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ORIGINAL PAPER

Asymmetry of Reuleaux polygons

HaiLin Jin[1]

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Abstract In this paper, we consider a measure of asymmetry for Reuleaux polygons, and show that the n-th ($n \geq 3$, n odd) regular Reuleaux polygons are the most symmetric ones among all n-th Reuleaux polygons. As a byproduct, we show that the Reuleaux triangles are the most asymmetric planar convex bodies of constant width.

Keywords Measure of asymmetry · Reuleaux triangle · Reuleaux polygons · constant width

Mathematics Subject Classification 52A38

1 Introduction

Measures of (central) symmetry, or as we prefer, asymmetry for convex bodies have been extensively investigated (see Besicovitch 1951; Groemer and Wallen 2001; Guo 2012; Guo and Jin 2011; Jin and Guo 2010, 2012; Lu and Pan 2005; Schneider 2009; Toth 2012). Among these researches, it is important to determine the extremal bodies in a class of convex bodies for a given asymmetry measure. A survey of results of this kind (up to 1963) has been published by Grünbaum (1963). In some of these investigations the definition of such measures is restricted to certain subsets of the class of all convex bodies. For example, in Besicovitch (1951), Besicovitch considered a

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✉ HaiLin Jin
jinhailin17@163.com

1 Department of Mathematics, Suzhou University of Science and Technology, Suzhou 215009, China

measure of asymmetry for domains of constant width in the euclidean plane $\mathbb{R}^2$, and showed that the most asymmetric domains are Reuleaux triangles.

Groemer and Wallen (2001) introduced a measure of convex domains of constant width, and determined the extremal bodies with respect to this asymmetry measure. More specifically, they also obtained that the most asymmetric domains are Reuleaux triangles.

Motivated by the work of Groemer and Wallen, replacing area by perimeter, Lu and Pan (2005) introduced another measure of asymmetry for convex domains of constant width. They showed that Reuleaux triangles are the most asymmetric domains of constant width in this sense.

In this paper, using the Lu–Pan measure of asymmetry, we show that the regular Reuleaux polygons have better symmetry than the irregular ones. Precisely, we prove the following theorem:

Theorem 1 *If K is a Reuleaux polygon of order n ($n \geq 5$, n odd), then*

$$\frac{n+1}{n-1} \leq \lambda(K) < 2,$$

where $\lambda(\cdot)$ denotes the Lu–Pan measure of asymmetry for convex bodies. Moreover, equality holds on the left-hand side if and only if K is regular.

From Theorem 1, we obtain the following result (see also Lu and Pan 2005):

Theorem 2 *Let K be a convex domain of constant width. Then*

$$1 \leq \lambda(K) \leq 2.$$

Equality holds on the left-hand side if and only if K is a circular disc. Equality holds on the right-hand side if and only if K is a Reuleaux triangle.

2 Preliminaries

Let C be a convex body, that is, a closed bounded convex subset of $\mathbb{R}^d$. Let $\mathcal{K}^d$ be the set of all d dimensional convex bodies. A convex body K is said to be of constant width if its width function, i.e., the support function of $K + (-K)$, is constant (see Chakerian and Groemer 1983; Heil and Martini 1993; Schneider 1993). Let $\mathcal{W}^d$ be the set of all convex bodies of constant width in $\mathcal{K}^d$. It is well-known that K is of constant width if and only if each boundary point of K is incident with (at least) one diameter (a chord of maximal length) of K.

By a diameter of $K \in \mathcal{K}^2$ of direction u we mean a line segment of direction u in K of maximal length. If $K \in \mathcal{W}^2$ then for any u there is exactly one diameter $D(u)$ of K of direction u, and the two lines that pass through the endpoints of $D(u)$ and are orthogonal to u are support lines of K. The diameter $D(u)$ splits K into two convex domains, say $K_+(u)$ and $K_-(u)$, where $K_+(u)$ lies in the 'positive' half-plane with respect to the line of direction u containing $D(u)$ (Groemer and Wallen 2001).

Groemer and Wallen (2001) defined the asymmetry function $\alpha(K)$ of $K \in \mathcal{W}^2$, by

$$\alpha(K) = \max\{A(K_+(u))/A(K_-(u)) : u \in S^1\}.$$

Here S^1 is the unit circle and $A(\cdot)$ is the area.

They proved that

$$1 \leq \alpha(K) \leq \alpha_0,$$

where $\alpha_0 = \frac{4\pi - 3\sqrt{3}}{2\pi - 3\sqrt{3}}$. Equality holds on the left-hand side if and only if K is a circular disc. Equality holds on the right-hand side if and only if K is a Reuleaux triangle.

