CrossMark

Computational aspects of Burnside rings, part I: the ring structure

Martin Kreuzer[1] · **Dilip P. Patil**[2]

Abstract The Burnside ring $\mathcal{B}(G)$ of a finite group G, a classical tool in group theory and representation theory, is studied from the point of view of computational commutative algebra. Starting from a table of marks, we describe efficient algorithms for computing a presentation, the image of the mark homomorphism, the prime ideals and the prime ideal graph, the singular locus, the conductor in its integral closure, the connected components of its spectrum, and its idempotents. On the way, we provide methods for identifying p-residual subgroups, direct products of subgroups of coprime order, commutator subgroups, and perfect subgroups.

Contents

✉ Martin Kreuzer
Martin.Kreuzer@uni-passau.de

Dilip P. Patil
patil@math.iisc.ernet.in

[1] Fakultät für Informatik und Mathematik, Universität Passau, 94030 Passau, Germany

[2] Department of Mathematics/CSA, Indian Institute of Science, Bangalore 560 012, India

1 Introduction

The Burnside ring is a classical tool in the theory of finite groups. It was introduced for the first time by Burnside (1911) in his book. Given a finite group G, it is usually defined as the Grothendieck ring of isomorphism classes of finite G-sets. It can be viewed as the representation ring of permutation representations of G.

Many papers have been written about the Burnside ring $\mathcal{B}(G)$, most of them from a group theoretical point of view. In this series of papers we change the perspective and consider the structure of $\mathcal{B}(G)$ as a commutative ring. As such, we can use the methods of computational commutative algebra to deal with it algorithmically. Since many properties of groups are reflected in the structure of this ring, we set ourselves the goal to find efficient algorithms for checking or computing them. For this endeavour, our starting point, i.e., the input of all algorithms, is the table of marks of a finite group. Although it is known that different groups can have the same table of marks (see for instance Thévenaz 1988), there is still a lot of information about the group contained in it. In particular, the table of marks determines a presentation of the Burnside ring and thus makes it possible to perform effective computations with this commutative ring.

Let us describe the contents in more detail. In Sect. 2 we recall the definition and the basic properties of the Burnside ring $\mathcal{B}(G)$. Given representatives $H_1, \ldots, H_s$ for the conjugacy classes of subgroups of G, it is a free $\mathbb{Z}$-module with basis $[G/H_1], \ldots, [G/H_s]$. Then, in Sect. 3, we provide the definition and the basic properties of the table of marks $\mathcal{T}(G)$. After we order $H_1, \ldots, H_s$ such that $\#H_1 \leq \cdots \leq \#H_s$, the matrix $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ is defined by $m_{ij} = \#(G/H_i)^{H_j}$, where the action is by left multiplication. We also observe that using only the knowledge of $\mathcal{T}(G)$, we cannot check whether one subgroup is conjugate to a normal subgroup of another, or identify the conjugacy class of the normalizer of a subgroup.

In Sect. 4 we introduce the mark homomorphism $\Phi_G : \mathcal{B}(G) \longrightarrow \mathbb{Z}^s$ and spell out a direct method for checking whether a tuple is in the image of Φ_G which avoids the use of possibly hard-to-compute standard congruences. Moreover, using this mark homomorphism, we write down an algorithm to compute a "canonical presentation" of $\mathcal{B}(G)$ of the form $\mathcal{B}(G) \cong \mathbb{Z}[x_1, \ldots, x_{s-1}]/I_G$. The defining ideal I_G is then identified in Sect. 5 as a vanishing ideal of a finite point set, called the set of mark points, and its primary decomposition allows us to write down all prime ideals of $\mathcal{B}(G)$ explicitly. In particular, we conclude that $\mathcal{B}(G)$ is a 1-dimensional, reduced, Cohen–Macaulay ring.

In Sect. 6 we begin a study of the Zariski topology on $\mathrm{Spec}(\mathcal{B}(G))$. The prime ideal graph (first introduced in Nicolson 1978) of $\mathcal{B}(G)$ has the s minimal primes as its vertices and a connecting edge between two vertices if and only if some maximal ideal

contains both minimal primes. Among others, we provide algorithms for computing the prime graph as a labeled graph, and for computing the prime graph of a finite nilpotent group. In particular, we show that the prime graph of a finite nilpotent group is a direct product of complete graphs, and we get an alternative proof of Dress' well-known connectedness theorem (cf. Dress 1969, Prop. 2) in this case.

The next section contains a characterization of the singular locus of $\mathcal{B}(G)$, an algorithm for its computation, as well as an algorithm for identifying the conjugacy class of the p-residual subgroup of a subgroup of G. Finally, in Sect. 8, we compute idempotents and quasi-idempotents of $\mathcal{B}(G)$ and determine the connected components of $\mathrm{Spec}(\mathcal{B}(G))$ effectively. The quasi-idempotents are the least multiples $r_i e_i$ of the idempotents e_i of the ghost ring $\mathbb{Z}^s$ such that $r_i e_i \in \mathrm{Im}(\Phi_G)$. Since we identify $\mathbb{Z}^s$ with the integral closure of $\mathcal{B}(G)$ in its full ring of quotients, they are the generators of the conductor of $\mathcal{B}(G)$. To compute them efficiently, we do not use the usual formula involving commutator subgroups (which cannot be identified from $\mathcal{T}(G)$) but a straightforward diagonalization algorithm for $\mathcal{T}(G)$.

To find the connected components of $\mathrm{Spec}(\mathcal{B}(G))$, we apply the naive breadth-first search algorithm. Then we use the characterization of Dress (1969) which characterizes perfect subgroups of G via the connected components of $\mathrm{Spec}(\mathcal{B}(G))$ to compute the conjugacy classes of perfect subgroups of G. For the last task, namely to compute the primitive idempotents of $\mathcal{B}(G)$, we can use the information about the connected components of $\mathcal{B}(G)$ and represent their characteristic functions as linear combinations of the rows of $\mathcal{T}(G)$. Again this results in an algorithm which relies only on the table of marks, in contrast to an implementation of the usual formula for the primitive idempotents (cf. Yoshida 1983) involving the Möbius function of the subgroup lattice.

All algorithms in this paper were implemented in the computer algebra system ApCoCoA (see The ApCoCoA Team 2013), using the tables of marks coming from GAP (see The GAP Group 2014). Our definitions and notation follow the books by Kreuzer and Robbiano (2000, 2005), unless explicitly noted otherwise. Good proofs of some standard results about Burnside rings that we need can be found in Karpilovsky (1995).

2 Definition and basic properties

In the following we let G be a finite group. A finite set X is called a G-**set** if there is an action of G from the left on X, i.e., a map $\varphi : G \times X \longrightarrow X$ such that $\varphi(gh, a) = \varphi(g, \varphi(h, a))$ and $\varphi(e, a) = a$ for $g, h \in G$, $a \in X$, and the neutral element $e \in G$. As usual, we write $g \cdot a = \varphi(g, a)$. The subsets $G \cdot a$ for $a \in X$ are called the **orbits** of the action. Choosing representatives $a_1, \ldots, a_n$ for the orbits, we have a disjoint union $X = G \cdot a_1 \uplus \cdots \uplus G \cdot a_n$. For every $a \in X$, the subgroup $G_a = \{g \in G \mid g \cdot a = a\}$ is called the **stabilizer** of a. There is a bijection between the set of left cosets G/G_{a_i} and $G \cdot a_i$ defined by $gG_{a_i} \mapsto g \cdot a_i$.

If we choose two representatives $a_i, b_i \in X$ of the same orbit $G \cdot a_i = G \cdot b_i$, the subgroups G_{a_i} and G_{b_i} are conjugate to each other, since $b_i = g \cdot a_i$ implies $G_{b_i} \cdot g \cdot a_i = g \cdot a_i$, and therefore $g^{-1}G_{b_i}g \cdot a_i = a_i$. This shows $g^{-1}G_{b_i}g \subseteq G_{a_i}$, and the reverse containment follows similarly. Thus, for $i \in \{1, \ldots, n\}$, the orbit $G \cdot a_i$

of φ may be identified with the set of all G/H where H ranges over the conjugacy class of the subgroup G_{a_i}.

Definition 2.1 The **Burnside ring** $\mathcal{B}(G)$ of G is defined to be the set of all formal differences of isomorphism classes of finite G-sets. Addition is given by disjoint union of G-sets, and multiplication by their cartesian product equipped with the diagonal action.

For a G-set X, let $[X]$ denote its isomorphism class. Since every finite G-set is the disjoint union of its orbits, the Burnside ring is generated as a $\mathbb{Z}$-module by the isomorphism classes of orbit types. As we saw above, these isomorphism classes can be identified with the sets $[G/H]$ where H ranges over the conjugacy class of a subgroup of G. For G-sets X, Y, the addition in $\mathcal{B}(G)$ is defined by $[X] + [Y] = [X \cup Y]$, and the multiplication is defined by $[X] \cdot [Y] = [X \times Y]$. Our first goal is to make these operations explicit in terms of the G-sets $[G/H]$. The following proposition describes the additive structure of $\mathcal{B}(G)$ explicitly.

Proposition 2.2 (Burnside) *Let G be a finite group.*

(a) *Two G-sets X, Y are isomorphic as G-sets if and only if $[X] = [Y]$ in $\mathcal{B}(G)$.*
(b) *Let $H_1, \ldots, H_s$ be representatives of the conjugacy classes of subgroups of G. Then $\mathcal{B}(G)$ is a free $\mathbb{Z}$-module with basis $\{[G/H_1], \ldots, [G/H_s]\}$.*

Proof See for instance Karpilovsky (1995, Thm. 2.1).   □

To get also the multiplicative structure of $\mathcal{B}(G)$ under control, we introduce the table of marks of a finite group next.

3 The table of marks

Let G be a finite group and X a G-set. For a subgroup H of G, we denote by $X^H = \{x \in X \mid h \cdot x = x \text{ for all } h \in H\}$ the set of fixed points of X under the action of H. The number $\#X^H$ is called the **mark** of H on X. If K is a further subgroup of G, we consider the action of K on G/H defined by $k \cdot gH = kgH$. Then the mark of K on G/H is denoted by

$$\mathrm{m}(H, K) = \#\{gH \in G/H \mid KgH = gH\} = \#(G/H)^K.$$

Notice that $KgH = gH$ is equivalent to $K \subseteq gHg^{-1}$, and also to $g^{-1}Kg \subseteq H$. This shows $\mathrm{m}(H, K) = 0$ if K is not conjugate to a subgroup of H.

