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ORIGINAL PAPER

From a single Sasakian manifold to a family of Sasakian manifolds

Gherici Beldjilali[1] · **Ahmed Mohammed Cherif**[2] · **Kaddour Zegga**[2]

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Abstract
The aim of this paper is to construct a 1-parameter family of Sasakian manifold starting from a single Sasakian manifold. Concrete examples are given.

Keywords Product manifolds · Sasakian manifolds · Kählerian manifolds.

Mathematics Subject Classification 53C15 · 53C25 · 53C55 · 53D25

1 Introduction

The product of manifolds plays a very important role in the construction of the new manifolds and also for the structures on manifolds. Bishop and O'Neill introduced the notion of warped product as a generalization of Riemannian product (Bishop and ONeill 1969).

The product of an almost contact manifold M and the real line $\mathbf{R}$ carries a natural almost complex structure. When this structure is integrable the almost contact structure is said to be normal.

By means of warped product there is a one-to-one correspondence between Sasakian and Kählerian structures [see Oubiña (1985)].

✉ Gherici Beldjilali
gherici.beldjilali@univ-mascara.dz

Ahmed Mohammed Cherif
med_cherif_ahmed@yahoo.fr

Kaddour Zegga
zegga.kaddour@yahoo.fr

1 Laboratory of Quantum Physics and Mathematical Modeling (LPQ3M), Department of Mathematics, University Mustapha Stambouli Mascara, Mascara, Algeria

2 Laboratory of Geometry, Analysis, Controle and Applications, Department of Mathematics, University Mustapha Stambouli Mascara, Mascara, Algeria

In Blair (2012), building on the work of Tanno (1969) (the homothetic deformation on contact metric manifold), Blair introduced the notion of $\mathcal{D}$-homothetic warping and he used it to prove the above correspondence by another way.

Recently, Beldjilali and Belkhelfa (2016) introduced the $\mathcal{D}$-homothetic bi-warping and they proved that every sasakian manifold M generates a 1-parameter family of Kählerian manifolds. Thereby generalizing the results of Oubiña (1985) and Blair (2012).

One of the goals of this paper is to explore other ways for constructing Sasakian manifolds. Results in our paper can be divided in three parts. In the first part, we state and proof our first construction result [see Beldjilali and Belkhelfa (2016)] by another way and we give an example. In the second part, we construct Sasakian structures from Kählerian structures built above using a new technique and we construct a concrete example. Finally, we combine the two previous steps together to provide a way to create a 1-parameter family of Sasakian manifolds from just one Sasakian manifold directly. This text is organized in the following way.

Section 2 is devoted to the background of the structures which will be used in the sequel.
In Sect. 3, we construct a 1-parameter family of Kählerian manifolds using the $\mathcal{D}$-homothetic bi-warping but in a different way than (Beldjilali and Belkhelfa 2016) and we give an example. In Sect. 4 We introduce certain Riemannian metric on the product of real line and the Kählerian manifold built in the Sect. 3 and we use it to construct a 1-parameter family of Sasakian manifolds. Finaly, in Sect. 5, we merge the two previous constructions to give a direct way to get a 1-parameter family of Sasakian manifolds from a single Sasakian manifold and we give a concrete example.

2 Preliminaries

2.1 Almost contact metric structures and Sasakian structures

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a (1, 1) tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi, \quad and \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

Such a manifold is said to be a contact metric manifold if $d\eta = \phi$, where $\phi(X, Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M. If, in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on M.

On the other hand, the almost contact metric structure of M is said to be normal if $[\varphi, \varphi](X, Y) = -2d\eta\ (X, Y)\xi$, for any X, Y, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that a Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1}$$

for any X,Y. Moreover, for a Sasakian manifold the following equation holds:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

From the formula (1) easily obtains

$$\nabla_X \xi = -\varphi X, \qquad (\nabla_X \eta)Y = -g(\varphi X, Y). \tag{2}$$

For more background on almost contact metric manifolds and recent study of η-Einstein manifolds, we recommend the reference Blair (2002) and Boyer et al. (2006).

2.2 Almost complex structures and Kählerian structures

An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. For an almost Hermitian manifold $(\overline{M}^{2n}, J, \overline{g})$ we thus have

$$J^2 = -1, \qquad \overline{g}(JX, JY) = \overline{g}(X, Y).$$

An almost complex stucture J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_j vanishes, with

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

For an almost Hermitian manifold $(\overline{M}, J, \overline{g})$, we define the fundamental Kähler form $\overline{\Omega}$ as:

$$\overline{\Omega}(X, Y) = \overline{g}(X, JY).$$

$(\overline{M}, J, \overline{g})$ is then called almost Kähler if $\overline{\Omega}$ is closed i.e. $\mathrm{d}\overline{\Omega} = 0$. It can be shown that this condition for $(\overline{M}, J, \overline{g})$ to be almost Kähler is equivalent to

$$\overline{g}\big((\overline{\nabla}_X J)Y, Z\big) + \overline{g}\big((\overline{\nabla}_Y J)Z, X\big) + \overline{g}\big((\overline{\nabla}_Z J)X, Y\big) = 0.$$

An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions: $d\overline{\Omega} = 0$ and $N_J = 0$. One can prove that these both conditions combined are equivalent with the single condition

$$\overline{\nabla} J = 0.$$

For more background on almost complex structure manifolds, we recommend the reference Yano and Kon (1984).

