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ORIGINAL PAPER

Constructing symplectomorphisms between symplectic torus quotients

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Abstract
We identify a family of torus representations such that the corresponding singular symplectic quotients at the 0-level of the moment map are graded regularly symplectomorphic to symplectic quotients associated to representations of the circle. For a subfamily of these torus representations, we give an explicit description of each symplectic quotient as a Poisson differential space with global chart as well as a complete classification of the graded regular diffeomorphism and symplectomorphism classes. Finally, we give explicit examples to indicate that symplectic quotients in this class may have graded isomorphic algebras of real regular functions and graded Poisson isomorphic complex symplectic quotients yet not be graded regularly diffeomorphic nor graded regularly symplectomorphic.

Keywords Symplectic reduction · Singular symplectic quotient · Hamiltonian torus action · Graded regular symplectomorphism

Mathematics Subject Classification Primary 53D20; Secondary 13A50 · 14L30

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1 Introduction

Let G be a compact Lie group and $G \to \mathrm{U}(V)$ a finite dimensional unitary representation of G. Here $\mathrm{U}(V)$ stands for the unitary group of V, i.e., the group of automorphisms preserving the hermitian inner product $\langle\cdot,\cdot\rangle$. To describe the orbit space V/G, i.e., the space of G-orbits in V, invariant theory is employed as follows. There exists a system of fundamental real homogeneous polynomial invariants $\varphi_1, \varphi_2, \dots, \varphi_m$; we refer to the system $\varphi_1, \varphi_2, \dots, \varphi_m$ as a *Hilbert basis*. This means that any real invariant polynomial $f \in \mathbb{R}[V]^G$ can be written as a polynomial in the φ's, i.e., there exists a polynomial $g \in \mathbb{R}[x_1, x_2, \dots, x_m]$ such that $f = g(\varphi_1, \varphi_2, \dots, \varphi_m)$. More generally, by a theorem of Schwarz (1975), for any smooth function $f \in \mathcal{C}^\infty(V)^G$ there exists $g \in \mathcal{C}^\infty(\mathbb{R}^m)$ such that $f = g(\varphi_1, \varphi_2, \dots, \varphi_m)$. The vector-valued map $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_m)$ gives rise to an embedding $\overline{\varphi}$ of V/G into euclidean space $\mathbb{R}^m$, which is called the *Hilbert embedding*. We denote its image by $X := \varphi(V)$. It turns out that $\overline{\varphi}$ is actually a *diffeomorphism* onto X, i.e., the pullback $\overline{\varphi}^*$ via $\overline{\varphi}$ induces an isomorphism of algebras $\mathcal{C}^\infty(X) := \{g\colon X \to \mathbb{R} \mid \exists G \in \mathcal{C}^\infty(\mathbb{R}^m)\colon g = G_{|X}\}$ and $\mathcal{C}^\infty(V/G) := \mathcal{C}^\infty(V)^G$. Moreover, the restriction of $\overline{\varphi}^*$ to the subalgebra $\mathbb{R}[X] := \{g\colon X \to \mathbb{R} \mid \exists G \in \mathbb{R}[x_1, x_2, \dots, x_m]\colon g = G_{|X}\}$ isomorphically to $\mathbb{R}[V/G] := \mathbb{R}[V]^G$ preserving the grading. Here we use the natural grading $\deg(x_i) := \deg(\varphi_i)$. We say that $\overline{\varphi}^*$ is a *graded regular diffeomorphism*. The algebra $\mathbb{R}[X]$ can be understood as the quotient of $\mathbb{R}[x_1, x_2, \dots, x_m]$ by the kernel of the restriction map, which we refer to as the ideal of *off-shell relations*. Its generators are assumed to be homogeneous in the natural grading. The real variety underlying $\mathbb{R}[X]$ is the Zariski closure $\overline{X}$ of X inside $\mathbb{R}^m$. The space X itself is not a real variety but a *semialgebraic set*, meaning that it is defined by polynomial equations as well as inequalities (Bochnak et al. 1998). How the inequalities cutting out X inside $\overline{X}$ are obtained has been explained in Procesi and Schwarz (1985).

The hermitian vector space V is equipped with the symplectic form $\omega = \mathrm{Im}\langle\cdot,\cdot\rangle$. Moreover, the action of G on V is Hamiltonian and admits a unique homogeneous quadratic moment map $J\colon\ V \to \mathfrak{g}^*$ where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of G. The zero fiber $Z := J^{-1}(0)$ of J is referred to as the *shell*. It is a real subvariety of V that is often singular, in which case it has a conical singularity at the origin. As the moment map J is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$, the group G acts on Z. The space $M_0 := Z/G$ of G-orbits in Z is called the (linear) *symplectic quotient*. By the work (Sjamaar and Lerman 1991) the smooth structure $\mathcal{C}^\infty(M_0)$ is given by the quotient $\mathcal{C}^\infty(V)^G/\mathcal{I}_Z^G$ where $\mathcal{I}_Z^G$ is the invariant part of the vanishing ideal $\mathcal{I}_Z := \{f \in \mathcal{C}^\infty(V) \mid f_{|Z} = 0\}$. Note that $\mathcal{C}^\infty(M_0)$ is in a canonical way a Poisson algebra containing the Poisson subalgebra $\mathbb{R}[M_0] := \mathbb{R}[V]^G/I_Z^G$, where $I_Z^G := \mathcal{I}_Z \cap \mathbb{R}[V]^G$. The image $Y := \varphi(Z)$ of Z under the Hilbert map is a semialgebraic subset of X. Its Zariski closure $\overline{Y}$ is described by the generators of the kernel in $\mathbb{R}[x_1, x_2, \dots, x_m]$ of the algebra morphism $x_i \mapsto \varphi_{i\,|Z} \in \mathcal{C}^\infty(M_0)$. We refer to it as the ideal of *on-shell* relations. The inequalities that cut out Y from $\overline{Y}$ are the same as those cutting out X from $\overline{X}$.

Let us now assume that we have two symplectic quotients M_0 and M_0' constructed from the representations $G \to \mathrm{U}(V)$ and $G' \to \mathrm{U}(V')$, respectively. By a *symplecto-*

morphism between M_0 and M_0' we mean a homeomorphism $F\colon M_0 \to M_0'$ such that the pullback F^* is an isomorphism of Poisson algebras $F^*\colon \mathcal{C}^\infty(M_0') \to \mathcal{C}^\infty(M_0)$. We say that F is *regular* if $F^*(\mathbb{R}[M_0']) \subseteq \mathbb{R}[M_0]$. A regular symplectomorphism is called *graded regular* if the map $(F^*)_{|\mathbb{R}[M_0']}\colon \mathbb{R}[M_0'] \to \mathbb{R}[M_0]$ preserves the grading. By the Lifting Theorem of Farsi et al. (2013), an isomorphism $f\colon \mathbb{R}[M_0'] \to \mathbb{R}[M_0]$ of Poisson algebras gives rise to a unique symplectomorphism if it is compatible with the inequalities.

When $G = \mathbb{T}^\ell$ is a torus, a representation V of complex dimension n can be described in terms of a weight matrix $A \in \mathbb{Z}^{\ell\times n}$; we use $M_0(A)$ to denote the symplectic quotient associated to the representation with weight matrix A. In Farsi et al. (2013, Theorem 7), it is demonstrated that for a weight matrix of the form $A = (D|C)$ where D is an $\ell\times\ell$ diagonal matrix with strictly negative entries on the diagonal and C is an $\ell\times 1$ matrix with strictly positive entries, the corresponding symplectic quotient by $\mathbb{T}^\ell$ is graded regularly symplectomorphic to the symplectic orbifold $\mathbb{C}/\mathbb{Z}_\eta$ where $\eta = \eta(A)$ is a quantity determined by the entries of A; see Definition 3. However, based on the explicit description of the ring $\mathbb{R}[\mathbb{C}]^{\mathbb{Z}_\eta}$ of real regular functions on the orbifold $\mathbb{C}/\mathbb{Z}_\eta$ given in the proof of (Farsi et al. 2013, Theorem 7), it is easy to see that $\mathbb{R}[\mathbb{C}]^{\mathbb{Z}_{\eta_1}}$ and $\mathbb{R}[\mathbb{C}]^{\mathbb{Z}_{\eta_2}}$ are isomorphic as algebras over $\mathbb{R}$ if and only if $\eta_1 = \eta_2$. Hence, an immediate corollary of Farsi et al. (2013, Theorem 7) is the following.

Corollary 1 *For $i = 1, 2$, let $A_i = (D_i|C_i)$ where each D_i is an $\ell_i \times \ell_i$ diagonal matrix with strictly negative entries on the diagonal and each C_i is an $\ell_i \times 1$ matrix with strictly positive entries. Then the symplectic quotients $M_0(A_1)$ and $M_0(A_2)$ are regularly diffeomorphic if and only if $\eta(A_1) = \eta(A_2)$, in which case they are graded regularly symplectomorphic.*

More recently, it was shown in Herbig et al. (2015, Theorem 1.1) that for general symplectic quotients, symplectomorphisms with symplectic orbifolds are rare, even if the graded regular requirements are dropped; see also Herbig and Seaton (2015). Hence, one cannot use isomorphisms with quotients by finite groups to approach a more general classification of higher-dimensional symplectic quotients by tori.

In this paper, we give a generalization of Corollary 1 as a step towards a general classification of linear symplectic quotients by tori into (graded) regular symplectomorphism classes. While Corollary 1 addresses a class of symplectic quotients by tori that can be reduced to quotients by finite groups, we consider here a class of symplectic quotients by tori that are graded regularly symplectomorphic to symplectic quotients by the circle $\mathbb{T}^1$. To state our main result, we say that a weight matrix $A \in \mathbb{Z}^{\ell\times(\ell+k)}$ is *Type II_k* if it can be expressed in the form $A = (D, c_1\boldsymbol{n}, \ldots, c_k\boldsymbol{n})$ with D a diagonal matrix with strictly negative diagonal entries, $\boldsymbol{n}$ a column matrix with strictly positive entries, and each $c_r \geq 1$. Our main result is that the symplectic quotient associated to a Type II_k matrix of any size is graded regularly symplectomorphic to a symplectic quotient by $\mathbb{T}^1$. Specifically, we have the following; see Definition 3 for the definitions of α and β.

Theorem 1 *Let $A \in \mathbb{Z}^{\ell\times(\ell+k)}$ be the Type II_k matrix corresponding to a faithful $\mathbb{T}^\ell$-representation V of dimension $n = \ell + k$. Then the symplectic quotient $M_0(A)$*

is graded regularly symplectomorphic to the $\mathbb{T}^1$*-symplectic quotient* $M_0(B)$ *where* $B = \big(-\alpha(A), c_1\beta(A), \ldots, c_k\beta(A)\big) \in \mathbb{Z}^{1\times(k+1)}$.

