Beitr Algebra Geom (2017) 58:93–129
DOI 10.1007/s13366-016-0311-9

ORIGINAL PAPER

Characterizations of zero-dimensional complete intersections

Martin Kreuzer[1] · **Le Ngoc Long**[1,2]

Received: 11 May 2016 / Accepted: 25 July 2016 / Published online: 29 July 2016

Abstract Given a 0-dimensional subscheme $\mathbb{X}$ of a projective space $\mathbb{P}^n_K$ over a field K, we characterize in different ways whether $\mathbb{X}$ is the complete intersection of n hypersurfaces. Besides a generalization of the notion of a Cayley–Bacharach scheme, these characterizations involve the Kähler and the Dedekind different of the homogeneous coordinate ring of $\mathbb{X}$ or its Artinian reduction. We also characterize arithmetically Gorenstein schemes in novel ways and bring in further tools such as the module of regular differential forms, the fundamental class, and the Jacobian module of $\mathbb{X}$. Throughout we strive to work over an arbitrary base field K and keep the scheme $\mathbb{X}$ as general as possible, thereby improving several known characterizations.

Keywords Zero-dimensional scheme · Complete intersection · Kähler different · Dedekind different · Arithmetically Gorenstein scheme · Cayley–Bacharach scheme · Hilbert function

Mathematics Subject Classification Primary 14M10; Secondary 13N05 · 13C40 · 13D40 · 14M05

✉ Martin Kreuzer
martin.kreuzer@uni-passau.de

Le Ngoc Long
nglong16633@gmail.com

1 Fakultät für Informatik und Mathematik, Universität Passau, 94030 Passau, Germany

2 Department of Mathematics, Hue University's College of Education, 34 Le Loi, Hue, Vietnam

1 Introduction

Given a 0-dimensional scheme $\mathbb{X}$ in a projective space $\mathbb{P}^n_K$ over a field K, it is an interesting question to characterize algebraically and geometrically when $\mathbb{X}$ is the complete intersection of n hypersurfaces. In general, characterizations of complete intersection rings using differentials, differents, and complementary modules have been studied by R. Berger, E. Kunz, H.-J. Nastold, G. Scheja, U. Storch, and others in the 1960s and 1970s. Many of these results we collected and unified in the book (Kunz 1986) which we use as our reference. In this paper we are specifically interested in zero-dimensional subschemes of projective spaces, for which the homogeneous coordinate ring is a 1-dimensional standard graded Cohen Macaulay ring.

In the case $n = 2$, i.e., for subschemes of the projective plane, Davis and Maroscia (1984) characterized complete intersections via the Cayley–Bacharach property and the symmetry of the Hilbert function of $\mathbb{X}$. Later it turned out that these conditions characterize arithmetically Gorenstein schemes for arbitrary $n \geq 2$ (see Davis et al. 1985; Kreuzer 1992). The attempt to refine this characterization by showing that $\mathbb{X}$ is the zero-set of a section of a vector bundle and then forcing the vector bundle to split into a direct sum of line bundles led to rather complicated and unwieldy conditions (see Kreuzer 1992 for the case $n = 3$ and Kreuzer et al. 2000 for the general case).

On the algebraic side, Wiebe (1969) proved for 0-dimensional local rings that they are complete intersections if and only if the 0-th Fitting ideal of the maximal ideal is non-zero. If we assume that the 0-dimensional scheme $\mathbb{X}$ is contained in the affine space $D_+(X_0)$, this characterization can be applied to the Artinian reduction $R/\langle x_0\rangle$ of the homogeneous coordinate ring $R = K[X_0, \ldots, X_n]/I_{\mathbb{X}}$ of $\mathbb{X}$. Similarly, assuming $\mathrm{char}(K) = 0$, Scheja and Storch (1975) characterized 0-dimensional local complete intersections by the non-vanishing of the Kähler different, i.e., the 0-th Fitting ideal of the Kähler differential module. Unfortunately, these characterizations do not allow finer distinctions such as the characterization of arithmetically Gorenstein schemes $\mathbb{X}$.

The main idea in this paper is to combine these two approaches and to use the Kähler differents of the algebras $R/K[x_0]$ and R/K together with some geometric properties to characterize 0-dimensional complete intersection schemes. For instance, we prove the following theorem which provides one possible answer to the question posed by Griffiths and Harris (1978): if $\mathbb{X}$ is a Cayley–Bacharach scheme, when is $\mathbb{X}$ a complete intersection?

Theorem 1.1 *Let $\mathbb{X}$ be a smooth 0-dimensional subscheme of $\mathbb{P}^n_K$. Then $\mathbb{X}$ is a complete intersection if and only if $\mathbb{X}$ is a Cayley–Bacharach scheme and the Hilbert function of the Kähler different $\vartheta_{\mathbb{X}}$ of $\mathbb{X}$ satisfies $HF_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$, where $r_{\mathbb{X}}$ is the regularity index of the Hilbert function of $\mathbb{X}$.*

As a consequence, we will see that this condition is also equivalent to the Kähler different being a principal ideal generated by a homogeneous non-zerodivisor of degree $r_{\mathbb{X}}$. The latter characterization is the graded version of a result given by Lenstra (1993) in the local case which in turn is a slight sharpening of a result used by Wiles (1995). Along the way, we prove a number of further characterizations of smooth 0-dimensional complete intersections.

Let us describe the contents of this paper in more detail. In Sect. 2 we recall the definition of the Kähler different. In fact, we introduce three Kähler differents which will be used later on. Assuming that $\mathbb{X}$ is contained in $D_+(X_0)$, as we always do, the residue class x_0 of X_0 in the homogeneous coordinate ring $R = K[X_0, \dots, X_n]/I_{\mathbb{X}}$ of $\mathbb{X}$ is a non-zerodivisor. Then we define the Kähler different $\vartheta_{\mathbb{X}} = F_0(\Omega^1_{R/K[x_0]})$, the reduced Kähler different $\bar{\vartheta}_{\mathbb{X}} = F_0(\Omega^1_{\overline{R}/K})$, where $\overline{R} = R/\langle x_0\rangle$, and the higher Kähler different $\vartheta^{(1)}_{\mathbb{X}} = F_1(\Omega^1_{R/K})$. After discussing the basic properties of these differents, we have a closer look at the Hilbert function of $\vartheta_{\mathbb{X}}$. Using some examples, we see that this is a tricky invariant and that even its eventual value, the Hilbert polynomial of $\vartheta_{\mathbb{X}}$, is in general difficult to determine.

Section 3 deals with the Dedekind different of a 0-dimensional scheme $\mathbb{X}$. To define it, we need to assume that $\mathbb{X}$ is locally Gorenstein. We make the construction in Herzog et al. (1971) explicit and embed the canonical module of $\mathbb{X}$ in its homogeneous coordinate ring to get the Dedekind complementary module whose inverse ideal is the Dedekind different. Notice that this construction depends on the choice of a system of traces for the local rings of $\mathbb{X}$. However, if $\mathbb{X}$ is smooth, we can use the canonical traces and get a well-defined result. The section ends with some relations between the Kähler and the Dedekind different of $\mathbb{X}$.

Many characterizations of 0-dimensional complete intersections use the Cayley–Bacharach property. In Sect. 4 we generalize the notion of a Caylay–Bacharach scheme (CB-scheme) to the case of a not necessarily reduced scheme $\mathbb{X}$ defined over an arbitrary base field K. Many concepts such as the degree of a point in $\mathbb{X}$ require careful adjustments in this generality. We also characterize CB-schemes via the existence of particular elements in the first homogeneous component of the Dedekind complementary module and use this result to generalize the characterization of arithmetically Gorenstein schemes via the Cayley–Bacharach property and the symmetry of their Hilbert function (see Kreuzer 1992).

The heart of the paper is contained in Sect. 5 where we characterize 0-dimensional complete intersections in several ways. The first characterization generalizes the above-mentioned result by Scheja and Storch (1975) and uses the non-vanishing of the reduced Kähler different. A second criterion uses a single value of the Hilbert function of $\vartheta_{\mathbb{X}}$ to distinguish complete intersections from arithmetically Gorenstein schemes. The third characterization answers the question of P. Griffiths and J. Harris for smooth 0-dimensional schemes by requiring a CB-scheme to have $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$. If we replace the Kähler different by the Dedekind different, we get a characterization of 0-dimensional arithmetically Gorenstein schemes, as Proposition 5.8 shows.

In the last section we present some characterizations which use the first Kähler different $\vartheta^{(1)}_{\mathbb{X}}$ of R/K. After collecting some properties of this different and its Hilbert function, we show that it can be used to characterize smooth 0-dimensional complete intersections in the same way as with $\vartheta_{\mathbb{X}}$ by the Cayley–Bacharach property and one non-zero value of its Hilbert function. Finally, we introduce the module of regular differential forms $\Omega_{\mathbb{X}}$ and the fundamental class $c_{\mathbb{X}} : \Omega^1_{R/K} \longrightarrow \Omega_{\mathbb{X}}$ of $\mathbb{X}$. Then smooth 0-dimensional complete intersections are also characterized by $c_{\mathbb{X}}(\Omega^1_{R/K}) = \vartheta^{(1)}_{\mathbb{X}}\,\Omega_{\mathbb{X}}$.

Lastly, smooth arithmetically Gorenstein schemes are characterized by the Cayley–Bacharach property and the dimension of their Jacobian module $J_{\mathbb{X}} = \Omega_{\mathbb{X}}/c_{\mathbb{X}}(\Omega^1_{R/K})$.

Unless mentioned explicitly otherwise, we adhere to the definitions and notation introduced in the books (Kreuzer and Robbiano 2000, 2005). The examples in this paper were calculated using a package implemented by the second author in the computer algebra system The ApCoCoA Team (2007).

2 Kähler differents of zero-dimensional schemes

In this paper we always work in the following setting. Let K be an arbitrary field, and let $P = K[X_0, \ldots, X_n]$ be the polynomial ring in $n+1$ indeterminates over K, equipped with the standard grading. Then P is the homogeneous coordinate ring of projective n-space $\mathbb{P}^n_K$.

Our main object of study is a (non-empty) 0-dimensional subscheme $\mathbb{X}$ of $\mathbb{P}^n_K$. Its homogeneous vanishing ideal in P is denoted by $I_{\mathbb{X}}$. Then $R = P/I_{\mathbb{X}}$ is the homogeneous coordinate ring of $\mathbb{X}$. It is a 1-dimensional standard graded Cohen–Macaulay K-algebra. Its homogeneous maximal ideal will be denoted by $\mathfrak{m}$.

Assumption In the following we assume that no point of the support of $\mathbb{X}$ is contained in the hypersurface at infinity $\mathcal{Z}(X_0)$. Consequently, the residue class x_0 of X_0 in R is a non-zerodivisor.

The ring $\overline{R} = R/\langle x_0\rangle$ is called the Artinian reduction of R. It is a 0-dimensional, local K-algebra, and hence a finite dimensional K-vector space of dimension $d = \deg(\mathbb{X})$. The maximal ideal of $\overline{R}$ is denoted by $\overline{\mathfrak{m}}$. It follows that R is a finite free $K[x_0]$-module of rank d. The modules of Kähler differentials of the three algebras R/K, $\overline{R}/K$, and $R/K[x_0]$ are related as follows (cf. Kunz 1986).

Proposition 2.1 *The element x_0 is a non-zerodivisor for the module $\Omega^1_{R/K[x_0]}$, and we have $\Omega^1_{R/K[x_0]}/x_0\,\Omega^1_{R/K[x_0]} \cong \Omega^1_{\overline{R}/K}$. Furthermore, there exists a canonical homogeneous exact sequence*

$$0 \longrightarrow Rdx_0 \longrightarrow \Omega^1_{R/K} \longrightarrow \Omega^1_{R/K[x_0]} \longrightarrow 0$$

Now the Fitting ideals of these three Kähler differential modules are given the following names. (For some basic properties of Fitting ideals we refer the reader to Kunz 1986, Appendix D.)

Definition 2.2 (a) The ideal $\vartheta_{R/K[x_0]} = F_0(\Omega^1_{R/K[x_0]})$ of R is called the **Kähler different** of $\mathbb{X}$ (or of $R/K[x_0]$). For short, we also write $\vartheta_{\mathbb{X}}$ instead of $\vartheta_{R/K[x_0]}$.

(b) The ideal $\vartheta_{\overline{R}/K} = F_0(\Omega^1_{\overline{R}/K})$ of $\overline{R}$ is called the **reduced Kähler different** of $\mathbb{X}$. We also write $\bar{\vartheta}_{\mathbb{X}}$ instead of $\vartheta_{\overline{R}/K}$ for short.

(c) The ideal $\vartheta^{(1)}_{R/K} = F_1(\Omega^1_{R/K})$ is called the **higher Kähler different** of $\mathbb{X}$ (or of R/K). This ideal is also denoted by $\vartheta^{(1)}_{\mathbb{X}}$ for short.

