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ORIGINAL PAPER

Clairaut anti-invariant submersions from locally product Riemannian manifolds

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Abstract
In this paper, we investigate some geometric properties of Clairaut submersions whose total space is a locally product Riemannian manifold.

Keywords Locally product Riemannian manifold · Riemannian submersion · Clairaut anti-invariant submersion

Mathematics Subject Classification Primary 53C15; Secondary 53C40

1 Introduction

Given a C^{∞}-submersion F from a (semi)-Riemannian manifold (N, g_N) onto a (semi)-Riemannian manifold (B, g_B), according to the circumstances on the map $F : (N, g_N) \to (B, g_B)$, we get the following:

Riemannian submersion (Falcitelli et al. 2004; O'Neill 1966; Gray 1967), almost Hermitian submersion (Watson 1976), paracontact paracomplex submersios (Gündüzalp and Şahin 2014), quaternionic submersion (Ianus et al. 2008), slant submersion (Akyol and Gündüzalp 2016; Gündüzalp 2013b; Gündüzalp and Akyol 2018; Şahin 2011), anti-invariant submersion (Beri et al. 2016; Şahin 2010), Clairaut submersion (Bishop 1972; Gündüzalp 2019; Taştan and Gerdan 2017; Lee et al. 2015; Allison 1996), conformal anti-invariant submersion (Akyol 2017; Akyol and Şahin 2016), etc.

In the present paper, we take into account Clairaut anti-invariant submersions from a locally product Riemannian manifold onto a Riemannian manifold. In Sect. 2, we recall some concepts, which are needed in the following section. In Sect. 3, we first obtain necessary and sufficient conditions for a curve on the manifold N of anti-

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invariant submersions to be geodesic. Then we present a new characterization for Clairaut anti-invariant submersions. Also, we present an example.

2 Preliminaries

2.1 Riemannian submersions

A C^∞-submersion $F : N \to B$ between two Riemannian manifolds (N, g_N) and (B, g_B) is called a Riemannian submersion if it satisfies conditions:

(i) the fibres $F^{-1}(b)$, $b \in B$, are r-dimensional Riemannian submanifolds of N, where $r = dim(N) - dim(B)$.

(ii) F_* preserves the lengths of horizontal vectors.
The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. We denote by $(ker F_*)$ the vertical distribution, by $(ker F_*)^\perp$ the horizontal distribution and by v and h the vertical and horizontal projection. A horizontal vector field X_1 on N is said to be *fundamental* if X_1 is F-related to a vector field X_{*1} on B.

A Riemannian submersion $F : N \to B$ defines two $(1, 2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on N, by the formulas:

$$\mathcal{T}_{X_1} X_2 = h\nabla_{vX_1} vX_2 + v\nabla_{vX_1} hX_2 \tag{1}$$

and

$$\mathcal{A}_{X_1} X_2 = v\nabla_{hX_1} hX_2 + h\nabla_{hX_1} vX_2 \tag{2}$$

for any $X_1, X_2 \in \chi(N)$ (see Falcitelli et al. 2004). Using (1) and (2), one can get

$$\nabla_{U_1} U_2 = \mathcal{T}_{U_1} U_2 + \hat{\nabla}_{U_1} U_2; \tag{3}$$
$$\nabla_{U_1} X_1 = \mathcal{T}_{U_1} X_1 + h(\nabla_{U_1} X_1); \tag{4}$$
$$\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + v(\nabla_{X_1} U_1), \tag{5}$$
$$\nabla_{X_1} X_2 = \mathcal{A}_{X_1} X_2 + h(\nabla_{X_1} X_2), \tag{6}$$

for any $X_1, X_2 \in \Gamma((ker F_*)^\perp)$, $U_1, U_2 \in \Gamma(ker F_*)$. In addition, if X_1 is basic then $h(\nabla_{U_1} X_1) = h(\nabla_{X_1} U_1) = \mathcal{A}_{X_1} U_1$.

The fundamental tensor fields $\mathcal{T}$, $\mathcal{A}$ satisfy:

$$\mathcal{T}_{U_1} U_2 = \mathcal{T}_{U_2} U_1, \quad U_1, U_2 \in \Gamma(ker F_*); \tag{7}$$
$$\mathcal{A}_{X_1} X_2 = -\mathcal{A}_{X_2} X_1 = \frac{1}{2} v[X_1, X_2], \quad X_1, X_2 \in \Gamma((ker F_*)^\perp). \tag{8}$$

2.2 Anti-invariant submersions

Let N be a n-dimensional smooth manifold. If it is endowed with a structure (P, g_N), where P is a (1, 1) tensor, and g_N is a Riemannian metric, satisfying

$$P^2 X_1 = X_1, \quad g_N(PX_1, X_2) = g_N(X_1, PX_2), \tag{9}$$

for any $X_1, X_2 \in \chi(N)$, it is called an almost product Riemannian manifold. An almost product Riemannian manifold N is called a locally product Riemannian manifold if

$$\nabla P = 0, \tag{10}$$

where ∇ is the Riemannian connection on N (Yano and Kon 1984).