Definition 3.1 Let s be the number of conjugacy classes of subgroups of G, and let $H_1, \ldots, H_s$ be subgroups of G which represent these conjugacy classes. We assume that these subgroups are numbered such that $\#H_i \leq \#H_j$ for $i \leq j$. Then the matrix $\mathcal{T}(G) \in \mathrm{Mat}_s(\mathbb{Z})$ whose entry in position (i, j) is $\mathrm{m}(H_i, H_j)$ is called the **table of marks** of G.

Let us collect some easy properties of the table of marks of G.

Remark 3.2 (a) Since $H_1 = \{e\}$ is the trivial subgroup of G, we have $\mathrm{m}(H_1, H_1) = \mathrm{ord}(G)$ and $\mathrm{m}(H_1, H_j) = 0$ for $j > 1$. Thus the first row of $\mathcal{T}(G)$ is always $(\#G, 0, \ldots, 0)$.

(b) Since $H_s = G$, we have $G/H_s = \{e\}$, and therefore $\mathrm{m}(H_s, H_j) = 1$ for $j = 1, \ldots, s$. Thus the last row of $\mathcal{T}(G)$ is always $(1, \ldots, 1)$.

(c) The first column of $\mathcal{T}(G)$ corresponds to the conjugacy class $[\{e\}]$ of the trivial subgroup which fixes all elements of $[G/H_i]$. Hence the first entry of the i-th row if $\mathcal{T}(G)$ is $\#(G/H_i)$.

(d) For $j > i$, the group H_j satisfies $\#H_j \geq \#H_i$. Thus it is not conjugate to a subgroup of H_i and we have $\mathrm{m}(H_i, H_j) = 0$. Hence the matrix $\mathcal{T}(G)$ is a lower triangular matrix.

The table of marks of a finite group can be made unique in the following way.

Remark 3.3 In Definition 3.1, we required that the conjugacy classes of subgroups of G and their representatives $H_1, \ldots, H_s$ are ordered such that $\#H_i \leq \#H_j$ for $i \leq j$. In fact, it would have been sufficient to require that $i \leq j$ if H_i is conjugate to a subgroup of H_j. But in both cases, the table of marks $\mathcal{T}(G)$ is not uniquely determined by G, since there may be several conjugacy classes of subgroups of the same order. To remedy this situation, we fix the following ordering of the rows and columns of $\mathcal{T}(G)$.

(a) First order $H_1, \ldots, H_s$ in increasing cardinality. Equivalently, the first column of $\mathcal{T}(G)$ will be ordered decreasingly.

(b) If $H_i, \ldots, H_j$ have the same number of elements, order them such that the diagonal entries of $\mathcal{T}(G)$ satisfy $m_{ii} \geq \cdots \geq m_{jj}$.

(c) If the rows of $\mathcal{T}(G)$ corresponding to $[G/H_i], \ldots, [G/H_j]$ have the same first element and the same diagonal element, reorder them descendingly with respect to the lexicographic order. Process these blocks from top to bottom in $\mathcal{T}(G)$.

In this way, we arrive at a table which we call the **canonical table of marks** of G. In the following, unless we explicitly mention otherwise, we shall always use the canonical table of marks of a finite group.

The table of marks of a finite group can be calculated using methods of computational group theory (see for instance Holt et al. 2005, Section 10.5 or Pfeiffer 1997). In this paper, it is considered as the *input* for all further algorithms. Whenever necessary, we used the computer algebra system GAP (see The GAP Group 2014) to determine it. Let us consider some examples.

Example 3.4 The table of marks of a finite group G has size 2×2 if and only if $G = C_p$ is a cyclic group of prime order p. In this case, we have $\mathcal{T}(C_p) = \begin{pmatrix} p & 0 \\ 1 & 1 \end{pmatrix}$.

Example 3.5 The table of marks of the symmetric group S_3 is

$$\mathcal{T}(S_3) = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Example 3.6 The table of marks of the cyclic group C_4 of order four is

$$\mathcal{T}(C_4) = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Example 3.7 The table of marks of the Klein four-group V_4 is

$$\mathcal{T}(V_4) = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The following proposition contains a small observation about the distribution of zeros in $\mathcal{T}(G)$.

Proposition 3.8 *Let r be the number of distinct prime factors of $\#G$. Then every column of $\mathcal{T}(G)$, except the first one, contains at least r zeros.*

Proof Let $p_1, \ldots, p_r$ be the distinct primes dividing $\#G$. Let $H_1, \ldots, H_r$ be subgroups of G such that H_i represents the conjugacy class of the p_i-Sylow subgroups of G. For every non-trivial subgroup U of G, there exists a prime p_i dividing $\#U$. By Cauchy's theorem, the subgroup U contains an element of order p_i. Now consider the action of U on the sets G/H_j for $j \neq i$. Since p_i does not divide the order of H_j, the group U is not conjugate to a subgroup of H_j. Hence the action has no fixed points, i.e., we have $\mathrm{m}(H_j, U) = 0$. Together with the zero in the first row of $\mathcal{T}(G)$, this provides the claimed zeros. $\square$

Examples 3.6 and 3.7 show that for a p-group G, the second column of $T(G)$ may contain one or more zeros, so the bound given in the proposition is sharp. We end this section with a collection of further properties of the table of marks of a finite group.

Proposition 3.9 *Let $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ be the table of marks of a finite group G, and let $H_1, \ldots, H_s$ be subgroups of G such that $[G/H_i]$ corresponds to the i-th row of $\mathcal{T}(G)$.*

(a) *For $i = 1, \ldots, s$, we have $m_{ii} = [N_G(H_i) : H_i]$, where $N_G(H_i)$ is the **normalizer** of H_i in G, i.e., the largest subgroup of G containing H_i as a normal subgroup.*

(b) *For $i = 1, \ldots, s$, the subgroup H_i of G is a normal subgroup if and only if $m_{i1} = m_{ii}$. More generally, the number m_{i1}/m_{ii} is the size of the conjugacy class of H_i in G.*

(c) *Let $i \in \{1, \ldots, s\}$. Every entry in the i-th row of $\mathcal{T}(G)$ is a multiple of m_{ii}. More precisely, the entry m_{ij} equals the product of m_{i1} by the number of conjugates of H_i containing H_j.*

(d) *The number of distinct conjugates of H_j contained in H_i is given by $(m_{ij}\, m_{j1})/(m_{jj}\, m_{i1})$.*

Proof Claim (a) follows from Solomon (1967, Lemma 2). The first part of claim (b) follows immediately from (a). The second part of (b) and (c) follow from the facts that $H_j g H_i = g H_i$ is equivalent to $H_j \subseteq g H_i g^{-1}$ and $\#N_G(H_i)$ elements g give the same conjugate $g H_i g^{-1}$. Claim (d) follows from a similar counting argument (see Pfeiffer 1997, Prop. 1.4).   □

Finally, we note that there exist certain types of information which cannot be derived from the table of marks.

Remark 3.10 Let $\mathcal{T}(G)$ be the table of marks of a finite group G.

(a) Using only the knowledge of $\mathcal{T}(G)$, it is not possible to check whether one subgroup of G is conjugate to a normal subgroup of another one (see Huerta-Aparicio et al. 2009, Theorem 4.2.5).

(b) It is also not possible to identify the normalizer of a subgroup (see Huerta-Aparicio et al. 2009, Theorem 4.2.6).

4 Computing a presentation

In the following we let G be a finite group and $\mathcal{T}(G)$ its table of marks. Our goal in this section is to compute a natural presentation of the Burnside ring $\mathcal{B}(G)$. The additive structure of $\mathcal{B}(G)$ is determined by the fact that the sets $[G/H]$, where H ranges over a set of representatives of the conjugacy classes of subgroups of G, form a $\mathbb{Z}$-basis of $\mathcal{B}(G)$. The multiplicative structure is obtained from the following observation.

Proposition 4.1 *Let $H_1, \ldots, H_s$ be subgroups of G which represent the distinct equivalence classes of subgroups of G under conjugation. Then the map*

$$\Phi_G : \mathcal{B}(G) \longrightarrow \prod_{i=1}^{s} \mathbb{Z} \quad \text{given by } [G/U] \mapsto (m(U, H_i))_{i=1,\ldots,s}$$

is an injective ring homomorphism. It is called the **mark homomorphism***.*

For a proof, see for instance Bouc (2000, Sec. 3.2). Since the image of Φ_G is a free $\mathbb{Z}$-submodule of $\mathbb{Z}^s$, we have the following easy Algorithm 1 to check whether or not a given tuple is in $\mathrm{Im}(\Phi_G)$.

Remark 4.2 In the classical theory of Burnside rings, containment in the image of Φ_G is usually checked via the so-called **standard congruences**. Typically, these congruences require the computation of a sum which extends over all cyclic subgroups of a certain group. Although it is possible to determine the cyclic subgroups of a given group from the table of marks, a more efficient version of this method could be based on the computation of the syzygies of the row vectors of $\mathcal{T}(G)$.

On the right-hand side of Φ_G, the $\mathbb{Z}$-module $\mathbb{Z}^s$ is a ring by component-wise multiplication. It is called the **ghost ring** of G. In order to have a more invariant representation of the ghost ring, we can use the set $\mathcal{C}(G)$ of equivalence classes of subgroups of G under conjugation and write the ghost ring as $\mathbb{Z}^{\mathcal{C}(G)}$.

Algorithm 1 (Checking for Mark Homomorphism Images)

Input: The table of marks $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ of a finite group G and a tuple $(a_1, \ldots, a_s) \in \mathbb{Z}^s$

Output: YES or NO depending on whether or not $(a_1, \ldots, a_s)$ is in $\mathrm{Im}(\Phi_G)$

1: If $(a_1, \ldots, a_s) = (0, \ldots, 0)$ then return YES.
2: Let $k = \max\{i \in \{1, \ldots, s\} \mid a_i \neq 0\}$.
3: **for** $i = k$ **to** 1 **do**
4:    If a_i is not an integer multiple of m_{ii} then return NO.
5:    If $a_i = b \cdot m_{ii}$ then subtract $b \cdot (m_{i1}, \ldots, m_{is})$ from $(a_1, \ldots, a_s)$.
6: **end for**
7: Check whether $(a_1, \ldots, a_s) = (0, \ldots, 0)$ and return the corresponding truth value.