Recall that the scheme $\mathbb{X}$ is called a **complete intersection** if its homogeneous vanishing ideal can be generated by n homogeneous polynomials, i.e., if there exist homogeneous polynomials $F_1, \ldots, F_n \in P$ such that $I_{\mathbb{X}} = \langle F_1, \ldots, F_n \rangle$. It is well-known that the Kähler different is given by $\vartheta_{\mathbb{X}} = \langle \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)} \rangle_R$ in this case, i.e., it is the principal ideal generated by the Jacobian determinant of $(F_1, \ldots, F_n)$. A similar description can be given for $\bar{\vartheta}_{\mathbb{X}}$.

Given a finitely generated graded module M over a standard graded K-algebra, we let $\mathrm{HF}_M : \mathbb{Z} \longrightarrow \mathbb{N}$ be the map defined by $\mathrm{HF}_M(i) = \dim_K(M_i)$. Recall that this map is called the **Hilbert function** of M. This is an integer function of polynomial type, i.e., it agrees with the value of the **Hilbert polynomial** $\mathrm{HP}_M(i)$ in large degrees $i \gg 0$. The number $\mathrm{ri}(M) = \min\{i \geq 0 \mid \mathrm{HF}_M(j) = \mathrm{HP}_M(j) \text{ for all } j \geq i\}$ is called the **regularity index** of M. Instead of HF_R, we also write $\mathrm{HF}_{\mathbb{X}}$ and call it the Hilbert function of $\mathbb{X}$.

Remark 2.3 The Hilbert function of $\mathbb{X}$ satisfies $\mathrm{HF}_{\mathbb{X}}(i) = 0$ for $i < 0$ and

$$1 = \mathrm{HF}_{\mathbb{X}}(0) < \mathrm{HF}_{\mathbb{X}}(1) < \cdots < \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) = d = \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} + 1) = \cdots$$

for some number $r_{\mathbb{X}} \geq 0$ which is called the **regularity index** of $\mathbb{X}$.

In the rest of this section we collect some general results and examples illustrating properties of the Hilbert function of the Kähler different of $\mathbb{X}$. Later we will see that the reduced Kähler different is much less interesting (it vanishes most of the time) and that one can prove similar things for the Hilbert function of the higher Kähler different (see Sect. 6).

Remark 2.4 Since x_0 is a non-zerodivisor of R and $\vartheta_{\mathbb{X}}$ is an ideal in R, the Hilbert function $\mathrm{HF}_{\vartheta_{\mathbb{X}}}$ is non-decreasing. Therefore it has an **initial degree** $\min\{i \in \mathbb{Z} \mid \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i) \neq 0\}$ and an **eventual value** $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(j)$ for $j \gg 0$ which is also known as the **Hilbert polynomial** of $\vartheta_{\mathbb{X}}$ and denoted by $\mathrm{HP}_{\vartheta_{\mathbb{X}}}$.

Recall that the minimal prime divisors of $I_{\mathbb{X}}$ are homogeneous prime ideals which correspond to points in $\mathbb{P}^n_K = \mathrm{Proj}(P)$. The set of these points is called the **support** of $\mathbb{X}$ and will be denoted by $\mathrm{Supp}(\mathbb{X}) = \{p_1, \ldots, p_s\}$. To each point p_i we have the associated local ring $\mathcal{O}_{\mathbb{X},p_i}$, its homogeneous vanishing ideal $\mathfrak{P}_i$ in P, and the associated homogeneous ideal $\mathfrak{p}_i$ in R.

Definition 2.5 (a) A point $p_i \in \mathrm{Supp}(\mathbb{X})$ is called a **reduced point** of $\mathbb{X}$, if $\mathcal{O}_{\mathbb{X},p_i}$ is a reduced ring.
(b) A point $p_i \in \mathrm{Supp}(\mathbb{X})$ is called a **smooth point** of $\mathbb{X}$, or $\mathbb{X}$ is called **smooth at** p_i, if $\mathcal{O}_{\mathbb{X},p_i}/K$ is a finite separable field extension.
(c) We say that $\mathbb{X}$ is **reduced** (resp. **smooth**) if it is reduced (resp. smooth) at all points of its support.

Clearly, if p_i is a smooth point of $\mathbb{X}$ then it is a reduced point of $\mathbb{X}$. The converse is true if K is a perfect field (cf. Kunz 1986, Propositions 5.18 and 7.12).

If $\mathbb{X}$ is a smooth complete intersection, then we know the Hilbert function of $\vartheta_{\mathbb{X}}$. More generally, if $\mathbb{X}$ is reduced then the Hilbert function of $\vartheta_{\mathbb{X}}$ can be described in terms of the subset $\mathbb{Y}$ of smooth points of $\mathbb{X}$, as the following proposition shows. Notice that for $\mathbb{Y} = \emptyset$ we have $I_{\mathbb{Y}/\mathbb{X}} = \mathfrak{m}$.

Proposition 2.6 *Let $\mathbb{X}$ be a 0-dimensional reduced complete intersection in $\mathbb{P}^n_K$. We write $I_{\mathbb{X}} = \langle F_1, \ldots, F_n \rangle$, where $F_j \in P$ is a homogeneous polynomial of degree d_j for $j = 1, \ldots, n$. (Recall that we have $r_{\mathbb{X}} = \sum_{i=1}^n d_i - n$ in this case.) Let $\mathbb{Y}$ be the subscheme of $\mathbb{X}$ consisting of all smooth points in Supp$(\mathbb{X})$. Then we have*

$$HF_{\vartheta_{\mathbb{X}}}(i) = \begin{cases} HF_{\mathbb{Y}}(i - r_{\mathbb{X}}) & \text{if } \mathbb{Y} \neq \emptyset \\ 0 & \text{if } \mathbb{Y} = \emptyset \end{cases}$$

for all $i \in \mathbb{Z}$.

Proof Let $I_{\mathbb{Y}/\mathbb{X}}$ be the ideal of $\mathbb{Y}$ in R and put $\Delta := \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)}$. Then we have $\vartheta_{\mathbb{X}} = \langle \Delta \rangle_R$. In the case $\mathbb{Y} = \emptyset$, we have $\Delta \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s = \langle 0 \rangle$ by Kunz (1986, Theorem 10.12). This implies $\vartheta_{\mathbb{X}} = \langle 0 \rangle$.

In the case $\mathbb{Y} \neq \emptyset$, we use Kunz (1986, Theorem 10.12) again to conclude that the element $\Delta + I_{\mathbb{Y}/\mathbb{X}}$ is a non-zerodivisor of $R_{\mathbb{Y}} = R/I_{\mathbb{Y}/\mathbb{X}}$ and that we have $\Delta \in \mathfrak{p}_j$ for all $j \in \{1, \ldots, s\}$ such that $p_j \notin \text{Supp}(\mathbb{Y})$. Now we fix the degree $i \geq 0$ and suppose $\text{HF}_{\mathbb{Y}}(i) = t$. Let $\{g_1 + I_{\mathbb{Y}/\mathbb{X}}, \ldots, g_t + I_{\mathbb{Y}/\mathbb{X}}\}$ be a K-basis of the vector space $(R_{\mathbb{Y}})_i$. Then the set $\{\Delta \cdot g_1 + I_{\mathbb{Y}/\mathbb{X}}, \ldots, \Delta \cdot g_t + I_{\mathbb{Y}/\mathbb{X}}\} \subseteq (R_{\mathbb{Y}})_{i+r_{\mathbb{X}}}$ is K-linearly independent. It follows that the vector space $(\Delta \cdot R)_{i+r_{\mathbb{X}}}$ has K-dimension greater than or equal to t, in other words, we have $\text{HF}_{\vartheta_{\mathbb{X}}}(i + r_{\mathbb{X}}) \geq \text{HF}_{\mathbb{Y}}(i)$.

On the other hand, we observe that $\Delta \cdot h = 0$ in R for every homogeneous element $h \in I_{\mathbb{Y}/\mathbb{X}} \setminus \{0\}$, since $\mathbb{X}$ is reduced. For every $f \in R_i$, we write $f = a_1 g_1 + \cdots + a_t g_t + h$ for some $a_1, \ldots, a_t \in K$ and $h \in (I_{\mathbb{Y}/\mathbb{X}})_i$. Then $\Delta \cdot f = \Delta \cdot (a_1 g_1 + \cdots + a_t g_t + h) = a_1 \Delta \cdot g_1 + \cdots + a_t \Delta \cdot g_t \in \langle \Delta \cdot g_1, \ldots, \Delta \cdot g_t \rangle_K$ (as $\Delta \cdot h = 0$ in R). Thus $(\vartheta_{\mathbb{X}})_{i+r_{\mathbb{X}}} = (\Delta \cdot R)_{i+r_{\mathbb{X}}} \subseteq \langle \Delta \cdot g_1, \ldots, \Delta \cdot g_t \rangle_K$, and hence $\text{HF}_{\vartheta_{\mathbb{X}}}(i + r_{\mathbb{X}}) \leq t = \text{HF}_{\mathbb{Y}}(i)$. Therefore the conclusion follows. □

In particular, if $\mathbb{X}$ is smooth, this proposition simplifies as follows.

Corollary 2.7 *Let $\mathbb{X}$ be a smooth 0-dimensional complete intersection in $\mathbb{P}^n_K$.*

(a) *The Kähler different of $\mathbb{X}$ is given by $\vartheta_{\mathbb{X}} = \left\langle \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)} \right\rangle_R$, where $\frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)}$ is a homogeneous non-zerodivisor of R of degree $r_{\mathbb{X}} = \sum_{i=1}^n d_i - n$.*
(b) *The Hilbert function of $\vartheta_{\mathbb{X}}$ satisfies $HF_{\vartheta_{\mathbb{X}}}(i) = HF_{\mathbb{X}}(i - r_{\mathbb{X}})$ for all $i \in \mathbb{Z}$.*

Remark 2.8 In the setting of the proposition, the regularity index of the Kähler different is $r_{\mathbb{X}} + r_{\mathbb{Y}}$. If we remove the condition that $\mathbb{X}$ is reduced, then we have $\text{HF}_{\vartheta_{\mathbb{X}}}(i) \geq \text{HF}_{\mathbb{Y}}(i - r_{\mathbb{X}})$ for all $i \in \mathbb{Z}$. This follows from the first part of the proof of the proposition.

The following example shows that, even if $\mathbb{X}$ is a complete intersection, the eventual value of the Hilbert function of $\vartheta_{\mathbb{X}}$ may not be equal to $\deg(\mathbb{X})$.

Example 2.9 Let K be a field with $\text{char}(K) \notin \{2, 3\}$, and let $\mathbb{X} \subseteq \mathbb{P}^2_K$ be the 0-dimensional complete intersection defined by the ideal $I_{\mathbb{X}} = \langle F, G \rangle$, where we have $F = X_1(X_1 - 2X_0)(X_1 + 2X_0)$ and $G = (X_2 - X_0)(X_1^2 + X_2^2 - 4X_0^2)$. Then $\deg(\mathbb{X}) = 9$ and $\text{Supp}(\mathbb{X}) = \{p_1, \ldots, p_7\}$, where $p_1 = (1 : 0 : 1)$, $p_2 = (1 : 0 : 2)$, $p_3 = (1 : 0 :$

$-2)$, $p_4 = (1:2:1)$, $p_5 = (1:2:0)$, $p_6 = (1:-2:1)$, and $p_7 = (1:-2:0)$. By $\mathfrak{P}_j$ we denote the associated homogeneous prime ideal in P of p_j for $j = 1,\dots,7$. The homogeneous primary decomposition of the ideal $I_{\mathbb{X}}$ is $I_{\mathbb{X}} = I_1 \cap \cdots \cap I_7$, where $I_j = \mathfrak{P}_j$ for $j \neq 5, 7$, $I_5 = \langle X_1 - 2X_0, X_2^2 \rangle$, and $I_7 = \langle X_1 + 2X_0, X_2^2 \rangle$. This means that $\mathbb{X}$ is not reduced at p_5 and p_7, and so $\mathbb{X}$ is not smooth at those points. In this case we have $\vartheta_{\mathbb{X}} = \langle \frac{\partial(F,G)}{\partial(x_1,x_2)} \rangle = \langle 4x_0x_1^2x_2 - 16x_0^2x_2^2 - 3x_1^2x_2^2 - 2x_0x_2^3 + 6x_2^4 \rangle$. In particular, the Jacobian determinant $\frac{\partial(F,G)}{\partial(x_1,x_2)}$ is a zerodivisor of R. This shows that the smoothness of $\mathbb{X}$ in the above corollary is a necessary hypothesis. Furthermore, we have

$$\begin{aligned} \mathrm{HF}_{\mathbb{X}} &: 1\ 3\ 6\ 8\ 9\ 9 \cdots \\ \mathrm{HF}_{\vartheta_{\mathbb{X}}} &: 0\ 0\ 0\ 0\ 1\ 3\ 6\ 7\ 7 \cdots \end{aligned}$$

Hence the Hilbert function of $\vartheta_{\mathbb{X}}$ can stabilize at a value $\neq \deg(\mathbb{X})$.