Definition 2.1 (Gündüzalp 2013a) Let (N, g_N, P) be an almost product Riemannian manifold and (B, g_B) a Riemannian manifold. Suppose that there exists a Riemannian submersion $F : N \to B$ such that $ker F_*$ is anti-invariant with respect to P, i.e., $P(ker F_*) \subseteq (ker F_*)^\perp$. At that time, we call F is an anti-invariant Riemannian submersion.

In this case, the horizontal distribution $(ker F_*)^\perp$ decomposed as

$$(ker F_*)^\perp = P(ker F_*) \oplus \eta, \tag{11}$$

where η is the complementary orthogonal distribution of $P(ker P_*)$ in $(ker F_*)^\perp$ and it is invariant with respect to P.

For any $X_1 \in \Gamma(ker F_*)^\perp$, we write

$$PX_1 = DX_1 + EX_1, \tag{12}$$

where DX_1 and EX_1 are vertical and horizontal components of PX_1. If $\eta = 0$, at that time an anti-invariant submersion is called a Lagrangian submersion.

3 Clairaut anti-invariant submersions

Let M be a revolution surface in R^3 with rotation axis d. $\forall x \in M$, we state by $r(x)$ the distance from x to d. Given a geodesic $c : J \subset R \to M$ on M, let $\varphi(s)$ be the angle between $c(s)$ and the meridian curve through $c(s)$, $s \in J$. A well-known Clairaut's theorem tells that for any geodesic c on M the product $r \sin \varphi$ is constant along c, i.e., it is independent of s. In the submersion theory, Bishop (1972) shows the concept of Clairaut submersion in the following way.

Definition 3.1 (Bishop 1972) A Riemannian submersion $F : (N, g_N, P) \to (B, g_B)$ is called a *Clairaut submersion* if there exists a positive function r on N such that, for any geodesic c on N, the function $(r \circ c) \sin \varphi$ is constant, where, for any s, $\varphi(s)$ is the angle between $\dot{c}(s)$ and the horizontal space at $c(s)$. Moreover, he gave a necessary and sufficient condition for a Riemannian submersion to be a Clairaut submersion.

Theorem 3.1 (Bishop 1972) *Let $F : (N, g_N, P) \to (B, g_B)$ be Riemannian submersion with connected fibres. Then, F is a Clairaut submersion with $r = e^g$ if and only if each fibre is completely umbilical and has the mean curvature vector field $H = -\nabla g$, where ∇g is the gradient of the function g with respect to g_N.*

Proposition 3.1 *Let F be an anti-invariant submersion from a locally product Riemannian manifold (N, g_N, P) onto a Riemannian manifold (B, g_B). If $c : J \subset R \to N$ is a regular curve and $U_1(s)$ and $X_1(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{c}(s) = W$ of $c(s)$, respectively, then c is a geodesic if and only if along c*

$$v\nabla_{\dot{c}}\mathcal{D}X_1 + \mathcal{A}_{X_1}PU_1 + \mathcal{T}_{U_1}PU_1 + (\mathcal{A}_{X_1} + \mathcal{T}_{U_1})EX_1 = 0, \tag{13}$$

and

$$h\nabla_{\dot{c}}EX_1 + h\nabla_{\dot{c}}PU_1 + (\mathcal{A}_{X_1} + \mathcal{T}_{U_1})\mathcal{D}X_1 = 0. \tag{14}$$

Proof From (10), we obtain

$$\nabla_{\dot{c}}\dot{c} = P(\nabla_{\dot{c}}P\dot{c}). \tag{15}$$

Since $\dot{c} = U_1 + X_1$, we can write

$$\nabla_{\dot{c}}\dot{c} = P(\nabla_{U_1+X_1}P(U_1 + X_1)). \tag{16}$$

By direct computations, we get

$$\nabla_{\dot{c}}\dot{c} = P(\nabla_{U_1}PU_1 + \nabla_{U_1}PX_1 + \nabla_{X_1}PU_1 + \nabla_{X_1}PX_1).$$

Using (12), we get

$$\nabla_{\dot{c}}\dot{c} = P(\nabla_{U_1}PU_1 + \nabla_{U_1}(DX_1 + EX_1) + \nabla_{X_1}PU_1 + \nabla_{X_1}(DX_1 + EX_1)).$$