The fact that Φ_G is a ring homomorphism implies that we can compute the product of two classes $[G/H_i], [G/H_j] \in \mathcal{B}(G)$ by multiplying the corresponding rows of $\mathcal{T}(G)$ component-wise and representing the result as a $\mathbb{Z}$-linear combination of the rows of $\mathcal{T}(G)$. Thus we get equalities

$$[G/H_i] \cdot [G/H_j] = \sum_{k=1}^{s} c_{ijk}[G/H_k]$$

with $c_{ijk} \in \mathbb{Z}$. In fact, since this product corresponds to the decomposition of the natural action of G on $G/H_i \times G/H_j$, the numbers c_{ijk} will be non-negative. Thus we arrive at the following algorithm for computing a presentation of $\mathcal{B}(G)$ using generators and relations.

Algorithm 2 (Computing a Presentation of the Burnside Ring)

Input: The table of marks $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ of a finite group G.

Output: A list $F = [f_1, \ldots, f_m]$ such that $\mathcal{B}(G) = \mathbb{Z}[x_1, \ldots, x_{s-1}]/\langle f_1, \ldots, f_m \rangle$.

1: Introduce new indeterminates $x_1, \ldots, x_{s-1}$ and an empty list $F = [\,]$.
2: Repeat the following steps for $1 \leq i \leq j \leq s - 1$. Then return the list F and stop.
3: Compute the component-wise product $v_{ij} = (m_{i1}m_{j1}, \ldots, m_{is}m_{js}) \in \mathbb{Z}^s$.
4: Represent v_{ij} as a $\mathbb{Z}$-linear combination $v_{ij} = \sum_{k=1}^{s} c_{ijk} \cdot (m_{k1}, \ldots, m_{ks})$ of the rows of $\mathcal{T}(G)$.
5: Append the polynomial $x_{s-i}x_{s-j} - \sum_{k=1}^{s-1} c_{ijk}x_{s-k} - c_{ijs}$ to the list F.

Corollary 4.3 *Let $\mathcal{T}(G) \in \mathrm{Mat}_s(\mathbb{Z})$ be the table of marks of G. Then Algorithm 2 computes a tuple $F = (f_1, \ldots, f_m)$ of polynomials in $\mathbb{Z}[x_1, \ldots, x_{s-1}]$ such that $\mathcal{B}(G) \cong \mathbb{Z}[x_1, \ldots, x_{s-1}]/\langle f_1, \ldots, f_m \rangle$.*

Under this isomorphism, the set $[G/H_i]$ corresponds to the residue class $\bar{x}_{s-i}$ if H_i is a subgroup of G in the conjugacy class corresponding to the i-th row of $\mathcal{T}(G)$.

Proof Let $H_1, \ldots, H_s$ be subgroups of G such that the conjugacy class $[G/H_i]$ corresponds to the i-th row of $\mathcal{T}(G)$ for $i = 1, \ldots, s$. From the description of Φ_G via component-wise multiplication and the fact that the last row of $\mathcal{T}(G)$ is $(1, \ldots, 1)$ it is clear that $[G/H_s]$ represents the multiplicative identity element of $\mathcal{B}(G)$. Since

the sets $[G/H_i]$ are a $\mathbb{Z}$-basis of $\mathcal{B}(G)$, it follows that $[G/H_1], \ldots, [G/H_{s-1}]$ is a system of $\mathbb{Z}$-algebra generators. We represent $[G/H_i]$ by an indeterminate x_{s-i}. Then the multiplicative structure of $\mathcal{B}(G)$ is determined by the relations resulting from the multiplication table of the algebra generators. These relations are precisely the polynomials computed in Steps 4 and 5 of the algorithm. $\qquad\qquad\square$

Let us compute this algebra structure in some examples.

Example 4.4 Let $G = S_3$ be the symmetric group on three symbols. Using the table of marks given in Example 3.5 and the algorithm, we compute:

$$(6, 0, 0, 0) \cdot (6, 0, 0, 0) = 6 \cdot (6, 0, 0, 0)$$
$$(6, 0, 0, 0) \cdot (3, 1, 0, 0) = 3 \cdot (6, 0, 0, 0)$$
$$(6, 0, 0, 0) \cdot (2, 0, 2, 0) = 2 \cdot (6, 0, 0, 0)$$
$$(3, 1, 0, 0) \cdot (3, 1, 0, 0) = (6, 0, 0, 0) + (3, 1, 0, 0)$$
$$(3, 1, 0, 0) \cdot (2, 0, 2, 0) = (6, 0, 0, 0)$$
$$(2, 0, 2, 0) \cdot (2, 0, 2, 0) = 2 \cdot (2, 0, 2, 0)$$

Therefore the algorithm computes the following presentation:

$$\mathcal{B}(S_3) \cong \mathbb{Z}[x_1, x_2, x_3]/\langle x_3^2 - 6x_3, \ x_3x_2 - 3x_3, \ x_3x_1 - 2x_3,$$
$$x_2^2 - x_3 - x_2, \ x_2x_1 - x_3, \ x_1^2 - 2x_1\rangle$$

Example 4.5 For a prime p, the Burnside ring of the cyclic group C_p with p elements satisfies $\mathcal{B}(C_p) \cong \mathbb{Z}[x]/\langle x^2 - p\,x\rangle$.

Example 4.6 The Burnside ring of the Klein four-group (see Example 3.7) has the presentation

$$\mathcal{B}(V_4) \cong \mathbb{Z}[x_1, x_2, x_3, x_4]/\langle x_4^2 - 4x_4, \ x_4x_3 - 2x_3, \ x_4x_2 - 2x_2, \ x_4x_1 - 2x_1, \ x_3^2 - 2x_3$$
$$x_3x_2 - x_4, \ x_3x_1 - x_4, \ x_2^2 - 2x_2, \ x_2x_1 - x_4, \ x_1^2 - 2x_1\rangle.$$

The presentation of $\mathcal{B}(G)$ computed by Algorithm 2 will be called the **canonical presentation** and the ideal will be called the **(canonical) defining ideal** of $\mathcal{B}(G)$ and denoted by $I(G)$. In the next section, we begin our study of the algebraic structure of $\mathcal{B}(G)$ based on the canoncial presentation. But before, let us end this section with a criterion which tells us when we can simplify the canonical presentation of $\mathcal{B}(G)$ somewhat.

Proposition 4.7 *Let $H_1, \ldots, H_s$ be subgroups of G which represent the distinct conjugacy classes of subgroups of G.*

(a) *For $i, j \in \{1, \ldots, s\}$, let $[G/H_i] \cdot [G/H_j] = \sum_{k=1}^{s} c_{ijk}[G/H_k]$ with $c_{ijk} \in \mathbb{N}$. Then $c_{ijk} \neq 0$ implies that H_k is conjugate to a subgroup of H_i and to a subgroup of H_j.*

(b) *If there exits a prime p dividing* $\mathrm{ord}(G)$ *such that G has a p-**complement**, i.e., a subgroup of complementary order to a p-Sylow subgroup of G, then the first row of $\mathcal{T}(G)$ is the component-wise product of two other rows of $\mathcal{T}(G)$.*

Proof Claim (a) follows from Solomon (1967, Lemma 1). To prove (b), we let H_i be a p-Sylow subgroup of G and H_j a subgroup of complementary order. Since no prime divides both the order of H_i and the order of H_j, no subgroup of G is conjugate to a subgroup of both H_i and H_j. Hence the claim follows from (a) by multiplying the rows corresponding to $[G/H_i]$ and $[G/H_j]$ in $\mathcal{T}(G)$. $\square$

In particular, part (b) of this proposition says that the canonical defining ideal of $\mathcal{B}(G)$ contains a polynomial of the form $x_{s-1} - x_i x_j$ with $i, j \in \{1, \ldots, s-2\}$. Therefore we can substitute x_{s-1} by $x_i x_j$ in $I(G)$ and get a **simplified presentation** $\mathcal{B}(G) \cong \mathbb{Z}[x_1, \ldots, x_{s-2}]/J(G)$ of $\mathcal{B}(G)$. We call $J(G)$ the **simplified defining ideal** of $\mathcal{B}(G)$. The following example shows that not every canonical presentation of a Burnside ring can be simplified.

Example 4.8 Consider the table of marks of the alternating group A_6. The second column corresponds to the conjugacy class of a subgroup generated by a double transposition. It contains zero entries in the following rows:

120	0	6	0	0	0	0	0	0	0	0	0	0	$\cdots$	0
120	0	0	6	0	0	0	0	0	0	0	0	0	$\cdots$	0
72	0	0	0	0	0	0	2	0	0	0	0	0	$\cdots$	0
40	0	4	4	0	0	0	0	0	0	0	4	0	$\cdots$	0

Since A_6 has no subgroups having 40, 72, or 120 elements, and therefore no row in $\mathcal{T}(A_6)$ starts with 9, 5, or 3, the first row of $\mathcal{T}(A_6)$ is not the product of other rows. Thus the canonical presentation of $\mathcal{B}(A_6)$ cannot be simplified.

5 Computing the prime spectrum

From the description of the additive and the multiplicative structure of $\mathcal{B}(G)$ it follows immediately that $\mathcal{B}(G)$ is a 1-dimensional reduced Cohen–Macaulay ring. The next proposition yields more precise information. It provides an important connection between the entries of the table of marks $T(G)$ of G and the defining ideal $I(G)$.

Proposition 5.1 *For $i = 1, \ldots, s$, let $p_i = (m_{s-1\,i}, m_{s-2\,i}, \ldots, m_{1i})$, and let $\mathbb{X}_G = \{p_1, \ldots, p_s\} \subset \mathbb{Z}^{s-1}$. Then the following claims hold.*

(a) *The **vanishing ideal***

$$\mathcal{I}(\mathbb{X}) = \{f \in \mathbb{Z}[x_1, \ldots, x_{s-1}] \mid f(p_1) = \cdots = f(p_s) = 0\}$$

of $\mathbb{X}_G$ equals the ideal $I(G)$.