The next example shows that, in general, the initial degree of the Hilbert function of $\vartheta_{\mathbb{X}}$ can be less than $r_{\mathbb{X}}$.

Example 2.10 Let $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ be the set of five $\mathbb{Q}$-rational points: $p_1 = (1:1:0)$, $p_2 = (1:1:1)$, $p_3 = (1:-1:1)$, $p_4 = (1:2:1)$, and $p_5 = (1:-2:1)$. We have $\mathrm{HF}_{\mathbb{X}}:\ 1\ 3\ 4\ 5\ 5 \cdots$ and $r_{\mathbb{X}} = 3$. Moreover, we have $\mathrm{HF}_{\vartheta_{\mathbb{X}}}:\ 0\ 0\ 1\ 1\ 3\ 4\ 5\ 5 \cdots$ and $\vartheta_{\mathbb{X}} = \langle x_0x_1 - x_1x_2, x_1^2x_2^2 - \frac{8}{5}x_2^4, x_1^3x_2 - \frac{5}{2}x_1x_2^3 \rangle$. Thus, in this case, the initial degree of the Hilbert function of $\vartheta_{\mathbb{X}}$ is less than $r_{\mathbb{X}} = 3$.

The following condition will play an important role in this paper.

Definition 2.11 A 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^n_K$ is called **arithmetically Gorenstein** if R is a Gorenstein ring.

Note that if $\mathbb{X}$ is a complete intersection then it is arithmetically Gorenstein, but the converse is not true in general, as the next example shows. Moreover, for an arithmetically Gorenstein scheme $\mathbb{X}$, later results will show that the initial degree of the Hilbert function of $\vartheta_{\mathbb{X}}$ is at least $r_{\mathbb{X}}$. The next example also illustrates that this initial degree can be strictly higher.

Example 2.12 Let $\mathbb{X} \subseteq \mathbb{P}^3_{\mathbb{F}_7}$ be the following set of five distinct $\mathbb{F}_7$-rational points on the twisted cubic curve: $p_1 = (1:0:0:0)$, $p_2 = (1:1:1:1)$, $p_3 = (1:-1:1:-1)$, $p_4 = (1:2:4:8)$, and $p_5 = (8:4:2:1)$. We have $\mathrm{HF}_{\mathbb{X}}:\ 1\ 4\ 5\ 5 \cdots$ and $r_{\mathbb{X}} = 2$. An application of Theorem 7 in Geramita and Orecchia (1981) shows that $\mathbb{X}$ is arithmetically Gorenstein. Moreover, a calculation gives us $\vartheta_{\mathbb{X}} = \langle x_2x_3^2 - 3x_3^3, x_1x_3^2, x_0x_3^2 - 3x_3^3, x_0^3 \rangle$ and $\mathrm{HF}_{\vartheta_{\mathbb{X}}}:\ 0\ 0\ 0\ 4\ 5\ 5 \cdots$. Hence $\mathbb{X}$ is not a complete intersection and the initial degree of $\mathrm{HF}_{\vartheta_{\mathbb{X}}}$ is $3 > r_{\mathbb{X}}$.

As mentioned above, the Kähler different of a complete intersection $\mathbb{X}$ is a principal ideal. The following example shows that the Kähler different can be principal without $\mathbb{X}$ being a complete intersection.

Example 2.13 Let $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ be the set consisting of six $\mathbb{Q}$-rational points: $p_1 = (1 : 0 : 0)$, $p_2 = (1 : 0 : 1)$, $p_3 = (1 : 1 : 0)$, $p_4 = (1 : 1 : 1)$, $p_5 = (1 : 2 : 0)$, and $p_6 = (1 : 2 : 1)$, and let $\mathbb{Y}$ be the fat point scheme defined by the saturated homogeneous ideal

$$I_{\mathbb{Y}} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_4 \cap \mathfrak{P}_5^2 \cap \mathfrak{P}_6^2.$$

Then $\mathbb{Y}$ is an almost complete intersection, i.e., $I_{\mathbb{Y}}$ is minimally generated by three homogeneous polynomials in P. In particular, $\mathbb{Y}$ is not a complete intersection. But in this case $\vartheta_{\mathbb{Y}}$ is the principal ideal generated by the homogeneous polynomial

$$x_0^5 - \tfrac{15}{4}x_0^2x_1^3 + \tfrac{55}{16}x_0x_1^4 - \tfrac{13}{16}x_1^5 + \tfrac{3}{4}x_1^3x_2^2 - \tfrac{7}{2}x_1^2x_2^3 + 5x_1x_2^4 - 2x_2^5.$$

Moreover, the Hilbert function of $\vartheta_{\mathbb{Y}}$ is $\mathrm{HF}_{\vartheta_{\mathbb{Y}}} :\ 0\,0\,0\,0\,0\,1\,3\,4\,4\cdots$ and its regularity index satisfies $\mathrm{ri}(\vartheta_{\mathbb{Y}}) = 7 < 8 = 2r_{\mathbb{Y}}$.

In the last part of this section we collect some results about the eventual value of $\mathrm{HF}_{\vartheta_{\mathbb{X}}}$, i.e., the Hilbert polynomial of $\vartheta_{\mathbb{X}}$.

Remark 2.14 If $\mathbb{X}$ is a fat point scheme in $\mathbb{P}^n_K$, then the Hilbert polynomial of the Kähler different $\vartheta_{\mathbb{X}}$ is exactly the number of reduced points of the scheme $\mathbb{X}$, and we have $\mathrm{ri}(\vartheta_{\mathbb{X}}) \leq n\, r_{\mathbb{X}}$ (see Kreuzer et al. 2015, Theorem 2.5).

Apart from some other special cases, to exactly determine the Hilbert polynomial of the Kähler different for an arbitrary 0-dimensional subscheme $\mathbb{X}$ of $\mathbb{P}^n_K$ is not an easy task. Hence we try at least to find (possibly sharp) bounds for it.

Proposition 2.15 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme, and let $\mathbb{X}_{\mathrm{sm}}$ be the set of smooth points in its support Supp$(\mathbb{X}) = \{p_1, \ldots, p_s\}$. Then we have*

$$\sum_{p_j \in \mathbb{X}_{\mathrm{sm}}} \dim_K(\mathcal{O}_{\mathbb{X},p_j}) \leq HP_{\vartheta_{\mathbb{X}}} \leq deg(\mathbb{X}) - (s - \#\mathbb{X}_{\mathrm{sm}}).$$

Proof Let $\mathfrak{P}_j \subseteq P$ be the associated prime ideal of p_j for $j = 1, \ldots, s$, and set

$$I := \bigcap_{p_j \in \mathrm{Supp}(\mathbb{X}) \setminus \mathbb{X}_{\mathrm{sm}}} \mathfrak{P}_j.$$

It follows from Kunz (1986, Theorem 10.12) that $\vartheta_{\mathbb{X}} \subseteq \mathfrak{p}_j = \mathfrak{P}_j / I_{\mathbb{X}}$ for every point $p_j \in \mathrm{Supp}(\mathbb{X}) \setminus \mathbb{X}_{\mathrm{sm}}$. Hence we get $\vartheta_{\mathbb{X}} \subseteq I/I_{\mathbb{X}}$, and consequently

$$\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i) \leq \mathrm{HF}_{I/I_{\mathbb{X}}}(i) = \mathrm{HF}_{\mathbb{X}}(i) - \mathrm{HF}_{P/I}(i) \leq \deg(\mathbb{X}) - \deg(\mathbb{Y})$$

for all $i \in \mathbb{N}$, where $\mathbb{Y}$ is the 0-dimensional subscheme of $\mathbb{P}^n_K$ defined by I. Observe that the scheme $\mathbb{Y}$ has degree $\deg(\mathbb{Y}) \geq s - \#\mathbb{X}_{\mathrm{sm}}$. Thus we obtain

$$\mathrm{HP}_{\vartheta_{\mathbb{X}}} \leq \deg(\mathbb{X}) - \deg(\mathbb{Y}) \leq \deg(\mathbb{X}) - (s - \#\mathbb{X}_{\mathrm{sm}}).$$

Next we prove the first inequality of $\mathrm{HP}_{\vartheta_{\mathbb{X}}}$. If $\mathbb{X}_{\mathrm{sm}} = \emptyset$, then there is nothing to prove, since we always have $\mathrm{HP}_{\vartheta_{\mathbb{X}}} \geq 0$. Now let us consider the case $\#\mathbb{X}_{\mathrm{sm}} \geq 1$. W.l.o.g. we may assume that $\mathbb{X}_{\mathrm{sm}} = \{p_1, \ldots, p_\varrho\}$ where $\varrho := \#\mathbb{X}_{\mathrm{sm}}$. Then Kunz (1986, Theorem 10.12) implies $\vartheta_{\mathbb{X}} \nsubseteq \mathfrak{p}_j$ for all $j = 1, \ldots, \varrho$. It follows from Homogeneous Prime Avoidance (see for instance Kreuzer and Robbiano 2005, Proposition 5.6.22) that there exists a homogeneous element $h \in (\vartheta_{\mathbb{X}})_m \setminus \{0\}$ for some $m \geq 0$ such that $h \notin \bigcup_{j=1}^{\varrho} (\mathfrak{p}_j)_m$. This implies that $h_{p_j} \neq 0$ in $\mathcal{O}_{\mathbb{X},p_j}$ for $j = 1, \ldots, \varrho$. Let $j \in \{1, \ldots, \varrho\}$, let $\varkappa_j = \dim_K(\mathcal{O}_{\mathbb{X},p_j})$, and let $\{e_{j1}, \ldots, e_{j\varkappa_j}\}$ be a K-basis of $\mathcal{O}_{\mathbb{X},p_j}$. For any non-zero element $a \in \mathcal{O}_{\mathbb{X},p_j}$, it is not difficult to verify that $\{ae_{j1}, \ldots, ae_{j\varkappa_j}\}$ is a K-basis of $\mathcal{O}_{\mathbb{X},p_j}$, then so is $\{h_{p_j} e_{jk} e_{j1}, \ldots, h_{p_j} e_{jk} e_{j\varkappa_j}\}$, where $1 \leq k \leq \varkappa_j$.

Now we consider the isomorphism of K-vector spaces $\imath : R_{r_{\mathbb{X}}} \to \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}$ given by $\imath(f) = (f_{p_1}, \ldots, f_{p_s})$, where $f_{p_j} \in \mathcal{O}_{\mathbb{X},p_j}$ is the germ of f at p_j for $j = 1, \ldots, s$ (cf. Kreuzer 1994, Lemma 1.1). For all $j = 1, \ldots, \varrho$ and for all $k_j = 1, \ldots, \varkappa_j$, we let $f_{jk_j} = \imath^{-1}((0, \ldots, 0, e_{jk_j}, 0, \ldots, 0)) \in R_{r_{\mathbb{X}}}$. Then we get

$$\langle hf_{11}, \ldots, hf_{1\varkappa_1}, \ldots, hf_{\varrho 1}, \ldots, hf_{\varrho\varkappa_\varrho} \rangle_K \subseteq (\vartheta_{\mathbb{X}})_{r_{\mathbb{X}}+m} \subseteq R_{r_{\mathbb{X}}+m}.$$

We show that $\{hf_{11}, \ldots, hf_{1\varkappa_1}, \ldots, hf_{\varrho 1}, \ldots, hf_{\varrho\varkappa_\varrho}\}$ is K-linearly independent. Remark that for $j_1, j_2 \in \{1, \ldots, \varrho\}$ and for $k_i \in \{1, \ldots, \varkappa_{j_i}\}$, where $i = 1, 2$, we have $f_{j_1k_1} \cdot f_{j_2k_2} \neq 0$ if $j_1 = j_2$ and $f_{j_1k_1} \cdot f_{j_2k_2} = 0$ if $j_1 \neq j_2$, and $hf_{j_1k_1}^2 \neq 0$ in $R_{2r_{\mathbb{X}}+m}$. Suppose for contradiction that there are $c_{11}, \ldots, c_{1\varkappa_1}, \ldots, c_{\varrho 1}, \ldots, c_{\varrho\varkappa_\varrho} \in K$, not all equal to zero, such that $\sum_{j=1}^{\varrho} \sum_{k_j=1}^{\varkappa_j} c_{jk_j} hf_{jk_j} = 0$. W.l.o.g. we may assume $c_{11} \neq 0$. We then have

$$hf_{11}^2 = \frac{1}{c_{11}} \left(\sum_{k_1=2}^{\varkappa_1} c_{1k_1} hf_{1k_1} f_{11} + \sum_{j=2}^{\varrho} \sum_{k_j=1}^{\varkappa_j} c_{jk_j} hf_{jk_j} f_{11} \right) = \frac{1}{c_{11}} \sum_{k_1=2}^{\varkappa_1} c_{1k_1} hf_{11} f_{1k_1}.$$

Thus, in $\mathcal{O}_{\mathbb{X},p_1}$, we get the equality $h_{p_1} e_{11}^2 = \frac{1}{c_{11}} \sum_{k_1=2}^{\varkappa_1} c_{1k_1} h_{p_1} e_{11} e_{1k_1}$, in contradiction to the fact that $\{h_{p_1} e_{11}^2, h_{p_1} e_{11} e_{12}, \ldots, h_{p_1} e_{11} e_{1\varkappa_1}\}$ is a K-basis of $\mathcal{O}_{\mathbb{X},p_1}$. Therefore we obtain

$$\begin{aligned} \mathrm{HP}_{\vartheta_{\mathbb{X}}} &\geq \dim_K \langle hf_{11}, \ldots, hf_{1\varkappa_1}, \ldots, hf_{\varrho 1}, \ldots, hf_{\varrho\varkappa_\varrho} \rangle_K \\ &= \sum_{j=1}^{\varrho} \varkappa_j = \sum_{p_j \in \mathbb{X}_{\mathrm{sm}}} \dim_K(\mathcal{O}_{\mathbb{X},p_j}) \end{aligned}$$

and the proposition is completely proved. □

Clearly, the lower bound for $\mathrm{HP}_{\vartheta_{\mathbb{X}}}$ is attained for a smooth scheme $\mathbb{X}$. Our next example shows that the upper bound for $\mathrm{HP}_{\vartheta_{\mathbb{X}}}$ is also sharp.