Theorem 1 can be thought of as a dimension reduction formula, allowing one to describe symplectic quotients by $\mathbb{T}^\ell$ associated to Type II_k weight matrices in terms of much simpler quotients by $\mathbb{T}^1$. In particular, it extends results concerning $\mathbb{T}^1$-symplectic quotients to this family of quotients by tori, e.g., the Hilbert series computations of Herbig and Seaton (2014) or the representability results of Watts (2016). The graded regular symplectomorphism given by Theorem 1 preserves several structures, and hence can be thought of as a symplectomorphism of symplectic stratified spaces, a graded isomorphism of the corresponding real algebraic varieties, etc., and it induces a graded Poisson isomorphism of the corresponding complex symplectic quotients, the complexifications treated as complex algebraic varieties with symplectic singularities; see Herbig et al. (2020)

The proof of Theorem 1 is given in Sect. 3 by indicating a Seshadri section for the action of the torus on the zero fiber of the moment map after complexifying and applying a result of Popov (1992); see Definition 2 and Theorem 5. The first proof we obtained of Theorem 1, however, was constructive for a smaller class of weight matrices, so-called *Type* I_k (see Definition 3), and used explicit descriptions of the corresponding symplectic quotients and algebras of real regular functions. Because this description has proven useful and may be of independent interest, we give this description and outline the constructive approach in Appendix A.

In the case of symplectic quotients of (real) dimension 2 considered in Corollary 1 (corresponding to Type I_1 weight matrices), the graded regular symplectomorphism class of $M_0(A)$ depends only on the constant $\eta(A)$, which is given by the sum $\alpha(A) + \beta(A)$ (see Definition 3). In the case of Type I_k weight matrices with $k > 1$, this is no longer the case; we show in Sect. 4 that the graded regular symplectomorphism classes of Type I_k symplectic quotients are classified by k, $\alpha(A)$, and $\beta(A)$. For Type II_k weight matrices, though the graded regular symplectomorphism class of $M_0(A)$ is certainly not determined by k and $\eta(A)$, the situation is more subtle, and such a classification would require very different techniques. In Sect. 5, we indicate this with examples of symplectic quotients associated to Type II_k weight matrices that fail to be graded regularly symplectomorphic, though the corresponding complex algebraic varieties are graded Poisson isomorphic, and hence the Hilbert series of real regular functions coincide.

2 Background on torus representations

In this section, we give a brief overview of the structures associated to (real linear) symplectic quotients by tori, specializing the constructions described in the Introduction. We refer the reader to Farsi et al. (2013), Herbig et al. (2009) for more details.

Let $G = \mathbb{T}^\ell$ and let V be a unitary G-module with $\dim_{\mathbb{C}} V = n$. Choosing a basis with respect to which the action of G is diagonal and letting $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ denote coordinates for V with respect to this basis, the action of G is given by

$$\boldsymbol{t}\boldsymbol{z} := \left(t_1^{a_{11}} t_2^{a_{21}} \dots t_\ell^{a_{\ell 1}} z_1, t_1^{a_{12}} t_2^{a_{22}} \dots t_\ell^{a_{\ell 2}} z_2, \dots, t_1^{a_{1n}} t_2^{a_{2n}} \dots t_\ell^{a_{\ell n}} z_n\right)$$

where $\boldsymbol{t} = (t_1, t_2, \dots, t_\ell) \in G$ and $A = (a_{ij}) \in \mathbb{Z}^{\ell \times n}$ is the *weight matrix* of the representation. Given a matrix $A \in \mathbb{Z}^{\ell \times n}$, we let V_A denote the n-dimensional representation of $\mathbb{T}^\ell$ whose weight matrix equals A and the corresponding basis for V_A. We let $\langle \cdot, \cdot \rangle$ denote the standard hermitian scalar product on V_A corresponding to this basis.

Letting $\boldsymbol{a}_j$ denote the jth column of A so that $A = (\boldsymbol{a}_1 \dots \boldsymbol{a}_n)$, it will be convenient to set

$$\boldsymbol{t}^{\boldsymbol{a}_j} := t_1^{a_{1j}} t_2^{a_{2j}} \dots t_\ell^{a_{\ell j}}$$

so that the action is given by

$$\boldsymbol{t}\boldsymbol{z} = \left(\boldsymbol{t}^{\boldsymbol{a}_1} z_1, \boldsymbol{t}^{\boldsymbol{a}_2} z_2, \dots, \boldsymbol{t}^{\boldsymbol{a}_n} z_n\right).$$

Row-reducing A over $\mathbb{Z}$ corresponds to changing coordinates $(t_1, \dots, t_2)$ for G, so we may assume that A is in reduced echelon form over $\mathbb{Z}$. Similarly, permuting the columns of A corresponds to reordering the basis for V_A.

With respect to the symplectic form given by $\omega(z, z') = \operatorname{Im}\langle z, z' \rangle$, the action of G on V_A is Hamiltonian and admits a unique homogeneous quadratic moment map $J_A \colon V_A \to \mathfrak{g}^*$; we will write $J = J_A$ when there is no potential for confusion. Identifying the Lie algebra $\mathfrak{t}^\ell$ of $\mathbb{T}^\ell$ with $\mathbb{R}^\ell$ using a basis for $\mathfrak{t}^\ell$ corresponding to the coordinates $(t_1, \dots, t_\ell)$ for $\mathbb{T}^\ell$ and the dual basis for $(\mathfrak{t}^\ell)^*$, $J = (J_1, \dots, J_\ell)$ can be expressed in terms of the component functions

$$J_i : V_A \longrightarrow \mathbb{R}, \quad J_i(z) := \frac{1}{2} \sum_{j=1}^{n} a_{ij} z_j \overline{z_j}, \quad j = 1, \dots, \ell. \tag{1}$$

As the action of $\mathbb{T}^\ell$ on $\mathfrak{t}^\ell$ is trivial, each component J_i is $\mathbb{T}^\ell$-invariant. Then the *shell* $Z = Z_A := J^{-1}(0)$ is the $\mathbb{T}^\ell$-invariant real algebraic variety in V_A corresponding to this family of quadratics. The *(real) symplectic quotient* is $M_0 = M_0(A) := Z_A/\mathbb{T}^\ell$. The *algebra of smooth functions* $\mathcal{C}^\infty(M_0)$ is defined by $\mathcal{C}^\infty(M_0) := \mathcal{C}^\infty(V)^G/\mathcal{I}_Z^G$ where $\mathcal{I}_Z$ is the vanishing ideal of Z in $\mathcal{C}^\infty(V)$ and $\mathcal{I}_Z^G := \mathcal{I}_Z \cap \mathcal{C}^\infty(V)^G$. The algebra $\mathcal{C}^\infty(M_0)$ inherits a Poisson structure from $\mathcal{C}^\infty(V)$, where the Poisson bracket is given on coordinates by $\{z_i, \overline{z_j}\} = -2\sqrt{-1}\delta_{ij}$, see Arms et al. (1990). Equipped with the algebra $\mathcal{C}^\infty(M_0)$ and its Poisson structure, M_0 is a *Poisson differential space*, see Farsi et al. (2013, Definition 5).

The algebra of real regular functions $\mathbb{R}[M_0]$ on M_0 is defined in terms of the real polynomial invariants $\mathbb{R}[V]^G$. Specifically, $\mathbb{R}[M_0] := \mathbb{R}[V]^G/I_Z^G$ where $I_Z^G := \mathcal{I}_Z^G \cap \mathbb{R}[V]^G$. The ideal I_Z^G is homogeneous with respect to the grading of $\mathbb{R}[V]$ by total degree so that $\mathbb{R}[M_0]$ is a graded algebra; it is as well a Poisson subalgebra of $\mathcal{C}^\infty(M_0)$. We refer to elements of $\mathbb{R}[V]^G$ as *off-shell invariants* and the corresponding classes in $\mathbb{R}[M_0]$ as *on-shell invariants*. Note that for $i = 1, \dots, n$, the real polynomials $z_i\overline{z_i}$ are always invariant. We will take advantage of the complex coordinate system on V for convenience, often expressing $\mathbb{R}[V]^G$ in terms of polynomials in the z_i and $\overline{z_i}$. By this, we mean that the real and imaginary parts of these polynomials are elements of $\mathbb{R}[V]^G$. Note that the real invariants $\mathbb{R}[V]^G$ can be computed in terms of the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ of V by Schwarz (1980, Proposition 5.8(1)), and $V \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic as a $\mathbb{T}^\ell$-module to $V \oplus V^*$.

Let $G_{\mathbb{C}} := (\mathbb{C}^\times)^\ell$ denote the complexification of G, and then the action of G on V extends to an action of $G_{\mathbb{C}}$. The categorical quotient $V /\!\!/ G_{\mathbb{C}} = \operatorname{Spec}\big(\mathbb{C}[V]^{G_{\mathbb{C}}}\big)$, which parameterizes the closed orbits in V, is an affine variety that is stratified by conjugacy classes of $G_{\mathbb{C}}$-isotropy groups (Luna 1973) and contains a unique open stratum $(V /\!\!/ G_{\mathbb{C}})_{\mathrm{pr}}$. Let $\pi\colon\ V \to V /\!\!/ G_{\mathbb{C}}$ denote the orbit map and $V_{\mathrm{pr}} = \pi^{-1}\big((V /\!\!/ G_{\mathbb{C}})_{\mathrm{pr}}\big)$. We recall the following from Schwarz (1995, (0.3), (7.3), (9.5), and (9.6)(3))

Definition 1 The $G_{\mathbb{C}}$-module V is *stable* if V contains an open dense subset of closed orbits and has *finite principal isotropy groups (FPIG)* if the isotropy groups of points in V_{pr} with closed orbits are finite.

Assume V has FPIG and let $k \geq 0$. We say V is *k-principal* if $V \smallsetminus V_{\mathrm{pr}}$ has complex codimension at least k in V. Letting $V_{(r)}$ denote the set of points in V with $G_{\mathbb{C}}$ isotropy group of complex dimension r, V is *k-modular* if for $r = 1, 2, \dots, \dim_{\mathbb{C}} G_{\mathbb{C}}$, $V_{(r)}$ has codimension at least $r + k$ in V. We say V is *k-large* if V is k-principal and k-modular.

Note that references differ about whether FPIG is assumed in the definitions of k-principal or k-modular, though it is always assumed in the definition of k-large. However, in the case considered here, stable implies FPIG.

Theorem 2 (Herbig et al. 2013, Theorem 3.2) *If $G = \mathbb{T}^\ell$ is a torus and V is a faithful $G_{\mathbb{C}}$-module, then V is stable if and only if it is 1-large.*

Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of the underlying real vector space of V, and then the $G_{\mathbb{C}}$-action extends to an action of $V_{\mathbb{C}}$; as a $G_{\mathbb{C}}$-module, $V_{\mathbb{C}}$ is isomorphic to $V \oplus V^*$. Let $\mu = \mu_A = J \otimes_{\mathbb{R}} \mathbb{C}\colon\ V_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}^*$ denote the complexification of the moment map, where $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}^*$ its dual, and let (μ) be the ideal in $\mathbb{C}[V_{\mathbb{C}}]$ generated by the components of μ with respect to a choice of basis for $\mathfrak{g}_{\mathbb{C}}^*$. Let $N = N_A$ denote the subscheme of $V_{\mathbb{C}}$ corresponding to (μ), called the *complex shell*. We recall the following, which holds for a general complex reductive group H.

Theorem 3 (Herbig et al. 2013, Theorem 2.2(3)) *If V is 1-large as an H-module, then the complex shell N is reduced and irreducible.*

Recall that a scheme is *reduced* if its local rings contain no nonzero nilpotent elements and *irreducible* if it is not the union of two proper Zariski closed subsets. Hence, Theorem 3 equivalently states that if V is 1-large, then (μ) is a prime ideal.

We will also make use of the following; see also Bulois (2018, Theorem 2.2). A reduced and irreducible scheme is *normal* if its local rings are integrally closed domains.

