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ORIGINAL PAPER

Some results on almost Ricci solitons and geodesic vector fields

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Abstract We show that a compact almost Ricci soliton whose soliton vector field is divergence-free is Einstein and its soliton vector field is Killing. Next we show that an almost Ricci soliton reduces to Ricci soliton if and only if the associated vector field is geodesic. Finally, we prove that a contact metric manifold is K-contact if and only if its Reeb vector field is geodesic.

Keywords Almost Ricci soliton · Contact metric structure · K-contact · Einstein Sasakian

Mathematics Subject Classification 53C25 · 53C44 · 53C21

1 Introduction

The notion of an almost Ricci soliton was introduced by Pigola et al. (2011) as a Riemannian manifold (M, g) satisfying the condition:

$$\pounds_V g + 2Ric = 2\lambda g, \tag{1}$$

where V is a smooth vector field on M, Ric denotes Ricci tensor of g, $\pounds_X$ denotes the Lie-derivative operator along V, and λ is a smooth real function on M. For λ constant, it reduces to a Ricci soliton. It is called a gradient almost Ricci soliton if $V = \nabla f$ (up to the addition of a Killing vector field). Barros et al. (2014) and Barros and Ribeiro

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Barros and Ribeiro (2012) have characterized compact almost Ricci solitons under some geometric conditions. In this article, we first prove the following rigidity result.

Theorem 1 *If a compact almost Ricci soliton (M, g, V, λ) has divergence-free soliton vector field V, then it is Einstein and V is Killing.*

Remark If a vector field is Killing, then it is divergence-free. However, the converse need not be true, in general. Theorem 1 provides a context under which the converse holds.

Now we show that the above result holds for a Ricci soliton without the compactness assumption, as follows.

Proposition 1 *A Ricci soliton with divergence-free soliton vector field V is Einstein and V is Killing.*

Next, we recall an operator $\Box$ which acts on a smooth vector field X such that $\Box X$ is a vector field with components $-(g^{jk}\nabla_j \nabla_k X^i + R^i_j X^j)$ in a local cordinate system x^i. Following Yano and Nagano (1961) we state

Definition 1 A vector field X is called a geodesic vector field if $\Box X = 0$.

This definition is different from and should not be confused with a vector field whose integral curves are geodesics. For details we refer to Yano and Nagano (1961) and Yano (1970). We note that the geodesic vector field condition is also equivalent to the condition $g^{jk}\pounds_X \Gamma^i_{jk} = 0$ which shows that X preserves geodesics on the average. Such vector fields were also studied by Stepanov and Shandra (2003) and generate an infinitesimal harmonic transformations. Obviously, a Killing vector field ($\pounds_X g_{ij} = 0$) and an affine Killing vector field ($\pounds_X \Gamma^i_{jk} = 0$) are special examples of a geodesic vector field. We now obtain a condition for an almost Ricci soliton to reduce to Ricci soliton as the following result.

Theorem 2 *An almost Ricci soliton (M, g, V, λ) reduces to Ricci soliton if and only if V is a geodesic vector field in the sense of Definition 1.*

Next, we recall the following result of Yano and Nagano (1961): "A vector field X on a compact orientable manifold M is Killing if and only if it is divergence-free and geodesic," and show that this result holds without the compactness condition on a contact metric manifold for X equal to the Reeb vector field. This provides a new characterization of a K-contact manifold as

Theorem 3 *A contact metric manifold is K-contact if and only if its Reeb vector field is geodesic in the sense of Definition 1.*

Finally, we prove the following result for an almost Ricci soliton as a contact metric manifold.

Proposition 2 *If a complete almost Ricci soliton (M, g, V, λ) has V as the Reeb vector field of a contact Riemannian manifold M with metric g, then (M, g) is compact Einstein and Sasakian.*

2 A brief review of contact geometry

A $(2m+1)$-dimensional smooth manifold is said to be contact if it has a global 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ on M. For a contact 1-form η there exists a unique vector field ξ such that $d\eta(\xi, X)= 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, we obtain a Riemannian metric g and a (1,1)-tensor field φ such that

$$d\eta(X, Y) = g(X, \varphi Y), \eta(X) = g(X, \xi), \varphi^2 = -I + \eta \otimes \xi \tag{2}$$

for arbitrary vector fields X, Y on M. Following Blair (2002) we recall the self-adjoint operator $h = \frac{1}{2}\pounds_\xi \varphi$. The tensors h and $h\varphi$ are trace-free, $h\xi = 0$ and $h\varphi = -\varphi h$. Following formula holds on a contact metric manifold:

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{3}$$

where ∇ is the Riemannian connection g. A contact metric structure is said to be K-contact if ξ is Killing with respect to g, equivalently. This condition is also equivalent to:

$$Q\xi = 2m\xi, \tag{4}$$

where Q denotes the Ricci operator defined by $g(QX, Y) = Ric(X, Y)$. The contact metric structure on M is said to be Sasakian if the almost Kaehler structure on the cone manifold $(M \times R^+, r^2 g + dr^2)$ over M, is Kaehler. Sasakian manifolds are K-contact and K-contact 3-manifolds are Sasakian.

3 Proofs of the results

Proof of Theorem 1 The g-trace of the almost Ricci soliton equation (1) and the hypothesis: $div V = 0$, provides

$$S = n\lambda, \tag{5}$$

where n is the dimension of M and S denotes the scalar curvature of g. As derived in Barros et al. (2014) and Sharma (2014), we have the following formula:

$$\pounds_V S = \Delta S - 2\lambda S + 2|Ric|^2 - 2(n-1)\Delta\lambda, \tag{6}$$

where $\Delta = div(grad)$ is the g-Laplacian. Following Barros et al. (2014) we integrate (6) using divergence theorem and use (5) obtaining

$$\int_M \left(|Ric|^2 - \lambda S - \frac{1}{2}\pounds_V S\right) dv_g = 0, \tag{7}$$

where dv_g denotes the volume element of (M, g). We note that $div(SV) = \nabla_i(SV^i) = (\nabla_i S)V^i + S\nabla_i V^i = \pounds_V S$, because $div V = 0$ by hypothesis.