Example 2.16 Let us go back to Example 2.9. The scheme $\mathbb{X}$ is a complete intersection with $\deg(\mathbb{X}) = 9$, and it is not smooth at two points p_5 and p_7. In this case we have $\varrho = 5$ and

$$\mathrm{HP}_{\vartheta_{\mathbb{X}}} = \deg(\mathbb{X}) - (s - \varrho) = 7 > 5 = \varrho = \sum_{p_j \in \mathbb{X}_{\mathrm{sm}}} \dim_K(\mathcal{O}_{\mathbb{X},p_j})$$

and $\mathrm{ri}(\vartheta_{\mathbb{X}}) = 7 < 8 = 2r_{\mathbb{X}}$.

3 Dedekind differents of zero-dimensional schemes

In this section we continue to use the notation introduced above. Recall that the graded R-module $\omega_R = \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])(-1)$ is called the **canonical module** of R. It is a finitely generated graded R-module with Hilbert function $\mathrm{HF}_{\omega_R}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-i)$ for all $i \in \mathbb{Z}$. (For further details about this module, we refer to Kreuzer 1994.)

It is known that, for a locally Gorenstein scheme $\mathbb{X}$, one can embed the canonical module of R as a fractional ideal into its homogeneous ring of quotients (see Herzog et al. 1971). Subsequently, we need to make this construction explicit. The presentation follows the construction given in Kreuzer (2000), generalizing it to the case at hand.

In a first step, we want to describe the homogeneous ring of quotients $Q^h(R)$ of R. It is defined as the localization of R with respect to the set of all homogeneous non-zerodivisors of R. In view of Kreuzer (1994, Lemma 1.1), there is a homogeneous injection of degree zero

$$\widetilde{\imath} : R \to \widetilde{R} := \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j] \hookrightarrow \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}] \qquad (*)$$

given by $\widetilde{\imath}(f) = (f_{p_1} T_1^i, \ldots, f_{p_s} T_s^i)$, where $f \in R_i$ for $i \geq 0$, and where $T_1, \ldots, T_s$ are indeterminates with $\deg(T_1) = \cdots = \deg(T_s) = 1$. In particular, we have $\widetilde{\imath}(x_0) = (T_1, \ldots, T_s)$. Now we can describe $Q^h(R)$ as follows.

Proposition 3.1 *The map $\widetilde{\imath}$ extends to an isomorphism of graded R-modules*

$$\Psi : Q^h(R) \xrightarrow{\sim} \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}],$$

where for every element $f/g \in Q^h(R)$ with $f \in R_k$ and a non-zerodivisor $g \in R_l$ we have

$$\Psi(\frac{f}{g}) = \frac{\widetilde{\imath}(f)}{\widetilde{\imath}(g)} = \left(\frac{f_{p_1}}{g_{p_1}} T_1^{k-l}, \ldots, \frac{f_{p_s}}{g_{p_s}} T_s^{k-l} \right).$$

In particular, we have $Q^h(R) \cong R_{x_0}$.

Proof For a non-zerodivisor $g \in R_i$, the element $g_{p_j} \in \mathcal{O}_{\mathbb{X},p_j}$ is a unit element for all $j = 1, \ldots, s$ (see Kreuzer 1998, Lemma 1.5). Let $f/g \in Q^h(R)$ with $f \in R_k$ and a non-zerodivisor $g \in R_l$. Then $\widetilde{\imath}(f) = (f_{p_1} T_1^k, \ldots, f_{p_s} T_s^k)$ and $\widetilde{\imath}(g) = (g_{p_1} T_1^l, \ldots, g_{p_s} T_s^l)$, so we get

$$\Psi\left(\frac{f}{g}\right) = \frac{\widetilde{\imath}(f)}{\widetilde{\imath}(g)} = \left(\frac{f_{p_1}}{g_{p_1}} T_1^{k-l}, \ldots, \frac{f_{p_s}}{g_{p_s}} T_s^{k-l} \right) \in \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}].$$

Thus the map $\Psi : Q^h(R) \to \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]$ is well defined. It is clearly true that Ψ is R-linear, homogeneous of degree zero. If $\Psi(\frac{f}{g}) = 0$, then $\frac{f_{p_j}}{g_{p_j}} = 0 \in \mathcal{O}_{\mathbb{X},p_j}$

for all $j = 1, \ldots, s$. This implies $f_{p_j} = 0$ for all $j = 1, \ldots, s$, and so $f = 0$, since the map $\widetilde{\imath}$ is injective. Hence the map Ψ is an injection. Now we show that the map Ψ is surjective. Let $(g_1, \ldots, g_s) \in \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]$. For $i \gg 0$ we have $\dim_K(R_i) = \deg(\mathbb{X}) = \dim_K(\prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}])_i$. Thus, for $i \gg 0$, $(T_1^i g_1, \ldots, T_s^i g_s)$ is of the form $\Psi(f)$ for some $f \in R$. Therefore the element $(g_1, \ldots, g_s)$ is the image of f/x_0^i, and the claim follows. □

Recall that a **trace map** of a finite algebra T/S is a T-basis of the module $\mathrm{Hom}_S(T, S)$. The second task we want to tackle is the construction of a trace map for the algebra $Q^h(R)/K[x_0, x_0^{-1}]$. For this we need to restrict our attention to a special class of 0-dimensional schemes.

Definition 3.2 A 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}_K^n$ is called **locally Gorenstein** if at each point $p_j \in \mathrm{Supp}(\mathbb{X})$ the local ring $\mathcal{O}_{\mathbb{X},p_j}$ is a Gorenstein ring.

The next proposition says that in the locally Gorenstein case the desired trace map exists.

Proposition 3.3 *Let $\mathbb{X} \subseteq \mathbb{P}_K^n$ be a 0-dimensional locally Gorenstein scheme with Supp$(\mathbb{X}) = \{p_1, \ldots, p_s\}$, and let $L_0 = K[x_0, x_0^{-1}]$. Then the following statements hold true.*

(a) *The algebra $Q^h(R)/L_0$ has a homogeneous trace map σ of degree zero.*
(b) *The map $\Sigma : Q^h(R) \to \underline{\mathrm{Hom}}_{L_0}(Q^h(R), L_0)$ given by $\Sigma(1) = \sigma$ is an isomorphism of graded $Q^h(R)$-modules.*
(c) *A homogeneous element $\sigma' \in \underline{\mathrm{Hom}}_{L_0}(Q^h(R), L_0)$ is a trace map of the algebra $Q^h(R)/L_0$ if and only if there exists a unit $u \in Q^h(R)$ such that $\sigma' = u \cdot \sigma$.*

Proof According to Proposition 3.1, we may identify $Q^h(R) = \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]$. Then we have

$$\begin{aligned}\underline{\mathrm{Hom}}_{L_0}(Q^h(R), L_0) &= \underline{\mathrm{Hom}}_{L_0}\left(\prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}], L_0\right)\\ &\cong \prod_{j=1}^{s} \underline{\mathrm{Hom}}_{L_0}(\mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}], L_0)\\ &\cong \prod_{j=1}^{s} \underline{\mathrm{Hom}}_{L_0}(L_0 \otimes_K \mathcal{O}_{\mathbb{X},p_j}, L_0)\\ &\cong \prod_{j=1}^{s} L_0 \otimes_K \mathrm{Hom}_K(\mathcal{O}_{\mathbb{X},p_j}, K).\end{aligned}$$

Since $\mathbb{X}$ is locally Gorenstein, the algebra $\mathcal{O}_{\mathbb{X},p_j}/K$ is a finite Gorenstein algebra for every $j \in \{1, \ldots, s\}$. It then follows from Kunz (1986, E.16) that there is a trace map $\overline{\sigma}_j \in \mathrm{Hom}_K(\mathcal{O}_{\mathbb{X},p_j}, K)$ such that $\mathrm{Hom}_K(\mathcal{O}_{\mathbb{X},p_j}, K) = \mathcal{O}_{\mathbb{X},p_j} \cdot \overline{\sigma}_j$ for $j = 1, \ldots, s$. By Kunz (1986, F.16), the map $\sigma_j = \overline{\sigma}_j \otimes \mathrm{id}_{L_0} : \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}] \to K[T_j, T_j^{-1}] \cong L_0$ is a homogeneous trace map of degree zero of the algebra $\mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]/L_0$.

Hence the **trace system** $\sigma = (\sigma_1, \ldots, \sigma_s)$ satisfies $\underline{\mathrm{Hom}}_{L_0}(Q^h(R), L_0) = Q^h(R) \cdot \sigma$, and claim (a) follows.

Since (c) follows from (b), it remains to prove claim (b). To this end, we show that $\mathrm{Ann}_{Q^h(R)}(\sigma) = \langle 0 \rangle$. Assume that $f \cdot \sigma = 0$ for some homogeneous element $f \in Q^h(R)$. We have $f \cdot \sigma(g) = \sigma(fg) = g \cdot \sigma(f) = 0$ for all $g \in Q^h(R)$. This implies $\alpha(f) = 0$ for every $\alpha \in \underline{\mathrm{Hom}}_{L_0}(Q^h(R), L_0)$. Since the algebra $R/K[x_0]$ is free of rank $\deg(\mathbb{X})$ and $Q^h(R) \cong R_{x_0} \cong R \otimes_{K[x_0]} L_0$, it follows that the algebra $Q^h(R)/L_0$ is also free of rank $\deg(\mathbb{X})$. Let $d = \deg(\mathbb{X})$, let $\{b_1, \ldots, b_d\}$ be a L_0-basis of $Q^h(R)$, and let $\{b_1^*, \ldots, b_d^*\}$ be the dual basis of $\{b_1, \ldots, b_d\}$. We write $f = \sum_{j=1}^d g_j b_j \in Q^h(R)$ with $g_1, \ldots, g_d \in L_0$. Then $g_j = b_j^*(\sum_{j=1}^d g_j b_j) = b_j^*(f) = 0$ for all $j = 1, \ldots, d$. Hence we obtain $f = 0$, and so $\mathrm{Ann}_{Q^h(R)}(\sigma) = \langle 0 \rangle$. □

When the scheme $\mathbb{X}$ is smooth, the algebra $Q^h(R)/K[x_0, x_0^{-1}]$ has a **canonical trace map**.

Corollary 3.4 *Let $\mathbb{X} \subseteq \mathbb{P}_K^n$ be a 0-dimensional smooth scheme, let $Q(R)$ denote the full ring of quotients of R, and set $L_0 := K[x_0, x_0^{-1}]$.*

(a) *The canonical trace map $Tr_{Q^h(R)/L_0}$ is a homogeneous trace map of degree zero of $Q^h(R)/L_0$. In particular, $Q^h(R)/L_0$ is étale.*
(b) *The canonical trace map $Tr_{Q(R)/K(x_0)}$ is a trace map of $Q(R)/K(x_0)$. In particular, $Q(R)/K(x_0)$ is étale.*

Proof For $j = 1, \ldots, s$, the algebra $\mathcal{O}_{\mathbb{X}, p_j}/K$ is a finite separable field extension, and hence the canonical trace map $\mathrm{Tr}_{\mathcal{O}_{\mathbb{X}, p_j}/K}$ (see Kunz 1986, F.3) is a trace map of $\mathcal{O}_{\mathbb{X}, p_j}/K$. If we choose $\overline{\sigma}_j = \mathrm{Tr}_{\mathcal{O}_{\mathbb{X}, p_j}/K}$ for $j = 1, \ldots, s$ in the construction of the trace map σ in Proposition 3.3(a), then σ is exactly the canonical trace map $\mathrm{Tr}_{Q^h(R)/L_0}$. The additional claim of (a) follows by Kunz (1986, F.8).

