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ORIGINAL PAPER

A note on flat nonholonomic Riemannian structures on three-dimensional Lie groups

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Abstract
We consider flat nonholonomic Riemannian manifolds, i.e., those whose associated parallel transport (induced by the nonholonomic connection) is path-independent. We first characterize flatness for structures on three-dimensional manifolds, and hence classify the flat left-invariant structures on simply connected Lie groups.

Keywords Nonholonomic Riemannian structure · Nonholonomic connection · Lie group

Mathematics Subject Classification 70G45 · 37J60

1 Introduction

The equivalence problem for three-dimensional nonholonomic Riemannian manifolds has been considered in Barrett et al. (2016). In particular, the left-invariant nonholonomic Riemannian structures on the three-dimensional simply connected Lie groups were classified, up to nonholonomic isometry and rescaling. Moreover, the equivalence classes were described in terms of isometric invariants. In this paper we consider the *flat* three-dimensional nonholonomic Riemannian manifolds, i.e., those whose associated parallel transport is path-independent. We first characterize flatness in three dimensions. Hence, by making use of the classification in Barrett et al. (2016), we are able to classify the flat left-invariant structures (on the three-dimensional simply connected Lie groups).

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The paper is organized as follows. In Sect. 2 we present the necessary elements of nonholonomic Riemannian geometry. Specifically, in Sect. 2.1 we recall the fundamental existence and uniqueness result for the nonholonomic connection, as well as the definitions of nonholonomic geodesics, nonholonomic isometries and the exterior covariant derivative associated to the nonholonomic connection. We also consider left-invariant nonholonomic Riemannian structures on Lie groups. Section 2.2 recalls the Schouten curvature tensor (which is canonically associated to every nonholonomic Riemannian structure) and some related tensors (obtained by contracting the Schouten tensor), whereas Sect. 2.3 introduces the parallel transport map associated to the nonholonomic connection and defines the notion of a flat nonholonomic Riemannian structure. In Sect. 3 we specialize to the case of three-dimensional nonholonomic Riemannian manifolds. Specifically, we recall the salient results of Barrett et al. (2016): the isometric invariants (Sect. 3.1) and the classification of left-invariant structures (Sect. 3.2). Section 4 contains the main results of the paper. In the first part (Sect. 4.1) we characterize, by means of the exterior covariant derivative, the flat nonholonomic Riemannian structures on three-dimensional manifolds. In the second part (Sect. 4.2) we use the foregoing characterization, together with the classification in Barrett et al. (2016), to obtain a classification of the flat left-invariant structures.

Throughout, we follow the summation convention on repeated indices. Unless stated otherwise, the following ranges on indices are used: $i, j, k = 1, \ldots, n$ (or $i, j, k = 0, 1, 2$ in Sect. 4) and $a, b, c = 1, \ldots, r$. We also assume that all manifolds, functions, vector fields, etc. are smooth, i.e., of class $\mathcal{C}^\infty$.

2 Nonholonomic Riemannian structures

2.1 Basic concepts

A *nonholonomic Riemannian structure* is a quadruple $\mathfrak{S} = (\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, where M is an n-dimensional (connected) manifold, $\mathcal{D}$ is a rank r nonintegrable distribution on M, $\mathcal{D}^\perp$ is a distribution complementary to $\mathcal{D}$ and $\mathbf{g}$ is a (positive definite) fiber metric on $\mathcal{D}$. We shall assume that $\mathcal{D}$ is *completely nonholonomic*, i.e., if $\mathcal{D}^1 \subseteq \mathcal{D}^2 \subseteq \cdots$ is the flag of $\mathcal{D}$, where

$$\mathcal{D}^1 = \mathcal{D}, \quad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i] \quad \text{for } i \geq 1,$$

then there exists $N \geq 2$ such that $\mathcal{D}^{N-1} \subsetneq T\mathsf{M}$ and $\mathcal{D}^N = T\mathsf{M}$. If $N = 2$, then $\mathcal{D}$ is called *strongly nonholonomic*. Every nonintegrable distribution on a three-dimensional manifold is strongly nonholonomic. It is well known that, if $\mathcal{D}$ is completely nonholonomic, then any two points in M can be joined by a $\mathcal{D}$-curve (i.e., an integral curve of $\mathcal{D}$). Let $\mathscr{P}$ be the projection onto $\mathcal{D}$ along $\mathcal{D}^\perp$ and let $\mathscr{Q}$ be the complementary projection. For convenience, we shall denote the projected Lie bracket $\mathscr{P}([\,\cdot\,,\,\cdot\,])$ by $[\![\,\cdot\,,\,\cdot\,]\!]$.

Associated to $\mathfrak{S}$ is a unique affine connection $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ (where $\Gamma(\mathcal{D})$ denotes the space of sections of $\mathcal{D}$), called the *nonholonomic connection*. Here "affine connection" means that ∇ is tensorial in its first argument and a derivation in the second. Like the Levi-Civita connection, the nonholonomic connection may be characterized as the unique connection that is both metric and torsion free, where the *torsion* of ∇ is the $(1, 2)$-tensor $T : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [\![X, Y]\!], \qquad X, Y \in \Gamma(\mathcal{D}).$$

Specifically, we have the following result (see, e.g., Langerock 2001).

Proposition 1 *Let* $\mathfrak{S} = (\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *be a nonholonomic Riemannian structure. There exists a unique affine connection* $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ *such that* $\nabla \mathbf{g} \equiv 0$ *and* $T \equiv 0$*, i.e.,*

$$Z[\mathbf{g}(X, Y)] = \mathbf{g}(\nabla_Z X, Y) + \mathbf{g}(X, \nabla_Z Y) \quad \textit{and} \quad \nabla_X Y - \nabla_Y X = [\![X, Y]\!]$$

for every $X, Y, Z \in \Gamma(\mathcal{D})$*. Furthermore,* ∇ *is characterized by* Koszul's formula:

$$\begin{aligned} 2\,\mathbf{g}(\nabla_X Y, Z) &= X[\mathbf{g}(Y, Z)] + Y[\mathbf{g}(X, Z)] - Z[\mathbf{g}(X, Y)] \\ &\quad + \mathbf{g}([\![X, Y]\!], Z) - \mathbf{g}([\![X, Z]\!], Y) - \mathbf{g}([\![Y, Z]\!], X) \end{aligned} \tag{1}$$

for every $X, Y, Z \in \Gamma(\mathcal{D})$*.*

A $\mathcal{D}$-curve γ is called a *nonholonomic geodesic* of $\mathfrak{S}$ if it is a geodesic of the nonholonomic connection ∇, i.e., $\nabla_{\dot\gamma}\dot\gamma = 0$.

Associated to ∇ is an exterior derivative. Let $\Omega^k(\mathcal{D}, \mathcal{D})$ be the space of $\mathcal{D}$-valued k-forms on $\mathcal{D}$. The *$\mathscr{P}$-exterior covariant derivative*, denoted $d^\nabla_{\mathscr{P}} : \Omega^k(\mathcal{D}, \mathcal{D}) \to \Omega^{k+1}(\mathcal{D}, \mathcal{D})$, is defined as follows:

(i) If $U \in \Omega^0(\mathcal{D}, \mathcal{D}) = \Gamma(\mathcal{D})$, then $d^\nabla_{\mathscr{P}} U(X) = \nabla_X U$ for every $X \in \Gamma(\mathcal{D})$.
(ii) If $\varphi \in \Omega^k(\mathcal{D}, \mathcal{D})$, $k \geq 1$, then

$$\begin{aligned} d^\nabla_{\mathscr{P}}\varphi(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i \nabla_{X_i}\varphi(X_0, \ldots, \widehat{X}_i, \ldots, X_k) \\ &\quad + \sum_{0 \leq i < j \leq k} (-1)^{i+j}\varphi([\![X_i, X_j]\!], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k) \end{aligned}$$

for every $X_0, \ldots, X_k \in \Gamma(\mathcal{D})$. Here $\widehat{X}_i$ indicates the omission of that element.

In particular, for a $\mathcal{D}$-valued 1-form φ, we have

$$d^\nabla_{\mathscr{P}}\varphi(X, Y) = \nabla_X\varphi(Y) - \nabla_Y\varphi(X) - \varphi([\![X, Y]\!]),$$

where $X, Y \in \Gamma(\mathcal{D})$. Note that the torsion of ∇ is exactly the $\mathscr{P}$-exterior covariant derivative of the identity map $\mathrm{id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$.