Theorem 4 (Herbig et al., Lemma 5.2) *Suppose $G = \mathbb{T}^\ell$ is a torus. If V is a stable and faithful $G_{\mathbb{C}}$-module, then both N and $N /\!\!/ G_{\mathbb{C}}$ are normal.*

The categorical quotient $N /\!\!/ G_{\mathbb{C}}$ in Theorem 4 is sometimes defined to be the *complex symplectic quotient*. In the case of small representations, however, it may fail to be the complexification of the real quotient and exhibit certain pathologies; see (Herbig et al. Section 2) for a discussion and an alternate definition of the complex symplectic quotient. In the case of 1-large representations, these technicalities can be avoided by the following, which holds for an arbitrary compact Lie group H and is proven in Herbig et al. with milder hypotheses.

Lemma 1 (Herbig et al., Lemma 2.8]) *Let H be a compact Lie group and V a unitary H-module. If V is 1-large as an $H_{\mathbb{C}}$-module, then*

$$\mathbb{R}[M_0] \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}[N]^{H_{\mathbb{C}}}.$$

The following definition and result, which holds for an arbitrary connected algebraic group H, will play an important role in the proof of Theorem 1 in Sect. 3.

Definition 2 (Popov and Vinberg 1994, Section 3.8) Let H be a connected algebraic group and let X be an irreducible H-variety. For $Y \subset X$, define $\mathbf{N}_H(Y) = \{g \in H \mid gY = Y\}$, the *normalizer* of Y. A subvariety $Y \subset X$ is a *Seshadri section* if $\overline{HY_0} = X$ for each irreducible component Y_0 of Y and $Hy \cap Y = \mathbf{N}_H(Y)y$ for any $y \in Y$.

Theorem 5 (Popov 1992, Corollary, page 169; Popov and Vinberg 1994, Theorem 3.14) *Let H be a connected algebraic group and let X be an irreducible normal H-variety. Suppose Y is a Seshadri section for the action of H on X such that* $\operatorname{codim}_X \overline{(X \smallsetminus HY)} \geq 2$. *Then Y is a Chevalley section, i.e., restriction of functions to Y defines an isomorphism $\mathbb{C}[X]^H \to \mathbb{C}[Y]^{\mathbf{N}_H(Y)}$.*

In this paper, we are primarily interested in the symplectic quotients $M_0(A)$ associated to a class of matrices that generalize the orbifold cases treated in Farsi et al. (2013, Theorem 7) and Corollary 1 and constitute a next step in the classification problem. To simplify the descriptions of these matrices, we introduce the following (non-standard) notation.

Definition 3 We say that an $\ell \times (\ell + k)$ weight matrix A is of *Type I_k* if it is of the form $A = \big(D, \overbrace{\boldsymbol{n}, \dots, \boldsymbol{n}}^{k}\big)$ where $D = \operatorname{diag}(-a_1, -a_2, \dots, -a_\ell)$ with each $a_i > 0$ and $\boldsymbol{n} = (n_1, n_2, \dots, n_\ell)^T$ with each $n_i > 0$. We will say that A is *Type II_k* if $A = (D, c_1\boldsymbol{n}, \dots, c_k\boldsymbol{n})$ with D and $\boldsymbol{n}$ as above and each $c_r \geq 1$. For a Type II_k weight matrix, we define

$$\alpha(A) := \operatorname{lcm}(a_1, \dots, a_\ell), \qquad m_i(A) := \frac{n_i \alpha(A)}{a_i} \qquad \text{for} \quad i = 1, \dots, \ell,$$

$$\beta(A) := \sum_{i=1}^{\ell} m_i(A), \qquad \text{and} \qquad \eta(A) := \alpha(A) + \beta(A).$$

We will often abbreviate $\alpha(A)$, $m_i(A)$, $\beta(A)$, and $\eta(A)$ as α, m_i, β, and η, respectively, when A is clear from the context.

For a weight matrix A of full rank, the representation V_A being faithful is equivalent to the nonzero $\ell \times \ell$ minors of A having no common factor, see Farsi et al. (2013). If A is Type II_k, then these minors are of the form $a_1 \dots a_\ell$ or $a_1 \dots a_{j-1} c_r n_j a_{j+1} \dots a_\ell$ for some $r = 1, \dots, k$, i.e., the product of the a_i or the same product with one a_j replaced with $c_r n_j$. The following is an immediate consequence.

Lemma 2 *Let A be a Type II_k weight matrix and $\ell > 1$. Then V_A is a faithful $\mathbb{T}^\ell$-module if and only if $\gcd(a_i, a_j) = 1$ for each $1 \le i < j \le n$, and for each $j = 1, \dots, \ell$, there is an $r \le k$ such that $\gcd(a_j, c_r n_j) = 1$.*

For a Type II_k weight matrix A, it is clear from the description of the moment map in Eq. (1) that the shell $J^{-1}(0)$ contains points with all nonzero coordinates. Hence by Herbig and Seaton (2015, Lemma 3 and Remark 2), if the $G_{\mathbb{C}}$-module V_A is faithful, then it is stable. It is easy to see that V_A is faithful as a G-module if and only if it is faithful as a $G_{\mathbb{C}}$-module. Combining this observation with Theorems 2, 3, and 4 yields the following.

Corollary 2 *Suppose A is a Type II_k weight matrix such that V_A is faithful as a $\mathbb{T}^\ell$-module. Then V_A is stable and 1-large as a $G_{\mathbb{C}}$-module, and the complex shell N_A is reduced, irreducible, and normal.*

If the hypotheses of Corollary 2 are satisfied, then by Herbig et al. (2013, Corollary 4.3), the ideal I_Z is generated by the components J_i of the moment map. Because the J_i are G-invariant in the case under consideration, we have

$$\mathbb{R}[M_0] = \mathbb{R}[V]^G/(J_1, \dots, J_\ell).$$

In particular, given Eq. (1), the quotient map $\mathbb{R}[V]^G \to \mathbb{R}[M_0]$ can be understood as defining the invariants $z_i\overline{z_i}$ for $i = 1, \dots, \ell$ in terms of the $z_i\overline{z_i}$ for $i = \ell+1, \dots, \ell+k$. By Lemma 1, an analogous statement holds for $\mathbb{C}[N]^{G_{\mathbb{C}}}$.

3 Proof of Theorem 1

In this section, we give the proof of our main result, Theorem 1, which is divided into several auxiliary results. Throughout this section, we consider a Type II_k weight matrix $A = (D, c_1\boldsymbol{n}, \dots, c_k\boldsymbol{n}) \in \mathbb{Z}^{\ell \times (\ell+k)}$ such that V_A is a faithful $\mathbb{T}^\ell$-module of dimension $n = \ell + k$. In addition, we let $B = \big(-\alpha(A), c_1\beta(A), \dots, c_k\beta(A)\big) \in \mathbb{Z}^{1\times(k+1)}$, see Definition 3. We assume throughout this section that $\ell > 1$; when $\ell = 1$, $A = B$ so that Theorem 1 is trivial. Note that $\alpha(A) = \alpha(B)$ and $\beta(A) = \beta(B)$ so that we may simply use α and β with no risk of confusion.

Our first result demonstrates that the $\mathbb{T}^1$-representation V_B is faithful.

Lemma 3 *Let $A = (D, c_1\boldsymbol{n}, \dots, c_k\boldsymbol{n}) \in \mathbb{Z}^{\ell\times(\ell+k)}$ be a Type II_k weight matrix. If V_A is a faithful $\mathbb{T}^\ell$-module, then $\gcd(\alpha, c_1\beta, \dots, c_k\beta) = 1$.*

Proof Suppose V_A is faithful, and let p be a prime that divides α and each $c_r\beta$ for contradiction. As p divides α, it divides some a_j; assume $p \mid a_1$ without loss of generality. By Lemma 2, it is not possible that $p \mid c_r$ for all r, so it must be that $p \mid \beta$. Similarly, $p \nmid a_i$ for each $i \neq 1$. Then $p \mid m_i = n_i\alpha/a_i$ for $i > 1$, so the fact that $p \mid \beta = \sum m_i$ implies that $p \mid m_1$. But as p does not divide any a_i except a_1, we have $\gcd(p, \alpha/a_1) = 1$. Hence $p \mid n_1$. As $p \mid a_1$ and $p \mid n_1$, p divides the first row of A, contradicting the fact that V_A is a faithful $\mathbb{T}^\ell$-module. □

To explain the relationship between the representations V_A and V_B, let $(u_1, \dots, u_n)$ denote coordinates for V_A and define the homomorphism $\psi\colon \mathbb{C}^\times \to (\mathbb{C}^\times)^\ell$ by

$$\psi(t) = (t^{\alpha/a_1}, \dots, t^{\alpha/a_\ell}). \tag{2}$$

As $\gcd(\alpha/a_1, \dots, \alpha/a_\ell) = 1$ by construction, ψ is injective and defines an action of $\mathbb{C}^\times$ on V_A with weight matrix $\big(\overbrace{-\alpha, \dots, -\alpha}^{\ell}, c_1\beta, \dots, c_k\beta\big)$. Hence, with respect to the action of $\psi(\mathbb{C}^\times)$, any (complex) $(k+1)$-dimensional subspace of V_A defined by imposing linear constraints on the first ℓ coordinates is isomorphic to V_B as a $\mathbb{C}^\times$-module.

We now indicate how to symplectically embed V_B as such a subspace of V_A while mapping the shell Z_B into the shell Z_A. A point $(u_1, \dots, u_n) \in V_A$ is in the shell Z_A if and only if $(|u_1|^2, \dots, |u_n|^2) \in \mathbb{R}^n_{\geq 0}$ is in the null space of the matrix A, see Eq. (1). Using coordinates $(z_1, \dots, z_{k+1})$ for V_B, define the linear map $\phi\colon\ V_B \to V_A$ by

$$\phi\colon\ (z_1, \dots, z_{k+1}) \longmapsto \left(\sqrt{\frac{m_1}{\beta}}z_1, \sqrt{\frac{m_2}{\beta}}z_1, \dots, \sqrt{\frac{m_\ell}{\beta}}z_1, z_2, z_3, \dots, z_{k+1}\right), \tag{3}$$

and then the image of ϕ is defined by the constraints

$$u_i = \sqrt{\frac{m_i}{m_1}}u_1, \quad i = 2, \dots, \ell. \tag{4}$$

We claim that ϕ preserves the symplectic forms, and the corresponding function $\mathbb{R}^{k+1}_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ given by

$$(|z_1|^2, \dots, |z_{k+1}|^2) \longmapsto \left(\frac{m_1}{\beta}|z_1|^2, \dots, \frac{m_\ell}{\beta}|z_1|^2, |z_2|^2, \dots, |z_{k+1}|^2\right) \tag{5}$$

maps the null space of B onto the null space of A.

Lemma 4 *The function $\phi\colon\ V_B \to V_A$ in Eq. (3) is a symplectic embedding that maps the shell $Z_B = J_B^{-1}(0)$ into the shell $Z_A = J_A^{-1}(0)$.*

Proof We compute

$$\phi^* \sum_{i=1}^n du_i \wedge d\overline{u_i} = \sum_{i=1}^{\ell} \frac{m_i}{\beta} dz_1 \wedge d\overline{z_1} + \sum_{i=2}^{k+1} dz_i \wedge d\overline{z_i} = \sum_{i=1}^{k+1} dz_i \wedge d\overline{z_i}$$

so that ϕ is a symplectic embedding. Let F denote the matrix of the linear extension $\mathbb{R}^{k+1} \to \mathbb{R}^n$ of the map defined in Eq. (5), and then F has upper-left $\ell \times 1$ block $(m_1/\beta, \ldots, m_\ell/\beta)^T$, the $k \times k$ identity in the lower-right block, and is zero elsewhere. Using the expression for the m_i in Definition 3 and recalling that $B = \big(-\alpha(A), c_1\beta(A), \ldots, c_k\beta(A)\big) \in \mathbb{Z}^{1\times(k+1)}$, one checks that

$$AF = \frac{1}{\beta}\boldsymbol{n}B.$$

Hence, the null space of B is contained in the null space of AF. It follows that ϕ maps Z_B into Z_A. □

Moreover, using the homomorphism ψ defined in Eq. (2), we have the following.