For (b), it suffices to show that $\mathrm{Tr}_{Q(R)/K(x_0)}$ is a trace map of $Q(R)/K(x_0)$. Since $\mathbb{X}$ is reduced, it follows from Kunz (1985, III, Proposition 4.23) and Bourbaki (1989, V, §1, Proposition 9) that

$$Q(R) \cong \prod_{j=1}^{s} Q(R/\mathfrak{p}_j) \cong \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X}, p_j}(T_j).$$

As above, $\mathcal{O}_{\mathbb{X}, p_j}/K$ is a finite separable field extension, and so $\mathcal{O}_{\mathbb{X}, p_j}$ and $K(x_0)$ are linearly disjoint over K (cf. Morandi 1996, V, Section 20). This implies $\mathcal{O}_{\mathbb{X}, p_j} \otimes_K K(x_0) \cong \mathcal{O}_{\mathbb{X}, p_j} K(x_0) = \mathcal{O}_{\mathbb{X}, p_j}(x_0)$. By letting $\Gamma = \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X}, p_j}$, we have the isomorphism $Q(R) \cong K(x_0) \otimes_K \Gamma$. Notice that $\mathrm{Tr}_{\Gamma/K} = (\mathrm{Tr}_{\mathcal{O}_{\mathbb{X}, p_1}/K}, \ldots, \mathrm{Tr}_{\mathcal{O}_{\mathbb{X}, p_s}/K})$ is a trace map of Γ/K, and $\mathrm{Tr}_{Q(R)/K(x_0)} = \mathrm{id}_{K(x_0)} \otimes_K \mathrm{Tr}_{\Gamma/K}$ (cf. Kunz 1986, F.5). Therefore $\mathrm{Tr}_{Q(R)/K(x_0)}$ is a trace map of $Q(R)/K(x_0)$, as desired. □

Now we are ready to introduce the Dedekind complementary module and the Dedekind different for a 0-dimensional locally Gorenstein scheme as follows.

Definition 3.5 Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional locally Gorenstein scheme, let $L_0 = K[x_0, x_0^{-1}]$, let σ be a fixed trace map of $Q^h(R)/L_0$. Then there is an injective homomorphism of graded R-modules

$$\begin{aligned} \Phi : \omega_R(1) &\hookrightarrow \underline{\mathrm{Hom}}_{L_0}(Q^h(R), L_0) = Q^h(R) \cdot \sigma \xrightarrow{\Sigma^{-1}} Q^h(R) \\ \varphi &\longmapsto \varphi \otimes \mathrm{id}_{L_0} \end{aligned}$$

The image of Φ is a homogeneous fractional R-ideal $\mathfrak{C}^\sigma_{\mathbb{X}}$ of L, it is called the **Dedekind complementary module** of $\mathbb{X}$ (or of $R/K[x_0]$) with respect to σ. Its inverse,

$$\delta^\sigma_{\mathbb{X}} = (\mathfrak{C}^\sigma_{\mathbb{X}})^{-1} = \{\, f \in Q^h(R) \mid f \cdot \mathfrak{C}^\sigma_{\mathbb{X}} \subseteq R \,\},$$

is called the **Dedekind different** of $\mathbb{X}$ (or of $R/K[x_0]$) with respect to σ.

In the next remark we collect some basic properties of the Dedekind complementary module.

Remark 3.6 (a) It follows from the isomorphism $\mathfrak{C}^\sigma_{\mathbb{X}} \cong \omega_R(1)$ and Kreuzer (1994, Proposition 1.3) that the graded R-module $\mathfrak{C}^\sigma_{\mathbb{X}}$ is finitely generated and

$$\mathrm{HF}_{\mathfrak{C}^\sigma_{\mathbb{X}}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-i-1) \quad \text{for all } i \in \mathbb{Z}.$$

(b) A system of generators of $\mathfrak{C}^\sigma_{\mathbb{X}}$ can be computed as follows. Let $<_\tau$ be a degree-compatible term ordering on the set of terms $\mathbb{T}^n$ of $K[X_1, \dots, X_n]$, and let $d = \deg(\mathbb{X})$. Then $\mathbb{T}^n \backslash \mathrm{LT}_\tau(I^{\mathrm{deh}}_{\mathbb{X}}) = \{\, T_1, \dots, T_d \,\}$ with $T_j = X_1^{\alpha_{j1}} \cdots X_n^{\alpha_{jn}}$ and $\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn}) \in \mathbb{N}^n$ for $j = 1, \dots, d$. W.l.o.g. we assume that $T_1 <_\tau \cdots <_\tau T_m$. Let $t_j = T_j + I_{\mathbb{X}} \in R$ and set $\deg(t_j) := \deg(T_j) = n_j$ for $j = 1, \dots, d$. Then we have $n_1 \le \cdots \le n_d \le r_{\mathbb{X}}$ and the set $\{\, t_1, \dots, t_d \,\}$ is a $K[x_0]$-basis of R (cf. Kreuzer and Robbiano 2005, Theorem 4.3.22). Let $\{t_1^*, \dots, t_d^*\}$ be the dual basis of $\{t_1, \dots, t_d\}$, and let $g_j = \Phi(t_j^*)$ for $j = 1, \dots, d$. We get $\mathfrak{C}^\sigma_{\mathbb{X}} = \langle g_1, \dots, g_d \rangle_{K[x_0]} \subseteq Q^h(R)$.

(c) When $\mathbb{X}$ is smooth, we also denote the Dedekind complementary module (respectively, the Dedekind different) with respect to the canonical trace map by $\mathfrak{C}_{\mathbb{X}}$ (respectively, $\delta_{\mathbb{X}}$).

Further properties of the Dedekind different of $\mathbb{X}$ are given in our next proposition.

Proposition 3.7 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional locally Gorenstein scheme, and let σ be a trace map for $Q^h(R)/L_0$.*

(a) *The Dedekind different $\delta^\sigma_{\mathbb{X}}$ is a homogeneous ideal of R and $x_0^{2r_{\mathbb{X}}} \in \delta^\sigma_{\mathbb{X}}$.*

(b) *The Hilbert function of $\delta^\sigma_{\mathbb{X}}$ satisfies $HF_{\delta^\sigma_{\mathbb{X}}}(i) = 0$ for $i < 0$, $HF_{\delta^\sigma_{\mathbb{X}}}(i) = deg(\mathbb{X})$ for $i \ge 2r_{\mathbb{X}}$, and*

$$0 \le HF_{\delta^\sigma_{\mathbb{X}}}(0) \le \cdots \le HF_{\delta^\sigma_{\mathbb{X}}}(2r_{\mathbb{X}}) = deg(\mathbb{X}).$$

(c) *The regularity index of $\delta^\sigma_{\mathbb{X}}$ satisfies $r_{\mathbb{X}} \le ri(\delta^\sigma_{\mathbb{X}}) \le 2r_{\mathbb{X}}$.*

Proof By Remark 3.6(a), we have $\mathrm{HF}_{\mathfrak{C}^\sigma_{\mathbb{X}}}(0) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-1) = \deg(\mathbb{X}) = \mathrm{HF}_{Q^h(R)}(0)$. This implies $R_0 \subseteq (\mathfrak{C}^\sigma_{\mathbb{X}})_0 = Q^h(R)_0$, and in particular, $1 \in \mathfrak{C}^\sigma_{\mathbb{X}}$. Hence $\delta^\sigma_{\mathbb{X}}$ is a homogeneous ideal of R.

Now let us write $\mathfrak{C}^\sigma_{\mathbb{X}} = \langle g_1, \ldots, g_d\rangle_{K[x_0]} \subseteq Q^h(R)$ as in Remark 3.6(b). Here g_j is homogeneous of degree $\deg(g_j) = -n_j$ (since Φ is homogeneous of degree zero). We claim that, for $j \in \{1, \ldots, d\}$, there is a homogeneous element $g'_j \in R_{r_{\mathbb{X}}}$ such that $g_j = x_0^{-r_{\mathbb{X}}-n_j} g'_j \in \mathfrak{C}^\sigma_{\mathbb{X}}$. Indeed, since $g_j \in Q^h(R) \cong R_{x_0}$, there exist $g''_j \in R$ and $d_j \in \mathbb{N}$ such that $g_j = x_0^{-d_j} g''_j$. If $\deg(g''_j) = d_j - n_j \le r_{\mathbb{X}}$, then we set $g'_j = x_0^{r_{\mathbb{X}}-d_j+n_j} g''_j \in R_{r_{\mathbb{X}}}$. If $\deg(g''_j) = d_j - n_j > r_{\mathbb{X}}$, then we write $g''_j = x_0^{d_j-n_j-r_{\mathbb{X}}} g'_j$ for some $g'_j \in R_{r_{\mathbb{X}}}$, since $R_i = x_0^{i-r_{\mathbb{X}}} R_{r_{\mathbb{X}}}$ for all $i \ge r_{\mathbb{X}}$. Thus we get $g_j = x_0^{-r_{\mathbb{X}}-n_j} g'_j$, as claimed. Consequently, we have $\mathfrak{C}^\sigma_{\mathbb{X}} = \langle x_0^{-r_{\mathbb{X}}-n_1} g'_1, \ldots, x_0^{-r_{\mathbb{X}}-n_m} g'_m\rangle_R$. Now it is easy to see that $x_0^{2r_{\mathbb{X}}} \in \delta^\sigma_{\mathbb{X}}$, since $n_j \le r_{\mathbb{X}}$ and $x_0^{2r_{\mathbb{X}}} \cdot (x_0^{-r_{\mathbb{X}}-n_j} g'_j) = x_0^{r_{\mathbb{X}}-n_j} g'_j \in R_{2r_{\mathbb{X}}-n_j}$ for all $j = 1, \ldots, d$. Hence claim (a) follows.

Next we shall prove claim (b). It is clear that $\mathrm{HF}_{\delta^\sigma_{\mathbb{X}}}(i) = 0$ for $i < 0$ and $\mathrm{HF}_{\delta^\sigma_{\mathbb{X}}}(i) \le \mathrm{HF}_{\delta^\sigma_{\mathbb{X}}}(i+1)$ for all $i \in \mathbb{Z}$, since $\delta^\sigma_{\mathbb{X}}$ is a homogeneous ideal of R by (a). Notice that $\mathrm{HF}_{\mathbb{X}}(i) = d = \deg(\mathbb{X})$ for all $i \ge r_{\mathbb{X}}$ and $\mathrm{HF}_{\delta^\sigma_{\mathbb{X}}}(i) \le \mathrm{HF}_{\mathbb{X}}(i)$ for all $i \in \mathbb{Z}$. So, the Hilbert function of $\delta^\sigma_{\mathbb{X}}$ satisfies $\mathrm{HF}_{\delta^\sigma_{\mathbb{X}}}(i) \le d$ for all $i \in \mathbb{Z}$. We write $\mathfrak{C}^\sigma_{\mathbb{X}} = \langle x_0^{-r_{\mathbb{X}}-n_1} g'_1, \ldots, x_0^{-r_{\mathbb{X}}-n_d} g'_d\rangle_R$ with $g'_1, \ldots, g'_d \in R_{r_{\mathbb{X}}}$ as above, and let $\{f_1, \ldots, f_d\}$ be a K-basis of $R_{r_{\mathbb{X}}}$. Then $f_i g'_j \in R_{2r_{\mathbb{X}}}$. There is $\widetilde{f}_{ij} \in R_{r_{\mathbb{X}}}$ such that $f_i g'_j = x_0^{r_{\mathbb{X}}} \widetilde{f}_{ij}$ for all $i, j \in \{1, \ldots, d\}$. Thus $(x_0^{r_{\mathbb{X}}} f_i)\cdot(x_0^{-r_{\mathbb{X}}-n_j} g'_j) = x_0^{-n_j} f_i g'_j = x_0^{r_{\mathbb{X}}-n_j} \widetilde{f}_{ij} \in R_{2r_{\mathbb{X}}-n_j}$ for all i, j. It follows that $\{x_0^{r_{\mathbb{X}}} f_1, \ldots, x_0^{r_{\mathbb{X}}} f_d\} \subseteq (\delta^\sigma_{\mathbb{X}})_{2r_{\mathbb{X}}} \subseteq R_{2r_{\mathbb{X}}}$. On the other hand, we see that

$$d = \mathrm{HF}_{\mathbb{X}}(2r_{\mathbb{X}}) = \dim_K \langle x_0^{r_{\mathbb{X}}} f_1, \ldots, x_0^{r_{\mathbb{X}}} f_d\rangle_K \le \mathrm{HF}_{\delta^\sigma_{\mathbb{X}}}(2r_{\mathbb{X}}) \le \mathrm{HF}_{\mathbb{X}}(2r_{\mathbb{X}}) = d.$$

Therefore we obtain the equalities $\mathrm{HF}_{\delta^\sigma_{\mathbb{X}}}(i) = d = \deg(\mathbb{X})$ for all $i \ge 2r_{\mathbb{X}}$.