(b) *The ideal $I(G)$ is reduced. Its minimal primes are the linear ideals $\mathfrak{p}_i = \langle x_1 - m_{s-1\,i}, \ldots, x_{s-1} - m_{1i}\rangle$, where $i \in \{1, \ldots, s\}$.*

Proof To prove (a), we note that $\mathbb{X}_G$ is a set of s points in an $(s-1)$-dimensional affine space. Since the table of marks is lower triangular and invertible, these points are in **linearly general position**, i.e., any subset of t points of $\mathbb{X}_G$ spans a $(t-1)$-dimensional affine space. Consequently, the vanishing ideal of $\mathbb{X}_G$ does not contain a linear polynomial and its leading term ideal with respect to a degree compatible term ordering is $\langle x_i x_j \mid i, j = 1, \ldots, s \rangle$, the monomial ideal generated by all terms of degree two. Now it is clear that the generators of $I(G)$ computed by Algorithm 2 are a Gröbner basis, and thus a system of generators, of $I(G)$ for such a term ordering.

Now we prove (b). The vanishing ideal of a set of points is the intersection of the vanishing ideals of the individual points, in our case of the ideals $\mathfrak{p}_i$. Since these ideals are clearly prime ideals of height $s-1$, it follows that $I(G)$ is reduced and the representation $I(G) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$ is its primary decomposition.   □

Part (a) of this proposition says that the ideal $I(G)$ is an **ideal of points**, i.e., the vanishing ideal of a finite set of points. Let us give this point set a name.

Definition 5.2 Let $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ be the table of marks of a finite group G. For $i = 1, \ldots, s$, let $p_i = (m_{s-1i}, m_{s-2i}, \ldots, m_{1i})$, and let $\mathbb{X}_G = \{p_1, \ldots, p_s\} \subset \mathbb{Z}^{s-1}$. Then $\mathbb{X}_G$ is called the **set of mark points** of G.

One question one may ask is whether and how the geometry of the point set $\mathbb{X}_G$ reflects group-theoretic properties of G. In view of the proposition and the fact that $\mathbb{Z}[x_1, \ldots, x_s]/\mathfrak{p}_i \cong \mathbb{Z}$, it is clear that $\mathcal{B}(G)$ is a one-dimensional reduced ring and that its prime ideals can be described as follows.

Corollary 5.3 *In the setting of the proposition, the spectrum* $\mathrm{Spec}(\mathcal{B}(G))$ *can be described as follows.*

(a) *There are s minimal primes $\bar{\mathfrak{p}}_1, \ldots, \bar{\mathfrak{p}}_s$ which can be computed from the table of marks as the residue class ideals of the ideals $\mathfrak{p}_i = \langle x_1 - m_{s-1i}, \ldots, x_{s-1} - m_{1i} \rangle$.*
(b) *The maximal ideals of $\mathcal{B}(G)$ are of the form $\bar{\mathfrak{m}}_{ip}$ where $\mathfrak{m}_{ip} = \mathfrak{p}_i + \langle p \rangle$ and $p \in \mathbb{N}$ is a prime number.*
(c) *A maximal ideal $\bar{\mathfrak{m}}_{ip}$ contains two minimal primes $\bar{\mathfrak{p}}_i, \bar{\mathfrak{p}}_j$ if and only if the reductions $\bar{p}_i, \bar{p}_j \in \mathbb{F}_p^{s-1}$ of the mark points p_i and p_j agree.*

Proof Claim (a) was already shown in the proposition and (b) follows from (a) and the isomorphism $\mathbb{Z}[x_1, \ldots, x_{s-1}]/\mathfrak{p}_i \cong \mathbb{Z}$ since every maximal ideal contains a minimal prime.

To prove (c), we note that $\mathfrak{p}_j \subset \mathfrak{m}_{ip}$ is equivalent to $x_k - m_{s-kj} \in \mathfrak{m}_{ip}$ for $k = 1, \ldots, s-1$. By reducing modulo p, we see that this is equivalent to

$$x_k - \bar{m}_{s-kj} \in \langle x_1 - \bar{m}_{s-1i}, \ldots, x_{s-1} - \bar{m}_{1i} \rangle$$

in $\mathbb{F}_p[x_1, \ldots, x_{s-1}]$, and therefore to $\bar{m}_{s-kj} = \bar{m}_{s-ki}$ for $k = 1, \ldots, s-1$.   □

Let us determine the prime spectrum of $\mathcal{B}(G)$ in a concrete case.

Example 5.4 Let us consider a group $G = \mathbb{Z}/p^n\mathbb{Z}$ where p is a prime number and $n \geq 1$. The subgroups $H_i = \langle \bar{p}^{n-i} \rangle$ with $i \in \{0, \ldots, n\}$ are normal and represent all

conjucagy classes of subgroups of G. Notice that $\#H_i = p^i$ for $i = 0, \ldots, n$. The table of marks of G is given by

$$
\mathcal{T}(G) = \begin{pmatrix}
p^n & 0 & 0 \cdots 0 & 0 & 0 \\
p^{n-1} & p^{n-1} & 0 \cdots 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
p^2 & p^2 & p^2 \cdots p^2 & 0 & 0 \\
p & p & p \cdots p & p & 0 \\
1 & 1 & 1 \cdots 1 & 1 & 1
\end{pmatrix}
$$

By the corollary, the minimal primes of $\mathcal{B}(G)$ are $\bar{\mathfrak{p}}_1, \ldots, \bar{\mathfrak{p}}_{n+1}$, where $\mathfrak{p}_{n+1-i} = \langle x_1 - p, x_2 - p^2, \ldots, x_i - p^i, x_{i+1}, \ldots, x_n \rangle$ for $i = 0, \ldots, n$. (Here we let $\mathfrak{p}_{n+1} = \langle x_1, \ldots, x_n \rangle$.)

Thus we see that $\mathrm{Spec}(\mathcal{B}(G))$ has $n + 1$ irreducible components, and using the prime p in part (c) of the corollary, it follows that $\mathrm{Spec}(\mathcal{B}(G))$ is connected.

Via Corollary 5.3.c, we can study the connected components of $\mathrm{Spec}(\mathcal{B}(G))$. The first result in this direction was shown by A. Dress who characterized them and showed that $\mathrm{Spec}(\mathcal{B}(G))$ is connected if and only if G is solvable (see Dress 1969). In the following section we want to examine and refine this result.

6 The prime ideal graph

Let G be a finite group, let $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ be its table of marks, let $\mathcal{B}(G)$ be its Burnside ring, let $I(G)$ be its canonical defining ideal, and let $\mathbb{X}_G$ be its set of mark points. Corollary 5.3(c) allows us to interpret the connected components of $\mathrm{Spec}(\mathcal{B}(G))$ as the connected components of the following graph.

Definition 6.1 Let $\mathbb{X}_G = \{p_1, \ldots, p_s\}$ be the set of mark points of a finite group G. Using $\mathbb{X}_G$ as the set of vertices, we define a graph Γ_G by letting (p_i, p_j) be an edge of Γ_G if and only if there exists a prime number p such that we have $\bar{p}_i = \bar{p}_j$ in $\mathbb{F}_p^{s-1}$. The graph Γ_G is called the **prime ideal graph** of G.

Thus two minimal primes $\bar{\mathfrak{p}}_i, \bar{\mathfrak{p}}_j$ of the Burnside ring $\mathcal{B}(G)$ have a common point in $\mathrm{Spec}(\mathcal{B}(G))$ if and only p_i and p_j are joined by an edge in Γ_G. The prime ideal graph of G was first considered in Nicolson (1978). To study it further, we use the following subgroups.

Lemma 6.2 *For every subgroup H of G and every prime number p, there exists a unique smallest normal subgroup $O^p(H)$ such that $H/O^p(H)$ is a p-group. It is called the p-**residual subgroup** of H.*

Proof This follows from the fact that, given two normal p-subgroups K_1, K_2 of H, the subgroup $K_1 \cap K_2$ is also normal in H and we have an injective group homomorphism $H/(K_1 \cap K_2) \hookrightarrow H/K_1 \times H/K_2$ given by $h(K_1 \cap K_2) \mapsto (hK_1, hK_2)$. $\square$

Notice that, in general, the conjugacy class of the p-residual subgroup $O^p(H)$ of H cannot be identified using the defining property directly, since one cannot check whether a subgroup of G is conjugate to a normal subgroup of H based on $\mathcal{T}(G)$ alone (see Remark 3.10.a). However, in Sect. 7 we provide an algorithm for performing this task using the prime ideal graph Γ_G. The following proposition says that Γ_G can be considered as a labelled graph.

Proposition 6.3 *Let $\mathbb{X}_G$ be the set of mark points and Γ_G the prime ideal graph of a finite group G. Let p_i, $p_j \in \mathbb{X}_G$ be two distinct points, and let H_i, H_j be subgroups of G corresponding to the i-th and j-th column of $\mathcal{T}(G)$, respectively.*

(a) *The points p_i, p_j are joined by an edge in Γ_G if and only if we have $O^p(H_i) = O^p(H_j)$.*
(b) *If the two points p_i, p_j are joined by an edge in Γ_G then there exists an integer $\alpha \in \mathbb{Z}$ such that $\#H_i = p^\alpha \cdot \#H_j$.*
(c) *There exists at most one prime number p such that $\mathfrak{m}_{i,p} = \mathfrak{m}_{j,p}$, i.e., there exists at most one edge in Γ_G joining p_i and p_j in Γ_G.*

Proof Claim (a) was shown in Dress (1969, Prop. 1.d). Claim (b) follows immediately from (a), and (c) is a consequence of (b). □

By attaching the corresponding prime p to every edge of Γ_G, we get the **labelled prime ideal graph** of G. Although it is straightforward, let us spell out the algorithm to compute the (labelled) prime ideal graph of G.

**Algorithm 3** (Computing the Labelled Prime Ideal Graph)
Input: The table of marks $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ of a finite group G.
Output: A set of triples (i, j, q) such that the pairs (i, j) are the edges of Γ_G and q is the label of (i, j).