Lemma 5 *The normalizer of $\phi(V_B)$ is given by $\mathbf{N}_{(\mathbb{C}^\times)^\ell}\big(\phi(V_B)\big) = \psi(\mathbb{C}^\times)$, and hence $\mathbf{N}_{\mathbb{T}^\ell}\big(\phi(V_B)\big) = \psi(\mathbb{T}^1)$.*

Proof Let $\boldsymbol{t} = (t_1, t_2, \ldots, t_\ell) \in (\mathbb{C}^\times)^\ell$, and then $\boldsymbol{t}$ preserves the constraints (4) if and only if $t_1^{a_1} = t_i^{a_i}$ for $i = 2, \ldots, \ell$. Choosing $t \in \mathbb{C}^\times$ such that $t^{\alpha/a_1} = t_1$ and recalling that $\gcd(\alpha/a_1, \ldots, \alpha/a_\ell) = 1$, it follows that $\boldsymbol{t}$ is of the form $(t^{\alpha/a_1}, \ldots, t^{\alpha/a_\ell})$. □

Complexifying the underlying real spaces, we consider the z_i and $w_i := \overline{z_i}$ as independent complex coordinates for $V_B \otimes_{\mathbb{R}} \mathbb{C}$ and u_i and $v_i := \overline{u_i}$ as independent complex coordinates for $V_A \otimes_{\mathbb{R}} \mathbb{C}$. Recall that $N_A = (J_A \otimes_{\mathbb{R}} \mathbb{C})^{-1}(0) \subset V_A \otimes_{\mathbb{R}} \mathbb{C}$ and $N_B = (J_B \otimes_{\mathbb{R}} \mathbb{C})^{-1}(0) \subset V_B \otimes_{\mathbb{R}} \mathbb{C}$ denotes the corresponding complex shells, with N_A defined by

$$-a_i u_i v_i + n_i \sum_{j=1}^{k} c_j u_{\ell+j} v_{\ell+j} = 0 \quad \text{for} \quad i = 1, \ldots, \ell. \tag{6}$$

and N_B by

$$-\alpha z_1 w_1 + \beta \sum_{j=1}^{k} c_j z_{j+1} w_{j+1} = 0. \tag{7}$$

We now demonstrate that the image of N_B under $\phi_{\mathbb{C}} = \phi \otimes_{\mathbb{R}} \mathbb{C}$ is a Seshadri section for the action of $(\mathbb{C}^\times)^\ell$ on N_A, see Definition 2, satisfying the hypotheses of Theorem 5.

Lemma 6 *The image $S := \phi_{\mathbb{C}}(N_B)$ of the complex shell N_B is a Seshadri section for the action of $(\mathbb{C}^\times)^\ell$ on the complex shell $N_A \subset V_A \otimes_{\mathbb{R}} \mathbb{C}$. Moreover, the (complex) codimension of $N_A \smallsetminus \overline{(\mathbb{C}^\times)^\ell S}$ in N_A is 2.*

Proof First observe that in the coordinates $(\boldsymbol{z}, \boldsymbol{w})$ for $V_B \otimes_{\mathbb{R}} \mathbb{C}$, $\phi_{\mathbb{C}}(\boldsymbol{z}, \boldsymbol{w}) = \big(\phi(\boldsymbol{z}), \phi(\boldsymbol{w})\big)$. By Lemma 3 and Corollary 2, both N_A and N_B are reduced, irreducible, and normal.

Fix a point $(\boldsymbol{u}, \boldsymbol{v}) \in N_A$, i.e., satisfying Eq. (6), and assume that each $u_i \neq 0$ for $i \leq \ell$. For $i = 2, \dots, \ell$, choose t_i such that

$$t_i^{-a_i} = \sqrt{\frac{m_i}{m_1}\frac{u_1}{u_i}}. \tag{8}$$

Letting $\boldsymbol{t} = (1, t_2, \dots, t_\ell) \in (\mathbb{C}^\times)^\ell$, we claim that $\boldsymbol{t}(\boldsymbol{u}, \boldsymbol{v}) \in S$. To see this, define

$$z_1 := u_1\sqrt{\beta/m_1}, \qquad w_1 = \frac{\sqrt{m_1\beta}}{\alpha u_1}\sum_{j=1}^{k} c_j u_{\ell+j} v_{\ell+j},$$
$$z_{i+1} = \boldsymbol{t}^{c_i \boldsymbol{n}} u_{i+\ell}, \qquad w_{i+1} = \boldsymbol{t}^{-c_i \boldsymbol{n}} v_{i+\ell}, \quad i = 1, \dots, k,$$

and then direct substitution of $(\boldsymbol{z}, \boldsymbol{w})$ into Eq. (7) demonstrates that $(\boldsymbol{z}, \boldsymbol{w}) \in N_B$. By Eq. (6), each v_i with $i = 1, \dots, \ell$ is given by

$$v_i = \frac{n_i}{a_i u_i}\sum_{j=1}^{k} c_j u_{\ell+j} v_{\ell+j} = \frac{m_i}{\alpha u_i}\sum_{j=1}^{k} c_j u_{\ell+j} v_{\ell+j}.$$

Using this fact, a simple computation confirms that

$$\boldsymbol{t}(\boldsymbol{u}, \boldsymbol{v}) = \big(\phi(\boldsymbol{z}), \phi(\boldsymbol{w})\big) \in S.$$

If $(\boldsymbol{u}, \boldsymbol{v}) \in N_A$ such that each $v_i \neq 0$ for $i \leq \ell$, then choosing $\boldsymbol{t}$ such that

$$t_i^{a_i} = \sqrt{\frac{m_i}{m_1}\frac{v_1}{v_i}}, \quad i = 2, \dots, \ell,$$

a similar computation demonstrates that $\boldsymbol{t}(\boldsymbol{u}, \boldsymbol{v}) \in S$ as well.

Taking the closure to account for points with some $u_i = 0$ and $v_j = 0$ for $i, j \leq \ell$, we have

$$\overline{(\mathbb{C}^\times)^\ell S} = N_A.$$

In particular, note that $N_A \smallsetminus (\mathbb{C}^\times)^\ell S$ consists of those points in N_A where some $u_i = 0$ and some $v_j = 0$ for $i, j \leq \ell$; in particular $N_A \smallsetminus (\mathbb{C}^\times)^\ell S$ is closed and has codimension 2 in N_A.

Now, the proof of Lemma 5 demonstrates that $\mathbf{N}_{(\mathbb{C}^\times)^\ell}(S) = \psi(\mathbb{C}^\times)$, where ψ is defined in Eq. 2. Indeed, for $\phi_{\mathbb{C}}(\boldsymbol{z}, \boldsymbol{w}) \in S$, $\boldsymbol{t}\phi_{\mathbb{C}}(\boldsymbol{z}, \boldsymbol{w}) \in S$ if and only if $\boldsymbol{t} \in \psi(\mathbb{C}^\times)$ is clear whenever $z_1 \neq 0$ or $w_1 \neq 0$. Hence, it remains only to show that $(\mathbb{C}^\times)^\ell \phi_{\mathbb{C}}(\boldsymbol{z}, \boldsymbol{w}) \cap S = \mathbf{N}_{(\mathbb{C}^\times)^\ell}(S)\phi_{\mathbb{C}}(\boldsymbol{z}, \boldsymbol{w})$ for a point $\phi_{\mathbb{C}}(\boldsymbol{z}, \boldsymbol{w}) \in S$ with $z_1 = w_1 = 0$. In this case, for arbitrary $\boldsymbol{t} \in (\mathbb{C}^\times)^\ell$, we choose an $s \in \mathbb{C}^\times$ such that $s^\beta = \boldsymbol{t}^{\boldsymbol{n}}$ and compute

$$\begin{aligned}\psi(s)\phi_{\mathbb{C}}(\boldsymbol{z},\boldsymbol{w}) &= \big(0,\ldots,0, s^{c_1\sum_i n_i\alpha/a_i} z_2,\ldots, s^{c_k\sum_i n_i\alpha/a_i} z_{k+1},\\ &\quad 0,\ldots,0, s^{-c_1\sum_i n_i\alpha/a_i} w_2,\ldots, s^{-c_k\sum_i n_i\alpha/a_i} w_{k+1}\big)\\ &= \big(0,\ldots,0, s^{c_1\beta} z_2,\ldots, s^{c_k\beta} z_{k+1},\\ &\quad 0,\ldots,0, s^{-c_1\beta} w_2,\ldots, s^{-c_k\beta} w_{k+1}\big) = \boldsymbol{t}\phi_{\mathbb{C}}(\boldsymbol{z},\boldsymbol{w}).\end{aligned}$$

Hence for such a point, $(\mathbb{C}^\times)^\ell \phi_{\mathbb{C}}(\boldsymbol{z},\boldsymbol{w}) = \mathbf{N}_{(\mathbb{C}^\times)^\ell}(S)\phi_{\mathbb{C}}(\boldsymbol{z},\boldsymbol{w})$. □

Combining Lemma 6 with Theorem 5 yields the following.

Corollary 3 *The restriction of functions on N_A to S defines an isomorphism $\mathbb{C}[N_A]^{(\mathbb{C}^\times)^\ell} \to \mathbb{C}[S]^{\mathbf{N}_{(\mathbb{C}^\times)^\ell}(S)}$.*

As S is isomorphic to the shell N_B via the embedding $\phi_{\mathbb{C}}$, which is equivariant with respect to the actions of $\mathbb{C}^\times$ and $\mathbf{N}_{(\mathbb{C}^\times)^\ell}(S)$ identified via ψ, it follows that $\phi_{\mathbb{C}}^*$ induces an isomorphism $\phi_{\mathbb{C}}^*\colon \mathbb{C}[S]^{\mathbf{N}_{(\mathbb{C}^\times)^\ell}(S)} \to \mathbb{C}[N_B]^{\mathbb{C}^\times}$. As $\phi_{\mathbb{C}}$ is a linear map, $\phi_{\mathbb{C}}^*$ preserves the grading. As the representations of $(\mathbb{C}^\times)^\ell$ and $\mathbb{C}^\times$ corresponding to A and B, respectively, are 1-large by Corollary 2, we have by Lemma 1 that $\mathbb{R}[Z_A]^{\mathbb{T}^\ell} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}[N_A]^{(\mathbb{C}^\times)^\ell}$ and $\mathbb{R}[Z_B]^{\mathbb{T}^1} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}[N_B]^{\mathbb{C}^\times}$. That is, ϕ^* induces a graded isomorphism of the algebras of real regular functions $\mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]$. By Lemma 4, this isomorphism is Poisson.

Summarizing, we have the following.