Two nonholonomic Riemannian structures $\mathfrak{S} = (\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $\mathfrak{S}' = (\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are said to be *NH-isometric* if there exists a diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ such that

$$\phi_*\mathcal{D} = \mathcal{D}', \quad \phi_*\mathcal{D}^\perp = \mathcal{D}'^\perp \quad \text{and} \quad \mathbf{g} = \phi^*\mathbf{g}'.$$

A map satisfying the above properties is called an *NH-isometry*. If ϕ is an NH-isometry, then $\nabla = \phi^*\nabla'$; consequently, ϕ establishes a one-to-one correspondence between the nonholonomic geodesics of $\mathfrak{S}$ and $\mathfrak{S}'$. Furthermore, ϕ preserves the projection operators: $\phi_*\mathscr{P}(X) = \mathscr{P}'(\phi_* X)$ and $\phi_*\mathscr{Q}(X) = \mathscr{Q}'(\phi_* X)$ for every $X \in \Gamma(T\mathsf{M})$.

Let G be a Lie group with Lie algebra $\mathfrak{g} = T_{\mathbf{1}}\mathsf{G}$. A nonholonomic Riemannian structure $\mathfrak{S} = (\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is said to be *left invariant* if every left translation L_g, $g \in \mathsf{G}$ is an NH-isometry. The simplest left-invariant structures are those whose nonholonomic geodesics are exactly the (left cosets of) one-parameter subgroups $t \mapsto g_0 \exp(tX_0)$, $X_0 \in \mathcal{D}_{\mathbf{1}}$, $g_0 \in \mathsf{G}$. These are the nonholonomic Riemannian analogues of bi-invariant Riemannian metrics. For such a structure, the nonholonomic connection ∇ is called a *Cartan–Schouten connection*. We have the following characterization of structures with Cartan–Schouten connections.

Proposition 2 (Barrett et al. 2016) *Let* $\mathfrak{S} = (\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *be a left-invariant nonholonomic Riemannian structure, with associated nonholonomic connection* ∇. *The following statements are equivalent*:

(i) ∇ *is Cartan–Schouten.*
(ii) $\nabla_X X = 0$ *for every left-invariant* $X \in \Gamma(\mathcal{D})$.
(iii) $\nabla_X Y = \frac{1}{2}[\![X, Y]\!]$ *for all left-invariant* $X, Y \in \Gamma(\mathcal{D})$.

2.2 The Schouten curvature tensor

As $\mathcal{D}$ is nonintegrable, the usual (Riemannian) curvature tensor cannot be defined for ∇. Instead, associated to every nonholonomic Riemannian structure is the $(1, 3)$-tensor field

$$\begin{aligned} &K : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), \\ &\quad K(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[\![X,Y]\!]}Z - [\![\mathscr{Q}([X, Y]), Z]\!], \end{aligned}$$

called the *Schouten curvature tensor* (see, e.g., Dragović and Gajić 2003). Let $\widehat{K}$ be the associated $(0, 4)$-tensor $\widehat{K}(W, X, Y, Z) = \mathbf{g}(K(W, X)Y, Z)$. We decompose $\widehat{K}$ into two components $\widehat{R}$ and $\widehat{C}$:

$$\widehat{R}(W, X, Y, Z) = \frac{1}{2}\big[\widehat{K}(W, X, Y, Z) - \widehat{K}(W, X, Z, Y)\big], \quad \widehat{C} = \widehat{K} - \widehat{R}.$$

$\widehat{R}$ behaves analogously to the Riemannian tensor, in that it satisfies the same symmetries (i.e., $\widehat{R}$ is skew-symmetric in the first pair and last pair of arguments, is symmetric if one swaps the first pair of arguments with the last pair, and satisfies the first Bianchi identity). Accordingly, we can define a $(0, 2)$-tensor, also called the *Ricci tensor*, as follows:

$$\mathrm{Ric} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to C^\infty(\mathsf{M}), \quad \mathrm{Ric}(X, Y) = \sum_a \widehat{R}(X_a, X, Y, X_a).$$

Here (X_a) is an orthonormal frame for $\mathcal{D}$. Likewise, let A be the (0, 2)-tensor

$$A : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \mathcal{C}^\infty(\mathsf{M}), \quad A(X, Y) = \sum_a \widehat{C}(X_a, X, Y, X_a).$$

Although Ric is symmetric, the tensor A is generally not; accordingly, let A_{sym} and A_{skew} be the symmetric and skew-symmetric parts of A, respectively. Since NH-isometries preserve ∇ and the projection operators $\mathscr{P}$ and $\mathscr{Q}$, it follows that K, $\widehat{K}$, $\widehat{R}$, $\widehat{C}$, Ric, A_{sym} and A_{skew} are all preserved under NH-isometry.

2.3 Parallel transport

The nonholonomic connection ∇ induces a parallel transport along $\mathcal{D}$-curves. A section V of $\mathcal{D}$ along a $\mathcal{D}$-curve γ is *parallel along* γ if $\nabla_{\dot\gamma} V = 0$. A vector field $X \in \Gamma(\mathcal{D})$ is *parallel* if $X \circ \gamma$ is parallel along γ for every $\mathcal{D}$-curve γ; clearly, X is parallel if and only if $\nabla X \equiv 0$.

Proposition 3 *Let $\gamma : [0, 1] \to M$ be a $\mathcal{D}$-curve and let $V_0 \in \mathcal{D}_{\gamma(0)}$. There exists a unique parallel section V of $\mathcal{D}$ along γ such that $V(0) = V_0$. (V is called the* parallel translate of V_0 *along γ.)*

Let $\gamma : [0, 1] \to \mathsf{M}$ be an $\mathcal{D}$-curve. The *parallel translation* $\Pi^t_\gamma : \mathcal{D}_{\gamma(0)} \to \mathcal{D}_{\gamma(t)}$, $t \in [0, 1]$ is specified by setting $\Pi^t_\gamma(V_0) = V(t)$, where V is the parallel translate of $V_0 \in \mathcal{D}_{\gamma(0)}$ along γ.

A *parallel frame* (X_a) for $\mathcal{D}$ is an orthonormal frame for $\mathcal{D}$ such that each X_a is parallel. The existence of a parallel frame is not guaranteed: it imposes quite severe restrictions on the structure. We say that a nonholonomic Riemannian structure $\mathfrak{S}$ is *flat on* $\mathcal{U}$ (where $\mathcal{U} \subseteq \mathsf{M}$ is open) if there exists a parallel frame for $\mathcal{D}$ defined on $\mathcal{U}$; if $\mathcal{U} = \mathsf{M}$, then we simply say $\mathfrak{S}$ is *flat*. Given an NH-isometry ϕ between structures $\mathfrak{S}$ and $\mathfrak{S}'$, if $\mathfrak{S}$ is flat on $\mathcal{U}$, then $\mathfrak{S}'$ is flat on $\phi(\mathcal{U})$. The converse does not hold: in three dimensions, there are many non-NH-isometric flat structures (see Sect. 4.2).

Proposition 4 *An orthonormal frame (X_a) for $\mathcal{D}$ is parallel if and only if $[\![X_a, X_b]\!] = 0$ for every $a, b = 1, \ldots, r$.*

Proof One implication is immediate, since ∇ is torsion free; the other follows from Koszul's formula (1). □

The following characterization of flatness is a straightforward generalization of a standard result.

Proposition 5 *$\mathfrak{S}$ is flat on $\mathcal{U} \subseteq M$ if and only if for any two points $p, q \in \mathcal{U}$ and for any $\mathcal{D}$-curve $\gamma : [0, 1] \to \mathcal{U}$ joining p to q, the parallel translation $\Pi^1_\gamma : \mathcal{D}_p \to \mathcal{D}_q$ does not depend on γ.*

For Riemannian manifolds, flatness is also characterized by the vanishing of the Riemannian curvature tensor. By contrast, the situation is more complicated in the nonholonomic Riemannian setting. The Schouten curvature tensor K does not generally characterize flatness (see, e.g., Dragović and Gajić 2003; however, in Sect. 4.1 we show that $K \equiv 0$ is a sufficient condition for flatness in three dimensions). Rather, one can define the Wagner curvature tensor (Dragović and Gajić 2003; Wagner 1935), the vanishing of which *does* characterize flatness. The construction of this tensor is quite sophisticated, and relies on the flag of the distribution. However, the construction is generally not intrinsic: it relies not only on the data $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, but also on some additional assumptions. Nonetheless, if the distribution is strongly nonholonomic, then these additional assumptions are automatically satisfied.