3 From Sasakian structure to Kählerian structures

In Beldjilali and Belkhelfa (2016), the authors introduced the notion of $\mathcal{D}$-homothetically bi-warped metric on $\overline{M} = M' \times M$ where M' is a Riemannian manifolds and M is an almost contact metric manifold by

$$\overline{g} = g' + f^2 g + f^2(h^2 - 1)\eta \otimes \eta. \tag{3}$$

where $fh \neq 0$ everywhere.

Using the Koszul formula for the Levi-Civita connection of a Riemannian metric, one can obtain the following:

Proposition 1 Beldjilali and Belkhelfa (2016) *Let ∇', ∇ and $\overline{\nabla}$ denote the Riemannian connections of g', g, and $\overline{g}$ respectively. For all X', Y' vector fields tangent to M' and independent of M and similarly for X, Y, we have*

$$\begin{aligned}
\overline{\nabla}_{X'}Y' &= \nabla'_{X'}Y', \\
\overline{g}(\overline{\nabla}_{X'}Y, Z) &= \overline{g}(\overline{\nabla}_Y X', Z) = -\overline{g}(\overline{\nabla}_Y Z, X') \\
&= fX'(f)g(Y, Z) + f\Big((h^2-1)X'(f) + fhX'(h)\Big)\eta(Y)\eta(Z), \\
\overline{g}(\overline{\nabla}_X Y, Z) &= \overline{g}(\nabla_X Y, Z) + f^2(h^2-1)\Big(\frac{1}{2}\big(g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)\big)\eta(Z) \\
&\quad + d\eta(X, Z)\eta(Y) + d\eta(Y, Z)\eta(X)\Big).
\end{aligned}$$

Our motivation is to consider the case where $M' = \mathbf{R}$ and M is a Sasakian manifold. For brevity we denote the unit tangent field to $\mathbf{R}$ by ∂_t In this case the proposition (1) gives

Proposition 2 *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. Let ∇ and $\overline{\nabla}$ denote the Riemannian connections of g and $\overline{g}$ respectively. For all X, Y vector fields tangent to M and independent of $\mathbf{R}$, we have*

$$\begin{aligned}
\overline{\nabla}_{\partial_t}X &= \overline{\nabla}_X\partial_t = \frac{f'}{f}X + \frac{h'}{h}\eta(X)\xi, \\
\overline{\nabla}_X Y &= \nabla_X Y + (1-h^2)\big(\eta(X)\varphi Y + \eta(Y)\varphi X\big) \\
&\quad - \Big(ff'g(\varphi X, \varphi Y) + fh(fh)'\eta(X)\eta(Y)\Big)\partial_t.
\end{aligned}$$

Next, we introduce a class of almost complex structure J on manifold $\overline{M}$:

$$J(a\partial_t, X) = \Big(fh\eta(X)\partial_t\ ,\ \varphi X - \frac{a}{fh}\xi\Big), \tag{4}$$

for any vector filds X of M where f, h, are functions on $\mathbf{R}$ and $fh \neq 0$ everywhere.

That $J^2 = -I$ is easily checked and for all $\overline{X} = (a\partial_t, X)$, $\overline{Y} = (b\partial_t, Y)$ on $\overline{M}$ we can see that $\overline{g}$ is an almost Hermitian with respect to J i.e.

$$\overline{g}\big(J\overline{X}, J\overline{Y}\big) = \overline{g}(\overline{X}, \overline{Y}).$$

Knowing that $(\overline{\nabla}_{\overline{X}} J)\overline{Y} = \overline{\nabla}_{\overline{X}}(J\overline{Y}) - J\overline{\nabla}_{\overline{X}}\overline{Y}$ with using the proposition (2) and formulas (1) and (2), we get the following proposition:

Proposition 3 *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. For all X, Y vectors fields on M, the non-zero components $\overline{\nabla} J$ are*

$$(\overline{\nabla}_X J)\partial_t = \frac{1}{f}(h - f')\varphi X$$
$$(\overline{\nabla}_X J)Y = (h - f')\Big(\frac{1}{h} g(X, Y)\xi - h\eta(Y)X + \frac{h^2 - 1}{h}\eta(X)\eta(Y)\xi + fg(X, \varphi Y)\partial_t\Big).$$

From the above proposition we have immediately that

Proposition 4 *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The Hermitian structure constructed in (3) and (4) is Kählerian if and only if $h = f'$*

Example 1 For this example, we rely on the example of Blair Blair (2002). We know that $\mathbf{R}^{2n+1}$ with coordinates (x^i, y^i, z), $i = 1..n$, admits the Sasakian structure

$$g = \frac{1}{4}\begin{pmatrix} \delta_{ij} + y^i y^j & 0 & -y^i \\ 0 & \delta_{ij} & 0 \\ -y^j & 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix},$$
$$\xi = 2\left(\frac{\partial}{\partial z}\right), \qquad \eta = \frac{1}{2}(dz - y^i dx^i).$$

So, using this structure, we can define a family of Kählerian structures $(J, \overline{g})$ on $\mathbf{R}^{2n+2}$ as follows

$$\overline{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} f^2(\delta_{ij} + f'^2 y^i y^j) & 0 & -\frac{1}{4} f^2 f'^2 y^i \\ 0 & 0 & \frac{\delta_{ij}}{4} f^2 & 0 \\ 0 & -\frac{1}{4} f^2 f'^2 y^j & 0 & \frac{1}{4} f^2 f'^2 \end{pmatrix}$$
$$J = \begin{pmatrix} 0 & -\frac{1}{2} y^j ff' & 0 & \frac{1}{2} ff' \\ 0 & 0 & \delta_{ij} & 0 \\ 0 & -\delta_{ij} & 0 & 0 \\ -\frac{2}{ff'} & 0 & y^j & 0 \end{pmatrix}$$