Corollary 4 *The restriction of functions to S and pulling back via $\phi_{\mathbb{C}}$ are both graded isomorphisms*

$$\mathbb{C}[N_B]^{\mathbb{C}^\times} \xrightarrow{\phi_{\mathbb{C}}^*} \mathbb{C}[S]^{\mathbf{N}_{(\mathbb{C}^\times)^\ell}(S)} \longrightarrow \mathbb{C}[N_A]^{(\mathbb{C}^\times)^\ell},$$

and the composition of these maps induces a graded Poisson isomorphism of the real algebras

$$\Psi\colon\ \mathbb{R}[M_0(A)] \longrightarrow \mathbb{R}[M_0(B)].$$

By Lemmas 4 and 6 and Corollary 4, it follows that ϕ induces an isomorphism between the Zariski closures of the real algebraic varieties defined by $\mathbb{R}[Z_A]^{\mathbb{T}^\ell}$ and $\mathbb{R}[Z_B]^{\mathbb{T}^1}$. To complete the proof of Theorem 1, it remains only to show that the semialgebraic conditions are preserved, i.e., the map ϕ induces a homeomorphism between the symplectic quotients.

Lemma 7 *The map ϕ induces a homeomorphism $M_0(B) = Z_B/\mathbb{T}^1 \to M_0(A) = Z_A/\mathbb{T}^\ell$.*

Proof It is clear from Lemma 5 that ϕ maps $\mathbb{T}^1$-orbits into $\mathbb{T}^\ell$-orbits injectively; in particular, the only point in Z_B with $z_1 = 0$ is the origin. As $\phi(Z_B) \subset (Z_A)$ by Lemma 4, it is sufficient to show that each element of Z_A is in the orbit of an element of $\phi(Z_B)$. So let $\boldsymbol{u} = (u_1, \ldots, u_n) \in Z_A$, and then for $i = 1, \ldots, \ell$, the fact that $J_i(\boldsymbol{u}) = 0$ and the expression for J_i in Eq. (1) imply

$$|u_i| = \sqrt{\frac{a_1 n_i}{a_i n_1}}|u_1| = \sqrt{\frac{m_i}{m_1}}|u_1|.$$

With this, each t_i for $i = 2, \ldots, \ell$ defined as in Eq. (8) is an element of $\mathbb{T}^1$. Hence, we may follow the proof of Lemma 6, defining $\boldsymbol{t} \in \mathbb{T}^\ell$ and $\boldsymbol{z} \in V_B$ in the same way, and then the same computations verify that $\boldsymbol{tu} = \phi(\boldsymbol{z})$ and $\boldsymbol{z} \in Z_B$. It follows that each $\mathbb{T}^\ell$-orbit in Z_A intersects $\phi(Z_B)$.

We leave it to the reader to show that the inverse homeomorphism is induced by the linear map

$$(u_1, u_2, \ldots, u_{k+\ell}) \mapsto \left(\sqrt{\frac{\beta}{m_1}}u_1, u_{\ell+1}, \ldots, u_{k+\ell}\right). \qquad \square$$

We illustrate Theorem 1 with the following.

Example 1 The weight matrix

$$A = \begin{pmatrix} -3 & 0 & 0 & 1 & 2 & 3 & 3 \\ 0 & -4 & 0 & 3 & 6 & 9 & 9 \\ 0 & 0 & -5 & 2 & 4 & 6 & 6 \end{pmatrix}$$

is Type II_4 with $\alpha = 60$, $n_1 = 1$, $n_2 = 3$, $n_3 = 2$, $c_1 = 1$, $c_2 = 2$, and $c_3 = c_4 = 3$. Hence, $m_1 = 20$, $m_2 = 45$, $m_3 = 24$, and $\beta = 89$, and the symplectic quotient $M_0(A)$ is graded regularly symplectomorphic to that associated to $(-60, 89, 178, 267, 267)$.

4 Classification for Type I_k matrices

In the case $k = 1$, Corollary 1 implies that two weight matrices A_1 and A_2 yield graded regularly symplectomorphic symplectic quotients if and only if $\eta(A_1) = \eta(A_2)$, i.e., if and only if $\alpha(A_1) + \beta(A_1) = \alpha(A_2) + \beta(A_2)$. For $k > 1$, this is no longer the case, as we demonstrate with the following. Recall that the *Krull dimension* of a commutative ring R is the supremum of the lengths of ascending chains of prime ideals in R.

Lemma 8 *Let* $A = \big(-\alpha, \overbrace{\beta, \ldots, \beta}^{k}\big)$ *and* $B = \big(-\alpha', \overbrace{\beta', \ldots, \beta'}^{k'}\big)$ *such that* V_A *and* V_B *are faithful* $\mathbb{T}^1$*-modules. If* $k \geq 2$ *and the symplectic quotients* $M_0(A)$ *and* $M_0(B)$ *are graded regularly diffeomorphic, then* $k = k'$, $\alpha = \alpha'$ *and* $\beta = \beta'$.

Proof First note that as V_A and V_B are faithful, $\gcd(\alpha, \beta) = \gcd(\alpha', \beta') = 1$. The existence of a graded regular diffeomorphism implies that $\mathbb{R}[M_0(A)]$ is graded isomorphic to $\mathbb{R}[M_0(B)]$. As the Krull dimensions of $\mathbb{R}[M_0(A)]$ and $\mathbb{R}[M_0(B)]$ are given by $2k$ and $2k'$, respectively, it follows that $k = k'$.

Let $\mathcal{Q}(A)$ denote the subalgebra of $\mathbb{R}[M_0(A)]$ that is generated by the quadratic monomials of the form $z_i\overline{z_i} + I_{Z_A}^G$ for $i = 1\ldots, k+1$ and $z_{1+i}\overline{z_{1+j}} + I_{Z_A}^G$ for $1 \leq i, j \leq k$, and define $\mathcal{Q}(B)$ identically as a subalgebra of $\mathbb{R}[M_0(B)]$. Note that $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ are obviously graded isomorphic. The lowest-degree element of $\mathbb{R}[M_0(A)]$ that is not an element of $\mathcal{Q}(A)$ has degree $\alpha+\beta$, and similarly for $\mathbb{R}[M_0(B)]$, so we can conclude that $\alpha + \beta = \alpha' + \beta'$. Finally, the number of monomials in $\mathbb{R}[M_0(A)]$ of degree $\alpha + \beta$ that are not elements of $\mathcal{Q}(A)$ is $\binom{\alpha+k-1}{k-1}$, and hence $\binom{\alpha+k-1}{k-1} = \binom{\alpha'+k-1}{k-1}$, i.e., $(\alpha + k - 1)!/\alpha! = (\alpha' + k - 1)!/\alpha'!$. As $k > 1$, it follows that $\alpha = \alpha'$, and hence $\beta = \beta'$. ◻

Corollary 5 *The graded regular symplectomorphism classes of symplectic quotients associated to Type I_k weight matrices with $k > 1$ are classified by the triple $(k, \alpha(A), \beta(A))$. Moreover, these graded regular symplectomorphism classes coincide with the graded regular diffeomorphism classes.*

It is not clear whether an analog to Lemma 8 is true for Type II_k matrices, but a proof using only the grading of $\mathbb{R}[M_0]$ as in Lemma 8 is not possible. First note that such a generalization would require restricting to specific representatives, e.g., requiring that $\gcd(c_1, \ldots, c_k) = 1$. Otherwise, it is possible that a $1 \times (k + 1)$ Type II_k matrix could be written in terms of α, β, and the c_i in more than one way, e.g., $(-1, 4, 12)$ could correspond to $\alpha = 1$, $\beta = 2$, $c_1 = 2$, and $c_2 = 6$ or to $\alpha = 1$, $\beta = 4$, $c_1 = 1$, and $c_2 = 3$. However, even with such a restriction, it is possible that $\mathbb{R}[M_0(A)]$ and $\mathbb{R}[M_0(B)]$ have the same Hilbert series yet fail to be graded regularly symplectomorphic. We will illustrate this in the next section.

5 The Hilbert series does not classify symplectic quotients by tori

If $R = \oplus_{i=0}^{\infty} R_i$ is a locally finite-dimensional $\mathbb{N}$-graded algebra over the field $\mathbb{K}$, recall that the *Hilbert series* of R is the generating function of the dimensions of the R_i as $\mathbb{K}$-vector spaces,

$$\mathrm{Hilb}_R(t) = \sum_{i=0}^{\infty} t^i \dim_{\mathbb{K}} R_i.$$

If R is finitely generated, then $\mathrm{Hilb}_R(t)$ is the Taylor series of a rational function that converges for $|t| < 1$.

The graded regular symplectomorphisms given by Theorem 1 were initially discovered by computing Hilbert series of the algebras of regular functions on symplectic quotients associated to large classes of weight matrices and looking for cases that coincide. While the Hilbert series has been a valuable heuristic to indicate potential graded regular symplectomorphisms and an important tool to distinguish between non-graded regularly symplectomorphic cases, one would likely guess that there are cases with the same Hilbert series that are not graded regularly symplectomorphic. In this section, we give examples to indicate that this is the case: the Hilbert series is not a fine enough invariant to distinguish graded regular symplectomorphism classes of symplectic quotients by tori. These examples further illustrate that two symplectic quotients can have several isomorphic structures yet fail to be graded regularly symplectomorphic.

Let $A = (-2, 3, 6)$ and $B = (-3, 2, 6)$. Note that these are both Type II_2 weight matrices; A corresponding to $\alpha = 2$, $\beta = 3$, $c_1 = 1$, and $c_2 = 2$; and B corresponding to $\alpha = 3$, $\beta = 2$, $c_1 = 1$, and $c_2 = 3$). Because the Hilbert series of symplectic quotients by $\mathbb{T}^1$ only depends on the absolute value of the weights (see Herbig and Seaton 2014, p. 47), the Hilbert series of $\mathbb{R}[M_0(A)]$ and $\mathbb{R}[M_0(B)]$ coincide. In particular, they are both given by

$$\frac{1 + t^3 + 2t^4 + t^5 + t^8}{(1 - t^5)(1 - t^3)(1 - t^2)^3}.$$

The off-shell invariants $\mathbb{R}[V_A]^{\mathbb{T}^1}$ are generated by

$$p_0 = z_1\overline{z_1}, \quad p_1 = z_2\overline{z_2}, \quad p_2 = z_3\overline{z_3}, \quad p_3 = z_2^2\overline{z_3}, \quad p_4 = z_3\overline{z_2}^2,$$
$$p_5 = z_1^3 z_3, \quad p_6 = \overline{z_1}^3\overline{z_3}, \quad p_7 = z_1^3 z_2^2, \quad p_8 = \overline{z_1}^3\overline{z_2}^2,$$

and the moment map determines p_0 via $2p_0 = 3p_1 + 6p_2$. The off-shell invariants $\mathbb{R}[V_B]^{\mathbb{T}^1}$ are generated by

$$q_0 = u_1\overline{u_1}, \quad q_1 = u_2\overline{u_2}, \quad q_2 = u_3\overline{u_3}, \quad q_3 = u_1^2 u_3, \quad q_4 = \overline{u_1}^2\overline{u_3},$$
$$q_5 = u_2^3\overline{u_3}, \quad q_6 = u_3\overline{u_2}^3, \quad q_7 = u_1^2 u_2^3, \quad q_8 = \overline{u_1}^2\overline{u_2}^3,$$

and the shell relation is given by $3q_0 = 2q_1 + 6q_2$.

Proposition 1 *For the weight matrices $A = (-2, 3, 6)$ and $B = (-3, 2, 6)$, the following hold true.*

(i.) The algebras $\mathbb{R}[M_0(A)] \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{R}[M_0(B)] \otimes_{\mathbb{R}} \mathbb{C}$ are graded Poisson isomorphic. Hence, the complex symplectic quotients are isomorphic as Poisson varieties.