We briefly describe Wagner's approach. Let $\mathfrak{S} = (\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian structure and let $\mathcal{D} = \mathcal{D}^1 \subsetneq \mathcal{D}^2 \subsetneq \cdots \subsetneq \mathcal{D}^{N-1} \subsetneq \mathcal{D}^N = T\mathsf{M}$, $N \geq 2$ be the flag of $\mathcal{D}$. The nonholonomic connection $\nabla^1 = \nabla$ induces a parallel transport along $\mathcal{D}^1$-curves. For each component $\mathcal{D}^i$, $i = 2, \ldots, N$ of the flag, Wagner constructs a connection $\nabla^i : \Gamma(\mathcal{D}^i) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$. Such a connection induces a parallel transport along $\mathcal{D}^i$-curves. Furthermore, ∇^i is defined in such a way that it extends ∇^{i-1} and the set of parallel tensors of ∇^i coincides with that of ∇^{i-1}. Finally, one gets a vector bundle connection ∇^N on $\mathcal{D}$ (whose corresponding parallel transport is along any curve in M) with an associated curvature tensor K^N; this is the Wagner curvature tensor. The vanishing of K^N characterizes the flatness of ∇^N, and hence (by construction of $\nabla^2, \ldots, \nabla^{N-1}$) the flatness of $\mathfrak{S}$. In this paper, we do not employ the Wagner curvature tensor, preferring a more direct approach: we characterize flatness by finding necessary and sufficient conditions for the existence of a rotation taking an arbitrary orthonormal frame for $\mathcal{D}$ to a parallel frame. The relation between this characterization and the Wagner tensor will be explored elsewhere. It is worth mentioning that Wagner also characterized the flatness of three-dimensional nonholonomic Riemannian manifolds (Wagner 1938); however, he stopped short of classifying these structures.

3 Nonholonomic Riemannian structures in three dimensions

Let M be a three-dimensional manifold and $\mathcal{D}$ a rank 2 completely nonholonomic distribution on M. There exists (locally) a contact form ω such that $\mathcal{D} = \ker \omega$. Clearly, ω is unique only up to a multiple by a nonvanishing function. Let (Y_0, Y_1, Y_2) be a (local) frame on M such that (Y_1, Y_2) is an orthonormal frame for $\mathcal{D}$ and Y_0 spans $\mathcal{D}^\perp$. Let c_{ij}^k be the functions (structure constants) given by $[Y_i, Y_j] = c_{ij}^k Y_k$. We assume, without loss of generality, that $c_{21}^0 = 1$. By imposing the normalization condition $|d\omega(Y_1, Y_2)| = 1$, we may fix the contact form ω up to sign. Evidently, the value of $|d\omega(Y_1, Y_2)|$ is independent of the choice of Y_1 and Y_2. Let $Z \in \Gamma(T\mathsf{M})$ denote the Reeb vector field of ω, i.e., Z is the unique vector field such that $\omega(Z) = 1$ and $d\omega(Z, \cdot) = 0$. The normalized contact form and the Reeb vector field depend only on the data $(\mathsf{M}, \mathcal{D}, \mathbf{g})$, hence are preserved (up to sign) under NH-isometry.

3.1 Isometric invariants

Several isometric invariants for nonholonomic Riemannian structures in three dimensions were introduced in Barrett et al. (2016). The first invariant $\vartheta \in \mathcal{C}^\infty(\mathsf{M})$ is defined as $\vartheta = \|\mathscr{P}(Z)\|^2$. Evidently, we have $\vartheta = 0$ exactly when $\mathcal{D}^\perp = \operatorname{span}\{Z\}$. Three curvature invariants (κ, χ_1 and χ_2) were also introduced. The first is defined as $\kappa = \frac{1}{2}\operatorname{tr}(\mathbf{g}^\sharp \circ \mathrm{Ric}^\flat)$; this invariant can be interpreted as the sectional curvature of $\mathcal{D}$. The second two invariants are defined to be the positive eigenvalue of $\mathbf{g}^\sharp \circ A^\flat_{sym}$ and the absolute value of the Pfaffian of $\mathbf{g}^\sharp \circ A^\flat_{skew}$, respectively; we have

$$\chi_1 = \sqrt{-\det(\mathbf{g}^\sharp \circ A^\flat_{sym})} \quad \text{and} \quad \chi_2 = \sqrt{\det(\mathbf{g}^\sharp \circ A^\flat_{skew})}.$$

In terms of the structure constants, the invariants take the form

$$\begin{aligned} \vartheta &= (c^0_{10})^2 + (c^0_{20})^2, \quad \kappa = \frac{1}{2}(c^2_{10} - c^1_{20}) - (c^1_{21})^2 - (c^2_{21})^2 - Y_1[c^2_{21}] + Y_2[c^1_{21}], \\ \chi_1 &= \frac{1}{2}\sqrt{(c^2_{10} + c^1_{20})^2 + (c^1_{10} - c^2_{20})^2}, \quad \chi_2 = \frac{1}{2}|c^1_{10} + c^2_{20}|. \end{aligned}$$

If $\mathfrak{S}$ is a left-invariant structure on a Lie group, then we may take Y_0, Y_1 and Y_2 to be left invariant. The contact form ω and Reeb vector field Z are also left invariant. In this case, the structure constants are in fact constant, as are the invariants ϑ, κ, χ_1 and χ_2.

3.2 Classification of invariant structures

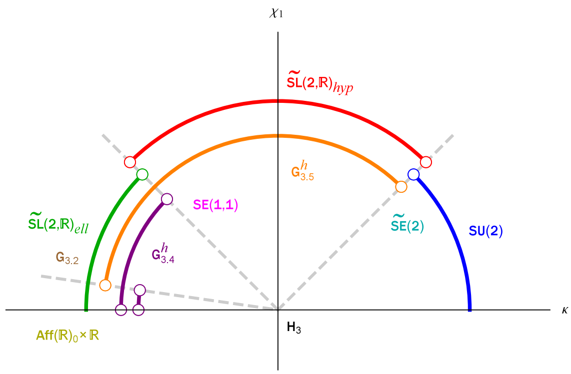
The left-invariant nonholonomic Riemannian structures on the three-dimensional simply connected Lie groups were recently classified (Barrett et al. 2016). We shall recall the salient aspects of this classification below.

Consider first the case when $\vartheta = 0$; then $\mathcal{D}^\perp = \operatorname{span}\{Z\}$. In this case the nonholonomic Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is completely specified by the associated sub-Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. The isometric invariants κ and χ_1 form a complete set of differential invariants for such structures (there are also discrete invariants; see Agrachev and Barilari 2012). Furthermore, by rescaling the metric, we may normalize (κ, χ_1) such that $\kappa = \chi_1 = 0$ or $\kappa^2 + \chi_1^2 = 1$. We recall below the classification of left-invariant structures (with $\vartheta = 0$) on three-dimensional simply connected Lie groups; see also Fig. 1. The complete list of three-dimensional Lie algebras and their associated simply connected Lie groups is given in the "Appendix".

Theorem 1 (Agrachev and Barilari 2012; Barrett et al. 2016) *Let $\mathfrak{S} = (\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $\mathfrak{S}' = (\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ be left-invariant nonholonomic Riemannian structures on three-dimensional simply connected Lie groups such that $\vartheta = \vartheta' = 0$, $\kappa = \kappa'$ and $\chi_1 = \chi_1'$.*

(i) *If $\kappa = \chi_1 = 0$, then $\mathfrak{S}$ is NH-isometric to (any structure on) the Heisenberg group H_3.*

Fig. 1 Normalized invariants for left-invariant nonholonomic Riemannian structures with vanishing ϑ



(ii) *If* $\chi_1 \neq 0$*, or* $\chi_1 = 0$ *and* $\kappa \geq 0$*, then* $\mathfrak{S}$ *is NH-isometric to* $\mathfrak{S}'$ *if and only if* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{g}'$*.*

(iii) *If* $\chi_1 = 0$ *and* $\kappa < 0$*, then* $\mathfrak{S}$ *is NH-isometric to the structure on* $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ *with elliptic-type distribution and metric (at identity) being (a rescaling of) the restriction of the Killing form to* $\mathcal{D}_1$*.*

Remark 1 There is a single equivalence class of structures with $\vartheta = 0$ on the affine group $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$. Remarkably, these are all NH-isometric to the structure on $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$ in item (iii).