1: Using $\mathrm{ord}(G) = m_{11}$, compute the prime factorization $\mathrm{ord}(g) = q_1^{\alpha_1} \cdots q_r^{\alpha_r}$ with distinct primes q_i.
2: Start with an empty list Γ.
3: **for** $i = 1, \ldots, r$ **do**
4:    Compute the reduced points $\bar{p}_j = (\bar{m}_{s-1\,j}, \ldots, \bar{m}_{1\,j}) \in \mathbb{F}_{q_i}$ for $j = 1, \ldots, s$.
5:    For all $1 \leq j < k \leq s$ such that $\bar{p}_j = \bar{p}_k$, append the triple (j, k, q_i) to Γ.
6: **end for**
7: Return the list Γ.

To get a better feeling for prime ideal graphs, let us have a look at some examples.

Example 6.4 For the cyclic group $C_8 = \mathbb{Z}/8\mathbb{Z}$, there is a unique normal subgroup of each order in $\{1, 2, 4, 8\}$. Hence the diagonal elements in the table of marks are equal to the elements in the first column and henceforth divisible by two, except for m_{44}, of course. Consequently, the mark points of C_8 all reduce to $(0, \ldots, 0)$ modulo two, and Γ_G is the following complete graph on four vertices.
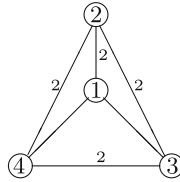
As we shall see later, prime ideal graphs which are complete graphs are typical for
p-groups. Also the following case of a cyclic group is easy to understand.

Example 6.5 Consider the cyclic group $C_{10} \cong C_2 \times C_5$. From Proposition 6.8 below,
we could determine its prime ideal graph as a direct product of graphs. Instead, let us
compute it in a more direct manner. Since C_{10} has unique normal subgroups of orders
1, 2, 5, and 10, its table of marks is given by

$$\mathcal{T}(C_{10}) = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence the prime ideal graph of C_{10} is the following labelled square.
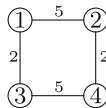
For non-commutative groups, the prime ideal graph has a more intricate structure.
Let us look at the simplest case first.

Example 6.6 The symmetric group S_3 has four conjugacy classes of subgroups: $\{e\}$,
C_2, A_3, and S_3. It table of marks was computed in Example 3.5. If we reduce it modulo
two and modulo three, we see that Γ_{S_3} is the following labelled graph.
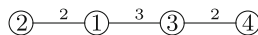

Our last introductory example has already some features of more general cases:
complete subgraphs representing Sylow subgroups which are attached to the vertex
"1" corresponding to the trivial subgroup, and "tentacles" attached to these subgraphs.

Example 6.7 The table of marks of the alternating group A_4 is given by

$$\mathcal{T}(A_4) = \begin{pmatrix} 12 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \\ 3 & 3 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Thus its prime ideal graph has the following shape.
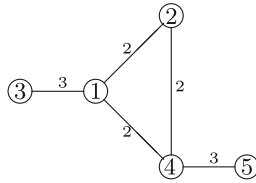
Notice that there are two complete subgraphs attached to node "1": one of them, labelled by "3", is a complete graph on two vertices, and the other one, labelled by "2", is the complete graph on three vertices. As we shall see later, they correspond to a 3-Sylow subgroup generated by a 3-cycle and the 2-Sylow subgroup V_4 generated by the double transpositions. It is interesting to observe that the three 2-element subgroups in V_2 are conjugate to each other in A_4 and thus contribute only one vertex to the complete subgraph. When we connect $\mathcal{B}(A_4)$ and $\mathcal{B}(V_4)$ using the restriction and induction maps, this will come to create tricky intricacies.

The next proposition explains the behaviour of $\mathcal{B}(G)$ and Γ_G under direct products of groups.

Proposition 6.8 *Let G_1, G_2 be two finite groups such that ord(G_1) and ord(G_2) are relatively prime.*

(a) *There is a canonical isomorphism of rings $\mathcal{B}(G_1) \otimes \mathcal{B}(G_2) \longrightarrow \mathcal{B}(G_1 \times G_2)$ given by $[G_1/H_1] \otimes [G_2/H_2] \mapsto [G_1 \times G_2/H_1 \times H_2]$.*
(b) *The table of marks of $G_1 \times G_2$ can be obtained by taking the Kronecker product $\mathcal{T}(G_1) \otimes \mathcal{T}(G_2)$ and reordering the rows and columns such that the subgroups representing the conjugacy classes of subgroups of $G_1 \times G_2$ are listed in increasing order.*
(c) *The graph $\Gamma_{G_1 \times G_2}$ is the **Cartesian product** of the graphs Γ_{G_1} and Γ_{G_2}, i.e., its set of vertices is the Cartesian product of the set of vertices $\{v_1, \ldots, v_s\}$ of Γ_{G_1} and the set of vertices $\{w_1, \ldots, w_t\}$ of Γ_{G_2}, and there is an edge from (v_i, w_j) to (v_k, w_l) if and only if $v_i = v_k$ and there is an edge in Γ_{G_2} from w_j to w_l or if $w_j = w_l$ and there is an edge in Γ_{G_1} from v_1 to v_k.*

Proof Part (a) is contained in Karpilovsky (1995, Thm. 2.9). It follows from the fact that every subgroup of $G_1 \times G_2$ is of the form $H_1 \times H_2$ with subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ (see Karpilovsky 1995, Lemma 2.8).

To prove part (b), we first note that subgroups $H_1 \times H_2$ and $K_1 \times K_2$ are conjugate in $G_1 \times G_2$ if and only if H_1, K_1 are conjugate in G_1 and H_2, K_2 are conjugate in G_2. Thus the size of $\mathcal{T}(G_1 \times G_2)$ is the product of the sizes of $\mathcal{T}(G_1)$ and $\mathcal{T}(G_2)$. Furthermore, we have m$(H_1 \times H_2, K_1 \times K_2) = $ m$(H_1, K_1) \cdot $ m(H_2, K_2). This implies the claim.

Finally we show (c). Let p be a prime dividing ord(G_2). Since it does not affect the prime ideal graph, we use the reordered form $\mathcal{T}(G_1) \otimes \mathcal{T}(G_2)$ of the table of marks of $G_1 \times G_2$. If we look at the last row of blocks of this matrix, we see that two columns can be equal modulo p only if the two corresponding columns of the last block are equal modulo p. Hence, if there is an edge from (v_i, w_j) to (v_k, w_l), then $\bar{w}_j = \bar{w}_l$.

Now there are two cases. If $i = k$, i.e., if we choose the same block column both times, the entries of $\bar{w}_j = \bar{w}_l$ are multiplied by the same numbers in each row. Hence the two columns in $\mathcal{T}(G_1) \otimes \mathcal{T}(G_2)$ are equal. But if we have $i \neq j$, i.e., if we choose columns from two distinct block columns, we can use the fact that each row of blocks is obtained by multiplying $\mathcal{T}(G_2)$ by a divisor of $\mathrm{ord}(G_1)$. Since p does not divide $\mathrm{ord}(G_1)$, the mod p reductions of the two columns of $\mathcal{T}(G_1) \otimes \mathcal{T}(G_2)$ cannot be equal.

For a prime p dividing $\mathrm{ord}(G_1)$, we argue in the same way, but using the matrix $\mathcal{T}(G_2) \otimes \mathcal{T}(G_1)$. Altogether, the claim follows. $\qquad\square$

The following example is an application of this proposition.

Example 6.9 Given the group $C_{30} \cong C_2 \times C_3 \times C_5$, we can apply the preceding proposition two times and get the following prime ideal graph.
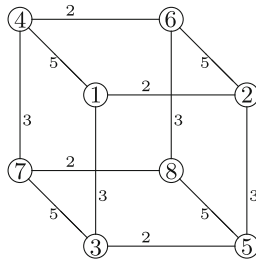
For a finite abelian group, the preceding proposition allows us to reduce the description of the prime ideal graph to the case of a finite abelian p-group for some prime p. More generally, we can characterize arbitrary finite p-groups using their labelled prime ideal graph. The next lemma is a crucial ingredient for this characterization.

Lemma 6.10 *Let p be a prime, let G be a finite p-group, and let $H, K \subset G$ be two non-trivial, proper subgroups of G. Then the prime p divides $m(H, K) = \#(G/H)^K$.*

Proof Let us write down the class equation for the operation of K on G/H:

$$\#(G/H) = \#(G/H)^K + \sum_{\{x \in G/H \mid K \neq K_x\}} [K : K_x]$$

Since H is a proper subgroup of G, the prime p divides $\#(G/H)$. Since K is a nontrivial p-group and $K_x \neq K$, the prime p divides all summands $[K : K_x]$. Hence p divides $\#(G/H)^K$. $\qquad\square$

Now we are ready to characterize p-groups as follows.

Proposition 6.11 *Let G be a finite group and p a prime number. Then the following conditions are equivalent.*

(a) *The group G is a p-group.*
(b) *The prime ideal graph is a complete graph and all of its edges have the label p.*

Proof Given a finite p-group G, the lemma shows that p divides all elements of $\mathcal{T}(G)$ except the ones in the last row. Hence the reductions mod p of all mark points are equal to $(0, \ldots, 0)$ and the prime ideal graph is a complete graph, all of whose edges are labelled by p.

Conversely, if all mod p reductions of the mark points are equal, they have to be equal to the last mark point, i.e., to zero. Thus all entries of the first $s - 1$ rows of $\mathcal{T}(G)$ are powers of p. In particular, the entry $m_{11} = \mathrm{ord}(G)$ is a power of p. □

Using the last two propositions, we can describe the prime ideal graph of finite abelian and, more generally, of finite nilpotent groups.

Remark 6.12 Let G be a finite nilpotent group.

(a) It is well-known that G is the direct product of its p-Sylow subgroups. Using Proposition 6.8, it follows that the prime ideal graph of G is the direct product of the prime ideal graphs of these p-Sylow subgroups. Every p-Sylow subgroup S is a p-group. Hence its prime ideal graph is the complete graph on s nodes, where s is the number of conjugacy classes of subgroups of S.
(b) For an abelian p-group G, the number s equals the number $N(G)$ of its subgroups. This number has been studied classically, although it is difficult to give a closed formula for $N(G)$. For an attempt in this direction, see Călugăreanu (2004). In simple cases it is, however, possible to evaluate $N(G)$ in closed form. For instance, we have $N(C_p \times C_p) = p + 3$ and $N(C_p \times C_p \times C_p) = 2p^2 + 2p + 4$.