(ii.) The algebras $\mathbb{R}[M_0(A)]$ and $\mathbb{R}[M_0(B)]$ are graded isomorphic. However, no graded isomorphism $\mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]$ preserves the inequalities describing the semialgebraic sets $M_0(A)$ and $M_0(B)$.

An immediate consequence of (*ii.*) is that the symplectic quotients $M_0(A)$ and $M_0(B)$ are not graded regularly symplectomorphic.

Proof of Proposition 1(i.) As in the proof of Lemma 6, we complexify the underlying real vector spaces to consider the z_i, $w_i := \overline{z_i}$, u_i, and $v_i := \overline{u_i}$ as independent complex variables. Then an easy-to-identify isomorphism over $\mathbb{C}$ is induced by the linear map $\phi\colon V_A \otimes_{\mathbb{R}} \mathbb{C} \to V_B \otimes_{\mathbb{R}} \mathbb{C}$ given by

$$\phi\colon (z_1, z_2, z_3, w_1, w_2, w_3) \longmapsto (\sqrt{-1}w_2, \sqrt{-1}w_1, z_3, \sqrt{-1}z_2, \sqrt{-1}z_1, w_3).$$

A simple computation demonstrates that ϕ is equivariant with respect to the two $\mathbb{C}^\times$-actions, implying that the corresponding map $\phi^*\colon\ \mathbb{C}[V_B \otimes_\mathbb{R} \mathbb{C}]^{\mathbb{C}^\times} \to \mathbb{C}[V_A \otimes_\mathbb{R} \mathbb{C}]^{\mathbb{C}^\times}$ is an isomorphism. Using coordinates $(u_1, u_2, u_3, v_1, v_2, v_3)$ for $V_B \otimes_\mathbb{R} \mathbb{C}$, we have

$$\begin{aligned}\phi^*(du_1 \wedge dv_1 + du_2 \wedge dv_2 + du_3 \wedge dv_3) &= -dw_2 \wedge dz_2 - dw_1 \wedge dz_1 + dz_3 \wedge dw_3\\ &= dz_1 \wedge dw_1 + dz_2 \wedge dw_2 + dz_3 \wedge dw_3\end{aligned}$$

so that ϕ is a symplectic embedding.

Identifying the real and complex invariants via $w_i = \overline{z_i}$ and $v_i = \overline{w_i}$, the map ϕ^* is given on generators by

$$\begin{aligned}
&\phi^* q_0 = -z_2 w_2 = -p_1, && \phi^* q_1 = -z_1 w_1 = -p_0,\\
&\phi^* q_2 = z_3 w_3 = p_2, && \phi^* q_3 = -z_3 w_2^2 = -p_4,\\
&\phi^* q_4 = -z_2^2 w_3 = -p_3, && \phi^* q_5 = -\sqrt{-1} w_1^3 w_3 = -\sqrt{-1} p_6,\\
&\phi^* q_6 = -\sqrt{-1} z_1^3 z_3 = -\sqrt{-1} p_5, && \phi^* q_7 = \sqrt{-1} w_1^3 w_2^2 = \sqrt{-1} p_8,\\
&\phi^* q_8 = \sqrt{-1} z_1^3 z_2^2 = \sqrt{-1} p_7,
\end{aligned}$$

so that $\phi^* J_B = J_A$. Hence ϕ^* induces an isomorphism $\mathbb{R}[M_0(B)] \otimes_\mathbb{R} \mathbb{C} \to \mathbb{R}[M_0(A)] \otimes_\mathbb{R} \mathbb{C}$, completing the proof. □

Clearly, the isomorphism ϕ^* does not restrict to a map $\mathbb{R}[M_0(B)] \to \mathbb{R}[M_0(A)]$ of the real algebras. Hence, to determine an isomorphism over $\mathbb{R}$, we need a more explicit description of $\mathbb{R}[M_0(A)]$ and $\mathbb{R}[M_0(B)]$.

Proof of Proposition 1(ii.) Using *Macaulay2* (Grayson and Stillman 2012), we compute the relations among the generators $p_1, p_2, \ldots, p_8$ of $\mathbb{R}[M_0(A)]$ to be

$$\begin{aligned}
&2p_0 - 3p_1 - 6p_2, \quad p_1^2 p_2 - p_4 p_3, \quad p_4 p_6 - p_2 p_8, \quad p_3 p_5 - p_2 p_7,\\
&\quad p_1^2 p_6 - p_3 p_8, \quad p_1^2 p_5 - p_4 p_7,\\
&\quad 324 p_1 p_2^3 + 216 p_2^4 + 27 p_1 p_4 p_3 + 162 p_2 p_4 p_3 - 8 p_5 p_6,\\
&\quad 27 p_1^3 p_3 + 324 p_1 p_2^2 p_3 + 216 p_2^3 p_3 + 162 p_4 p_3^2 - 8 p_6 p_7,\\
&\quad 27 p_1^3 p_4 + 324 p_1 p_2^2 p_4 + 216 p_2^3 p_4 + 162 p_4^2 p_3 - 8 p_5 p_8,\\
&\quad 432 p_2^5 - 81 p_1^2 p_4 p_3 - 432 p_1 p_2 p_4 p_3 - 648 p_2^2 p_4 p_3 + 24 p_1 p_5 p_6 - 16 p_2 p_5 p_6,\\
&\quad 27 p_1^5 + 162 p_1^2 p_4 p_3 + 324 p_1 p_2 p_4 p_3 + 216 p_2^2 p_4 p_3 - 8 p_7 p_8,\\
&\quad 324 p_1 p_2^2 p_3 p_6 + 216 p_2^3 p_3 p_6 - 8 p_6^2 p_7 + 27 p_1 p_3^2 p_8 + 162 p_2 p_3^2 p_8,\\
&\quad 324 p_1 p_2^2 p_4 p_5 + 216 p_2^3 p_4 p_5 + 27 p_1 p_4^2 p_7 + 162 p_2 p_4^2 p_7 - 8 p_5^2 p_8,\\
&\quad 432 p_2^4 p_3 p_6 + 24 p_1 p_6^2 p_7 - 16 p_2 p_6^2 p_7 - 81 p_1^2 p_3^2 p_8 - 432 p_1 p_2 p_3^2 p_8 - 648 p_2^2 p_3^2 p_8,\\
&\quad 432 p_2^4 p_4 p_5 - 81 p_1^2 p_4^2 p_7 - 432 p_1 p_2 p_4^2 p_7 - 648 p_2^2 p_4^2 p_7 + 24 p_1 p_5^2 p_8 - 16 p_2 p_5^2 p_8.
\end{aligned}$$

Similarly, the relations among the generators $q_1, q_2, \ldots, q_8$ of $\mathbb{R}[M_0(B)]$ are given by

$$3q_0 - 2q_1 - 6q_2, \quad 4q_1^2 q_2 + 24 q_1 q_2^2 + 36 q_2^3 - 9 q_3 q_4, \quad q_4 q_6 - q_2 q_8,$$

$$q_3q_5 - q_2q_7, \qquad 4q_1^2q_6 + 24q_1q_2q_6 + 36q_2^2q_6 - 9q_3q_8,$$
$$4q_1^2q_5 + 24q_1q_2q_5 + 36q_2^2q_5 - 9q_4q_7,$$
$$108q_1q_2^3 + 216q_2^4 + 9q_1q_3q_4 - 54q_2q_3q_4 - 4q_5q_6,$$
$$q_1^3q_4 - q_5q_8, \qquad q_1^3q_3 - q_6q_7,$$
$$108q_2^5 - 9q_1^2q_3q_4 + 18q_1q_2q_3q_4 - 27q_2^2q_3q_4 + 4q_1q_5q_6 + 16q_2q_5q_6,$$
$$4q_1^5 + 24q_1q_5q_6 + 36q_2q_5q_6 - 9q_7q_8,$$
$$108q_1q_2^2q_3q_6 + 216q_2^3q_3q_6 - 4q_6^2q_7 + 9q_1q_3^2q_8 - 54q_2q_3^2q_8,$$
$$108q_1q_2^2q_4q_5 + 216q_2^3q_4q_5 + 9q_1q_4^2q_7 - 54q_2q_4^2q_7 - 4q_5^2q_8,$$
$$108q_2^4q_3q_6 + 4q_1q_6^2q_7 + 16q_2q_6^2q_7 - 9q_1^2q_3^2q_8 + 18q_1q_2q_3^2q_8 - 27q_2^2q_3^2q_8,$$
$$108q_2^4q_4q_5 - 9q_1^2q_4^2q_7 + 18q_1q_2q_4^2q_7 - 27q_2^2q_4^2q_7 + 4q_1q_5^2q_8 + 16q_2q_5^2q_8.$$

Define the map $\Psi\colon\ \mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]$ by

$$\Psi(p_1) = q_1 + 3q_2, \qquad \Psi(p_2) = -\frac{3}{2}q_2, \qquad \Psi(p_3) = q_4,$$
$$\Psi(p_4) = -\frac{27}{8}q_3, \qquad \Psi(p_5) = q_6, \qquad \Psi(p_6) = -\frac{81}{16}q_5,$$
$$\Psi(p_7) = -\frac{2}{3}q_8, \qquad \Psi(p_8) = -\frac{729}{64}q_7.$$

A tedious though elementary computation demonstrates that Ψ maps the ideal of relations of the p_i into the ideal of relations of the q_i, and Ψ^{-1} similarly maps the ideal of relations of the q_i into the ideal of relations of the p_j. Therefore, $\Psi\colon\ \mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]$ is an isomorphism. Note that $p_2 = z_3\overline{z_3} \geq 0$, while $\Psi(p_2) = -3q_2/2 \leq 0$ so that Ψ does not preserve the inequalities.

To show that any graded isomorphism $\mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]$ fails to preserve the inequalities, suppose for contradiction that $\Phi\colon\ \mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]$ is such a graded isomorphism. Let $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ denote the subalgebras of $\mathbb{R}[M_0(A)]$ and $\mathbb{R}[M_0(B)]$, respectively, that are generated by elements of degree at most four. Then Φ restricts to an isomorphism $\Phi|_{\mathcal{Q}(A)}\colon\ \mathcal{Q}(A) \to \mathcal{Q}(B)$.