Consider now the case when $\vartheta > 0$. Define a canonical frame (X_0, X_1, X_2) on M as follows: $X_0 = \mathscr{Q}(Z)$, $X_1 = \mathscr{P}(Z)/\|\mathscr{P}(Z)\|$ and $X_2 \in \Gamma(\mathcal{D})$ is the unique unit vector field orthogonal to X_1 such that $d\omega(X_1, X_2) = 1$ (here $\|\cdot\|$ is the norm on $\mathcal{D}$ induced by $\mathbf{g}$). Since ω and Z are specified up to sign, so are X_0 and X_1; on the other hand, X_2 is uniquely specified. Accordingly, if ϕ is an NH-isometry between nonholonomic Riemannian structures $\mathfrak{S}$ and $\mathfrak{S}'$, then $\phi_* X_0 = \pm X_0'$, $\phi_* X_1 = \pm X_1'$ and $\phi_* X_2 = X_2'$. The commutator relations of (X_0, X_1, X_2) are given by

$$\begin{cases} [X_1, X_0] = c^1_{10} X_1 + c^2_{10} X_2 \\ [X_2, X_0] = c^0_{20} X_0 + c^1_{20} X_1 + c^2_{20} X_2 \\ [X_2, X_1] = X_0 + c^1_{21} X_1 + c^2_{21} X_2. \end{cases} \tag{2}$$

The commutator relations (2) uniquely determine (up to sign) the nonholonomic Riemannian structure. Let $\mathbf{C}$ be the matrix-valued function on M given by

$$\mathbf{C} = \begin{bmatrix} 0 & c^1_{10} & c^2_{10} \\ c^0_{20} & c^1_{20} & c^2_{20} \\ 1 & c^1_{21} & c^2_{21} \end{bmatrix}.$$

In the case of a left-invariant structure, we have that the canonical frame is left invariant. It follows that the structure constants c^k_{ij}, and hence $\mathbf{C}$, are constant. Furthermore, it turns out that NH-isometries must preserve the group structure:

Proposition 6 *Let $\mathfrak{S}$ and $\mathfrak{S}'$ be left-invariant nonholonomic Riemannian structures on three-dimensional simply connected Lie groups G and G', respectively; assume that $\vartheta, \vartheta' > 0$. If $\mathfrak{S}$ is NH-isometric to $\mathfrak{S}'$ with respect to an NH-isometry $\phi : \mathsf{G} \to \mathsf{G}'$, then $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, where $L_{\phi(\mathbf{1})}$ is a left translation and $\phi' : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism.*

It follows from Proposition 6 that NH-isometries preserve the Killing form $\mathcal{K}$. Hence we can introduce three further invariants ϱ_0, ϱ_1 and ϱ_2, defined as $\varrho_i = -\frac{1}{2}\mathcal{K}(X_i(\mathbf{1}), X_i(\mathbf{1}))$. In terms of the structure constants, we have

$$\begin{aligned}\varrho_0 &= -\frac{1}{2}\left[(c_{10}^1)^2 + 2c_{20}^1 c_{10}^2 + (c_{20}^2)^2\right], \quad \varrho_1 = c_{10}^2 - \frac{1}{2}(c_{21}^2)^2, \\ \varrho_2 &= -\frac{1}{2}\left[(c_{20}^0)^2 + 2c_{20}^1 + (c_{21}^1)^2\right].\end{aligned}$$

For structures on the unimodular Lie groups, ϑ, ϱ_0, ϱ_1 and ϱ_2 form a complete set of invariants. On the non-unimodular Lie groups (except for $\mathsf{G}_{3.5}^h$, $h = 1$) there exist at most two structures with the same invariants ϑ, ϱ_0, ϱ_1 and ϱ_2. On the other hand, there are infinitely many structures on $\mathsf{G}_{3.5}^h$, $h = 1$ with the same invariants ϑ, ϱ_0, ϱ_1 and ϱ_2 (and at most two with the same invariants ϑ, κ and χ_2).

Theorem 2 (Barrett et al. 2016) *Let $\mathfrak{S}$ be a left-invariant nonholonomic Riemannian structure on a three-dimensional simply connected Lie group, rescaled such that $\vartheta = 1$. Then $\mathfrak{S}$ is NH-isometric to exactly one of the equivalence class representatives listed in Tables 1 and 2.*

Table 1 Left-invariant nonholonomic Riemannian structures (with $\vartheta = 1$) on the unimodular Lie groups

Lie group	Equivalence classes (**C**)	Invariants $\varrho_0, \varrho_1, \varrho_2$	Conditions	Designation
H_3	$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$0,\ 0,\ 0$		$\mathfrak{S}^{\mathsf{H}_3}$
$\widetilde{\mathsf{SE}}(2)$	$\begin{bmatrix} 0 & -\sqrt{\alpha_1\alpha_2} & \alpha_1 \\ -1 & -(1+\alpha_2) & \sqrt{\alpha_1\alpha_2} \\ 1 & 1 & 0 \end{bmatrix}$	$\alpha_1,\ \alpha_1,\ \alpha_2$	$\alpha_1, \alpha_2 \geq 0$ $\alpha_1^2 + \alpha_2^2 \neq 0$	$\mathfrak{S}_{\alpha_1,\alpha_2}^{\widetilde{\mathsf{SE}}(2)}$
$\mathsf{SE}(1,1)$	$\begin{bmatrix} 0 & -\sqrt{\alpha_1\alpha_2} & -\alpha_1 \\ -1 & -(1-\alpha_2) & \sqrt{\alpha_1\alpha_2} \\ 1 & 1 & 0 \end{bmatrix}$	$-\alpha_1,\ -\alpha_1,\ -\alpha_2$	$\alpha_1, \alpha_2 \geq 0$ $\alpha_1^2 + \alpha_2^2 \neq 0$	$\mathfrak{S}_{\alpha_1,\alpha_2}^{\mathsf{SE}(1,1)}$
$\mathsf{SU}(2)$	$\begin{bmatrix} 0 & -\delta & \alpha_1 \\ -1 & -(1+\alpha_2) & \delta \\ 1 & 1 & 0 \end{bmatrix}$	$-\delta^2 + \alpha_1(1+\alpha_2)$, $\alpha_1,\ \alpha_2$	$\alpha_1, \alpha_2 > 0,\ \delta \geq 0$ $\delta^2 - \alpha_1\alpha_2 < 0$	$\mathfrak{S}_{\alpha_1,\alpha_2,\delta}^{\mathsf{SU}(2)}$
$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$	$\begin{bmatrix} 0 & -\delta & -\alpha_1 \\ -1 & -(1-\alpha_2) & \delta \\ 1 & 1 & 0 \end{bmatrix}$	$-\delta^2 - \alpha_1(1-\alpha_2)$, $-\alpha_1,\ -\alpha_2$	$\alpha_1, \alpha_2 > 0,\ \delta \geq 0$ $\delta^2 - \alpha_1\alpha_2 < 0$	$\mathfrak{S}_{\alpha_1,\alpha_2,\delta}^{\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}}$
$\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$	$\begin{bmatrix} 0 & -\delta & -\gamma_1 \\ -1 & -(1-\gamma_2) & \delta \\ 1 & 1 & 0 \end{bmatrix}$	$-\delta^2 - \gamma_1(1-\gamma_2)$, $-\gamma_1,\ -\gamma_2$	$\gamma_1, \gamma_2 \in \mathbb{R},\ \delta \geq 0$ $\delta^2 - \gamma_1\gamma_2 > 0$	$\mathfrak{S}_{\gamma_1,\gamma_2,\delta}^{\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}}$

Table 2 Left-invariant nonholonomic Riemannian structures (with $\vartheta = 1$) on the non-unimodular Lie groups