Finally, claim (c) is an immediate consequence of the claims (a) and (b). □

In the last part of this section we present some relations between the Kähler different and the Dedekind different of a locally Gorenstein 0-dimensional scheme $\mathbb{X}$.

Proposition 3.8 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional smooth scheme. Then we have*

$$\delta^n_{\mathbb{X}} \subseteq \vartheta_{\mathbb{X}} \subseteq \delta_{\mathbb{X}}.$$

Proof Notice that the algebra $R/K[x_0]$ is free of rank $\deg(\mathbb{X})$, so it is flat. By Corollary 3.4, we know that the algebra $Q^h(R)/K[x_0, x_0^{-1}]$ is étale. Then Kunz (1986, G.11) yields $\delta_{\mathbb{X}} = \delta_N$, where δ_N is the **Noether different** of $R/K[x_0]$ (as defined in Kunz 1986, G.1). Thus the desired chain of inclusions follows from Kunz (1986, Proposition 10.18). □

Let us point out that, if $\mathbb{X}$ is a smooth complete intersection, then Proposition 5.2 below and Kunz (1986, Proposition 10.17) show that $\vartheta_{\mathbb{X}} = \delta_{\mathbb{X}}$. Moreover, the above inclusions can be false if the scheme $\mathbb{X}$ is not a smooth scheme. The following example implies that, in general, the Kähler and Dedekind differents do not agree even when $\mathbb{X}$ is a complete intersection.

Example 3.9 Let $\mathbb{X}$ be the 0-dimensional complete intersection given in Example 2.9. We know already that $\mathbb{X}$ is not smooth at p_5 and p_7 and that the Kähler different is a principal ideal generated by a non-zero homogeneous element of degree $r_{\mathbb{X}}$. In this case the canonical trace map $\mathrm{Tr}_{\mathcal{O}_{\mathbb{X},p_j}/K} = \mathrm{id}_K$ is a trace map of $\mathcal{O}_{\mathbb{X},p_j}/K$ for $j \neq 5, 7$. Moreover, we observe that $\mathcal{O}_{\mathbb{X},p_5} \cong K[X_1, X_2]/\langle X_1 - 2, X_2^2\rangle = K \oplus x_2 K$, and so $\{1, x_2\}$ is a K-basis of $\mathcal{O}_{\mathbb{X},p_5}$. In particular, $\overline{\sigma}_5 : \mathcal{O}_{\mathbb{X},p_5} \to K$ given by $\overline{\sigma}_5(x_2) = 1$ and $\overline{\sigma}_5(1) = 0$ is a trace map of $\mathcal{O}_{\mathbb{X},p_5}/K$. The trace map $\overline{\sigma}_7 : \mathcal{O}_{\mathbb{X},p_7} \to K$ of $\mathcal{O}_{\mathbb{X},p_7}/K$ can be found in the same way. Using these trace maps, we get a trace system $\sigma : Q^h(R) \to K[x_0, x_0^{-1}]$ of the algebra $Q^h(R)/K[x_0, x_0^{-1}]$. Now we compute the Dedekind different and get

$$\delta^{\sigma}_{\mathbb{X}} = \langle x_1^4 - 4x_0x_1^2x_2 - 32x_0^2x_2^2 + 2x_1^2x_2^2 - 4x_0x_2^3 + 12x_2^4\rangle.$$

Note that the homogeneous element $x_1^4 - 4x_0x_1^2x_2 - 32x_0^2x_2^2 + 2x_1^2x_2^2 - 4x_0x_2^3 + 12x_2^4$ is a non-zerodivisor of R. So, we have $\mathrm{HP}_{\vartheta_{\mathbb{X}}} = 7 < 9 = \mathrm{HP}_{\delta^{\sigma}_{\mathbb{X}}} = \mathrm{HP}_{(\delta^{\sigma}_{\mathbb{X}})^n}$. Hence we get $(\delta^{\sigma}_{\mathbb{X}})^n \nsubseteq \vartheta_{\mathbb{X}}$ and $\vartheta_{\mathbb{X}} \nsubseteq \delta^{\sigma}_{\mathbb{X}}$.

The last inclusion in the proposition can be a strict inclusion for 0-dimensional arithmetically Gorenstein schemes, as our next example demonstrates.

Example 3.10 Let $\mathbb{X} = \{p_1, \ldots, p_5\} \subseteq \mathbb{P}^3_{\mathbb{F}_7}$ be the set of five distinct K-rational points given in Example 2.12. We know that $\mathbb{X}$ is arithmetically Gorenstein, but not a complete intersection. We also have $\delta_{\mathbb{X}} = \langle x_0^2 - 2x_0x_3 - 3x_1x_3 - 2x_3^2\rangle$ and $\vartheta_{\mathbb{X}} = \langle x_2x_3^2 - 3x_3^3, x_1x_3^2, x_0x_3^2 - 3x_3^3, x_0^3\rangle$. Thus we get $\vartheta_{\mathbb{X}} \subsetneq \delta_{\mathbb{X}}$.

Finally, we relate the Hilbert functions of the Kähler different and the Dedekind different of a smooth scheme $\mathbb{X}$ as follows.

Corollary 3.11 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional smooth scheme. Then we have $HP_{\vartheta_{\mathbb{X}}} = HP_{\delta_{\mathbb{X}}} = deg(\mathbb{X})$ and $r_{\mathbb{X}} \leq ri(\vartheta_{\mathbb{X}}) \leq (n+1)r_{\mathbb{X}}$.*

Proof The equalities of Hilbert polynomials follow from Propositions 3.7 and 3.8. Also, it is clear that $r_{\mathbb{X}} \leq \mathrm{ri}(\vartheta_{\mathbb{X}})$. Since $\mathbb{X}$ is smooth, we can argue similarly as in the proof of Proposition 2.15 to get a homogeneous element $h \in (\vartheta_{\mathbb{X}})_m \setminus \bigcup_{j=1}^{s} (\mathfrak{p}_j)_m$ for some $m \geq 0$, where $\mathfrak{p}_j$ is the homogeneous prime ideal of R corresponding to $p_j \in \mathrm{Supp}(\mathbb{X})$. So, h is a non-zerodivisor of R. According to Geramita and Maroscia (1984, Proposition 1.1), we can find a minimal system $\{F_1, \ldots, F_r\}$ of generators of $I_{\mathbb{X}}$ such that $\deg(F_j) \leq r_{\mathbb{X}} + 1$ for all $j = 1, \ldots, r$. Hence $\vartheta_{\mathbb{X}}$ is generated in degree $\leq nr_{\mathbb{X}}$. If $m > nr_{\mathbb{X}}$ and $\langle(\vartheta_{\mathbb{X}})_{nr_{\mathbb{X}}}\rangle_R \subseteq \bigcup_{j=1}^{s} \mathfrak{p}_j$, then $\langle(\vartheta_{\mathbb{X}})_{nr_{\mathbb{X}}}\rangle_R \subseteq \mathfrak{p}_j$ for some $j \in \{1, \ldots, s\}$, and hence the element h cannot exist. Thus h can be chosen such that $\deg(h) = m \leq nr_{\mathbb{X}}$. Moreover, if $\{f_1, \ldots, f_{\deg(\mathbb{X})}\}$ is a K-basis of $R_{r_{\mathbb{X}}}$, then $\{hf_1, \ldots, hf_{\deg(\mathbb{X})}\}$ is a K-basis of $(\vartheta_{\mathbb{X}})_{r_{\mathbb{X}}+m}$. Therefore we have $\mathrm{ri}(\vartheta_{\mathbb{X}}) \leq (n+1)r_{\mathbb{X}}$. □

4 Cayley–Bacharach Schemes

In many previous characterizations of 0-dimensional complete intersections, a particular geometric condition has played a leading role: the Cayley–Bacharach property. In this section we define and study the notion of a Cayley–Bacharach scheme in a substantially more general context than has been done hitherto. We work with an arbitrary 0-dimensional subscheme $\mathbb{X}$ of a projective space $\mathbb{P}^n_K$ over an arbitrary base field K. As in the previous sections, let the support of $\mathbb{X}$ be given by $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$. We denote the residue field of $\mathbb{X}$ at p_j by $K(p_j) = \mathcal{O}_{\mathbb{X},p_j}/\mathfrak{m}_{\mathbb{X},p_j}$, and set $\varkappa_j = \dim_K K(p_j)$.

Definition 4.1 A subscheme $\mathbb{Y} \subseteq \mathbb{X}$ is called a p_j-**subscheme** if the following conditions are satisfied:

(a) $\mathcal{O}_{\mathbb{Y},p_k} = \mathcal{O}_{\mathbb{X},p_k}$ for $k \neq j$.
(b) The map $\mathcal{O}_{\mathbb{X},p_j} \twoheadrightarrow \mathcal{O}_{\mathbb{Y},p_j}$ is an epimorphism.

A p_j-subscheme $\mathbb{Y} \subseteq \mathbb{X}$ is called **maximal** if $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - \varkappa_j$.

Notice that a maximal p_j-subscheme of a 0-dimensional scheme $\mathbb{X}$ in $\mathbb{P}^n_K$ which has K-rational support, i.e., for which all closed points of $\mathbb{X}$ are K-rational, is nothing but a subscheme $\mathbb{Y} \subseteq \mathbb{X}$ of degree $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$ with $\mathcal{O}_{\mathbb{Y},p_j} \neq \mathcal{O}_{\mathbb{X},p_j}$.

Proposition 4.2 *Let* $\Gamma = \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}$, *and let* $\mathfrak{S}(\mathcal{O}_{\mathbb{X},p_j}) = \mathrm{Ann}_{\mathcal{O}_{\mathbb{X},p_j}}(\mathfrak{m}_{\mathbb{X},p_j})$ *be the socle of* $\mathcal{O}_{\mathbb{X},p_j}$. *There is a 1-1 correspondence*

$$\left\{ \begin{array}{l} \textit{maximal } p_j - \textit{subschemes} \\ \textit{of the scheme} \mathbb{X} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{ideals } \langle (0, \dots, 0, s_j, 0, \dots, 0) \rangle_\Gamma \subseteq \Gamma \\ \textit{with } s_j \in \mathfrak{S}(\mathcal{O}_{\mathbb{X},p_j}) \setminus \{0\} \end{array} \right\}$$

Proof Let $\mathbb{Y} \subseteq \mathbb{X}$ be a maximal p_j-subscheme, let $I_{\mathbb{Y}/\mathbb{X}}$ denote the saturated ideal of $\mathbb{Y}$ in R, and let $\alpha_{\mathbb{Y}/\mathbb{X}} = \min\{i \in \mathbb{N} \mid (I_{\mathbb{Y}/\mathbb{X}})_i \neq \langle 0 \rangle\}$. Furthermore, we let $\{e_{j1}, \dots, e_{j\varkappa_j}\} \subseteq \mathcal{O}_{\mathbb{X},p_j}$ be such that their residue classes form a K-basis of $K(p_j)$, and let $f_{\mathbb{Y}} \in (I_{\mathbb{Y}/\mathbb{X}})_{\alpha_{\mathbb{Y}/\mathbb{X}}} \setminus \{0\}$. Since $R_i = x_0^{i - r_{\mathbb{X}}} R_{r_{\mathbb{X}}}$ for $i \geq r_{\mathbb{X}}$, we get from Kreuzer (1994, Lemma 1.2) that $\alpha_{\mathbb{Y}/\mathbb{X}} \leq r_{\mathbb{X}}$. Using the map $\widetilde{\imath}$ given by $(*)$, we write $\widetilde{\imath}(f_{\mathbb{Y}}) = (0, \dots, 0, s_j T_j^{\alpha_{\mathbb{Y}/\mathbb{X}}}, 0, \dots, 0) \in \widetilde{R}$. Clearly, we have $s_j \neq 0$.

