbb{C} = \mathbb{R}(i)$ of complex numbers one verifies that the mapping φ from $(F, \cdot)$ onto $(U_F, \cdot)$, which associates to $x + yi$ the element $(x + yi)/\sqrt{x^2 + y^2}$ of norm 1, is a group homomorphism with kernel $P = \{x \in K : x > 0\}$. Since P is the positive cone of the additive group of K this shows that F can be endowed with a cyclic order of degree 2. $\square$

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