One immediate consequence of this remark is that the spectrum of the Burnside ring of a finite nilpotent group is connected. To continue with a more refined study of the prime ideal graph and the connected components of the spectrum of $\mathcal{B}(G)$, we need to introduce additional tools first.

7 The singular locus

In this section we determine the regular and the singular points in the spectrum of $\mathcal{B}(G)$. Recall that the Burnside ring is 1-dimensional and reduced. Hence its localizations at the minimal primes $\bar{\mathfrak{p}}_i$ are fields and thus regular. The following proposition classifies the maximal ideals which correspond to regular points of $\mathrm{Spec}(\mathcal{B}(G))$. We use the notation introduced in the previous sections.

Proposition 7.1 *Let $i \in \{1, \ldots, s\}$, and let $p \in \mathbb{N}$ be a prime number. For the maximal ideal $\bar{\mathfrak{m}}_{ip}$ of $\mathcal{B}(G)$, the following conditions are equivalent.*

(a) *The localization $\mathcal{B}(G)_{\bar{\mathfrak{m}}_{ip}}$ is a regular local ring.*
(b) *The minimal prime $\bar{\mathfrak{p}}_i$ is the only minimal prime of $\mathcal{B}(G)$ contained in $\bar{\mathfrak{m}}_{ip}$.*

Proof If $\mathcal{B}(G)_{\bar{\mathfrak{m}}_{ip}}$ is a regular local ring, it is an integral domain, and therefore the zero ideal is its only minimal prime. Conversely, if the localization $\mathcal{B}(G)_{\bar{\mathfrak{m}}_{ip}}$ contains only one minimal prime, the fact that it is a reduced ring implies that it is an integral domain and that $\bar{\mathfrak{p}}_i \, \mathcal{B}(G)_{\bar{\mathfrak{m}}_{ip}}$ is the zero ideal. From the fact that $\bar{\mathfrak{m}}_{ip} = \bar{\mathfrak{p}}_i + \langle p \rangle$ is follows that the maximal ideal of $\mathcal{B}(G)_{\bar{\mathfrak{m}}_{ip}}$ is generated by one element, namely by $\bar{p}$, and hence that $\mathcal{B}(G)_{\bar{\mathfrak{m}}_{ip}}$ is a regular local ring. □

As a consequence, we can characterize the singular locus $\mathrm{Sing}(\mathcal{B}(G))$ of the spectrum of $\mathcal{B}(G)$ as follows.

Proposition 7.2 *Let $i \in \{1, \ldots, s\}$, and let $p \in \mathbb{N}$ be a prime number. For the maximal ideal $\bar{\mathfrak{m}}_{ip}$ of $\mathcal{B}(G)$, the following conditions are equivalent.*

(a) *We have $\bar{\mathfrak{m}}_{ip} \in \mathrm{Sing}(\mathcal{B}(G))$.*
(b) *The ring $\mathcal{B}(G)_{\bar{\mathfrak{m}}_{ip}}$ has a non-trivial zerodivisor.*
(c) *There is an edge in the prime ideal graph Γ_G starting at the mark point p_i which is labeled by the prime p.*
(d) *The prime p divides $[N_G(O^p(H_i)) : O^p(H_i)]$.*

Proof The equivalence of (a) and (b) was shown in the proof of the preceding proposition. The equivalence of (a) and (c) follows from Corollary 5.3.c and the preceding proposition.

Next we show that (c) implies (d). Let $j \in \{1, \ldots, s\}$ be such that $O^p(H_i)$ is conjugate to H_j. By Dress (1969, Prop. 1.d), there exists an edge (p_i, p_j) in Γ_G which is labeled by p. In particular, the reductions $\bar{p}_i, \bar{p}_j \in \mathbb{F}_p^{s-1}$ are the same. Notice that, by (c), we may assume that $i \neq j$. Hence the first non-zero entry $m_{jj} = [N_G(O^p(H_i)) : O^p(H_i)]$ in the j-th column of $\mathcal{T}(G)$ has to be congruent to zero modulo p.

Finally, we prove that (d) implies (c). Let $j \in \{1, \ldots, s\}$ be such that $O^p(H_i)$ is conjugate to H_j. Since p divides $m_{jj} = [N_G(H_j) : H_j]$, the p-Sylow subgroup of $N_G(H_j)/H_j$ is non-trivial. Hence H_j is a normal subgroup of a subgroup U of G such that U/H_j is a non-trivial p-group, and therefore we have $H_j = O^p(U)$. Letting $k \in \{1, \ldots, s\}$ be the index such that U is conjugate to H_k, we see that (p_j, p_k) is an edge in Γ_p which is labeled by p, and thus also (p_i, p_k) is such an edge.　□

From this description of the set of edges of Γ_G labeled by a fixed prime p we obtain that the corresponding graph is a disjoint union of complete graphs. Each of these complete graphs corresponds to exactly one singular maximal ideal of $B(G)$. This yields Algorithm 4 for computing $\mathrm{Sing}(\mathcal{B}(G))$.

Given one of the complete subgraphs of Γ_G labeled p, the vertex with the lowest index j, i.e. the conjugacy class $[H_j]$ corresponding to the smallest group H_j, is the one such that $O^p(H_i)$ is conjugate to H_j for all p_i in this subgraph. Hence we get Algorithm 5 for identifying the conjugacy class of $O^p(H_i)$.

8 Idempotents and quasi-idempotents

As in the previous sections, we let G be a finite group and we assume that we are given its table of marks $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$.

Definition 8.1 The $\mathbb{Q}$-algebra $\mathcal{B}_{\mathbb{Q}}(G) = \mathbb{Q} \otimes \mathcal{B}(G)$ is called the **Burnside algebra** of G.

By Proposition 5.1, the Burnside algebra of G is given by

$$\mathcal{B}_{\mathbb{Q}}(G) = \mathbb{Q}[x_1, \ldots, x_{s-1}]/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s)$$

Algorithm 4 (Computing Sing($\mathcal{B}(G)$))
Input: The table of marks $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ of a finite group G
Output: A set S of pairs (i, p) such that the corresponding maximal ideals $\bar{\mathfrak{m}}_{ip}$ are precisely the singular prime ideals of $\mathcal{B}(G)$).

1: Let $S = \emptyset$ and let D be the set of prime numbers dividing the order of G.
2: **for** $p \in D$ **do**
3:    Using Algorithm 3, compute the labeled prime ideal graph of G.
4:    Let $V = \{1, 2, \ldots, s\}$.
5:    **while** $V \neq \emptyset$ **do**
6:       Choose $i \in V$, remove it from V, and let $L = \{p_i\}$.
7:       **repeat**
8:          Compute the set M of all $p_j \notin L$ such that for some $p_k \in L$ there is an edge (p_j, p_k) of Γ_G labeled p.
9:          Append M to L.
10:      **until** $M = \emptyset$
11:      Append all pairs (j, p) such that $p_j \in L$ to S and remove those j from V.
12:   **end while**
13: **end for**
14: **return** S

Algorithm 5 (Identifying $O^p(H_i)$)
Input: The table of marks $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ of a finite group G, a prime p, and $i \in \{1, \ldots, s\}$
Output: The index $\ell \in \{1, \ldots, s\}$ such that $[O^p(H_i)] = [H_\ell]$

1: Using Algorithm 3, compute the labeled prime ideal graph of G.
2: Let $L = \{p_i\}$.
3: **repeat**
4:    Compute the set M of all $p_j \notin L$ such that for some $p_k \in L$ there is an edge (p_j, p_k) of Γ_G labeled p.
5:    Append M to L.
6: **until** $M = \emptyset$
7: **return** $\ell = \min\{j \mid p_j \in L\}$

---

where $\mathfrak{p}_i = \langle x_1 - m_{s-1\,i}, \ldots, x_{s-1} - m_{1i}\rangle$, for $i \in \{1, \ldots, s\}$. Thus the Chinese Remainder Theorem implies that $\mathcal{B}_{\mathbb{Q}}(G)$ is isomorphic to $\mathbb{Q}^s$. Since this ring consists only of zero-divisors and units, it is the full ring of quotients of $\mathcal{B}(G)$.

Proposition 8.2 *In the above setting, let* $\gamma_G : \mathcal{B}_{\mathbb{Q}}(G) \longrightarrow \mathbb{Q}^s$ *be the composition*

$$\mathcal{B}_{\mathbb{Q}}(G) \cong \mathbb{Q}[x_1, \ldots, x_{s-1}/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s) \cong \prod_{j=1}^{s} \mathbb{Q}[x_1, \ldots, x_{s-1}]/\mathfrak{p}_j \cong \mathbb{Q}^s$$

(a) *The map γ_G is an isomorphism of $\mathbb{Q}$-algebras and the canonical diagram*

$$
\begin{array}{ccc}
\mathcal{B}(G) & \longhookrightarrow & \mathcal{B}_{\mathbb{Q}}(G) \\
\Phi_G \downarrow & & \downarrow \gamma_G \\
\mathbb{Z}^s & \longhookrightarrow & \mathbb{Q}^s
\end{array}
$$

is commutative.

(b) *The ghost ring $\mathbb{Z}^{\mathcal{C}(G)}$ is isomorphic to the integral closure of the Burnside ring $\mathcal{B}(G)$ in its full ring of quotients.*

Proof First we show (a). The map Φ_G is given by $\Phi_G([G/H_i]) = (m_{i1}, \ldots, m_{is})$ for $i = 1, \ldots, s$. For $i = s$, i.e., for $[G/H_s] = [G/G]$, the diagram commutes since this is the unit element in $\mathcal{B}(G)$ and all maps are algebra homomorphisms. Now let $i \in \{1, \ldots, s-1\}$ and $j = s - i$. Then $[G/H_i]$ is the residue class of $x_{s-i} = x_j$ under the canonical presentation. The Chinese Remainder Theorem maps $\bar{x}_j$ to $(\bar{x}_j, \ldots, \bar{x}_j)$ where the residue classes have to be taken appropriately. The isomorphism $\mathbb{Q}[x_1, \ldots, x_{s-1}]/\mathfrak{p}_k \to \mathbb{Q}$ maps $\bar{x}_j$ to $\bar{m}_{s-jk} = \bar{m}_{ik}$. Therefore the image of $[G/H_i]$ under γ_G is $(m_{i1}, \ldots, m_{is})$, and the diagram commutes.