Again using *Macaulay2* (Grayson and Stillman 2012), the algebra $\mathcal{Q}(A)$ generated by $p_1, p_2, \ldots, p_6$ has relations generated by

$$R_1 = p_1^2p_2 - p_3p_4, \qquad R_2 = 27\big(4p_2^3(3p_1 + 2p_2) + (p_1 + 6p_2)p_3p_4\big) - 8p_5p_6,$$
$$R_3 = -81p_1^2p_3p_4 - 8(-54p_2^5 + 54p_1p_2p_3p_4 + 81p_2^2p_3p_4 - 3p_1p_5p_6 + 2p_2p_5p_6),$$
$$R_4 = 27p_3p_4(p_1^3 + 12p_1p_2^2 + 8p_2^3 + 6p_3p_4) - 8p_1^2p_5p_6,$$

and the algebra $\mathcal{Q}(B)$ generated by $q_1, q_2, \ldots, q_6$ has relations generated by

$$R_1' = 4q_2(q_1 + 3q_2)^2 - 9q_3q_4,$$
$$R_2' = 108q_2^3(q_1 + 2q_2) + 9(q_1 - 6q_2)q_3q_4 - 4q_5q_6,$$
$$R_3' = 108q_2^5 - 9(q_1^2 - 2q_1q_2 + 3q_2^2)q_3q_4 + 4(q_1 + 4q_2)q_5q_6,$$

$$R'_4 = 9q_1^3 q_3 q_4 - 4(q_1 + 3q_2)^2 q_5 q_6.$$

As Φ preserves the grading, it must be of the form

$$\begin{aligned} &\Phi(p_1) = c_{11}q_1 + c_{12}q_2, \quad \Phi(p_2) = c_{21}q_1 + c_{22}q_2, \quad \Phi(p_3) = c_{33}q_3 + c_{34}q_4,\\ &\Phi(p_4) = c_{43}q_3 + c_{44}q_4, \quad \Phi(p_5) = c_{55}q_5 + c_{56}q_6, \quad \Phi(p_6) = c_{65}q_5 + c_{66}q_6,\\ &\Phi(p_7) = c_{77}q_7 + c_{78}q_8, \quad \Phi(p_8) = c_{87}q_7 + c_{88}q_8. \end{aligned} \tag{9}$$

Using the fact that Φ preserves the grading and maps the ideal of relations for the p_i into the ideal of relations for the q_i, we must have

$$\Phi(R_1) = k_1 R'_1, \qquad \text{and} \qquad \Phi(R_2) = k_2 R'_2 + k_3 q_1 R'_1 + k_4 q_2 R'_2$$

for some $k_1, k_2, k_3, k_4 \in \mathbb{R}$. Computing the q_1^3, q_3^2, and $q_1^2 q_2$ coefficients of each side of the first equation and the $q_1^2 q_2^2$, q_2^4, $q_1 q_2^3$, $q_2 q_3 q_4$, and $q_1 q_3 q_4$ coefficients of each side of the second equation yields the system

$\Phi(R_1)$:	q_1^3 :	$c_{11}^2 c_{21} = 0,$
	q_3^2 :	$c_{33}c_{43} = 0,$
	$q_1^2 q_2$:	$c_{11}(2c_{12}c_{21} + c_{11}c_{22}) = 4k_1,$
$\Phi(R_2)$:	$q_1^2 q_2^2$:	$81c_{21}c_{22}(3c_{12}c_{21} + 3c_{11}c_{22} + 4c_{21}c_{22}) = k_2(6k_3 + k_4),$
	$q_1 q_2^3$:	$9c_{22}^2(9c_{12}c_{21} + 3c_{11}c_{22} + 8c_{21}c_{22}) = k_2(9 + 3k_3 + 2k_4),$
	q_2^4 :	$3c_{22}^3(3c_{12} + 2c_{22}) = k_2(6 + k_4),$
	$q_1 q_3 q_4$:	$3(c_{11} + 6c_{21})(c_{34}c_{43} + c_{33}c_{44}) = k_2(1 - k_3),$
	$q_2 q_3 q_4$:	$3(c_{12} + 6c_{22})(c_{34}c_{43} + c_{33}c_{44}) = -k_2(6 + k_4),$

Every solution of this system not corresponding to $\Phi(p_i) = 0$ for some i satisfies $c_{11} = -2c_{22}/3$, $c_{12} = -2c_{22}$, and $c_{21} = 0$. Hence, though $p_1 \geq 0$ and $p_2 \geq 0$, either $c_{22} > 0$ so that $\Phi(p_1) = -2c_{22}(q_1/3 + q_2) < 0$ for any nonzero q_1 or q_2, or $c_{22} < 0$ so that $\Phi(p_2) = c_{22}q_2 < 0$ for any nonzero q_2. In either case, Φ does not preserve the inequalities describing the semilagebraic sets $M_0(A)$ and $M_0(B)$. □

For another example, consider $A' = (-2, 1, 1)$ and $B' = (-1, 2, 1)$. Both are weight matrices of type Type II_2; the former has $\alpha = 2$, $\beta = 1$, $c_1 = 1 = c_2$, while the latter has $\alpha = 1$, $\beta = 1$, $c_1 = 2$, and $c_2 = 1$. As above, $\mathbb{R}[M_0(A')]$ and $\mathbb{R}[M_0(B')]$ have the same Hilbert series, given by

$$\frac{1 + 2t^2 + 4t^3 + 2t^4 + t^6}{(1 - t^3)^2(1 - t^2)^2}.$$

The quadratic off-shell invariants of the action with weight matrix A' are spanned by $z_1\overline{z_1}$, $z_2\overline{z_2}$, $z_3\overline{z_3}$, $z_2\overline{z_3}$, and $z_3\overline{z_2}$ with relation $(z_2\overline{z_2})(z_3\overline{z_3}) = (z_2\overline{z_3})(z_3\overline{z_2})$, and the moment map determines $z_1\overline{z_1}$ in terms of $z_2\overline{z_2}$, $z_3\overline{z_3}$. For the action with weight matrix B', the quadratic off-shell invariants are generated by $u_1\overline{u_1}$, $u_2\overline{u_2}$, $u_3\overline{u_3}$, u_1u_3, and $\overline{u_1u_3}$ with relation $(u_1u_3)(\overline{u_1u_3}) = (u_1\overline{u_1})(u_3\overline{u_3})$, and the moment map expresses $u_1\overline{u_1} = 2u_2\overline{u_2} + u_3\overline{u_3}$. Considering only the Poisson brackets of the quadratics, computations similar to those above demonstrate that any graded Poisson isomorphism $\Phi\colon \mathbb{R}[M_0(B')] \to \mathbb{R}[M_0(A')]$ must map $u_3\overline{u_3} \mapsto cz_2\overline{z_2} + (c-1)z_3\overline{z_3} + \sqrt{-1}dz_3\overline{z_2}$ where $c \in \{0, 1\}$ and $d \neq 0$. For each $z_2, z_3 \in \mathbb{C}$, there is a $z_1 \in \mathbb{C}$ such that $(z_1, z_2, z_3) \in Z_{A'}$ so that $z_3\overline{z_2}$ is not bounded by inequalities. As $u_3\overline{u_3}, z_2\overline{z_2}, z_3\overline{z_3} \geq 0$, it follows that Φ cannot preserve the inequalities.

Finally, we consider a closely related example that is not of Type I_k nor II_k for any k. Let

$$A'' = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B'' = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

To see that the Hilbert series of $\mathbb{R}[M_0(A'')]$ and $\mathbb{R}[M_0(B'')]$ coincide note that the cotangent-lifted weight matrix corresponding to A'',

$$\left(\begin{array}{cccc|cccc} -1 & 0 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 & 0 & 1 & -1 & -1 \end{array}\right),$$

can be transformed into that of B'',

$$\left(\begin{array}{cccc|cccc} -1 & 0 & 1 & 1 & 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right)$$

by transposing the column pairs $(1, 4)$, $(3, 7)$, $(5, 8)$ and row-reducing over $\mathbb{Z}$. The common Hilbert series is given by

$$\frac{1 + 2t^2 + 2t^3 + 2t^4 + t^6}{(1-t^3)^2(1-t^2)^2}.$$

The quadratic off-shell invariants associated to A'' are $z_1\overline{z_1}$, $z_2\overline{z_2}$, $z_3\overline{z_3}$, $z_4\overline{z_4}$, $z_3\overline{z_4}$, and $z_4\overline{z_3}$, the moment map expresses $z_1\overline{z_1}$ and $z_2\overline{z_2}$ in terms of $z_3\overline{z_3}$ and $z_4\overline{z_4}$, and we have the relation $(z_3\overline{z_4})(z_4\overline{z_3}) = (z_3\overline{z_3})(z_4\overline{z_4})$. Similarly, the quadratic off-shell invariants associated to B'' are $z_1\overline{z_1}$, $z_2\overline{z_2}$, $z_3\overline{z_3}$, $z_4\overline{z_4}$, z_1z_3, and $\overline{z_1z_3}$, the moment map expresses $z_1\overline{z_1}$ and $z_2\overline{z_2}$ in terms of $z_3\overline{z_3}$ and $z_4\overline{z_4}$, and we have the relation $(z_1z_3)(\overline{z_1z_3}) = (z_2\overline{z_2} + z_3\overline{z_3})(z_3\overline{z_3})$. Hence, computations identical to those for A' and B' demonstrate that the only Poisson isomorphisms between the algebras $\mathbb{R}[M_0(A'')]$ and $\mathbb{R}[M_0(B'')]$ do not satisfy the semialgebraic conditions, and hence do not correspond to a graded regular symplectomorphism.

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A Constructive approach to Theorem 1

We first obtained a proof of Theorem 1 for Type I_k matrices by determining an explicit description of the symplectic quotient M_0 and algebra $\mathbb{R}[M_0]$ of regular functions. This description may be of independent interest and illustrates the structure of these spaces, so we include it here. The proofs of these results are cumbersome computations and hence only summarized.

Proposition 2 *Let* $A = \big(D, \overbrace{\boldsymbol{n}, \ldots, \boldsymbol{n}}^{k}\big) \in \mathbb{Z}^{\ell \times (\ell+k)}$ *be a type* I_k *weight matrix such that* V_A *is a faithful* $\mathbb{T}^\ell$*-module. Then a generating set for the algebra* $\mathbb{R}[V_A]^{\mathbb{T}^\ell}$ *of invariants is given by*

1. *the* ℓ *quadratic monomials* $r_i := z_i\overline{z_i}$ *for* $i = 1, \ldots, \ell$*,*
2. *the* k^2 *quadratic monomials* $p_{i,j} := z_{\ell+i}\overline{z_{\ell+j}}$ *for* $1 \leq i, j \leq k$*,*
3. *the* $\binom{\alpha+k-1}{k-1}$ *degree* η *monomials* $q_{\boldsymbol{s}} := \prod_{i=1}^{\ell} z_i^{m_i} \prod_{i=1}^{k} z_{\ell+i}^{s_i}$ *where* $\boldsymbol{s} = (s_1, \ldots, s_k)$ *and the* s_i *are any choice of nonnegative integers such that* $\sum_{i=1}^{k} s_i = \alpha$*, and*
4. *the* $\alpha + k - 1$ *choose* $k - 1$ *degree* η *monomials* $\overline{q_{\boldsymbol{s}}}$ *for each choice of* $\boldsymbol{s}$*.*

For a generating set for $\mathbb{R}[M_0(A)]$*, the generators in (1) can be omitted using the on-shell relations.*

Proof A simple computation demonstrates that each of the monomials listed in Proposition 2 is invariant. To prove the proposition, one first establishes the result when $k = 1$ by induction on ℓ; the base case is simple, and the inductive step is accomplished by comparing the invariants of A to those corresponding to submatrices formed by removing a single row and the resulting column of zeros. For general k, consider the map $\phi\colon \mathbb{R}[z_1, \ldots, z_{\ell+k}, \overline{z_1}, \ldots, \overline{z_{\ell+k}}] \to \mathbb{R}[w_1, \ldots, w_{\ell+1}, \overline{w_1}, \ldots, \overline{w_{\ell+1}}]$ that maps $z_i \mapsto w_i$ and $\overline{z_i} \mapsto \overline{w_i}$ for $i \leq \ell$, $z_{\ell+i} \mapsto w_{\ell+1}$, and $\overline{z_{\ell+i}} \mapsto \overline{w_{\ell+1}}$. It is easy to see that ϕ maps A-invariants onto $(D, \mathbf{n})$-invariants, and then the proof is completed by considering the preimages of the $(D, \mathbf{n})$-invariants, a case with $k = 1$. □