Using (3)–(6), we have

$$\begin{aligned}\nabla_{\dot{c}}\dot{c} = {} & P(h(\nabla_{\dot{c}}PU_1 + \nabla_{\dot{c}}EX_1) + (\mathcal{A}_{X_1} + \mathcal{T}_{U_1})(DX_1 + EX_1)\\ & + v\nabla_{\dot{c}}DX_1 + \mathcal{A}_{X_1}PU_1 + \mathcal{T}_{U_1}PU_1).\end{aligned}$$

Taking the vertical and horizontal pieces of this equation. we have

$$v\nabla_{\dot{c}}\mathcal{D}X_1 + \mathcal{A}_{X_1}PU_1 + \mathcal{T}_{U_1}PU_1 + (\mathcal{A}_{X_1} + \mathcal{T}_{U_1})EX_1 = vP\nabla_{\dot{c}}\dot{c} \tag{17}$$

and

$$h\nabla_{\dot{c}}EX_1 + h\nabla_{\dot{c}}PU_1 + (\mathcal{A}_{X_1} + \mathcal{T}_{U_1})\mathcal{D}X_1 = hP\nabla_{\dot{c}}\dot{c}. \tag{18}$$

From (17) and (18), it is simple to see that c is a geodesic if and only if (13) and (14) hold. □

Theorem 3.2 *Let F be an anti-invariant submersion from a locally product Riemannian manifold (N, g_N, P) onto a Riemannian manifold (B, g_B). At that time F is a Clairaut submersion with $r = e^g$ if and only if along c the following equation holds*

$$g_N(h\nabla_{\dot{c}}EX_1 + (\mathcal{A}_{X_1} + \mathcal{T}_{U_1})\mathcal{D}X_1, PU_1) = g_N(\nabla g, X_1)\|U_1\|^2, \tag{19}$$

where $U_1(s)$ and $X_1(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{c}(s)$ of the geodesic $c(s)$ on N, severally.

Proof Let $c(s)$ be a geodesic with speed $\sqrt{b}$ on N, at that time, we get

$$b = \|\dot{c}(s)\|^2.$$

Thence, we conclude that

$$g_N(X_1(s), X_1(s)) = b\cos^2\varphi(s), \quad g_N(U_1(s), U_1(s)) = b\sin^2\varphi(s), \tag{20}$$

where $\varphi(s)$ is the angle between $\dot{c}(s)$ and the horizontal space at $c(s)$. Differentiating the second expression in (20), we get

$$\frac{d}{ds}g_N(U_1(s), U_1(s)) = 2g_N(\nabla_{\dot{c}(s)}U_1(s), U_1(s)) = 2b\cos\varphi(s)\sin\varphi(s)\frac{d\varphi}{ds}(s). \tag{21}$$

Thus, using (9) and (10), we obtain

$$g_N(h\nabla_{\dot{c}(s)}PU_1(s), PU_1(s)) = b\cos\varphi(s)\sin\varphi(s)\frac{d\varphi}{ds}(s). \tag{22}$$

By (14), we arrive at along c,

$$-g_N(h\nabla_{\dot{c}}EX_1 + (\mathcal{A}_{X_1} + \mathcal{T}_{U_1})\mathcal{D}X_1, PU_1) = b\cos\varphi\sin\varphi\frac{d\varphi}{ds}. \tag{23}$$

Moreover, F is a Clairaut anti-invariant submersion with $r = e^g$ if and only if

$$\frac{d}{ds}(e^g\sin\varphi) = 0 \Leftrightarrow e^g\left(\frac{dg}{ds}\sin\varphi + \cos\varphi\frac{d\varphi}{ds}\right) = 0.$$

Striking recent equation with non-zero element $b\sin\varphi$, we obtain

$$\frac{dg}{ds}b\sin^2\varphi + b\cos\varphi\sin\varphi\frac{d\varphi}{ds} = 0. \tag{24}$$

From (23) and (24), we have

$$g_N(h\nabla_{\dot{c}}EX_1 + (\mathcal{A}_{X_1} + \mathcal{T}_{U_1})\mathcal{D}X_1, PU_1) = \frac{dg}{ds}(c(s))\|U_1\|^2. \tag{25}$$

Since $\frac{dg}{ds}(c(s)) = \dot{c}[g] = g_N(\nabla g, \dot{c}) = g_N(\nabla g, X_1)$, the claim (19) follows from (25). □

Theorem 3.3 *Let F be a Clairaut anti-invariant submersion from a locally product Riemannian manifold (N, g_N, P) onto a Riemannian manifold (B, g_B) with $r = e^g$. At that time, we get*

$$\mathcal{A}_{PU_3} PX_1 = X_1(g)U_3 \tag{26}$$

for $X_1 \in \eta$ and $U_3 \in ker F_\star$ such that PU_3 is basic.