Claim (b) follows from (a) since $\mathbb{Z}^s$ is integrally closed im $\mathbb{Q}^s$. □

After identifying the mark homomorphism $\Phi_G : \mathcal{B}(G) \longrightarrow \mathbb{Z}^s$ as the embedding of the Burnside ring into its integral closure, we recall the notion of the **conductor** of $\mathcal{B}(G)$. This is defined as the ideal

$$\mathfrak{f}(\mathcal{B}(G)) = \{a \in \mathcal{B}(G) \mid a \cdot \mathbb{Z}^s \subset \mathrm{Im}(\Phi_G)\}$$

Equivalently, $\mathfrak{f}(\mathcal{B}(G))$ can be described as the annihilator of $\mathbb{Z}^s/\mathrm{Im}(\Phi_G)$ or as the largest ideal of $\mathcal{B}(G)$ which is also an ideal of $\mathbb{Z}^s$.

Clearly, an element $a \in \mathcal{B}(G)$ is contained in the conductor if and only if $a\, e_i \in \mathrm{Im}(\Phi_G)$ for $i = 1, \ldots, s$. If we write $\Phi_G(a) = (a_1, \ldots, a_s)$, then this is equivalent to $a_i\, e_i \in \mathrm{Im}(\Phi_G)$ for $i = 1, \ldots, s$. Keeping in mind that $\mathrm{Im}(\Phi_G)$ is a free $\mathbb{Z}$-module of rank s and that, consequently, $\mathbb{Z}^s/\mathrm{Im}(\Phi_G)$ is a torsion group, we are led to the following definition.

Definition 8.3 For $i = 1, \ldots, s$, let r_i be the smallest positive integer such that $r_i e_i \in \mathrm{Im}(\Phi)$. Then the elements $\Phi_G^{-1}(r_1 e_1), \ldots, \Phi_G^{-1}(r_s e_s)$ are called the **quasi-idempotents** of $\mathcal{B}(G)$ and the numbers r_i are called its **quasi-idempotent indices**.

By what we remarked above, the quasi-idempotents of $\mathcal{B}(G)$ generate the conductor $\mathfrak{f}(\mathcal{B}(G))$. Algorithm 6 computes the quasi-idempotent indices of $\mathcal{B}(G)$.

Algorithm 6 (Computing the Quasi-Idempotent Indices)

Input: The table of marks $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ of a finite group G.

Output: A tuple $(r_1, \ldots, r_s) \in \mathbb{N}^s$ containing the quasi-idempotent indices of $\mathcal{B}(G)$.

1: **for** $i = 1$ **to** s **do**
2:     Let $(v_1, \ldots, v_s) = (m_{i1}, \ldots, m_{is})$ and $r_i = m_{ii}$.
3:     **repeat**
4:         Let $j = \max\{k \in \{1, \ldots, i-1\} \mid v_k \neq 0\}$.
5:         Multiply $(v_1, \ldots, v_s)$ by $\mathrm{lcm}(|v_j|, m_{jj})/|v_j|$ and let $r_i = v_i$.
6:         Add or subtract a suitable multiple of $(m_{j1}, \ldots, m_{js})$ to $(v_1, \ldots, v_s)$ to get $v_j = 0$.
7:     **until** $(v_1, \ldots, v_s) = (0, \ldots, 0)$
8: **end for**
9: **return** the tuple $(r_1, \ldots, r_s)$.

It is easy to see that this algorithm is correct and performs nothing but a step-wise diagonalization of $\mathcal{T}(G)$. To understand its performance better, we apply it to a concrete example.

Example 8.4 The table of marks of the alternating group A_5 is given by

$$
\mathcal{T}(A_5) = \begin{pmatrix}
60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
10 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
6 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
5 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

Now let us use Algorithm 6 to compute the quasi-idempotents of $\mathcal{B}(A_5)$.

(1) For $i = 1$, we have $(v_1, \ldots, v_s) = 60e_1$, an therefore $r_1 = 60$.
(2) For $i = 2$, we have to multiply $(30, 2, 0, \ldots, 0)$ by 2 and subtract the first row of $\mathcal{T}(A_5)$. Hence we get $r_2 = 4$.
(3) For $i = 3$, we have to multiply $(20, 0, 2, 0, \ldots, 0)$ by 3 and subtract the first row of $\mathcal{T}(A_5)$. We obtain $r_3 = 6$.
(4) For $i = 4$, we first have to multiply $(15, 3, 0, 3, 0, \ldots, 0)$ by two and subtract three times the second row of $\mathcal{T}(A_5)$. The intermediate result is $(-60, 0, 0, 6, 0, \ldots, 0)$. Now we add the first row and conclude that $r_4 = 6$.
(5) For $i = 5$, we multiply $(12, 0, 0, 0, 2, 0, \ldots, 0)$ by 5, subtract the first row of $\mathcal{T}(A_5)$ and get $r_5 = 10$.
(6) Continuing in this way, we also find $r_6 = r_7 = 2$, $r_8 = 3$, and $r_9 = 1$.

In particular, the result $r_9 = 1$ says that $\mathcal{B}(A_5)$ contains the non-trivial idempotent element $\Phi_G^{-1}(e_9)$, and hence $\mathrm{Spec}(\mathcal{B}(A_5))$ is not connected.

The quasi-idempotent indices could also have been computed using the following formula.

Remark 8.5 For a positive integer n, let $\mathrm{sqfr}(n)$ be the **squarefree part** of n, i.e., the product of the distinct prime numbers dividing n. Nicolson (1978) proved the formula $r_i = \mathrm{sqfr}([H_i : H_i']) \cdot m_{ii}$, where H_i' is the commutator subgroup of H_i. Since it is possible (although a bit tedious) to identify the row of $\mathcal{T}(G)$ corresponding to H_i', we could have computed r_i directly using this formula. However, we think that Algorithm 6 is much more efficient.

From the quasi-idempotent indices, we can compute the actual quasi-idempotents of $\mathcal{B}(G)$ as follows.

Remark 8.6 To compute the quasi-idempotents of $\mathcal{B}(G)$ from the table of marks, we can, for instance, use either of the following two ways.

(a) In Algorithm 6, start with the tuple e_i (representing the fact that we initialize v to contain the i-th row of $\mathcal{T}(G)$) and keep track of the operations performed by Steps 5 and 6 of this algorithm, so that the final tuple $(c_{i1}, \ldots, c_{is})$ contains the coefficients of the representation of $r_i e_i$ in terms of the rows of $\mathcal{T}(G)$.
(b) After running Algorithm 6, write $r_i e_i$ as a linear combination of the rows of $\mathcal{T}(G)$, i.e., write $r_i e_i = \sum_{j=1}^{s} c_{ij}(m_{j1}, \ldots, m_{js})$ with $c_{ij} \in \mathbb{N}$.

In both cases, the elements $q_i = \sum_{j=1}^{s} c_{ij}[G/H_j]$ with $1 \leq i \leq s$ are the quasi-idempotents of $\mathcal{B}(G)$.

Now let us turn to the actual idempotents of $\mathcal{B}(G)$. Recall that an idempotent e of $\mathcal{B}(G)$ is called a **primitive idempotent** if it is not of the form $e = e' + e''$ with idempotents e', e'' such that $e'e'' = 0$. The following proposition was shown in Dress (1969, Prop. 2). It characterizes the primitive idempotents of $\mathcal{B}(G)$.

Proposition 8.7 *The primitive idempotents of $\mathcal{B}(G)$ are in 1–1 correspondence with the connected components of $\mathrm{Spec}(\mathcal{B}(G))$. These, in turn, are in 1–1 correspondence with the indices $i \in \{1, \ldots, s\}$ such that H_i is a perfect subgroup of G.*

This proposition suggests that we should first look for a way to compute the connected components of $\mathrm{Spec}(\mathcal{B}(G))$. Based on the algorithm for computing the prime ideal graph (see Algorithm 3), this can be achieved by a simple breadth-first search. Of course, for large graphs, many more efficient algorithms are well-known.

Algorithm 7 (Computing the Connected Components of $\mathrm{Spec}(\mathcal{B}(G))$)
Input: The table of marks $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ of a finite group G.
Output: A list of lists $[C_1, \ldots, C_t]$ such that each C_i contains the indices j of the minimal primes $\bar{\mathfrak{p}}_j$ in one connected component of $\mathrm{Spec}(\mathcal{B}(G))$.

1: Using Algorithm 3, compute the prime ideal graph of G.
2: Let $L = [1, 2, \ldots, s]$ and $M = [\,]$.
3: **repeat**
4:     Choose i in L and remove it from L.
5:     Let $C = [i]$.
6:     **repeat**
7:         Find all j in L such that (i, j) is an edge of the prime ideal graph.
8:         Remove these elements j from L and append them to C.
9:     **until** no new element j is appended to C.
10:     Put C into the list M.
11: **until** $L = [\,]$
12: **return** the list M.

Let us apply this algorithm in the case of the group A_5 discussed above.

Example 8.8 For the group A_5, the table of marks given in Example 8.4 and Algorithm 3 indicate that there are two connected components of $\mathrm{Spec}(\mathcal{B}(G))$. To check this, we follow the steps of Algorithm 7.

(1) First compute the prime ideal graph and get $\Gamma_G = \{(1, 2), (1, 3), (1, 5), (2, 4), (3, 6), (4, 8), (5, 7)\}$.

(2) Let $L = [1, 2, \ldots, 9]$ and $M = [\ ]$.

(4), (5) Choose $i = 1$, let $L = [2, \ldots, 9]$, and let $C = [1]$.

(6)–(9) Then iteration stops when $C = [1, 2, \ldots, 8]$ and $L = [9]$.

(10) Let $M = [C]$.

(4), (5) Choose $i = 9$, let $L = [\ ]$, and let $C' = [9]$.

(10) Let $M = [C, C']$.