We claim that $s_j \in \mathfrak{S}(\mathcal{O}_{\mathbb{X},p_j})$. Indeed, if otherwise, then there is an element $a \in \mathfrak{m}_{\mathbb{X},p_j}$ such that $as_j \neq 0$. Suppose there are $c_1, \dots, c_{\varkappa_j+1} \in K$ such that

$$c_1 e_{j1} s_j + \dots + c_{\varkappa_j} e_{j\varkappa_j} s_j + c_{\varkappa_j+1} a s_j = (c_1 e_{j1} + \dots + c_{\varkappa_j} e_{j\varkappa_j} + c_{\varkappa_j+1} a) s_j = 0.$$

If $c_1 \overline{e}_{j1} + \dots + c_{\varkappa_j} \overline{e}_{j\varkappa_j} \neq 0$ in $K(p_j)$, then $c_1 e_{j1} + \dots + c_{\varkappa_j} e_{j\varkappa_j}$ is a unit element, so is $c_1 e_{j1} + \dots + c_{\varkappa_j} e_{j\varkappa_j} + c_{\varkappa_j+1} a$ (as $a \in \mathfrak{m}_{\mathbb{X},p_j}$). It follows from the above equality that $s_j = 0$, it is impossible. So, we must have $c_1 \overline{e}_{j1} + \dots + c_{\varkappa_j} \overline{e}_{j\varkappa_j} = 0$. This implies $c_1 = \dots = c_{\varkappa_j} = 0$, since $\{\overline{e}_{j1}, \dots, \overline{e}_{j\varkappa_j}\}$ is a K-basis of $K(p_j)$. From this we deduce $c_{\varkappa_j+1} a s_j = 0$, hence $c_{\varkappa_j+1} = 0$ (as $as_j \neq 0$). Therefore the set $\{e_{j1} s_j, \dots, e_{j\varkappa_j} s_j, as_j\}$ is K-linearly independent. Let

$$f_a = \widetilde{\imath}^{-1}((0, \dots, 0, a T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \quad \text{and} \quad f_{as_j} = \widetilde{\imath}^{-1}((0, \dots, 0, as_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)).$$

Then $f_a, f_{as_j} \in R_{r_{\mathbb{X}}}$ and $x_0^{\alpha_{\mathbb{Y}/\mathbb{X}}} f_{as_j} = f_a f_{\mathbb{Y}}$, and so Kreuzer (1994, Lemma 1.2) yields $f_{as_j} \in \langle f_{\mathbb{Y}} \rangle_R^{\mathrm{sat}} \subseteq I_{\mathbb{Y}/\mathbb{X}}$. Similarly, we have

$$f_{jk_j} = \widetilde{\imath}^{-1}((0, \dots, 0, e_{jk_j} s_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in (I_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}}$$

for $k_j = 1, \dots, \varkappa_j$. Thus we get

$$\dim_K (I_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \geq \dim_K \left\langle f_{j1}, \dots, f_{j\varkappa_j}, f_{as_j} \right\rangle_K = \varkappa_j + 1$$

and hence $\deg(\mathbb{Y}) < \deg(\mathbb{X}) - \varkappa_j$, a contradiction.

Next we consider $f \in (I_{\mathbb{Y}/\mathbb{X}})_i \setminus \{0\}$ with $i \geq \alpha_{\mathbb{Y}/\mathbb{X}}$. The previous claim also tells us that $f_{p_j} \in \mathfrak{G}(\mathcal{O}_{\mathbb{X},p_j}) \setminus \{0\}$. If $f_{p_j} \in \mathfrak{G}(\mathcal{O}_{\mathbb{X},p_j}) \setminus \langle s_j \rangle_{\mathcal{O}_{\mathbb{X},p_j}}$, then it is not difficult to check that $\{ f_{p_j}, e_{j1}s_j, \dots, e_{j\varkappa_j}s_j \}$ is K-linearly independent. This implies $\deg(\mathbb{Y}) < \deg(\mathbb{X}) - \varkappa_j$, and it is impossible. Hence we have $f_{p_j} \in \langle s_j \rangle_{\mathcal{O}_{\mathbb{X},p_j}}$.

Let $g \in R_i \setminus \{0\}$ with $i \geq \alpha_{\mathbb{Y}/\mathbb{X}}$ be such that $\widetilde{\imath}(g) = (0, \dots, 0, g_{p_j} T_j^i, 0, \dots, 0)$ and $g_{p_j} \in \langle s_j \rangle_{\mathcal{O}_{\mathbb{X},p_j}}$. We are able to write $g_{p_j} = as_j$ for some $a \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{\mathbb{X},p_j}$. Using a similar argument as the previous part we get $g \in \langle f_{\mathbb{Y}} \rangle_R^{\mathrm{sat}} \subseteq I_{\mathbb{Y}/\mathbb{X}}$. Therefore the image of $I_{\mathbb{Y}/\mathbb{X}}$ in $\Gamma = \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}$ is $\langle (0, \dots, 0, s_j, 0, \dots, 0) \rangle_\Gamma$ with $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X},p_j}) \setminus \{0\}$, as was to be shown.

Conversely, let $(0, \dots, 0, s_j, 0, \dots, 0) \in \Gamma$ with $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X},p_j}) \setminus \{0\}$, and let $f = \widetilde{\imath}^{-1}((0, \dots, 0, s_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$. We set $\mathbb{Y} := \mathcal{Z}(f) \subseteq \mathbb{X}$. Then we have $I_{\mathbb{Y}/\mathbb{X}} = \langle f \rangle_R^{\mathrm{sat}}$. Obviously, the scheme $\mathbb{Y}$ is a p_j-subscheme of $\mathbb{X}$. It suffices to prove $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - \varkappa_j$. Let $f_{jk_j} = \widetilde{\imath}^{-1}((0, \dots, 0, e_{jk_j} s_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$ and $g_{jk_j} = \widetilde{\imath}^{-1}((0, \dots, 0, e_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$ for $k_j = 1, \dots, \varkappa_j$. We see that $x_0^{r_{\mathbb{X}}} f_{jk_j} = g_{jk_j} f$ for every $k_j \in \{1, \dots, \varkappa_j\}$, and so Kreuzer (1994, Lemma 1.2) implies $f_{jk_j} \in (I_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}}$. Thus we get the inequality

$$\dim_K (I_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \geq \dim_K \langle f_{j1}, \dots, f_{j\varkappa_j} \rangle_K = \varkappa_j.$$

Moreover, for $h \in (I_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$, there is a number $m \in \mathbb{N}$ such that $x_0^m h \in \langle f \rangle_R$. This clearly forces $h_{p_j} = as_j$ for some $a \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{\mathbb{X},p_j}$ and $h_{p_k} = 0$ for $k \neq j$. Let us write $as_j = \sum_{k_j=1}^{\varkappa_j} c_{jk_j} e_{jk_j} s_j$ for some $c_{j1}, \dots, c_{j\varkappa_j} \in K$. Then $\widetilde{\imath}(h) = \widetilde{\imath}(\sum_{k_j=1}^{\varkappa_j} c_{jk_j} f_{jk_j})$. Since the map $\widetilde{\imath}$ is injective, we have $h = \sum_{k_j=1}^{\varkappa_j} c_{jk_j} f_{jk_j} \in \langle f_{j1}, \dots, f_{j\varkappa_j} \rangle_K$. This implies $\dim_K (I_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \leq \varkappa_j$, and therefore this inequality becomes an equality. Hence we obtain $\dim_K (I_{\mathbb{Y}/\mathbb{X}})_{i+r_{\mathbb{X}}} = \varkappa_j$ for $i \geq 0$ or $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - \varkappa_j$, as desired. □

Corollary 4.3 *A* 0*-dimensional scheme* $\mathbb{X} \subseteq \mathbb{P}_K^n$ *contains a subscheme of degree* $\deg(\mathbb{X}) - 1$ *if and only if it has a* K*-rational point.*

Proof Due to Proposition 4.2, it suffices to show that if $\mathbb{Y} \subseteq \mathbb{X}$ is a subscheme of degree $\deg(\mathbb{X}) - 1$ with $\mathcal{O}_{\mathbb{Y},p_j} \neq \mathcal{O}_{\mathbb{X},p_j}$, then p_j is a K-rational point. Suppose that p_j is not K-rational, i.e., $\dim_K K(p_j) = \varkappa_j \geq 2$. We may write $K(p_j) = K \oplus K v_1 \oplus \cdots \oplus K v_{\varkappa_j - 1}$, where $\{1, v_1, \dots, v_{\varkappa_j-1}\}$ is a K-basis of $K(p_j)$. Let $f \in$

$(I_{\mathbb{Y}/\mathbb{X}})_{r_\mathbb{X}}\setminus\{0\}$. Then $\widetilde{\imath}(f) = (0,\dots,0,s_jT^{r_\mathbb{X}},0,\dots,0)$ for some $s_j \in \mathcal{O}_{\mathbb{X},p_j}\setminus\{0\}$. It is not difficult to check that s_j, v_1s_j are K-linear independent. By setting $g := \widetilde{\imath}^{-1}((0,\dots,0,v_1s_jT^{r_\mathbb{X}},0,\dots,0)) \in R_{r_\mathbb{X}}$, we have $g \in I_{\mathbb{Y}/\mathbb{X}}$ and $\dim_K\langle f,g\rangle_K = 2$. Hence we get $2 = \dim_K\langle f,g\rangle_K \le \dim_K(I_{\mathbb{Y}/\mathbb{X}})_{r_\mathbb{X}} = 1$, a contradiction. □

Let $\mathbb{Y}$ be a maximal p_j-subscheme of $\mathbb{X}$, and let $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X},p_j})\setminus\{0\}$ be a socle element corresponding to a non-zero element of $(I_{\mathbb{Y}/\mathbb{X}})_{\alpha_{\mathbb{Y}/\mathbb{X}}}$. We also say that s_j is a socle element of $\mathcal{O}_{\mathbb{X},p_j}$ corresponding to $\mathbb{Y}$. Let $\{e_{j1},\dots,e_{j\varkappa_j}\} \subseteq \mathcal{O}_{\mathbb{X},p_j}$ be elements whose residue classes form a K-basis of $K(p_j)$. For $a \in \mathcal{O}_{\mathbb{X},p_j}$, we set

$$\mu(a) := \min\{\, i \in \mathbb{N} \mid (0,\dots,0,aT_j^i,0,\dots,0) \in \widetilde{\imath}(R)\,\}$$

and

$$\mu_{\mathbb{Y}/\mathbb{X}} := \max\{\, \mu(e_{jk_j}s_j) \mid k_j = 1,\dots,\varkappa_j\,\}.$$

Lemma 4.4 *The number $\mu_{\mathbb{Y}/\mathbb{X}}$ depends neither on the choice of the socle element s_j nor on the specific choice of $\{e_{j1},\dots,e_{j\varkappa_j}\}$.*

Proof First we show that $\mu(a+b) \le \max\{\,\mu(a),\mu(b)\,\}$ for all $a,b \in \mathcal{O}_{\mathbb{X},p_j}$. It suffices to consider the case $\mu(a) \le \mu(b)$. Let

$$f = \widetilde{\imath}^{-1}((0,\dots,0,aT_j^{\mu(a)},0,\dots,0)) \text{ and } g = \widetilde{\imath}^{-1}((0,\dots,0,bT_j^{\mu(b)},0,\dots,0)).$$

Then we have $f,g \in R$ and

$$\widetilde{\imath}(x_0^{\mu(b)-\mu(a)}f + g) = (0,\dots,0,(a+b)T_j^{\mu(b)},0,\dots,0).$$

It follows that $\mu(a+b) \le \mu(b)$.

Now let $s'_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X},p_j})\setminus\{0\}$ such that $s'_j = as_j$ for some unit $a \in \mathcal{O}_{\mathbb{X},p_j}$. Note that $s_j = a^{-1}s'_j$. We set $d_j := \max\{\,\mu(e_{jk_j}s_j) \mid k_j = 1,\dots,\varkappa_j\,\}$ and $d'_j := \max\{\,\mu(e_{jk_j}s'_j) \mid k_j = 1,\dots,\varkappa_j\,\}$. We want to prove the equality $d_j = d'_j$. By symmetry, we need only show that $d'_j \le d_j$. For $l \in \{1,\dots,\varkappa_j\}$, we write $e_{jl}a = a^l_{j1}e_{j1} + \cdots + a^l_{j\varkappa_j}e_{j\varkappa_j} \mod \mathfrak{m}_{\mathbb{X},p_j}$ for $a^l_{j1},\dots,a^l_{j\varkappa_j} \in K$, not all equal to zero. We deduce $e_{jl}s'_j = e_{jl}as_j = a^l_{j1}e_{j1}s_j + \cdots + a^l_{j\varkappa_j}e_{j\varkappa_j}s_j$. Thus we have

$$\begin{aligned}\mu(e_{jl}s'_j) &= \mu(a^l_{j1}e_{j1}s_j + \cdots + a^l_{j\varkappa_j}e_{j\varkappa_j}s_j)\\ &\le \max\{\,\mu(a^l_{jk_j}e_{jk_j}s_j) \mid k_j = 1,\dots,\varkappa_j\,\}\\ &= \max\{\,\mu(e_{jk_j}s_j) \mid k_j = 1,\dots,\varkappa_j\,\} = d_j.\end{aligned}$$

Therefore we obtain $d'_j = \max\{\,\mu(e_{jk_j}s'_j) \mid k_j = 1,\dots,\varkappa_j\,\} \le d_j$.