Proof From Theorem 3.1, we obtain

$$\mathcal{T}_{U_1} U_2 = -g_N(U_1, U_2)\nabla g, \tag{27}$$

where $U_1, U_2 \in ker F_\star$. If we crash this equation by PU_3, $U_3 \in ker F_\star$ such that PU_3 is fundamental and from (3), we get

$$g_N(\nabla_{U_1} U_2, PU_3) = -g_N(U_1, U_2)g_N(\nabla g, PU_3).$$

Thus, we have

$$g_N(\nabla_{U_1} PU_3, U_2) = g_N(U_1, U_2)g_N(\nabla g, PU_3),$$

since $g_N(U_2, PU_3) = 0$.

By (10), we get

$$g_N(P\nabla_{U_1} U_3, U_2) = g_N(U_1, U_2)g_N(\nabla g, PU_3).$$

Using (9), we arrive at

$$g_N(\nabla_{U_1} U_3, PU_2) = g_N(U_1, U_2)g_N(\nabla g, PU_3).$$

Again, using (3), we obtain

$$g_N(\mathcal{T}_{U_1} U_3, PU_2) = g_N(U_1, U_2)g_N(\nabla g, PU_3).$$

Thus, by (27),

$$-g_N(U_1, U_3)g_N(\nabla g, PU_2) = g_N(U_1, U_2)g_N(\nabla g, PU_3) \tag{28}$$

If take $U_1 = U_3$ and exchange U_1 with U_2 in (28), we provide

$$-\|U_2\|^2 g_N(\nabla g, PU_1) = g_N(U_1, U_2)g_N(\nabla g, PU_2). \tag{29}$$

Using (28) with $U_1 = U_3$ and (29), we get

$$-g_N(\nabla g, PU_1) = \frac{g_N^2(U_1, U_2)}{\|U_2\|^2\|U_1\|^2} g_N(\nabla g, PU_1). \tag{30}$$

On the other hand, using (10), we obtain

$$g_N(\nabla_{U_2} PU_3, PX_1) = g_N(P\nabla_{U_2} U_3, PX_1)$$

for any $X_1 \in \eta$. Thus, using (9), we get

$$g_N(\nabla_{U_2} PU_3, PX_1) = g_N(\nabla_{U_2} U_3, X_1).$$

Using (3) and (27), we obtain

$$g_N(\nabla_{U_2} PU_3, PX_1) = -g_N(U_2, U_3) g_N(\nabla g, X_1). \tag{31}$$

Since PU_3 is fundamental and from $h\nabla_{U_2} PU_3 = \mathcal{A}_{PU_3} U_2$, we have

$$g_N(h\nabla_{U_2} PU_3, PX_1) = g_N(\mathcal{A}_{PU_3} U_2, PX_1). \tag{32}$$

Using (31),(32) and the anti-symmetry of $\mathcal{A}$, we find

$$g_N(\mathcal{A}_{PU_3} PX_1, U_2) = g_N(\nabla g, X_1) g_N(U_3, U_2). \tag{33}$$

Since $\mathcal{A}_{PU_3} PX_1$, U_2 and U_3 are vertical and ∇g is horizontal, we derive (26).

Now, if $\nabla g \in Pker F\star$, then from (30) and the equality situation of Schwarz inequality, we get the following. □

Corollary 3.1 *Let F be a Clairaut anti-invariant submersion from a locally product Riemannian manifold (N, g_N, P) onto a Riemannian manifold (B, g_B) with $r = e^g$. If $\nabla g \in Pker F\star$, at that time, either g is constant on $Pker F\star$ or the fibres of F are 1-dimensional.*

Furthermore, while the function g is constant, $\nabla g \equiv 0$. Hence, by Theorem 3.1 and Corollary 3.1, we get that:

Corollary 3.2 *Let F be a Clairaut anti-invariant submersion from a locally product Riemannian manifold (N, g_N, P) onto a Riemannian manifold (B, g_B) with $r = e^g$ and $\nabla g \in Pker F\star$. If $dim(ker F*) > 1$, at that time, the fibres of F are completely geodesic if and only if $\mathcal{A}_{PU_3} PX_1 = 0$ for $U_3 \in ker F_*$ such that PU_3 is fundamental and $X_1 \in \eta$.*

In addition, if the anti-invariant submersion F in Theorem 3.3 is Lagrangian, at that time, $\mathcal{A}_{PU_3} PX_1 = 0$ always zero, since $\eta = \{0\}$. Hence, we obtain that:

Corollary 3.3 *Let F be a Clairaut Lagrangian submersion from a locally product Riemannian manifold (N, g_N, P) onto a Riemannian manifold (B, g_B) with $r = e^g$. Then either the fibres of F are one dimensional or they are totally geodesic.*

Now, we present example of a Clairaut submersion.