4 From Kählerian strucrures to Sasakian structures

Now, consider $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold and $(\overline{M} = \mathbf{R} \times M, J, \overline{g})$ the 1-parameter family of Kählerian structures defined above by

$$\begin{cases} \overline{g} = dt^2 + f^2 g + f^2(f'^2 - 1)\eta \otimes \eta, \\ J(a\frac{\partial}{\partial t}, X) = \Big(ff'\eta(X)\frac{\partial}{\partial t}\ ,\ \varphi X - \frac{a}{ff'}\xi\Big), \end{cases} \tag{5}$$

where $f = f(t) \in C^\infty(\mathbf{R})$ and $ff' \neq 0$ anywhere.
The fundamental 2-form $\overline{\Omega}$ of $(J, \overline{g})$ is

$$\overline{\Omega}\Big(\big(a\frac{\partial}{\partial t}, X\big), \big(b\frac{\partial}{\partial t}, Y\big)\Big) = \overline{g}\Big(\big(a\frac{\partial}{\partial t}, X\big), J\big(b\frac{\partial}{\partial t}, Y\big)\Big),$$

we can check that is very simply as follows:

$$\begin{aligned} \overline{\Omega} &= f(2f'\,\mathrm{d}t \wedge \eta + f\phi) \\ &= \mathrm{d}\big(f^2\eta\big), \end{aligned}$$

where d denotes the exterior derivative and ϕ the fundamental 2-form of (φ, ξ, η, g). Putting

$$\theta = f^2\eta,$$

and define on $\tilde{M} = \mathbf{R} \times \overline{M}$ a Riemannian metric $\tilde{g}$ by

$$\tilde{g} = \overline{g} + \tilde{\eta} \otimes \tilde{\eta} \qquad with \qquad \tilde{\eta} = dr + \theta, \tag{6}$$

where r is the standard coordinate with respect to the frame ∂_r on $\mathbf{R}$. For all X, Y vector fields on M, we have:

$$\tilde{g}(X, Y) = \overline{g}(X, Y) + \theta(X)\theta(Y), \quad \tilde{g}(X, \partial_r) = \theta(X), \quad \tilde{g}(\partial_r, \partial_r) = 1.$$

We denote by $\overline{\nabla}$ (resp. $\tilde{\nabla}$) the covariant derivative with respect to the metric $\overline{g}$ on $\overline{M}$ (resp. $\tilde{g}$ on $\tilde{M}$). From the Koszul formula, we have the following:

Proposition 5 *For all X, Y, Z vector fields on $\overline{M}$, we have:*

1) $\tilde{g}(\tilde{\nabla}_{\partial_r}\partial_r, \partial_r) = \tilde{g}(\tilde{\nabla}_{\partial_r}\partial_r, X) = 0;$
2) $\tilde{g}(\tilde{\nabla}_{\partial_r}X, \partial_r) = \tilde{g}(\tilde{\nabla}_X\partial_r, \partial_r) = 0;$
3) $\tilde{g}(\tilde{\nabla}_{\partial_r}X, Y) = \tilde{g}(\tilde{\nabla}_X\partial_r, Y) = \overline{g}(X, JY);$
4) $\tilde{g}(\tilde{\nabla}_X Y, \partial_r) = \frac{1}{2}\big[X\theta(Y) + Y\theta(X) + \theta([X, Y])\big];$
5) $\tilde{g}(\tilde{\nabla}_X Y, Z) = \overline{g}(\overline{\nabla}_X Y, Z) + \frac{1}{2}\big[X(\theta(Y)) + Y(\theta(X)) + \theta([X, Y])\big]\theta(Z)$
$+\overline{g}(X, JZ)\theta(Y) + \overline{g}(Y, JZ)\theta(X).$

Proof For **1)**, we have:

$$\tilde{g}(\tilde{\nabla}_{\partial_r}\partial_r, \partial_r) = \frac{1}{2}\partial_r \tilde{g}(\partial_r, \partial_r) = 0,$$

because $\tilde{g}(\partial_r, \partial_r) = 1$. We compute

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{\partial_r}\partial_r, X) &= \partial_r \tilde{g}(\partial_r, X) + \partial_r \tilde{g}(X, \partial_r) - X\tilde{g}(\partial_r, \partial_r) \\ &\quad + \tilde{g}(X, [\partial_r, \partial_r]) + \tilde{g}(\partial_r, [X, \partial_r]) - \tilde{g}(\partial_r, [\partial_r, X]), \end{aligned}$$

as $\tilde{g}(\partial_r, X) = \tilde{\eta}(X)$, $\tilde{g}(\partial_r, \partial_r) = 1$, $[\partial_r, \partial_r] = 0$ and $[X, \partial_r] = [\partial_r, X] = 0$, we obtain

$$\tilde{g}(\tilde{\nabla}_{\partial_r}\partial_r, X) = 2\partial_r \tilde{\eta}(X) = 0,$$

because $\tilde{\eta}(X) = \theta(X) \in C^\infty(\overline{M})$ does not depend on r.
2) we have:

$$\tilde{g}(\tilde{\nabla}_{\partial_r} X, \partial_r) = \partial_r \tilde{g}(X, \partial_r) - \tilde{g}(X, \tilde{\nabla}_{\partial_r}\partial_r) = 0,$$

from **1)**. Since $[X, \partial_r] = 0$,

$$\tilde{g}(\tilde{\nabla}_X \partial_r, \partial_r) = \tilde{g}(\tilde{\nabla}_{\partial_r} X, \partial_r) = 0.$$