Lie group	Equivalence classes (**C**)	Invariants $\varrho_0, \varrho_1, \varrho_2$	Conditions	Designation
$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$	$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -(1-\gamma) & \alpha \\ 1 & 1-\gamma & -\alpha \end{bmatrix}$	$-\frac{1}{2}\alpha^2,\ -\frac{1}{2}\alpha^2,\ -\frac{1}{2}\gamma^2$	$\alpha \geq 0,\ \gamma \in \mathbb{R}$ $\alpha^2 + \gamma^2 \neq 0$	$\mathfrak{S}_{\alpha,\gamma}^{\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}}$
$\mathsf{G}_{3.2}$	$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -\frac{1}{4}(\beta-2)^2 & 0 \\ 1 & 1-\beta & 0 \end{bmatrix}$	$0,\ 0,\ -\frac{1}{4}\beta^2$	$\beta \neq 0$	$\mathfrak{S}_{\beta}^{\mathsf{G}_{3.2}}$
	$\begin{bmatrix} 0 & -\alpha & \frac{1}{4}\beta^2 \\ -1 & -\frac{(2\alpha+\beta)^2}{\beta^2} & \alpha+\beta \\ 1 & \frac{4\alpha+\beta}{\beta} & -\beta \end{bmatrix}$	$-\frac{1}{4}\beta^2,\ -\frac{1}{4}\beta^2,\ -\frac{4\alpha^2}{\beta^2}$	$\alpha \geq 0,\ \beta \neq 0$ $(1-\operatorname{sgn}(\alpha))\beta \geq 0$	$\mathfrak{S}_{\alpha,\beta}^{\mathsf{G}_{3.2}}$
$\mathsf{G}_{3.4}^h$	$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -\frac{h^2(\beta-2)^2-\beta^2}{4h^2} & 0 \\ 1 & 1-\beta & 0 \end{bmatrix}$	$0,\ 0,\ -\frac{(h^2+1)\beta^2}{4h^2}$	$\beta \neq 0$	$\mathfrak{S}_{\beta}^{\mathsf{G}_{3.4}^h}$
	$\begin{bmatrix} 0 & -\alpha & \frac{(h^2-1)\beta^2}{4h^2} \\ -1 & -\frac{h^2(2\alpha+\beta)^2-\beta^2}{(h^2-1)\beta^2} & \alpha+\beta \\ 1 & \frac{h^2(4\alpha+\beta)-\beta}{(h^2-1)\beta} & -\beta \end{bmatrix}$	$-\frac{(h^2+1)\beta^2}{4h^2},\ -\frac{(h^2+1)\beta^2}{4h^2},$ $-\frac{4h^2(h^2+1)\alpha^2}{(h^2-1)^2\beta^2}$	$\alpha \geq 0,\ \beta \neq 0$ $(1-\operatorname{sgn}(\alpha))\beta \geq 0$	$\mathfrak{S}_{\alpha,\beta}^{\mathsf{G}_{3.4}^h}$
$\mathsf{G}_{3.5}^h$	$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -\frac{h^2(\beta-2)^2+\beta^2}{4h^2} & 0 \\ 1 & 1-\beta & 0 \end{bmatrix}$	$0,\ 0,\ -\frac{(h^2-1)\beta^2}{4h^2}$	$\beta \neq 0$	$\mathfrak{S}_{\beta}^{\mathsf{G}_{3.5}^h}$
	$\begin{bmatrix} 0 & -\alpha & \frac{(h^2+1)\beta^2}{4h^2} \\ -1 & -\frac{h^2(2\alpha+\beta)^2+\beta^2}{(h^2+1)\beta^2} & \alpha+\beta \\ 1 & \frac{h^2(4\alpha+\beta)+\beta}{(h^2+1)\beta} & -\beta \end{bmatrix}$	$-\frac{(h^2-1)\beta^2}{4h^2},\ -\frac{(h^2-1)\beta^2}{4h^2},$ $-\frac{4h^2(h^2-1)\alpha^2}{(h^2+1)^2\beta^2}$	$\alpha \geq 0,\ \beta \neq 0$ $(1-\operatorname{sgn}(\alpha))\beta \geq 0$	$\mathfrak{S}_{\alpha,\beta}^{\mathsf{G}_{3.5}^h}$

We have the following characterization of structures with a Cartan–Schouten connection.

Proposition 7 *Let $\mathfrak{S}$ be a left-invariant nonholonomic Riemannian structure on a three-dimensional Lie group* G.

(i) *Suppose* G *is unimodular; then* ∇ *is Cartan–Schouten if and only if* $\vartheta = 0$. (*When* G *is non-unimodular and* ∇ *is Cartan–Schouten, we have* $\vartheta > 0$.)
(ii) *Suppose* G *is non-unimodular; then* ∇ *is Cartan–Schouten if and only if* $\varrho_0 = \varrho_1 = \chi_2 = 0$ *and* $\varrho_2 + \frac{1}{2}\vartheta = 2\kappa$.

Accordingly, every equivalence class of structures on a unimodular Lie group in Fig. 1 has a Cartan–Schouten connection. On the other hand, for structures on a non-unimodular group, the equivalence classes $\mathfrak{S}_{0,1}^{\mathsf{Aff}(\mathbb{R})_0\times\mathbb{R}}$, $\mathfrak{S}_1^{\mathsf{G}_{3.2}}$, $\mathfrak{S}_1^{\mathsf{G}_{3.4}^h}$ and $\mathfrak{S}_1^{\mathsf{G}_{3.5}^h}$ (see Table 2) are exactly those with a Cartan–Schouten connection.

4 Flat structures

In this section we consider the flat nonholonomic Riemannian structures on three-dimensional manifolds. We first characterize flatness in three dimensions; this is followed by a classification of the flat left-invariant structures.

4.1 Characterization

Let $\mathfrak{S} = (\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian structure on a three-dimensional manifold M. Let (X_0, X_1, X_2) be a local frame defined on an open neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ such that X_0 spans $\mathcal{D}^\perp$ and (X_1, X_2) is an orthonormal frame for $\mathcal{D}$. The structure constants of (X_0, X_1, X_2) are denoted by c_{ij}^k; we may assume that $c_{21}^0 = 1$. The dual frame (ν^0, ν^1, ν^2) satisfies the structure equations $d\nu^k = \sum_{0\le i<j\le 2} c_{ij}^k \nu^j \wedge \nu^i$.

Lemma 1 *$\mathfrak{S}$ is locally flat on $\mathcal{U}$ if and only if there exists $\theta \in \mathcal{C}^\infty(\mathcal{U})$ such that*

$$X_1[\theta] = c_{21}^1 \quad \textit{and} \quad X_2[\theta] = c_{21}^2.$$

If such a function θ exists, then the rotated frame $(\cos\theta X_1 - \sin\theta X_2, \sin\theta X_1 + \cos\theta X_2)$ *is parallel.*

Proof Let (Y_1, Y_2) be another orthonormal frame for $\mathcal{D}$. There exists an orthogonal transformation taking (X_1, X_2) to (Y_1, Y_2). That is, there exist $\sigma \in \{-1, 1\}$ and $\theta \in \mathcal{C}^\infty(\mathcal{U})$ such that

$$\begin{cases} Y_1 = \sigma\cos\theta X_1 - \sin\theta X_2 \\ Y_2 = \sigma\sin\theta X_1 + \cos\theta X_2. \end{cases}$$

By Proposition 4, the frame (Y_1, Y_2) is a parallel frame for $\mathcal{D}$ if and only if $[\![Y_2, Y_1]\!] = 0$. We have $[\![Y_2, Y_1]\!] = (X_1[\theta] - \sigma c^1_{21})X_1 + (X_2[\theta] - \sigma c^2_{21})X_2$, and so (Y_1, Y_2) is parallel exactly when $X_1[\theta] = \sigma c^1_{21}$ and $X_2[\theta] = \sigma c^2_{21}$. Changing the sign of θ if necessary, we may take $\sigma = 1$. □

Lemma 2 *There exists a rotation* (Y_1, Y_2) *of* (X_1, X_2) *such that* $[\![Y_2, Y_1]\!] = 0$ *if and only if the following equations hold*:

$$\begin{cases} (c^1_{10} - c^2_{20})c^1_{21} + (c^2_{10} + c^1_{20})c^2_{21} + c^0_{20}c^1_{10} - \frac{1}{2}c^0_{10}(c^2_{10} + c^1_{20}) + c^0_{10}\kappa \\ \quad = -\frac{1}{2}X_1[c^2_{10} + c^1_{20}] + X_1[\kappa] + X_2[c^1_{10}] \\ (c^2_{10} + c^1_{20})c^1_{21} - (c^1_{10} - c^2_{20})c^2_{21} - c^0_{10}c^2_{20} + \frac{1}{2}c^0_{20}(c^2_{10} + c^1_{20}) + c^0_{20}\kappa \\ \quad = \quad \frac{1}{2}X_2[c^2_{10} + c^1_{20}] + X_2[\kappa] - X_1[c^2_{20}]. \end{cases} \tag{3}$$