Example 3.1 Let N be a Euclidean 3-space defined by $N = \{(x_1, x_2, x_3) \in R^3 : (x_1, x_2) \neq (0, 0) \text{ and } x_3 \neq 0\}$.

We consider the product structure (P, g_N) on N given by $g_N = e^{2x_3}(dx_1)^2 + e^{2x_3}(dx_2)^2 + (dx_3)^2$ and $P(a, b, c) = (a, -b, c)$.

A P-basis can be given by $\{e_1 = e^{-x_3}\frac{\partial}{\partial x_1}, e_2 = e^{-x_3}\frac{\partial}{\partial x_2}, e_3 = \frac{\partial}{\partial x_3}\}$.
Let B be $\{(t, x_3) \in R^2\}$. We select the metric g_B on B, $g_B = e^{2x_3}(dt)^2 + (dx_3)^2$.

Now, we defined a map $F : (N, P, g_N) \to (B, g_B)$ by

$$F(x_1, x_2, x_3) = \left(\frac{x_1 + x_2}{\sqrt{2}}, x_3\right).$$

At that time, by direct calculations, we get

$$ker F_* = span\left\{U_1 = \frac{e_1 - e_2}{\sqrt{2}}\right\}$$

and

$$(ker F_*)^{\perp} = span\left\{X_1 = \frac{e_1 + e_2}{\sqrt{2}}, X_2 = \frac{\partial}{\partial x_3}\right\}.$$

Then, it is simple to see that F is a Riemannian submersion. Furthermore $PU_1 = X_1$implies that $P(ker F_*) \subset (ker F_*)^{\perp}$. Consequently, F is anti-invariant Riemannian submersion. Furthermore, the fibres of F are frankly completely umbilical, from they are 1-dimensional. In this place, we will find that a $g \in C^{\infty}(N)$ filling $\mathcal{T}_{U_1}U_1 = -\nabla g$. The Riemannian connection ∇ of the metric tensor g_N is given by

$$\begin{aligned} 2g_N(\nabla_{U_1}U_2, U_3) &= U_1 g_N(U_2, U_3) + U_2 g_N(U_1, U_3) - U_3 g_N(U_2, U_1) \\ &\quad - g_N([U_2, U_3], U_1) - g_N([U_1, U_3], U_2) + g_N(U_3, [U_1, U_2]), \end{aligned}$$

for any $U_1, U_2, U_3 \in \chi(N)$. Using the above formula for the Riemannian metric g_N, we can simply calculate that

$$\nabla_{e_1}e_1 = \nabla_{e_2}e_2 = -\frac{\partial}{\partial x_3}$$

and

$$\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = 0.$$

Hence, we get

$$\begin{aligned} \nabla_{U_1}U_1 &= \frac{1}{2}(\nabla_{e_1}e_1 - \nabla_{e_1}e_2 - \nabla_{e_2}e_1 + \nabla_{e_2}e_2) \\ &= -\frac{\partial}{\partial x_3}. \end{aligned}$$

By (3), we have

$$\mathcal{T}_{U_1} U_1 = -\frac{\partial}{\partial x_3}.$$

Moreover, for any $g \in C^{\infty}(N)$ the gradient of g with respect to g_N is given by

$$\nabla g = \sum_{i,j}^{3} g_N^{ij} \frac{\partial g}{\partial x_i} \frac{\partial}{\partial x_j} = e^{-2x_3} \frac{\partial g}{\partial x_1} \frac{\partial}{\partial x_1} + e^{-2x_3} \frac{\partial g}{\partial x_2} \frac{\partial}{\partial x_2} + \frac{\partial g}{\partial x_3} \frac{\partial}{\partial x_3}.$$

At that time, it is simple to see that $\nabla g = \frac{\partial}{\partial x_3}$ for the function $g = x_3$ and $\mathcal{T}_{U_1} U_1 = -\nabla g = -x_3$. In addition to, for all $U_2 \in \Gamma(ker F_*)$, we obtain

$$\mathcal{T}_{U_2} U_2 = -\|U_2\|^2 \nabla g.$$

Hence, by Theorem 3.1, the submersion F is Clairaut.

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