The algorithms returns the pair $M = [[1, \ldots, 8], [9]]$. Hence $\operatorname{Spec}(\mathcal{B}(G))$ has two connected components, namely the set of prime ideals containing one of $\bar{\mathfrak{p}}_1, \ldots, \bar{\mathfrak{p}}_8$, and the set of prime ideals containing $\bar{\mathfrak{p}}_9$.

This example and the following one illustrate the observation in Dress (1969, p. 216), that $\operatorname{Spec}(\mathcal{B}(G))$ has two connected components, one of which is a single point, if and only if the group G is minimally simple.

Example 8.9 For the group A_6, Algorithm 7 finds four connected components of $\operatorname{Spec}(\mathcal{B}(G))$, namely $C = \{\bar{\mathfrak{p}}_1, \ldots, \bar{\mathfrak{p}}_{19}\}$ and three isolated points. So, although the group A_6 is simple, it is not minimally simple.

Notice that, by the proposition above, every connected component of $\operatorname{Spec}(\mathcal{B}(G))$ contains exactly one minimal prime $\bar{\mathfrak{p}}_i$ such that H_i is a perfect group. Moreover, this group is the smallest subgroup H_i for which $\bar{\mathfrak{p}}_i$ is in that connected component. Hence we can find the conjugacy classes of perfect subgroups of G using the following Algorithm 8.

Algorithm 8 (Computing the Perfect Subgroups of G)

Input: The table of marks $\mathcal{T}(G) = (m_{ij}) \in \operatorname{Mat}_s(\mathbb{Z})$ of a finite group G.

Output: A tuple $(j_1, \ldots, j_t)$ such that the subgroups H_{j_i} are precisely the perfect subgroups among $H_1, \ldots, H_s$.

1: Let P be the empty tuple.

2: Using Algorithm 7, compute the lists $C_1, \ldots, C_t$ such that C_i contains the indices j of all minimal primes $\bar{\mathfrak{p}}_j$ in the i-th connected component of $\operatorname{Spec}(\mathcal{B}(G))$.

3: **for** $i = 1$ **to** t **do**

4:     Append the minimal number j_i in C_i to P.

5: **end for**

6: **return** the tuple $P = (j_1, \ldots, j_t)$.

The correctness of this algorithm follows from Dress (1969, Prop. 2), since a subgroup U of G which has no non-trivial solvable quotients is necessarily perfect, so that the perfect subgroups are the smallest elements of the equivalence classes corresponding to the connected components of $\operatorname{Spec}(\mathcal{B}(G))$.

Corollary 8.10 *Using Algorithm 8, we can check effectively whether the group G is solvable: it suffices to check whether H_1, the trivial subgroup, is the only perfect subgroup of G.*

One last job remains to be done: we would like to have an algorithm for computing the primitive idempotents in terms of the basis of $\mathcal{B}(G)$. Knowing the connected

components of $\mathrm{Spec}(\mathcal{B}(G))$, they are easy to compute because their image in the ghost ring is the sum of the corresponding primitive idempotents e_i of $\mathbb{Z}^s$. Thus we get Algorithm 9.

Algorithm 9 (Computing Primitive Idempotents of $\mathcal{B}(G)$)

Input: The table of marks $\mathcal{T}(G) = (m_{ij}) \in \mathrm{Mat}_s(\mathbb{Z})$ of a finite group G and a set $C \subseteq \{1, \ldots, s\}$ corresponding to a connected component of $\mathrm{Spec}(\mathcal{B}(G))$.

Output: A tuple $(c_1, \ldots, c_s)$ such that the primitive idempotent corresponding to C is $c_1[G/H_1] + \cdots + c_s[G/H_s]$.

1: Let $(\varepsilon_1, \ldots, \varepsilon_s)$ be the **characteristic function** of C, i.e., let $\varepsilon_i = 1$ if $i \in C$ and $\varepsilon_i = 0$ if $i \notin C$.
2: Write the tuple $(\varepsilon_1, \ldots, \varepsilon_s)$ as a $\mathbb{Z}$-linear combination of the rows of $\mathcal{T}(G)$. Let $(c_1, \ldots, c_s)$ be the coefficient vector.
3: **return** the tuple $(c_1, \ldots, c_s)$.

Another way to describe the primitive idempotents of $\mathcal{B}(G)$ is the formula

$$e_{G,H_i}^{\pi} = (\#N_G(H_i))^{-1} \cdot \sum_{D \leq N_G(H_i)} \#D \cdot \chi(D, H_i) \cdot [G/D]$$

proved in Yoshida (1983). However, for effectively evaluating this formula, in addition to identifying the perfect subgroups H_i of G, we need to determine $N_G(H_i)$, and this is in general not possible using only the knowledge of $\mathcal{T}(G)$ (see Remark 3.10). Moreover, the formula uses the Möbius function of the subgroup lattice of G which is not easy to compute. Thus we believe that Algorithm 8 is much more efficient.

For an example, let us return to the alternating group A_5 once again.

Example 8.11 For the group A_5 discussed above, the connected components of $\mathrm{Spec}(\mathcal{B}(G))$ were shown to be indexed by $C_1 = \{1, 2, \ldots, 8\}$ and $C_2 = \{9\}$. Let us compute the corresponding primitive idempotents.

(1) To find the idempotent for C_1, we have to write the tuple $(1, 1, \ldots, 1, 0)$ as a linear combination of the rows of $\mathcal{T}(G)$. The result is the coefficient tuple $(1, -2, -1, 0, 0, 1, 1, 1, 0)$. Hence the first primitive idempotent is

$$e_{C_1} = [G/H_1] - 2[G/H_2] - [G/H_3] + [G/H_6] + [G/H_7] + [G/H_8]$$

(2) For the component C_2, we have to write $(0, 0, \ldots, 0, 1)$ as a linear combination of the rows of $\mathcal{T}(G)$. Alternatively, using $e_{C_2} = 1 - e_{C_1}$, we get

$$e_{C_2} = -[G/H_1] + 2[G/H_2] + [G/H_3] - [G/H_6] - [G/H_7] - [G/H_8] + [G/H_9]$$

In this example the spectrum of $\mathcal{B}(G)$ consists of a connected component and a single irreducible component. As shown in Raggi-Cárdenas and Valero-Elizondo (2005), this is equivalent to G being perfect and all subgroups of G being solvable. Let us see an example of this type where G is not a simple group.

Example 8.12 The table of marks of the group $G = \mathrm{SL}_2(\mathbb{F}_5)$ is given by

$$\mathcal{T}(\mathrm{SL}_2(\mathbb{F}_5)) = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 60 & 60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 30 & 30 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20 & 20 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 15 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 12 & 12 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 10 & 10 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 6 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 5 & 5 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The second row of this table shows that H_2 is a normal subgroup, whence G is not simple. The prime divisors of $\#G = 120$ are 2, 3, and 5. Besides $H_1 = \{e\}$, the only other perfect subgroup of G turns out to be G itself: there are no subgroups of index 2 or 3, and the subgroups of index 5 are equal to their normalizer.

Thus $\mathrm{Spec}(\mathcal{B}(G))$ has two connected components. When we reduce the table modulo 2, 3, and 5, we see that $\bar{\mathfrak{p}}_1, \ldots, \bar{\mathfrak{p}}_{11}$ are all in one connected component, and the irreducible component of $\bar{\mathfrak{p}}_{12}$ is the second connected component of $\mathcal{B}(G)$.

Finally, we compute the primitive idempotents. For the first component we have to write $(1, 1, \ldots, 1, 0)$ as a linear combination of the rows of $\mathcal{T}(G)$. We get

$$e_{C_1} = [G/H_2] - 2[G/H_4] - [G/H_6] + [G/H_9] + [G/H_{10} + [G/H_{11}]$$

and the other primitive idempotent is then $e_{C_2} = [G/H_{12}] - e_{C_1}$.

References

Bouc, S.: Burnside rings. In: Hazewinkel, M. (ed.) Handbook of Algebra, vol. 2, pp. 739–804. Elsevier, Amsterdam (2000)

Burnside, W.: Theory of Groups of Finite Order, 2nd edn. Cambridge University Press, Cambridge (1911)

Călugăreanu, G.: The total number of subgroups of a finite Abelian group. Sci. Math. Japan **60**, 157–167 (2004)

Dress, A.: A characterization of solvable groups. Math. Z. **110**, 213–217 (1969)

Holt, D., Eick, B., O'Brien, E.: Handbook of Computational Group Theory. Chapman & Hall/CRC, Boca Raton (2005)

Huerta-Aparicio, L., Molina-Rueda, A., Raggi-Cárdenas, A., Valero-Elizondo, L.: On some invariants preserved by isomorphisms of tables of marks. Rev. Colomb. Mat. **43**, 165–174 (2009)

Karpilovsky, G.: Group Representations, vol. 4, part III, North Holland Mathematics Studies, vol. 182. Elsevier, Amsterdam (1995)

Kreuzer, M., Robbiano, L.: Computational Commutative Algebra 1. Springer, Heidelberg (2000)

Kreuzer, M., Robbiano, L.: Computational Commutative Algebra 2. Springer, Heidelberg (2005)

Nicolson, D.M.: On the graph of prime ideals of the Burnside ring of a finite group. J. Algebra **51**, 335–353 (1978)

Nicolson, D.M.: The orbit of the regular G-set under the full automorphism group of the Burnside ring of a finite group G. J. Algebra **51**, 288–299 (1978)

Pfeiffer, G.: The subgroups of M_{24}, or how to compute the table of marks of a finite group. Exp. Math. **6**, 247270 (1997)

Raggi-Cárdenas, A.G., Valero-Elizondo, L.: Groups with isomorphic Burnside rings. Arch. Math. **84**, 193–197 (2005)

Solomon, L.: The Burnside algebra of a finite group. J. Comb. Theory **2**, 603–615 (1967)

The ApCoCoA Team, ApCoCoA: Approximate Computations in Commutative Algebra (2013). http://www.apcocoa.org

The GAP Group, GAP—groups, algorithms, and programming, version 4.7.6 (2014). http://www.gap-system.org

Thévenaz, J.: Isomorphic Burnside rings. Commun. Algebras **16**, 1945–1947 (1988)

Yoshida, T.: Idempotents of Burnside rings and Dress induction theorem. J. Algebra **80**, 90–105 (1983)