Proof Let $(Y_1, Y_2) = (\cos\theta X_1 - \sin\theta X_2, \sin\theta X_1 + \cos\theta X_2)$ be a rotation of (X_1, X_2), where $\theta \in \mathcal{C}^\infty(\mathcal{U})$. By Lemma 1, we have that $[\![Y_2, Y_1]\!] = 0$ if and only if $X_1[\theta] = c^1_{21}$ and $X_2[\theta] = c^2_{21}$. We claim that there exists $\theta \in \mathcal{C}^\infty(\mathcal{U})$ satisfying the conditions $X_1[\theta] = c^1_{21}$, $X_2[\theta] = c^2_{21}$ if and only if (3) hold. If such a function θ exists, then

$$\begin{aligned} d\theta &= X_0[\theta]\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2 \\ &= ([X_2, X_1][\theta] - c^1_{21}X_1[\theta] - c^2_{21}X_2[\theta])\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2 \\ &= (X_2[c^1_{21}] - X_1[c^2_{21}] - (c^1_{21})^2 - (c^2_{21})^2)\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2 \\ &= (\kappa - \tfrac{1}{2}(c^2_{10} - c^1_{20}))\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2. \end{aligned}$$

The right-hand side is independent of θ; accordingly, let $\varpi = (\kappa - \frac{1}{2}(c^2_{10} - c^1_{20}))\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2$. Then $d\varpi = f_{01}\,\nu^0 \wedge \nu^1 + f_{02}\,\nu^0 \wedge \nu^2$, where

$$\begin{cases} f_{01} = c^1_{10}c^1_{21} + c^2_{10}c^2_{21} - c^0_{10}(\frac{1}{2}(c^2_{10} - c^1_{20}) - \kappa) \\ \qquad + X_1[\frac{1}{2}(c^2_{10} - c^1_{20}) - \kappa] + X_0[c^1_{21}] \\ f_{02} = c^1_{20}c^1_{21} + c^2_{20}c^2_{21} - c^0_{20}(\frac{1}{2}(c^2_{10} - c^1_{20}) - \kappa) \\ \qquad + X_2[\frac{1}{2}(c^2_{10} - c^1_{20}) - \kappa] + X_0[c^2_{21}]. \end{cases}$$

Using $d^2 = 0$ on the structure equations for $d\nu^1$ and $d\nu^2$, we get

$$\begin{cases} X_0[c^1_{21}] = -c^1_{20}c^0_{10} + c^1_{10}c^0_{20} - c^2_{20}c^1_{21} + c^1_{20}c^2_{21} + X_1[c^1_{20}] - X_2[c^1_{10}] \\ X_0[c^2_{21}] = -c^2_{20}c^0_{10} + c^2_{10}c^0_{20} + c^2_{10}c^1_{21} - c^1_{10}c^2_{21} + X_1[c^2_{20}] - X_2[c^2_{10}]. \end{cases}$$

Hence

$$\begin{cases} f_{01} = (c_{10}^1 - c_{20}^2)c_{21}^1 + (c_{10}^2 + c_{20}^1)c_{21}^2 + c_{20}^0 c_{10}^1 - \frac{1}{2}c_{10}^0(c_{10}^2 + c_{20}^1) + c_{10}^0\kappa \\ \qquad + \frac{1}{2}X_1[c_{10}^2 + c_{20}^1] - X_1[\kappa] - X_2[c_{10}^1] \\ f_{02} = (c_{10}^2 + c_{20}^1)c_{21}^1 - (c_{10}^1 - c_{20}^2)c_{21}^2 - c_{20}^2 c_{10}^0 + \frac{1}{2}c_{20}^0(c_{10}^2 + c_{20}^1) + c_{20}^0\kappa \\ \qquad - \frac{1}{2}X_2[c_{10}^2 + c_{20}^1] - X_2[\kappa] + X_1[c_{20}^2]. \end{cases}$$

Suppose θ exists, so that $d\varpi = d^2\theta = 0$; then $f_{01} = f_{02} = 0$, which yields the equations (3). Conversely, if (3) hold, then $d\varpi = 0$, i.e., ϖ is closed. Hence ϖ is locally exact: there exists an open neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ and $\theta \in \mathcal{C}^\infty(\mathcal{U}')$ such that $\varpi = d\theta$. The rotated frame (Y_1, Y_2) then satisfies $[\![Y_2, Y_1]\!] = 0$. □

Using the condition for flatness in Lemma 2 (which depends on the choice of an orthonormal frame for $\mathcal{D}$), we shall derive an invariant characterization of the flat structures. We have the decomposition $T\mathsf{M} = \mathcal{D} \oplus \text{span}\{Z\}$, where Z is the Reeb vector field of the normalized contact form ω. Let $\mathscr{R} : T\mathsf{M} \to \text{span}\{Z\}$ be the projection onto the distribution spanned by Z. In particular, we have $\mathscr{R}([X_2, X_1]) = d\omega(X_1, X_2)Z$.

Theorem 3 *$\mathfrak{S}$ is locally flat on $\mathcal{U}$ if and only if*

$$d_{\mathscr{P}}^{\nabla} F = F \circ \rho \quad \text{on}\, \mathcal{U}. \tag{4}$$

Here $F = \mathbf{g}^\sharp \circ (\text{Ric}^\flat + A_{sym}^\flat + A_{skew}^\flat)$ and $\rho \in \Omega^2(\mathcal{D}, \mathcal{D})$ is given by $\rho(X_1, X_2) = \mathscr{P}(\mathscr{R}([X_2, X_1]))$.

Proof Since $(\mathbf{g}^\sharp \circ \text{Ric}^\flat)(X_a) = \text{Ric}(X_a, X_1)X_1 + \text{Ric}(X_a, X_2)X_2$, $\text{Ric}(X_a, X_a) = \kappa$ and $\text{Ric}(X_a, X_b) = 0$ for $a \neq b$, we have

$$\begin{aligned} d_{\mathscr{P}}^{\nabla}(\mathbf{g}^\sharp \circ \text{Ric}^\flat)(X_1, X_2) &= \nabla_{X_1}(\mathbf{g}^\sharp \circ \text{Ric}^\flat)(X_2) - \nabla_{X_2}(\mathbf{g}^\sharp \circ \text{Ric}^\flat)(X_1) \\ &\quad - (\mathbf{g}^\sharp \circ \text{Ric}^\flat)([\![X_1, X_2]\!]) \\ &= \nabla_{X_1}(\kappa X_2) - \nabla_{X_2}(\kappa X_1) + (c_{21}^1 \kappa X_1 + c_{21}^2 \kappa X_2). \end{aligned}$$

It is not difficult to show that $\nabla_{X_1} X_2 = -c_{21}^1 X_1$ and $\nabla_{X_2} X_1 = c_{21}^2 X_2$. Hence

$$\begin{aligned} d_{\mathscr{P}}^{\nabla}(\mathbf{g}^\sharp \circ \text{Ric}^\flat)(X_1, X_2) &= X_1[\kappa]X_2 - \kappa c_{21}^1 X_1 - X_2[\kappa]X_1 - \kappa c_{21}^2 X_2 \\ &\quad + (c_{21}^1 \kappa X_1 + c_{21}^2 \kappa X_2) \\ &= -X_2[\kappa]X_1 + X_1[\kappa]X_2. \end{aligned}$$

Similar calculations yield

$$\begin{aligned} d_{\mathscr{P}}^{\nabla}(\mathbf{g}^\sharp \circ A^\flat)(X_1, X_2) &= (X_1[c_{20}^2] - \tfrac{1}{2}X_2[c_{10}^2 + c_{20}^1])X_1 + (X_2[c_{10}^1] \\ &\quad - \tfrac{1}{2}X_1[c_{10}^2 + c_{20}^1])X_2 + 2\,(\mathbf{g}^\sharp \circ A_{sym}^\flat)([\![X_2, X_1]\!]). \end{aligned}$$

Here $A = A_{sym} + A_{skew}$. In addition, we have $(\mathbf{g}^\sharp \circ \mathrm{Ric}^\flat \circ \rho)(X_1, X_2) = -c^0_{20}\kappa X_1 + c^0_{10}\kappa X_2$ and

$$(\mathbf{g}^\sharp \circ A^\flat \circ \rho)(X_1, X_2) = -(c^0_{10}c^1_{10} + \tfrac{1}{2}c^0_{20}(c^2_{10} + c^1_{20}))X_1 \\ -(c^0_{20}c^2_{20} + \tfrac{1}{2}c^0_{10}(c^2_{10} + c^1_{20}))X_2.$$

Let f_{01} and f_{02} be defined as in the proof of Lemma 2. There exists a parallel frame for $\mathcal{D}$ if and only if $f_{01} = f_{02} = 0$. From the foregoing calculations, it follows that

$$(d^\nabla_{\mathscr{P}} F - F \circ \rho)(X_1, X_2) = f_{02}X_1 - f_{01}X_2.$$

(As $d^\nabla_{\mathscr{P}} F - F \circ \rho$ is skew-symmetric, it is fully determined by its evaluation of $X_1 \wedge X_2$.) This proves the result. □