Lu and Pan (2005) modified the definition of $\alpha(K)$, defined another measure $\lambda(K)$ of asymmetry for $K \in \mathcal{W}^2$ as follows:

$$\lambda(K) = \max\{L(K_+(u))/L(K_-(u)) : u \in S^1\},$$

where $L(K_+(u))$ and $L(K_-(u))$ are, respectively, the lengths of the arcs $\mathrm{bd}(K) \cap \mathrm{bd}(K_+(u))$ and $\mathrm{bd}(K) \cap \mathrm{bd}(K_-(u))$.

They proved that

$$1 \leq \lambda(K) \leq 2,$$

Equality holds on the left-hand side if and only if K is a circular disc. Equality holds on the right-hand side if and only if K is a Reuleaux triangle.

3 Proof of Theorems 1–2

Let $K \in \mathcal{W}^2$ and $V \subset \mathrm{bd}(K)$. The set V is called a pinching set if each diameter of K is incident with (at least) one point of V. A convex body K of constant width is called a Reuleaux polygon if it admits a finite pinching set. In fact, each Reuleaux polygon contains a polygon with the vertices being same as the Reuleaux polygon. In this case we say that the polygon generates the Reuleaux polygon. For example, each Reuleaux triangle can be generated by an equilateral triangle.

Let K be a Reuleaux polygon generated by the polygon with vertices $e_1 e_2 \cdots e_n$, n odd. It is obvious that each diameter of K meets at least one of $\{e_1, e_2, \ldots, e_n\}$. Define

$$\lambda(K, e_i) = \max\{L(K_+(u))/L(K_-(u)) : u \in S^1, e_i \in D(u)\}.$$

Clearly, we have $\lambda(K) = \max\{\lambda(K, e_i), i = 1, 2, \ldots n\}$.

Proof of Theorem 1 Let the width of K be ω. (1) For simplicity, we start with the case $n = 5$. By the definition of Reuleaux polygons, $|e_1 e_3| = |e_1 e_4| = |e_2 e_4| = |e_2 e_5| =$

$|e_3e_5| = \omega$. We shall denote these vertices $e_i (i = 1, 2, 3, 4, 5)$ by cyclic index. That is $e_i = e_j$ if $i \equiv j \pmod 5$. Then,

$$\lambda(K, e_i) = \max\left\{\frac{|\widehat{e_ie_{i+1}}| + |\widehat{e_{i+1}e_{i+2}}| + |\widehat{e_{i+2}e_{i+3}}|}{|\widehat{e_{i+3}e_{i+4}}| + |\widehat{e_{i+4}e_i}|}, \frac{|\widehat{e_{i+2}e_{i+3}}| + |\widehat{e_{i+3}e_{i+4}}| + |\widehat{e_{i+4}e_i}|}{|\widehat{e_ie_{i+1}}| + |\widehat{e_{i+1}e_{i+2}}|}\right\},$$

$i = 1, 2, 3, 4, 5$. Here $|\widehat{e_ie_{i+1}}|$ denotes the length of the circle arc between e_i and e_{i+1} in the boundary of K. By Barber's Theorem (Chakerian and Groemer 1983), we have $|\widehat{e_1e_2}| + |\widehat{e_2e_3}| + |\widehat{e_3e_4}| + |\widehat{e_4e_5}| + |\widehat{e_5e_1}| = \omega\pi$. Therefore, we have

$$\lambda(K) = \beta(K) - 1,$$

where

$$\beta(K) = \max\left\{\frac{\omega\pi}{|\widehat{e_ie_{i+1}}| + |\widehat{e_{i+1}e_{i+2}}|}, i = 1, 2, 3, 4, 5\right\}.$$

Set $a_i = \frac{\omega\pi}{|\widehat{e_ie_{i+1}}|+|\widehat{e_{i+1}e_{i+2}}|}, i = 1, 2, 3, 4, 5$. By the inequality between harmonic and arithmetic means, we have

$$\frac{5}{2} = 5(a_1^{-1} + a_2^{-1} + a_3^{-1} + a_4^{-1} + a_5^{-1})^{-1} \leq \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5}.$$

Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5$.
So,

$$\beta(K) = \max\{a_i, i = 1, 2, 3, 4, 5\} \geq \frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} \geq \frac{5}{2}.$$

Equality holds if and only if $|\widehat{e_1e_2}| = |\widehat{e_2e_3}| = |\widehat{e_3e_4}| = |\widehat{e_4e_5}| = |\widehat{e_5e_1}|$.

Therefore, we have $\lambda(K) \geq \frac{3}{2}$, and equality holds if and only if K is a regular Reuleaux polygon.