3) By the Koszul formula and the definition of $\tilde{g}$, we have:

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{\partial_r} X, Y) &= \partial_r \tilde{g}(X, Y) + X\tilde{g}(Y, \partial_r) - Y\tilde{g}(\partial_r, X) \\ &\quad + \tilde{g}(Y, [\partial_r, X]) + \tilde{g}(X, [Y, \partial_r]) - \tilde{g}(\partial_r, [X, Y]) \\ &= \partial_r \overline{g}(X, Y) + \partial_r(\theta(X)\theta(Y)) + X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]) \\ &= X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]), \end{aligned}$$

here $[\partial_r, X] = [Y, \partial_r] = 0$ and $\partial_r \overline{g}(X, Y) = \partial_r(\theta(X)\theta(Y)) = 0$. We conclude that:

$$\tilde{g}(\tilde{\nabla}_{\partial_r} X, Y) = \mathrm{d}\theta(X, Y) = \overline{\Omega}(X, Y) = \overline{g}(X, JY).$$

We compute:

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X \partial_r, Y) &= X\tilde{g}(\partial_r, Y) + \partial_r \tilde{g}(Y, X) - Y\tilde{g}(X, \partial_r) \\ &\quad + \tilde{g}(Y, [X, \partial_r]) + \tilde{g}(\partial_r, [Y, X]) - \tilde{g}(X, [\partial_r, Y]) \\ &= X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]) \\ &= 2\mathrm{d}\theta(X, Y), \end{aligned}$$

that is, $\tilde{g}(\tilde{\nabla}_X \partial_r, Y) = \overline{g}(X, JY)$.
4) We have:

$$\tilde{g}(\tilde{\nabla}_X Y, \partial_r) = X\tilde{g}(Y, \partial_r) - \tilde{g}(Y, \tilde{\nabla}_X \partial_r)$$

$$
\begin{aligned}
&= X(\theta(Y)) - \overline{g}(X, JY)\\
&= X(\theta(Y)) - \mathrm{d}\theta(X, Y)\\
&= X(\theta(Y)) - \frac{1}{2}\big[X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])\big]\\
&= \frac{1}{2}\big[X(\theta(Y)) + Y(\theta(X)) + \theta([X, Y])\big].
\end{aligned}
$$

5) First we shall use the Koszul formula for a Riemann metric $\tilde{g}$ and the Levi-Civita connection $\tilde{\nabla}$:

$$
\begin{aligned}
2\tilde{g}(\tilde{\nabla}_X Y, Z) = {} & X\tilde{g}(Y, Z) + Y\tilde{g}(Z, X) - Z\tilde{g}(X, Y)\\
& +\tilde{g}(Z, [X, Y]) + \tilde{g}(Y, [Z, X]) - \tilde{g}(X, [Y, Z]),
\end{aligned}
$$

by the definition of the metric $\tilde{g}$, we have:

$$
\begin{aligned}
2\tilde{g}(\tilde{\nabla}_X Y, Z) = {} & X\overline{g}(Y, Z) + X(\theta(Y)\theta(Z)) + Y\overline{g}(Z, X) + Y(\theta(Z)\theta(X))\\
& -Z\overline{g}(X, Y) - Z(\theta(X)\theta(Y)) + \overline{g}(Z, [X, Y]) + \theta(Z)\theta([X, Y])\\
& +\overline{g}(Y, [Z, X]) + \theta(Y)\theta([Z, X]) - \overline{g}(X, [Y, Z]) - \theta(X)\theta([Y, Z]),
\end{aligned}
$$

by the Koszul formula for $\overline{g}$ and $\overline{\nabla}$, we get:

$$
\begin{aligned}
2\tilde{g}(\tilde{\nabla}_X Y, Z) = {} & 2\overline{g}(\overline{\nabla}_X Y, Z) + X(\theta(Y))\theta(Z) + \theta(Y)X(\theta(Z))\\
& +Y(\theta(Z))\theta(X) + \theta(Z)Y(\theta(X)) - Z(\theta(X))\theta(Y)\\
& -\theta(X)Z(\theta(Y)) + \theta(Z)\theta([X, Y]) + \theta(Y)\theta([Z, X])\\
& -\theta(X)\theta([Y, Z]),
\end{aligned}
$$

from the formula:

$$
2\mathrm{d}\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]),
$$

we obtain:

$$
\begin{aligned}
2\tilde{g}(\tilde{\nabla}_X Y, Z) = {} & 2\overline{g}(\overline{\nabla}_X Y, Z) + [X(\theta(Y)) + Y(\theta(X)) + \theta([X, Y])]\theta(Z)\\
& +2\theta(Y)d\theta(X, Z) + 2\theta(X)d\theta(Y, Z),
\end{aligned}
$$

finally, by the condition $\tilde{\Omega} = d\theta$ and the definition of $\tilde{\Omega}$, we obtain:

$$
\begin{aligned}
\tilde{g}(\tilde{\nabla}_X Y, Z) = {} & \overline{g}(\overline{\nabla}_X Y, Z) + \theta(X)\overline{g}(Y, JZ) + \theta(Y)\overline{g}(X, JZ)\\
& +\frac{1}{2}[X(\theta(Y)) + Y(\theta(X)) + \theta([X, Y])]\theta(Z).
\end{aligned}
$$