Corollary 1 *Suppose $\vartheta = 0$; then $\mathfrak{S}$ is locally flat on $\mathcal{U}$ if and only if $d^\nabla_{\mathscr{P}} F \equiv 0$ on $\mathcal{U}$.*

Proof If $\vartheta = 0$, then $\mathscr{P}(Z) = 0$, and so $\rho \equiv 0$. Hence $\mathfrak{S}$ is flat exactly when $d^\nabla_{\mathscr{P}} F$ vanishes. □

Corollary 2 *If $K \equiv 0$ on $\mathcal{U}$, then $\mathfrak{S}$ is locally flat on $\mathcal{U}$.*

Proof If $K \equiv 0$, then Ric, A_{sym} and A_{skew} all vanish identically. Hence $F \equiv 0$, and the condition $d^\nabla_{\mathscr{P}} F = F \circ \rho$ is trivially satisfied. □

4.2 Classification

Let $\mathfrak{S} = (\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a left-invariant nonholonomic Riemannian structure on a three-dimensional simply connected Lie group G. Suppose that (X_0, X_1, X_2) is a left-invariant frame on G such that X_0 spans $\mathcal{D}^\perp$ and (X_1, X_2) is an orthonormal frame for $\mathcal{D}$. Let c^k_{ij} be the structure constants of the frame; as before, we take $c^0_{21} = 1$. With respect to (X_1, X_2), we have

$$d^\nabla_{\mathscr{P}} F(X_1, X_2) = 2\,(\mathbf{g}^\sharp \circ A^\flat_{sym})([\![X_2, X_1]\!]) = \begin{bmatrix} c^2_{10} + c^1_{20} & -(c^1_{10} - c^2_{20}) \\ c^2_{20} - c^1_{10} & -(c^2_{10} + c^1_{20}) \end{bmatrix} \begin{bmatrix} c^1_{21} \\ c^2_{21} \end{bmatrix} \quad (5)$$

and

$$(F \circ \rho)(X_1, X_2) = \begin{bmatrix} \kappa + \frac{1}{2}(c^2_{10} + c^1_{20}) & c^2_{20} \\ -c^1_{10} & \kappa - \frac{1}{2}(c^2_{10} + c^1_{20}) \end{bmatrix} \begin{bmatrix} -c^0_{20} \\ c^0_{10} \end{bmatrix}. \quad (6)$$

We shall consider the following cases: (I) $\vartheta = 0$; (II-a) $\vartheta > 0$ and G is unimodular; (II-b) $\vartheta > 0$ and G is non-unimodular.

Theorem 4 (Case I: $\vartheta = 0$) *$\mathfrak{S}$ is flat if and only if G is unimodular (hence ∇ is a Cartan–Schouten connection) or $\mathsf{G} = \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.*

Proof By Corollary 1, $\mathfrak{S}$ is flat if and only if $d^{\nabla}_{\mathscr{P}} F$ vanishes. From (5), this happens exactly when $A^{\flat}_{sym}([\![X_2, X_1]\!]) = 0$. Suppose $\chi_1 > 0$. Then $A^{\flat}_{sym}$ is invertible, and $[\![X_2, X_1]\!] = 0$ implies that every left-invariant frame for $\mathcal{D}$ is parallel; that $c^1_{21} = c^2_{21} = 0$; and that ∇ is a Cartan–Schouten connection. Moreover, G must be unimodular. Conversely, every such structure is clearly flat.

On the other hand, suppose $\chi_1 = 0$. Then $A^{\flat}_{sym} \equiv 0$, and so every such structure in this case is flat. From Theorem 1 (see also Fig. 1), these are the following structures (up to NH-isometry and rescaling):

- the structure on $\mathsf{SU}(2)$ with metric (at identity) being the Killing form restricted to $\mathcal{D}_{\mathbf{1}}$ (when $\kappa > 0$);
- any structure on the Heisenberg group H_3 (when $\kappa = 0$);
- the structure on $\widetilde{\mathsf{SL}}(2, \mathbb{R})$ with elliptic-type distribution and metric (at identity) being the Killing form restricted to $\mathcal{D}_{\mathbf{1}}$ (when $\kappa < 0$); also, any structure on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ (see Remark 1).

With the exception of those on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, these are all structures on unimodular Lie groups with Cartan–Schouten connections. □

Remark 2 The case when $\chi_1 = 0$ was essentially proved in Agrachev and Barilari (2012). Indeed, it was shown that for a sub-Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ on a three-dimensional simply connected manifold (which may be viewed as a nonholonomic Riemannian structure with $\vartheta = 0$) with constant κ and $\chi_1 = 0$, there exists a rotation $(\widehat{X}_1, \widehat{X}_2)$ of the frame (X_1, X_2) such that $[\widehat{X}_1, Z] = \kappa \widehat{X}_2$, $[\widehat{X}_2, Z] = -\kappa \widehat{X}_1$ and $[\widehat{X}_2, \widehat{X}_1] = Z$, i.e., such that $(\widehat{X}_1, \widehat{X}_2)$ is parallel.

Theorem 5 (Case II-a: $\vartheta > 0$, G unimodular) *$\mathfrak{S}$ is flat if and only if $\varrho_0 = \vartheta \varrho_1$ and $\varrho_2 = 0$ (in which case $\kappa^2 = \chi_1^2$). Consequently, any flat structure is NH-isometric up to rescaling to exactly one of the following structures*:

$$\mathfrak{S}^{SE(1,1)}_{\alpha_1,0} \quad (when\ \varrho_0 < 0), \qquad \mathfrak{S}^{H_3} \quad (when\ \varrho_0 = 0), \qquad \mathfrak{S}^{\widetilde{SE}(2)}_{\alpha_1,0} \quad (when\ \varrho_0 > 0).$$

Proof We take (X_0, X_1, X_2) to be the canonical frame (described in Sect. 3.2). Since G is unimodular, we have $c^2_{21} = 0$ and $c^1_{21} = -c^0_{20} = \|\mathscr{P}(Z)\|$, whence $[\![X_2, X_1]\!] = d\omega(X_1, X_2)\mathscr{P}(Z) = \rho(X_1, X_2)$. Furthermore, we have $\chi_2 = 0$, i.e., $A_{skew} \equiv 0$. The condition (4) becomes

$$\begin{aligned} &2\,(\mathbf{g}^{\sharp} \circ A^{\flat}_{sym})([\![X_2, X_1]\!]) = (\mathbf{g}^{\sharp} \circ (\mathrm{Ric}^{\flat} + A^{\flat}_{sym}))([\![X_2, X_1]\!]) \\ \iff\quad & \mathrm{Ric}^{\flat}([\![X_2, X_1]\!]) = A^{\flat}_{sym}([\![X_2, X_1]\!]) \\ \iff\quad & \mathrm{Ric}^{\flat}(X_1) = A^{\flat}_{sym}(X_1) \\ \iff\quad & A_{sym}(X_1, X_1) = \kappa \text{ and } A_{sym}(X_1, X_2) = 0. \end{aligned}$$

In terms of the structure constants, this is equivalent to the conditions $c^1_{10} = 0$, $c^1_{20} = -(c^0_{20})^2$, which are in turn equivalent to $\varrho_0 = \vartheta \varrho_1$ and $\varrho_2 = 0$. Furthermore, this implies that $\kappa^2 = \chi_1^2$. In terms of the classification in Theorem 2, these are the following structures:

- the equivalence classes $\mathfrak{S}^{\mathsf{SE}(1,1)}_{\alpha_1,\alpha_2}$ with $\alpha_2 = 0$ (when $\varrho_0 < 0$);
- the equivalence class $\mathfrak{S}^{\mathsf{H}_3}$ (when $\varrho_0 = 0$);
- the equivalence classes $\mathfrak{S}^{\widetilde{\mathsf{SE}}(2)}_{\alpha_1,\alpha_2}$ with $\alpha_2 = 0$ (when $\varrho_0 > 0$).