Now, we prove that $\lambda(K) < 2$. In fact, we need to prove $a_i < 3, i = 1, 2, 3, 4, 5$. This is equivalent to $b_i > \frac{\pi}{3}\omega$, where $b_i = |\widehat{e_ie_{i+1}}| + |\widehat{e_{i+1}e_{i+2}}|, i = 1, 2, 3, 4, 5$, and $e_6 = e_1, e_7 = e_2$. So, we only need to prove $\angle e_i + \angle e_{i+1} > \frac{\pi}{3}, i = 1, 2, 3, 4, 5$, where $\angle e_1 = \angle e_3e_1e_4, \angle e_2 = \angle e_4e_2e_5, \angle e_3 = \angle e_5e_3e_1, \angle e_4 = \angle e_1e_4e_2, \angle e_5 = \angle e_2e_5e_3$.

We only prove $\angle e_1 + \angle e_2 > \frac{\pi}{3}$.

Construct the triangle $\triangle e_1e_4e_5'$ such that $\triangle e_1e_4e_5' \cong \triangle e_2e_4e_5$, and e_5', e_3 lie on different sides of the line e_1e_4. Consider $\triangle e_3e_4e_5$ and $\triangle e_3e_4e_5'$, and notice $\angle e_3e_4e_5' > \angle e_3e_4e_5$, we have $|e_3e_5'| > |e_3e_5| = \omega$. Since $|e_3e_5'| > |e_1e_3| = |e_1e_5'| = \omega$, and we have $\angle e_3e_1e_5' > \frac{\pi}{3}$, which implies $\angle e_1 + \angle e_2 > \frac{\pi}{3}$.

Therefore, we have proved $\frac{3}{2} \leq \lambda(K) < 2$. The case for $n = 5$ follows.

(2) Now we consider the general case $n \geq 5$, and n is odd. Set $n = 2m+1$, $m \geq 2$. Then by the definition of Reuleaux polygon, we have $|e_i e_{m+i}| = \omega, i = 1, 2, \ldots, 2m+1$, where $e_{k+2m+1} = e_k$, $k = 1, 2, \ldots, 2m+1$. Therefore,

$$\lambda(K, e_i) = \max \left\{ \frac{\sum_{k=i}^{i+m} |\widehat{e_k e_{k+1}}|}{\sum_{k=i+m+1}^{i+2m} |\widehat{e_k e_{k+1}}|}, \frac{\sum_{k=i+m}^{i+2m} |\widehat{e_k e_{k+1}}|}{\sum_{k=i}^{i+m-1} |\widehat{e_k e_{k+1}}|} \right\}, \ i = 1, 2, \ldots, 2m+1.$$

So,

$$\lambda(K) = \max \left\{ \frac{\sum_{k=i}^{i+m} |\widehat{e_k e_{k+1}}|}{\sum_{k=i+m+1}^{i+2m} |\widehat{e_k e_{k+1}}|}, \quad i = 1, 2, \ldots, 2m+1 \right\}.$$

By the inequality between harmonic and arithmetic means, using the same technique as in step (1), we obtain $\lambda(K) \geq \frac{m+1}{m}$, and equality holds if and only if K is a regular Reuleaux polygon.

Now we prove $\lambda(K) < 2$. To do this we need to show

$$\sum_{k=i+m+1}^{i+2m} |\widehat{e_k e_{k+1}}| > \frac{\pi}{3}\omega, \quad i = 1, 2, \ldots, 2m+1.$$

This is equivalent to

$$\sum_{k=i}^{i+m-1} \angle e_k > \frac{\pi}{3}, \quad i = 1, 2, \ldots, 2m+1,$$

where $\angle e_k := \angle e_{k+m} e_k e_{k+m+1}$.

We only give a proof for the case $i = 1$. We translate $\triangle e_j e_{m+j} e_{m+j+1}$, $j = 2, \ldots, m$, to $\triangle e_1 e'_{m+j} e'_{m+j+1}$, respectively, (i.e., $\triangle e_j e_{m+j} e_{m+j+1} \cong \triangle e_1 e'_{m+j} e'_{m+j+1}$) such that $e'_{m+j} \in \mathrm{bd} B(e_1, \omega)$, $j = 2, \ldots, m+1$, where $B(e_1, \omega) := \{x \in \mathbb{R}^2 : |xe_1| \leq \omega\}$ denotes the circle with center e_1 and radius ω, and $e'_{m+2} = e_{m+2}$.

If $|e_{m+1} e'_{2m+1}| > \omega = |e_{m+1} e_{2m+1}|$, then we obtain $\sum_{k=1}^{m} \angle e_k = \angle e_{m+1} e_1 e'_{2m+1} > \frac{\pi}{3}$.

In what follows we prove

$$|e_{m+1} e'_{2m+1}| > |e_{m+1} e_{2m+1}|. \quad (*)$$

Comparing $\triangle e_{m+1} e_{2m} e_{2m+1}$ and $\triangle e_{m+1} e'_{2m} e'_{2m+1}$, the inequality above holds, if the following conditions hold:

(i) $|e_{2m} e_{2m+1}| = |e'_{2m} e'_{2m+1}|$;
(ii) $\angle e_{m+1} e_{2m} e_{2m+1} < \angle e_{m+1} e'_{2m} e'_{2m+1}$;
(iii) $|e_{m+1} e_{2m}| < |e_{m+1} e'_{2m}|$.