Lemma 1 *We choose an orthonormal basis $\{e_1, ..., e_{2n}\}$ of the tangent space $T_x\overline{M}$ at each point $x \in \overline{M}$, then*

$$\{\partial_r, e_1 - \theta(e_1)\partial_r, ..., e_{2n} - \theta(e_{2n})\partial_r\}$$

is an orthonormal basis of $T_{(r,x)}\tilde{M}$, where $r \in \mathbf{R}$.

Proof For any $i, j \in \{1, ..., 2n\}$, we have:

$$\begin{aligned}\tilde{g}\big(e_i - \theta(e_i)\partial_r, e_j - \theta(e_j)\partial_r\big) &= \tilde{g}(e_i, e_j) - \theta(e_j)\tilde{g}(e_i, \partial_r) - \theta(e_i)\tilde{g}(\partial_r, e_j)\\ &\quad +\theta(e_i)\theta(e_j)g(\partial_r, \partial_r)\\ &= \overline{g}(e_i, e_j) + \theta(e_i)\theta(e_j) - \theta(e_i)\theta(e_j) - \theta(e_i)\theta(e_j)\\ &\quad +\theta(e_i)\theta(e_j)\\ &= \overline{g}(e_i, e_j) = \delta_{ij}.\end{aligned}$$

We compute:

$$\begin{aligned}\tilde{g}\big(e_i - \theta(e_i)\partial_r, \partial_r\big) &= \tilde{g}(e_i, \partial_r) - \theta(e_i)\tilde{g}(\partial_r, \partial_r)\\ &= \theta(e_i) - \theta(e_i) = 0.\end{aligned}$$

So that, for all vector $v \in T_{(r,x)}\tilde{M}$, there exist constants $a, b_1, ..., b_{2n}$ such that:

$$v = a\partial_r + \sum_{i=1}^{2n} b_i\big(e_i - \theta(e_i)\partial_r\big). \tag{7}$$

Note that, $a = \tilde{g}(v, \partial_r)$ and $b_i = \tilde{g}\big(v, e_i - \theta(e_i)\partial_r\big)$ for all $i = 1, .., 2n$. From the Proposition 5, and the Lemma 1, we get the following:

Proposition 6 *For all X, Y vector fields on M, we have:*

1) $\tilde{\nabla}_{\partial_r}\partial_r = 0$;
2) $\tilde{\nabla}_{\partial_r}X = \tilde{\nabla}_X\partial_r = -JX + \theta(JX)\partial_r$;
3) $\tilde{\nabla}_X Y = \overline{\nabla}_X Y - \theta(Y)\big(JX - \theta(JX)\partial_r\big) - \theta(X)\big(JY - \theta(JY)\partial_r\big) + \frac{1}{2}\big[(\overline{\nabla}_X\theta)Y + (\overline{\nabla}_Y\theta)X\big]\partial_r.$

Proof Let $\{e_1, ..., e_{2n}\}$ be an orthonormal frame on $\overline{M}$. From the proposition 5, we have: **1)**

$$\begin{aligned}&\tilde{g}\big(\tilde{\nabla}_{\partial_r}\partial_r, \partial_r\big) = 0,\\ &\tilde{g}\big(\tilde{\nabla}_{\partial_r}\partial_r, e_i - \theta(e_i)\partial_r\big) = \tilde{g}\big(\tilde{\nabla}_{\partial_r}\partial_r, e_i) - \theta(e_i)\tilde{g}(\tilde{\nabla}_{\partial_r}\partial_r, \partial_r\big) = 0.\end{aligned}$$

2)

$$\tilde{g}\big(\tilde{\nabla}_{\partial_r}X, \partial_r\big) = 0,$$

$$\begin{aligned}\tilde{g}\big(\tilde{\nabla}_{\partial_r}X, e_i-\theta(e_i)\partial_r\big) &= \tilde{g}(\tilde{\nabla}_{\partial_r}X, e_i)-\theta(e_i)\tilde{g}(\tilde{\nabla}_{\partial_r}X, \partial_r)\\ &= \overline{g}(X, Je_i) = -\overline{g}(JX, e_i),\end{aligned}$$

so that:

$$\begin{aligned}\tilde{\nabla}_{\partial_r}X &= -\overline{g}(JX, e_i)e_i + \overline{g}(JX, e_i)\theta(e_i)\partial_r\\ &= -JX + \theta(JX)\partial_r,\end{aligned}$$

with the same method we find that $\tilde{\nabla}_X\partial_r = -JX + \theta(JX)\partial_r$.
3) We compute:

$$\begin{aligned}\tilde{g}\big(\tilde{\nabla}_X Y, \partial_r\big) &= \frac{1}{2}\big[X\theta(Y) + Y\theta(X) + \theta([X, Y])\big],\\ \tilde{g}\big(\tilde{\nabla}_X Y, e_i-\theta(e_i)\partial_r\big) &= \tilde{g}(\tilde{\nabla}_X Y, e_i) - \theta(e_i)\tilde{g}(\tilde{\nabla}_X Y, \partial_r)\\ &= \overline{g}(\overline{\nabla}_X Y, e_i) + \frac{1}{2}\big[X(\theta(Y)) + Y(\theta(X)) + \theta([X, Y])\big]\theta(e_i)\\ &\quad +\overline{g}(X, Je_i)\theta(Y) + \overline{g}(Y, Je_i)\theta(X)\\ &\quad -\frac{1}{2}\theta(e_i)\big[X\theta(Y) + Y\theta(X) + \theta([X, Y])\big]\\ &= \overline{g}(\overline{\nabla}_X Y, e_i) - \overline{g}(JX, e_i)\theta(Y) - \overline{g}(JY, e_i)\theta(X),\end{aligned}$$

we conclude that:

$$\begin{aligned}\tilde{\nabla}_X Y &= \frac{1}{2}\big[X\theta(Y) + Y\theta(X) + \theta([X, Y])\big]\partial_r\\ &\quad +\overline{g}(\overline{\nabla}_X Y, e_i)e_i - \overline{g}(\overline{\nabla}_X Y, e_i)\theta(e_i)\partial_r\\ &\quad -\overline{g}(JX, e_i)\theta(Y)e_i + \overline{g}(JX, e_i)\theta(Y)\theta(e_i)\partial_r\\ &\quad -\overline{g}(JY, e_i)\theta(X)e_i + \overline{g}(JY, e_i)\theta(X)\theta(e_i)\partial_r\\ &= \frac{1}{2}\big[X\theta(Y) + Y\theta(X) + \theta([X, Y])\big]\partial_r\\ &\quad +\overline{\nabla}_X Y - \theta(\overline{\nabla}_X Y)\partial_r\\ &\quad -\theta(Y)JX + \theta(Y)\theta(JX)\partial_r\\ &\quad -\theta(X)JY + \theta(X)\theta(JY)\partial_r.\end{aligned}$$

Now, we define on $\tilde{M}$ a structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ as follows

$$\tilde{\xi} = \partial_r, \quad \tilde{\varphi}\partial_r = 0, \quad \tilde{\varphi}X = JX - \theta(JX)\partial_r, \tag{8}$$

for all X vector field on $\overline{M}$.

Proposition 7 *The manifold $(\tilde{M}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ constructed as above is an almost contact metric manifold. Specifically, it is a Sasakian manifold as long as $(\overline{M}, J, \overline{g})$ is a Kählerian manifold.*

Proof We have $\tilde{\eta} = dr + \theta$ and $\tilde{\xi} = \partial_r$, so $\tilde{\eta}(\tilde{\xi}) = 1$. As $\tilde{\varphi}\partial_r = 0$, we get $\tilde{\varphi}^2\partial_r = 0$, on the other hand, $-\partial_r + \eta(\partial_r)\partial_r = 0$. Let X vector field on $\overline{M}$, we compute:

$$\begin{aligned}\tilde{\varphi}^2 X &= \tilde{\varphi}\big(JX - \theta(JX)\partial_r\big)\\ &= \tilde{\varphi}(JX) - \theta(JX)\tilde{\varphi}\partial_r\\ &= J^2X - \theta(J^2X)\partial_r\\ &= -X + \theta(X)\partial_r\\ &= -X + \tilde{\eta}(X)\tilde{\xi}.\end{aligned}$$

Let X, Y vector fields on $\overline{M}$, we have

$$\begin{aligned}\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) &= \tilde{g}\big(JX - \theta(JX)\partial_r, JY - \theta(JY)\partial_r\big)\\ &= \tilde{g}\big(JX, JY) - \theta(JY)\tilde{g}(JX, \partial_r) - \theta(JX)\tilde{g}(\partial_r, JY)\\ &\quad +\theta(JX)\theta(JY)\tilde{g}(\partial_r, \partial_r),\end{aligned}$$

by the definition of the metric $\tilde{g}$ with $\tilde{\eta} = dr + \theta$, we obtain

$$\begin{aligned}\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) &= \overline{g}(JX, JY) + \theta(JX)\theta(JY) - \theta(JX)\theta(JY)\\ &\quad -\theta(JX)\theta(JY) + \theta(JX)\theta(JY)\\ &= \overline{g}(JX, JY),\end{aligned}$$

as $\overline{g}(JX, JY) = \overline{g}(X, Y)$ and $\tilde{g}(X, Y) = \overline{g}(X, Y) + \theta(X)\theta(Y)$, we conclude that

$$\begin{aligned}\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) &= \tilde{g}(X, Y) - \theta(X)\theta(Y)\\ &= \tilde{g}(X, Y) - \tilde{\eta}(X)\tilde{\eta}(Y).\end{aligned}$$

As $\tilde{\varphi}\partial_r = 0$, $\tilde{g}(X, \partial_r) = \theta(X) = \tilde{\eta}(X)$, $\tilde{\eta}(\partial_r) = 1$ and $\tilde{g}(\partial_r, \partial_r) = 1$, we get

$$\begin{aligned}\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}\partial_r) &= \tilde{g}(X, \partial_r) - \tilde{\eta}(X)\tilde{\eta}(\partial_r) = 0,\\ \tilde{g}(\tilde{\varphi}\partial_r, \tilde{\varphi}\partial_r) &= \tilde{g}(\partial_r, \partial_r) - \tilde{\eta}(\partial_r)\tilde{\eta}(\partial_r) = 0.\end{aligned}$$

This confirms that $(\tilde{M}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost contact metric manifold.