Proposition 3 *Let* $A = \big(D, \overbrace{\boldsymbol{n}, \ldots, \boldsymbol{n}}^{k}\big) \in \mathbb{Z}^{\ell \times (\ell+k)}$ *be a type* I_k *weight matrix such that* V_A *is a faithful* $\mathbb{T}^\ell$*-module. The (off-shell) relations among the* r_i*,* $p_{i,j}$*,* $q_{\boldsymbol{s}}$*, and* $\overline{q_{\boldsymbol{s}}}$ *are generated by the following.*

1. $p_{g,h}p_{i,j} - p_{g,j}p_{i,h}$ *for* $1 \leq g, h, i, j \leq k$ *with* $g \neq i$ *and* $h \neq j$*.*
2. $p_{g,h}q_{\boldsymbol{s}} - p_{i,h}q_{\boldsymbol{s}'}$ *where* $s'_g = s_g + 1$*,* $s'_i = s_i - 1$*, and* $s'_j = s_j$ *for* $j \neq g, i$*. Note that we must have* $s_i \geq 1$*.*
3. $p_{g,h}\overline{q_{\boldsymbol{s}}} - p_{g,i}\overline{q_{\boldsymbol{s}'}}$ *where* $s'_g = s_g + 1$*,* $s'_i = s_i - 1$*, and* $s'_j = s_j$ *for* $j \neq g, i$*. Note that we must have* $s_i \geq 1$*.*
4. $q_{\boldsymbol{s}}q_{\boldsymbol{s}'} - q_{\boldsymbol{t}}q_{\boldsymbol{t}'}$ *where* $\boldsymbol{s} + \boldsymbol{s}' = \boldsymbol{t} + \boldsymbol{t}'$ *and* $\boldsymbol{s} \neq \boldsymbol{t}$*.*

5. $\overline{q_{\boldsymbol{s}}}\,\overline{q_{\boldsymbol{s}'}} - \overline{q_{\boldsymbol{t}}}\,\overline{q_{\boldsymbol{t}'}}$ *where* $\boldsymbol{s} + \boldsymbol{s}' = \boldsymbol{t} + \boldsymbol{t}'$ *and* $\boldsymbol{s} \neq \boldsymbol{t}$.
6. $\prod_{i=1}^{\ell} r_i^{m_i} \prod_{j=1}^{\alpha} p_{g_j,h_j} - q_{\boldsymbol{s}}\overline{q_{\boldsymbol{s}'}}$ *where the vector* $(g_1, \ldots, g_\alpha)$ *contains each value* g *exactly* s_g *times and the vector* $(h_1, \ldots, h_\alpha)$ *contains each value* h *exactly* s'_h *times.*

On-shell, the monomials additionally satisfy the defining relations of the moment map, $-a_i r_i + n_i \sum_{j=1}^{k} p_{j,j}$ *for* $i = 1, \ldots, \ell$.

Proof One verifies that each of these relations holds by direct computation using the definitions of the monomials given in Proposition 2. The proof that all relations are generated by these is by induction on k. For the case $k = 1$, there is only one nontrivial relation, $p_{1,1}^{\alpha} \prod_{i=1}^{\ell} r_i^{m_i} - q_{(\alpha)}\overline{q_{(\alpha)}}$; a simple yet tedious consideration of cases demonstrates that this generates all relations. The induction step is demonstrated by considering the preimages of invariants under the map $\mathbb{C}[z_1, \ldots, z_{\ell+k+1}] \to \mathbb{C}[z_1, \ldots, z_{\ell+k}]$ given by $(z_1, \ldots, z_{\ell+k+1}) \mapsto (z_1, \ldots, z_{\ell+k} + z_{\ell+k+1})$. □

One then verifies the following by direct computation.

Proposition 4 *Let* $A = \big(D, \overbrace{\boldsymbol{n}, \ldots, \boldsymbol{n}}^{k}\big) \in \mathbb{Z}^{\ell \times (\ell+k)}$ *be a type* I_k *weight matrix such that* V_A *is a faithful* $\mathbb{T}^\ell$*-module. The Poisson brackets of the Hilbert basis elements given in Proposition 2 are as follows. Note that the indices* g, h, i, j *need not be distinct unless otherwise noted.*

- $\{r_g, r_h\} = \{r_g, p_{h,i}\} = \{q_{\boldsymbol{s}}, q_{\boldsymbol{s}'}\} = \{\overline{q_{\boldsymbol{s}}}, \overline{q_{\boldsymbol{s}'}}\} = 0.$
- $\{r_i, q_{\boldsymbol{s}}\} = -\frac{2}{\sqrt{-1}} m_i q_{\boldsymbol{s}}.$
- $\{r_i, \overline{q_{\boldsymbol{s}}}\} = \frac{2}{\sqrt{-1}} m_i \overline{q_{\boldsymbol{s}}}.$
- $\{p_{g,h}, p_{i,j}\} = \begin{cases} \frac{2}{\sqrt{-1}} p_{i,h}, & g = j \text{ and } h \neq i, \\ -\frac{2}{\sqrt{-1}} p_{g,j}, & g \neq j \text{ and } h = i, \\ \frac{2}{\sqrt{-1}} (p_{h,h} - p_{g,g}) & g = j \text{ and } h = i, \text{ and } g \neq h \\ 0, & g \neq j \text{ and } h \neq i \text{ or } g = j = h = i. \end{cases}.$
- $\{p_{g,h}, q_{\boldsymbol{s}}\} = \begin{cases} -\frac{2}{\sqrt{-1}} s_g q_{\boldsymbol{s}'}, & s_g > 0, \\ 0, & s_g = 0, \end{cases}$
  where $s'_g = s_g - 1$, $s'_h = s_h + 1$, *and* $s'_i = s_i$ *for* $i \neq g, h$.
- $\{p_{g,h}, \overline{q_{\boldsymbol{s}}}\} = \begin{cases} \frac{2}{\sqrt{-1}} s_g \overline{q_{\boldsymbol{s}'}}, & s_g > 0, \\ 0, & s_g = 0, \end{cases}$
  where $s'_g = s_g - 1$, $s'_h = s_h + 1$, *and* $s'_i = s_i$ *for* $i \neq g, h$.
- $\{q_{\boldsymbol{s}}, \overline{q_{\boldsymbol{s}'}}\} = \frac{2}{\sqrt{-1}} q_{\boldsymbol{s}}\overline{q_{\boldsymbol{s}'}} \left(\sum_{i=1}^{\ell} \frac{m_i^2}{r_i} + \sum_{j=1}^{k} \frac{s_j s'_j}{p_{j,j}} \right)$, *which we note is polynomial as the* r_i *and* $p_{j,j}$ *divide* $q_{\boldsymbol{s}} q_{\boldsymbol{s}'}$.

The above results give an explicit description of the Poisson algebra of regular functions. It remains only to determine the semialgebraic description of the symplectic quotient.

Proposition 5 *Let* $A = \big(D, \overbrace{\boldsymbol{n}, \ldots, \boldsymbol{n}}^{k}\big) \in \mathbb{Z}^{\ell\times(\ell+k)}$ *be a type* I_k *weight matrix associated such that* V_A *is a faithful* $\mathbb{T}^\ell$*-module. Using the real Hilbert basis given by the real and imaginary parts of the monomials listed in Proposition 2, the image of the Hilbert embedding is described by the relations given in Proposition 3 as well as the inequalities* $r_i \geq 0$ *for* $i = 1, \ldots, \ell$ *and* $p_{j,j} \geq 0$ *for* $j = 1, \ldots, k$.

Proof From the definition of the monomials, it is easy to see that these inequalities are satisfied. For the converse, choose values of the r_i, $p_{i,j}$, and q_s such that each $r_i \geq 0$, each $p_{i,i} \geq 0$, and the remaining values are arbitrary elements of $\mathbb{C}$ such that the each $p_{i,j} = \overline{p_{j,i}}$ and relations in Proposition 3 are satisfied. It is then easy to see that the values $|r_i|$, $|p_{i,j}|$ for $i \neq j$, and $|q_s|$ are determined by the $p_{i,i}$. Specifically, using the relations of Proposition 3(1), we have

$$|p_{i,j}| = \sqrt{p_{i,i}\,p_{j,j}},$$

using the moment map, we have

$$|r_i| = \frac{n_i}{a_i}\sum_{j=1}^{k} p_{j,j}$$

and using the relations of Proposition 3(6), we have

$$q_s = \sqrt{\prod_{i=1}^{\ell}\left(\frac{n_i}{a_i}\right)^{m_i}\left(\sum_{j=1}^{k} p_{i,i}\right)^{\sum_{i=1}^{\ell} m_i}}\left(\prod_{j=1}^{k} p_{i,i}^{s_i}\right)^{\alpha/2}.$$

Similarly, using the relations of Proposition 3(3), one checks that the arguments of the q_s where s has only one nonzero coordinate (which must be equal to α) determine the arguments of the $p_{i,j}$ and the other $q_{s'}$. It follows that one can find a point $(z_1, \ldots, z_n)$ mapped via the Hilbert embedding to these values of r_i, $p_{i,j}$, and q_s by choosing the modulus of each $z_{\ell+i}$ to be $\sqrt{p_{i,i}}$, the modulus of each z_i for $i \leq \ell$ to be determined by the moment map, the argument of each z_i for $i \leq \ell$ to be 0, and the argument of each $z_{\ell+i}$ to be the argument of $q_{(0,\ldots,0,\alpha,0,\ldots,0)}$ where α occurs in the ith position. □

With this explicit description of $M_0(A)$ and $\mathbb{R}[M_0(A)]$ the following can be verified by direct computation.

Theorem 6 *Let* $A \in \mathbb{Z}^{\ell\times(\ell+k)}$ *be a Type* I_k *matrix such that* V_A *is a faithful* $\mathbb{T}^\ell$*-module, and let* $B = \big(-\alpha(A), \beta(A), \ldots, \beta(A)\big) \in \mathbb{Z}^{1\times(k+1)}$*. Using coordinates* $(w_1, \ldots, w_{k+1})$ *for* V_B*, define the map* $\Phi\colon \mathbb{C}[V_A]^{\mathbb{T}^\ell} \to \mathbb{C}[V_B]$ *by*

$$r_i \longmapsto \frac{m_i(A)}{\beta(A)} w_1\overline{w_1}, \qquad 1 \leq i \leq \ell,$$
$$p_{ij} \longmapsto w_{i+1}\overline{w_{i+1}}, \qquad 1 \leq i, j \leq k,$$

$$q_s \longmapsto \sqrt{\beta(A)^{-\beta(A)} \prod_{j=1}^{\ell} m_j(A)^{m_j(A)}} \; w_1^{\beta(A)} \prod_{j=1}^{k} w_{j+1}^{s_j},$$

$$\overline{q_s} \longmapsto \sqrt{\beta(A)^{-\beta(A)} \prod_{j=1}^{\ell} m_j(A)^{m_j(A)}} \; \overline{w_1}^{\beta(A)} \prod_{j=1}^{k} \overline{w_{j+1}}^{s_j}.$$

Then Φ is a well-defined homomorphism $\Phi\colon \mathbb{C}[V_A]^{\mathbb{T}^\ell} \to \mathbb{C}[V_B]^{\mathbb{T}^1}$ inducing an isomorphism $\mathbb{R}[M_0(A)] \to \mathbb{R}[M_0(B)]$ and a graded regular symplectomorphism between $M_0(A)$ and $M_0(B)$.

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