□

Theorem 6 (Case II-b: $\vartheta > 0$, G non-unimodular)

(i) *If $G = Aff(\mathbb{R})_0 \times \mathbb{R}$, then $\mathfrak{S}$ is flat.*
(ii) *Suppose $\chi_2 = 0$; $\mathfrak{S}$ is flat if and only if ∇ is a Cartan–Schouten connection.*
(iii) *Suppose $\chi_2 > 0$; any flat structure is NH-isometric up to rescaling to exactly one of the following structures*:

$$\mathfrak{S}^{G_{3.2}}_{\alpha,\beta} \qquad with \quad \alpha = -\frac{\beta}{8}\left(1 \pm \sqrt{1-4\beta^2}\right),\ -\tfrac{1}{2} \leq \beta < 0$$

$$\mathfrak{S}^{G^h_{3.4}}_{\alpha,\beta},\ 0 < h < 1 \qquad with \quad \alpha = -\frac{(h^2-1)\beta}{8h^2}\left(1 \pm \sqrt{1-4\beta^2}\right),\ 0 < \beta \leq \tfrac{1}{2}$$

$$\mathfrak{S}^{G^h_{3.4}}_{\alpha,\beta},\ 1 < h \qquad with \quad \alpha = -\frac{(h^2-1)\beta}{8h^2}\left(1 \pm \sqrt{1-4\beta^2}\right),\ -\tfrac{1}{2} \leq \beta < 0$$

$$\mathfrak{S}^{G^h_{3.5}}_{\alpha,\beta} \qquad with \quad \alpha = -\frac{(h^2+1)\beta}{8h^2}\left(1 \pm \sqrt{1-4\beta^2}\right),\ -\tfrac{1}{2} \leq \beta < 0.$$

Proof Considering the equivalence class representatives in Table 2, a direct (but tedious) calculation, using the condition that (5) equals (6), yields the result. We illustrate with the case of $\mathsf{G}_{3.2}$. Take (X_0, X_1, X_2) to be the canonical frame. Consider first the family of equivalence classes $\mathfrak{S}^{\mathsf{G}_{3.2}}_{\beta}$; we have

$$\begin{aligned}
&(d^{\nabla}_{\mathscr{P}} F - F \circ \rho)(X_1, X_2) \\
&\quad = \begin{bmatrix} -\frac{1}{4}(\beta-2)^2 & 0 \\ 0 & \frac{1}{4}(\beta-2)^2 \end{bmatrix} \begin{bmatrix} 1-\beta \\ 0 \end{bmatrix} - \begin{bmatrix} -(\beta-1)^2 & 0 \\ 0 & -\frac{3}{4}\beta(\beta-\frac{4}{3}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\quad = \begin{bmatrix} \frac{1}{4}\beta^2(\beta-1) \\ 0 \end{bmatrix}.
\end{aligned}$$

Hence a member of the equivalence class $\mathfrak{S}^{\mathsf{G}_{3.2}}_{\beta}$ is flat exactly when $\beta = 1$. From Sect. 3.2, this is exactly the structure on $\mathsf{G}_{3.2}$ with a Cartan–Schouten connection. On the other hand, consider the family of equivalence classes $\mathfrak{S}^{\mathsf{G}_{3.2}}_{\alpha,\beta}$; then

$$(d^{\nabla}_{\mathscr{P}} F - F \circ \rho)(X_1, X_2) = \begin{bmatrix} -\frac{\alpha}{\beta^3}(16\alpha^2 + 4\alpha\beta + \beta^4) \\ \frac{1}{4\beta}(16\alpha^2 + 4\alpha\beta + \beta^4) \end{bmatrix}.$$

Thus the structure is flat if and only if $16\alpha^2 + 4\alpha\beta + \beta^4 = 0$. It is not difficult to show that this occurs exactly when $\alpha = -\frac{\beta}{8}(1 \pm \sqrt{1-4\beta^2})$ and $-\frac{1}{2} \leq \beta < 0$. □

Remark 3 Every left-invariant nonholonomic Riemannian structure on H_3 and $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ is flat. On the other hand, apart from those with Cartan–Schouten connections, there are no flat structures on the semisimple groups $\mathsf{SU}(2)$ and $\widetilde{\mathsf{SL}}(2,\mathbb{R})$. Every other group (apart from H_3 and $\mathsf{Aff}(\mathbb{R})_0\times\mathbb{R}$) admits a one-parameter family of flat structures, up to equivalence.

Remark 4 The classical Chaplygin problem (see, e.g., Fedorov, Y., García-Naranjo 2010; Barrett et al. 2016) may be modelled by a left-invariant nonholonomic Riemannian structure on the Euclidean group $\mathsf{SE}(2)$. The reduced dynamics on the Lie algebra $\mathfrak{se}(2)$ exhibit three qualitatively different cases (of increasing analytical complexity): the "Chaplygin skate" (when $\vartheta = 0$); the "Chaplygin sleigh" (when $\vartheta > 0$ and $\kappa^2 = \chi_1^2$); and the "hydrodynamic Chaplygin sleigh" (when $\vartheta > 0$ and $\kappa^2 \neq \chi_1^2$). The equivalence classes of structures corresponding to the Chaplygin sleigh are $\mathfrak{S}^{\widetilde{\mathsf{SE}}(2)}_{\alpha_1,0}$. Accordingly, the flat structures for the Chaplygin problem correspond precisely to the case of the Chaplygin skate and the Chaplygin sleigh.

Appendix Real Lie algebras of dimension three

The classification of real three-dimensional Lie algebras is well known. Our preference is for the Bianchi–Behr enumeration (MacCallum 1999; Krasiński et al. 2003; Mubarakzyanov 1963). In terms of an appropriate ordered basis (E_1, E_2, E_3), the commutator relations are given by

$$\begin{cases} [E_2, E_3] = n_1 E_1 - a E_2 \\ [E_3, E_1] = a E_1 + n_2 E_2 \\ [E_1, E_2] = n_3 E_3. \end{cases}$$

The coefficients a, n_1, n_2 and n_3 for each type of algebra may be found in Table 3, together with the (unique) simply connected Lie group corresponding to each algebra.

Table 3 Bianchi–Behr classification of real three-dimensional Lie algebras

Type	Bianchi	a	n_1	n_2	n_3	Simply connected Lie group	Unimodular
$3\mathfrak{g}_1$	I	0	0	0	0	$\mathbb{R}^3$	•
$\mathfrak{g}_{2.1}\oplus\mathfrak{g}_1$	III	1	1	−1	0	$\mathsf{Aff}(\mathbb{R})_0\times\mathbb{R}$	
$\mathfrak{g}_{3.1}$	II	0	1	0	0	H_3	•
$\mathfrak{g}_{3.2}$	IV	1	1	0	0	$\mathsf{G}_{3.2}$	
$\mathfrak{g}_{3.3}$	V	1	0	0	0	$\mathsf{G}_{3.3}$	
$\mathfrak{g}_{3.4}^0$	VI_0	0	1	−1	0	$\mathsf{SE}(1,1)$	•
$\mathfrak{g}_{3.4}^h$	VI_h	$h>0$ $h\neq1$	1	−1	0	$\mathsf{G}_{3.4}^h$	
$\mathfrak{g}_{3.5}^0$	VII_0	0	1	1	0	$\widetilde{\mathsf{SE}}(2)$	•
$\mathfrak{g}_{3.5}^h$	VII_h	$h>0$	1	1	0	$\mathsf{G}_{3.5}^h$	
$\mathfrak{g}_{3.6}$	VIII	0	1	1	−1	$\widetilde{\mathsf{SL}}(2,\mathbb{R})$	•
$\mathfrak{g}_{3.7}$	IX	0	1	1	1	$\mathsf{SU}(2)$	•

Remark 5 The Abelian group $\mathbb{R}^3$ and the group $\mathsf{G}_{3.3}$ do not admit completely nonholonomic left-invariant distributions. Accordingly, there do not exist left-invariant nonholonomic Riemannian structures on these groups.

Remark 6 Apart from $\widetilde{\mathsf{SL}}(2, \mathbb{R})$ (the universal cover of $\mathsf{SL}(2, \mathbb{R})$) there exists, up to Lie group automorphism, at most one completely nonholonomic left-invariant distribution on each three-dimensional simply connected Lie group. On $\widetilde{\mathsf{SL}}(2, \mathbb{R})$ there exist exactly two such distributions up to automorphism, according as whether the Killing form restricted to the distribution (at identity) is definite or indefinite. Following Agrachev and Barilari (2012), if the Killing form is definite on a given distribution, we shall say that the distribution is of *elliptic type*, and denote the group as $\widetilde{\mathsf{SL}}(2, \mathbb{R})_{ell}$. On the other hand, when the Killing form is indefinite on the distribution, we shall say that it is of *hyperbolic type*, and write $\widetilde{\mathsf{SL}}(2, \mathbb{R})_{hyp}$ for the group.

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