On the other hand, the manifold $(\tilde{M}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is Sasakian if it satisfies:

$$(\tilde{\nabla}_{\tilde{\xi}}\tilde{\varphi})\tilde{\xi} = \tilde{g}(\tilde{\xi}, \tilde{\xi})\tilde{\xi} - \tilde{\eta}(\tilde{\xi})\tilde{\xi} \tag{9}$$

$$(\tilde{\nabla}_{\tilde{\xi}}\tilde{\varphi})X = \tilde{g}(\tilde{\xi}, X)\tilde{\xi} - \tilde{\eta}(X)\tilde{\xi} \tag{10}$$

$$(\tilde{\nabla}_X\tilde{\varphi})\tilde{\xi} = \tilde{g}(X, \tilde{\xi})\tilde{\xi} - \tilde{\eta}(\tilde{\xi})X \tag{11}$$

$$(\tilde{\nabla}_X\tilde{\varphi})Y = \tilde{g}(X, Y)\tilde{\xi} - \tilde{\eta}(Y)X, \tag{12}$$

for all X, Y vectors fields on $\overline{M}$. It is easy to prove that conditions (9), (10), and (11) are satisfied, and we have:

$$(\tilde{\nabla}_{\tilde{\xi}}\tilde{\varphi})\tilde{\xi} = \tilde{\nabla}_{\tilde{\xi}}\tilde{\varphi}\tilde{\xi} - \tilde{\varphi}(\tilde{\nabla}_{\tilde{\xi}}\tilde{\xi}) = \tilde{g}(\tilde{\xi}, \tilde{\xi})\tilde{\xi} - \tilde{\eta}(\tilde{\xi})\tilde{\xi} = 0,$$

$$\begin{aligned}(\tilde{\nabla}_{\tilde{\xi}}\tilde{\varphi})X &= \tilde{\nabla}_{\tilde{\xi}}\tilde{\varphi}X - \tilde{\varphi}(\tilde{\nabla}_{\tilde{\xi}}X)\\ &= \tilde{\nabla}_{\tilde{\xi}}JX - \tilde{\nabla}_{\tilde{\xi}}(\theta(JX)\tilde{\xi}) + \tilde{\varphi}^2(X)\\ &= -\tilde{\varphi}(JX) - \theta(JX)\tilde{\nabla}_{\tilde{\xi}}\tilde{\xi} - X + \theta(X)\tilde{\xi}\\ &= -(J^2X - \theta(J^2X)\tilde{\xi}) - X + \theta(X)\tilde{\xi}\\ &= X + \theta(X)\tilde{\xi} - X + \theta(X)\tilde{\xi} = 0,\end{aligned}$$

so that, $(\tilde{\nabla}_{\tilde{\xi}}\tilde{\varphi})X = \tilde{g}(\tilde{\xi}, X)\tilde{\xi} - \tilde{\eta}(X)\tilde{\xi} = 0$. We compute:

$$\begin{aligned}(\tilde{\nabla}_X\tilde{\varphi})\tilde{\xi} &= \tilde{\nabla}_X\tilde{\varphi}\tilde{\xi} - \tilde{\varphi}(\tilde{\nabla}_X\tilde{\xi})\\ &= \tilde{\varphi}^2X = -X + \theta(X)\tilde{\xi},\end{aligned}$$

so, $(\tilde{\nabla}_X\tilde{\varphi})\tilde{\xi} = \tilde{g}(X, \tilde{\xi})\tilde{\xi} - \tilde{\eta}(\tilde{\xi})X = -X + \theta(X)\tilde{\xi}$.
For the condition (12), we compute:

$$(\tilde{\nabla}_X\tilde{\varphi})Y = \tilde{\nabla}_X\tilde{\varphi}Y - \tilde{\varphi}(\tilde{\nabla}_XY), \tag{13}$$

the first term of (13), is given by:

$$\begin{aligned}\tilde{\nabla}_X\tilde{\varphi}Y &= \tilde{\nabla}_X(JY - \theta(JY)\tilde{\xi})\\ &= \tilde{\nabla}_XJY - X(\theta(JY))\tilde{\xi} - \theta(JY)\tilde{\nabla}_X\tilde{\xi},\end{aligned}$$

using the proposition 5, we have

$$\begin{aligned}\tilde{\nabla}_X\tilde{\varphi}Y &= \overline{\nabla}_XJY - \theta(JY)\tilde{\varphi}X - \theta(X)\tilde{\varphi}JY + \frac{1}{2}\big[(\overline{\nabla}_X\theta)JY + (\overline{\nabla}_{JY}\theta)X\big]\tilde{\xi}\\ &\quad -X(\theta(JY))\tilde{\xi} + \theta(JY)\tilde{\varphi}X\\ &= \overline{\nabla}_XJY + \theta(X)Y - \theta(X)\theta(Y)\tilde{\xi} - \frac{1}{2}X(\theta(JY))\tilde{\xi} - \frac{1}{2}\theta(\overline{\nabla}_XJY)\tilde{\xi}\\ &\quad +\frac{1}{2}(JY)(\theta(X))\tilde{\xi} - \frac{1}{2}\theta(\overline{\nabla}_{JY}X)\tilde{\xi},\end{aligned} \tag{14}$$

the second term of (13), is given by:

$$\begin{aligned}-\tilde{\varphi}(\tilde{\nabla}_XY) &= -\tilde{\varphi}(\overline{\nabla}_XY) + \theta(Y)\tilde{\varphi}^2X + \theta(X)\tilde{\varphi}^2Y\\ &= -J\overline{\nabla}_XY + \theta(J\overline{\nabla}_XY)\tilde{\xi} - \theta(Y)X + 2\theta(X)\theta(Y)\tilde{\xi} - \theta(X)Y,\end{aligned} \tag{15}$$