Now (i) is true from the fact that $\triangle e_m e_{2m} e_{2m+1} \cong \triangle e_1 e'_{2m} e'_{2m+1}$.

For (ii), we calculate $\angle e_{m+1}e_{2m}e_{2m+1} = \angle e_{2m-1}e_{2m}e_{2m+1} - \angle e_{m+1}e_{2m}e_{2m-1} = \frac{\pi-\angle e_m}{2} + \frac{\pi-\angle e_{m-1}}{2} - \angle e_{2m} - \angle e_{m+1}e_{2m}e_{2m-1}$,

$\angle e_{m+1}e'_{2m}e'_{2m+1} = \angle e'_{2m-1}e'_{2m}e'_{2m+1} - \angle e_{m+1}e'_{2m}e'_{2m-1} = \frac{\pi-\angle e_m}{2} + \frac{\pi-\angle e_{m-1}}{2} - \angle e_{m+1}e'_{2m}e'_{2m-1}$.

Thus, (ii) holds if

$$\angle e_{m+1}e_{2m}e_{2m-1} > \angle e_{m+1}e'_{2m}e'_{2m-1}. \quad (**)$$

Notice that

$$\angle e_{m+1}e_{2m}e_{2m-1} = \sum_{j=m+1}^{2m-2} \angle e_j e_{2m} e_{j+1}$$

and

$$\angle e_{m+1}e'_{2m}e'_{2m-1} = \sum_{j=m+1}^{2m-2} \angle e'_j e'_{2m} e'_{j+1},$$

where $e'_{m+1} = e_{m+1}, e'_{m+2} = e_{m+2}$, and we should prove that $\angle e_j e_{2m} e_{j+1} > \angle e'_j e_{2m} e'_{j+1}$, $j = m+1, \ldots, 2m-2$. Translate the 4-gon $e_j e_{j+1} e_{2m} e_1$ to the 4-gon $e'_j e'_{j+1} e''_{2m} e_1$. So, $|e_1 e''_{2m}| = |e_1 e_{2m}| < \omega$, which implies that $e''_{2m} \in \operatorname{int} B(e_1, \omega)$, interior of $B(e_1, \omega)$. Notice that $e'_j, e'_{j+1}, e'_{2m} \in \operatorname{bd} B(e_1, \omega)$, therefore, $\angle e_j e_{2m} e_{j+1} = \angle e'_j e''_{2m} e'_{j+1} > \angle e'_j e'_{2m} e'_{j+1}$.

It remains to show (iii). We claim that $|e_{m+1}e_i| < |e_{m+1}e'_i|$, $i = m+2, \ldots, 2m$.

First, we have $|e_{m+1}e_{m+2}| = |e_{m+1}e'_{m+2}|$.

Second, we consider $\triangle e_{m+1}e_{m+2}e_{m+3}$ and $\triangle e_{m+1}e'_{m+2}e'_{m+3}$. We have $|e_{m+1}e_{m+3}| < |e_{m+1}e'_{m+3}|$. Using the same technique as in (ii), we can prove $|e_{m+1}e_i| < |e_{m+1}e'_i|$, $i = m+2, \ldots, 2m$, by induction. □

Proof of Theorem 2 For each convex domain K of constant width ω, there exists Reuleaux polygons K_i, $i = 1, 2, \ldots$, such that $K_i \to K$, as $i \to \infty$ with respect to the Hausdorff metric. Since $L(\cdot)$ is continuous, we have $1 \le \lambda(K) \le 2$.

If K is a circular disc, then $\lambda(K) = 1$. Conversely, if $\lambda(K) = 1$, then for every $u \in S^1$ we have $L(K_-(u)) = L(K_+(u))$, which implies that K is centrally symmetric (see Groemer 1996 Theorem 4.5.9, or Groemer and Wallen 2001 p. 519). But since K is of constant width it must be a circular disc.

If K is a Reuleaux triangle, then $\lambda(K) = 2$. Conversely, if $\lambda(K) = 2$, then there exists a direction u and a diameter $|e_1e_2| = D(u)$ of K such that $L(K_-(u)) = \frac{\pi}{3}\omega$, $L(K_+(u)) = \frac{2\pi}{3}\omega$. Set $K_+(u)' := B(e_1, \omega) \cap B(e_2, \omega) \cap K_+(u)$. Then $K_+(u) \subset K_+(u)'$. Since $L(K_+(u)) = \frac{2\pi}{3}\omega$, we have that $K_+(u) = K_+(u)'$. This implies that K is Reuleaux triangle. □

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