Using this and (5) in Eq. (7) we obtain

$$\int_M \left(|Ric|^2 - \frac{S^2}{n}\right) dv_g = 0. \tag{8}$$

But we also have the relation:

$$|Ric|^2 - \frac{S^2}{n} = \left|Ric - \frac{S}{n}g\right|^2. \tag{9}$$

Consequently, Eq. (8) assumes the form

$$\int_M \left|Ric - \frac{S}{n}g\right|^2 dv_g = 0. \tag{10}$$

which implies that $Ric = \frac{S}{n}g$, i.e. g is Einstein. This can also be written, in view of (5), as $Ric = \lambda g$. Using this in (1) shows that V is Killing, completing the proof.

Proof of Proposition 1 Equation (5) is applicable in this case as well. As λ is constant for Ricci soliton, S is also constant. Hence formulas (5) and (6) imply $|Ric|^2 = \frac{S^2}{n}$. Using this in relation (9) we obtain $Ric = \frac{S}{n}g = \lambda g$, i.e. g is Einstein. Finally, (1) implies that V is Killing, completing the proof.

Proof of Theorem 2 First we take the g-trace of Eq. (1) and differentiate so as to get

$$\nabla_j \nabla_i V^i + \nabla_j S = n \nabla_j \lambda. \tag{11}$$

Next, writing (1) as $\nabla_j V^i + \nabla^i V_j + 2R^i_j = 2\lambda \delta^i_j$, differentiating it, and using the twice contracted Bianchi second identity: $\nabla_i R^i_j = \frac{1}{2}\nabla_j S$ gives

$$\nabla_i \nabla_j V^i + \nabla_i \nabla^i V_j + \nabla_j S = 2\nabla_j \lambda. \tag{12}$$

Subtracting Eq. (11) from (12) we immediately obtain

$$R_{kj} V^k + \nabla_i \nabla^i V_j = (2 - n)\nabla_j \lambda$$

i.e. $\Box V = (n-2)\nabla\lambda$. Hence λ is constant if and only if V is geodesic, completing the proof.

Remark It has been kindly brought to my attention by the referee that the last formula in the above proof follows from equation (2.9) of Barros et al. (2014) and also from equation (3.6) of Ghosh (2015) and Weitzenbock's formula: $\Delta X = \bar{\Delta} X + QX$ where $\bar{\Delta}$ is the rough Laplacian. In order to make this paper self-contained, we have included the above proof.

Proof of Theorem 3 By definition, we have

$$(\Box\xi)^i = -(\nabla^j \nabla_j \xi^i + R^i_j \xi^j). \tag{13}$$

Next, by Ricci identity we have

$$\nabla_j \nabla_i \xi^j - \nabla_i \nabla_j \xi^j = R_{ik}\xi^k.$$

But $\nabla_j \xi^j = 0$ for a contact metric, and hence the above equation reduces to

$$\nabla_j \nabla_i \xi^j = R_{ik}\xi^k. \tag{14}$$

On the other hand, we have

$$\begin{aligned}\nabla_j \nabla_i \xi^j &= \nabla^j \nabla_i \xi_j \\ &= 2\nabla^j \varphi_{ij} + \nabla^j \nabla_j \xi_i \\ &= -2\nabla_j \varphi^j_i + \nabla^j \nabla_j \xi_i,\end{aligned}$$

where we used formula (3) in the second step. The use of formula $\nabla_j \varphi^j_i = -2m\xi_i$ for a contact metric manifold (see Tanno 1988) in the above equation provides

$$\nabla_j \nabla_i \xi^j = 4m\xi_i + \nabla^j \nabla_j \xi_i \tag{15}$$

Equations (14) and (15) show that

$$(\Box\xi)^i = -2(R^i{}_j \xi^j - 2m\xi^i)$$

This consequence, in conjunction with the K-contact condition (4), completes the proof.

Proof of Proposition 2 By hypothesis $V = \xi$, and hence (1) becomes

$$(\pounds_\xi g)(X, Y) + 2Ric(X, Y) = 2\lambda g(X, Y) \tag{16}$$

Now using the formula (3) and the property $h\varphi = -\varphi h$ we derive

$$(\pounds_\xi g)(X, Y) = 2\lambda g(h\varphi X, Y) \tag{17}$$

Hence Eq. (16) becomes

$$h\varphi X + QX = \lambda X \tag{18}$$

Substituting ξ for X shows that $Q\xi = \lambda\xi$. Hence, applying the following result of Ghosh (2014): "A complete contact Ricci almost soliton whose soliton vector field is point-wise collinear with the Reeb vector field is Einstein if the Reeb vector field is an eigenvector of the Ricci operator" we conclude that g is Einstein. Tracing equation

(18) and noting that $div.\xi = 0$ gives $S = \lambda(2m+1)$. Hence we get $Ric = \lambda g$. Substituting it back in (16) gives $\pounds_\xi g = 0$, i.e g is K-contact. As (M, g) is complete and the Einstein constant is $2m$, by Myers' theorem, (M, g) is compact. Applying the following result of Boyer and Galicki Boyer and Galicki (2001): "A compact Einstein K-contact manifold is Sasakian", we conclude that g is Sasakian, completing the proof.

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