Substituting the formulas (14) and (15) in (13), we obtain:

$$\begin{aligned}(\tilde{\nabla}_X\tilde{\varphi})Y &= (\overline{\nabla}_XJ)Y + \frac{1}{2}\big[(JY)(\theta(X)) - X(\theta(JY)) - \theta([JY, X])\big]\tilde{\xi}\\ &\quad -\theta(\overline{\nabla}_XJY)\tilde{\xi} - \theta(Y)X + \theta(X)\theta(Y)\tilde{\xi} + \theta(J\overline{\nabla}_XY)\tilde{\xi}\\ &= (\overline{\nabla}_XJ)Y + \mathrm{d}\theta(JY, X)\tilde{\xi} - \theta((\overline{\nabla}_XJ)Y)\tilde{\xi} - \theta(Y)X + \theta(X)\theta(Y)\tilde{\xi}\end{aligned}$$

$$= (\overline{\nabla}_X J)Y + \overline{g}(X,Y)\tilde{\xi} - \tilde{\eta}((\overline{\nabla}_X J)Y)\tilde{\xi} - \tilde{\eta}(Y)X + \tilde{\eta}(X)\tilde{\eta}(Y)\tilde{\xi}$$
$$= (\overline{\nabla}_X J)Y - \tilde{\eta}((\overline{\nabla}_X J)Y)\tilde{\xi} + \tilde{g}(X,Y)\tilde{\xi} - \tilde{\eta}(Y)X,$$

that is:

$$(\tilde{\nabla}_X\tilde{\varphi})Y = -\tilde{\varphi}^2(\overline{\nabla}_X J)Y + \tilde{g}(X,Y)\tilde{\xi} - \tilde{\eta}(Y)X,$$

and since $(\overline{M}, J, \overline{g})$ is a Kählerian manifold i.e. $\overline{\nabla} J = 0$ then

$$(\tilde{\nabla}_X\tilde{\varphi})Y = \tilde{g}(X,Y)\tilde{\xi} - \tilde{\eta}(Y)X. \tag{16}$$

This completes the proof.

5 From Sasakian strucrure to 1-parameter family Sasakian structures

In this section, we merge the two previous constructions to give a direct way to get a 1-parameter family of Sasakian manifolds from a single Sasakian manifold i.e. instead of going from the Sasaki case to the Kählerian case and then to the Sasaki case again, one can construct a direct transfer bridge to the family of Sasakienne structures from a single Sasakian structure.

The main theorem in this paper is the following:

Theorem 1 *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The product $\tilde{M} = \mathbf{R}^2 \times M$ provided with the almost contact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ such that*

$$\tilde{g} = dt^2 + f^2 g + f^2(f'^2 - 1)\eta\otimes\eta + \tilde{\eta}\otimes\tilde{\eta},$$
$$\tilde{\eta} = dr + f^2\eta, \qquad \tilde{\xi} = \partial_r,$$

and for all X vector field on M

$$\begin{cases} \tilde{\varphi}\partial_r = 0, \\ \tilde{\varphi}\partial_t = \frac{-1}{ff'}\big(\xi - f^2\partial_r\big), \\ \tilde{\varphi}X = \varphi X + ff'\eta(X)\partial_t, \end{cases}$$

where $f = f(t)$ and $ff' \neq 0$ everywhere, is a 1-parameter family of Sasakian manifold.

Proof Follows from Propositions (4) and (7).

Example 2 Basing on Example 1 and taking dimension 3, we get the Sasakian structure

$$g = \frac{1}{4}\begin{pmatrix} 1+y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix},$$
$$\xi = 2\frac{\partial}{\partial z}, \qquad \eta = \frac{1}{2}(dz - ydx).$$

So, using Theorem 1, we can define a 1-parameter family of Sasakian structures $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on $\mathbf{R}^4$ as follows

$$\tilde{g} = \frac{1}{4}\begin{pmatrix} 4 & 0 & -2yf^2 & 0 & 2f^2 \\ 0 & 4 & 0 & 0 & 0 \\ -2yf^2 & 0 & f^2\big(1 + y^2(f^2 + f'^2)\big) & 0 & -yf^2(f^2 + f'^2) \\ 0 & 0 & 0 & f^2 & 0 \\ 2f^2 & 0 & -yf^2(f^2 + f'^2) & 0 & f^2(f^2 + f'^2) \end{pmatrix},$$

$$\tilde{\varphi} = \begin{pmatrix} 0 & \frac{f}{f'} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}ff'y & 0 & \frac{1}{2}ff' \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -\frac{2}{ff'} & 0 & y & 0 \end{pmatrix},$$

$$\tilde{\xi} = \frac{\partial}{\partial r}, \qquad \tilde{\eta} = dr + \frac{1}{2}f^2(dz - ydx).$$

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