Finally, let $\{e'_{j1},\dots,e'_{j\varkappa_j}\} \subseteq \mathcal{O}_{\mathbb{X},p_j}$ be another set whose residue classes form a K-basis of $K(p_j)$, and let $d''_j := \max\{\,\mu(e'_{jk_j}s_j) \mid k_j = 1,\dots,\varkappa_j\,\}$. We can argue similarly as above to get $\mu(e'_{jk_j}s_j) \le d_j$ for $k_j \in \{1, \dots, \varkappa_j\}$. This implies $d''_j \le d_j$, and hence $d''_j = d_j$. □

Using this lemma, we can now generalize the definition of the degree of a point in $\mathbb{X}$ as follows.

Definition 4.5 Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme. For $p_j \in \mathrm{Supp}(\mathbb{X})$, the **degree of** p_j **in** $\mathbb{X}$ is defined as

$$\deg_{\mathbb{X}}(p_j) := \min\{\, \mu_{\mathbb{Y}/\mathbb{X}} \mid \mathbb{Y} \text{ is a maximal } p_j\text{--subscheme of } \mathbb{X}\,\}.$$

Let us check that this definition agrees with the usual one in the case of a scheme $\mathbb{X}$ with K-rational support, and thus for instance in the case of an algebraically closed base field K.

Remark 4.6 Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme with $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$.

(a) If $\mathbb{X}$ has K-rational support then $\varkappa_1 = \cdots = \varkappa_s = 1$. In this case we have

$$\deg_{\mathbb{X}}(p_j) = \min\{\, \alpha_{\mathbb{Y}}/\mathbb{X} \mid \mathbb{Y} \text{ is a maximal } p_j\text{--subscheme of } \mathbb{X}\,\}.$$

If, in addition, $\mathbb{X}$ is reduced, then

$$\deg_{\mathbb{X}}(p_j) = \alpha_{\mathbb{X}\setminus\{p_j\}/\mathbb{X}} = \min\{i \in \mathbb{N} \mid (I_{\mathbb{X}\setminus\{p_j\}/\mathbb{X}})_i \ne \langle 0\rangle\}$$

for all $j = 1, \dots, s$. In other words, the degree $\deg_{\mathbb{X}}(p_j)$ agrees with the degree of a point in the sense of Geramita et al. (1993, Definition 2.1).

(b) We have $0 \le \deg_{\mathbb{X}}(p_j) \le r_{\mathbb{X}}$ for all $j = 1, \dots, s$. In particular, if $\mathbb{X}$ is a reduced scheme which has K-rational support, then there always exists a point $p_j \in \mathbb{X}$ with maximal degree $\deg_{\mathbb{X}}(p_j) = r_{\mathbb{X}}$ (cf. Geramita et al. 1993, Proposition 1.14).

In case all points of $\mathrm{Supp}(\mathbb{X})$ have the maximum possible degree $r_{\mathbb{X}}$, we have the following notion.

Definition 4.7 A 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^n_K$ is called a **Cayley–Bacharach scheme** (in short, **CB-scheme**) if every point $p_j \in \mathrm{Supp}(\mathbb{X})$ has the maximum possible degree $\deg_{\mathbb{X}}(p_j) = r_{\mathbb{X}}$.

First of all, we give an example which shows that a 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^n_K$ which does not have K-rational support can be a CB-scheme.

Example 4.8 Let $K = \mathbb{Q}$, and let $\mathbb{X}$ be the 0-dimensional subscheme of $\mathbb{P}^2_K$ of degree 14 with support $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_{12}\}$, where $p_1 = (1:0:0)$, $p_2 = (1:1:0)$, $p_3 = (1:1:1)$, $p_4 = (1:0:1)$, $p_5 = (1:-1:1)$, $p_6 = (1:1:-1)$, $p_7 = (1:0:-1)$, $p_8 = (1:2:0)$, $p_9 = (1:2:1)$, $p_{10} = (1:2:-1)$, p_{11} corresponds to $\mathfrak{P}_{11} = \langle 2X_0^2 + X_1^2, X_2\rangle$, and p_{12} corresponds to $\mathfrak{P}_{12} = \langle X_1, X_0^2 + 7X_2^2\rangle$. Clearly, $\mathbb{X}$ does not have $\mathbb{Q}$-rational support, since two points p_{11} and p_{12} are not K-rational. A calculation gives us

$$\begin{array}{ll} \mathrm{HF}_{\mathbb{X}}: & 1\ 3\ 6\ 10\ 14\ 14\cdots \\ \mathrm{HF}_{\mathbb{X}\setminus\{p_j\}}: & 1\ 3\ 6\ 10\ 13\ 13\cdots \quad (j=1,\dots,10) \\ \mathrm{HF}_{\mathbb{X}\setminus\{p_{11}\}}: & 1\ 3\ 6\ 10\ 12\ 12\cdots \\ \mathrm{HF}_{\mathbb{X}\setminus\{p_{12}\}}: & 1\ 3\ 6\ \ 9\ 12\ 12\cdots. \end{array}$$

We have $\alpha_{\mathbb{X}\setminus\{p_j\}/\mathbb{X}} = r_{\mathbb{X}\setminus\{p_j\}} = r_{\mathbb{X}} = 4$ for $j = 1, \dots, 11$. This implies $\deg_{\mathbb{X}}(p_j) = 4$ for $j = 1, \dots, 11$. We also see that $\alpha_{\mathbb{X}\setminus\{p_{12}\}/\mathbb{X}} = 3 < r_{\mathbb{X}\setminus\{p_{12}\}} = r_{\mathbb{X}} = 4$ and $\mathrm{HF}_{I_{\mathbb{X}\setminus\{p_{12}\}/\mathbb{X}}}(3) = 1 < \varkappa_{12} = \dim_{\mathbb{Q}} \mathcal{O}_{\mathbb{X},p_{12}} = 2$. Let $\{e_{12\,1}, e_{12\,2}\}$ be the K-basis of $\mathcal{O}_{\mathbb{X},p_{12}}$ given by $e_{12,1} = 1$ and $e_{12,2} = x_2$. Here we use the isomorphism $\mathcal{O}_{\mathbb{X},p_{12}} \cong K[X_1, X_2]/\langle X_1, 1+7X_2^2\rangle = K \oplus x_2K$. Then we find $\mu(e_{12,1}) = 3$ and $\mu(e_{12,2}) = 4$. This implies $\deg_{\mathbb{X}}(p_{12}) = 4$. Hence $\mathbb{X}$ is a CB-scheme.

Our next remark points out an important difference between the more general definition of a CB-scheme given here and the classical definition based on hypersurfaces passing through all points of $\mathbb{X}$ but one.

Remark 4.9 Given a 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^n_K$, we consider the following two statements:

(a) The scheme $\mathbb{X}$ is a CB-scheme.
(b) Every hypersurface of degree $r_{\mathbb{X}} - 1$ which contains all but one point of $\mathbb{X}$ automatically contains $\mathbb{X}$.

Clearly, if $\mathbb{X}$ has K-rational support, then the statements (a) and (b) are equivalent. In general, we observe that (b) implies (a), but (a) does not imply (b). For example, the reduced 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ given in Example 4.8 is a CB-scheme. But $\alpha_{\mathbb{X}\setminus\{p_{12}\}/\mathbb{X}} = 3 < r_{\mathbb{X}\setminus\{p_{12}\}} = r_{\mathbb{X}} = 4$, so statement (b) is not satisfied.

The following proposition gives a simple criterion for detecting whether a given 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^n_K$ is a CB-scheme.

Proposition 4.10 *A 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^n_K$ is a CB-scheme if and only if, for all $p_j \in Supp(\mathbb{X})$, every maximal p_j-subscheme $\mathbb{Y} \subseteq \mathbb{X}$ satisfies*

$$\dim_K (I_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}-1} < \varkappa_j.$$

Proof We always have $\dim_K (I_{\mathbb{Y}_j/\mathbb{X}})_i \le \varkappa_j$ for $i \ge 0$. Also, $\dim_K (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}-1} = \varkappa_j$ if and only if $\mu_{\mathbb{Y}/\mathbb{X}} \le r_{\mathbb{X}} - 1$. This is equivalent to $\deg_{\mathbb{X}}(p_j) \le r_{\mathbb{X}} - 1$. Hence the conclusion follows. □

Let us see an example for the applicability of this proposition.

Example 4.11 Let $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ be the 0-dimensional scheme of degree 8 with support $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_7\}$, where $p_1 = (1:0:0)$, $p_2 = (1:1:0)$, $p_3 = (1:1:1)$, $p_4 = (1:0:1)$, $p_5 = (1:-1:1)$, $p_6 = (1:2:3)$, and p_7 corresponds to

$\mathfrak{P}_7 = \langle 2X_0^2 + X_1^2, X_2\rangle$. We have $\varkappa_1 = \cdots = \varkappa_6 = 1$ and $\varkappa_7 = 2$. The Hilbert functions of $\mathbb{X}$ and its subschemes are computed as follows

$$\begin{aligned} &\mathrm{HF}_{\mathbb{X}}: && 1\,3\,6\,8\,8\ldots \\ &\mathrm{HF}_{\mathbb{X}\setminus\{p_j\}}: && 1\,3\,6\,7\,7\cdots \quad (j = 1,\ldots,6) \\ &\mathrm{HF}_{\mathbb{X}\setminus\{p_7\}}: && 1\,3\,6\,6\,6\ldots. \end{aligned}$$

From this we deduce $(I_{\mathbb{X}\setminus\{p_j\}/\mathbb{X}})_{r_{\mathbb{X}}-1} = (I_{\mathbb{X}\setminus\{p_j\}/\mathbb{X}})_2 = \langle 0\rangle$ for all $j = 1,\ldots,7$. By Proposition 4.10, $\mathbb{X}$ is a CB-scheme.

Next let us consider the subscheme $\mathbb{Y} = \mathbb{X}\setminus\{p_6\}$ of $\mathbb{X}$. The support of $\mathbb{Y}$ is $\mathrm{Supp}(\mathbb{Y}) = \{p_1,\ldots,p_5,p_7\}$. The Hilbert functions of $\mathbb{Y}$ and its subschemes are

$$\begin{aligned} &\mathrm{HF}_{\mathbb{Y}}: && 1\,3\,6\,7\,7\ldots \\ &\mathrm{HF}_{\mathbb{Y}\setminus\{p_j\}}: && 1\,3\,5\,6\,6\ldots \quad (j = 1,3,5) \\ &\mathrm{HF}_{\mathbb{Y}\setminus\{p_j\}}: && 1\,3\,6\,6\,6\ldots \quad (j = 2,4) \\ &\mathrm{HF}_{\mathbb{Y}\setminus\{p_7\}}: && 1\,3\,5\,5\,5\ldots. \end{aligned}$$

We see that $\dim_K(I_{\mathbb{Y}\setminus\{p_j\}/\mathbb{Y}})_{r_{\mathbb{Y}}-1} = \dim_K(I_{\mathbb{Y}\setminus\{p_j\}/\mathbb{Y}})_2 = 1 = \dim_K \mathcal{O}_{\mathbb{Y},p_j}$ for $j = 1,3,5$. Thus the subscheme $\mathbb{Y}$ is not a CB-scheme by Proposition 4.10.

A key result for characterizing 0-dimensional schemes which are CB-schemes is Kreuzer (1994, Theorem 2.4) which shows that this property is equivalent to the existence of special elements in the initial homogeneous component of the canonical module. In our more general setting, this result can be generalized as follows.

Proposition 4.12 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional locally Gorenstein scheme with support Supp$(\mathbb{X}) = \{p_1,\ldots,p_s\}$, and let σ be a trace map of $Q^h(R)/K[x_0,x_0^{-1}]$. Then $\mathbb{X}$ is a CB-scheme if and only if for every $j \in \{1,\ldots,s\}$ there exists a non-zero element $g_j^* \in (\mathfrak{C}^{\sigma}_{\mathbb{X}})_{-r_{\mathbb{X}}}$ such that $g_j^* = x_0^{-2r_{\mathbb{X}}}\widetilde{g}_j^*$ with $\widetilde{g}_j^* \in R_{r_{\mathbb{X}}}$ and $(\widetilde{g}_j^*)_{p_j} \in \mathcal{O}_{\mathbb{X},p_j}\setminus\mathfrak{m}_{\mathbb{X},p_j}$.*

Proof Since $\mathbb{X}$ is locally Gorenstein, there is for each point p_j a uniquely determined maximal p_j-subscheme $\mathbb{Y}_j \subseteq \mathbb{X}$ corresponding to a socle element $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X},p_j})$ of $\mathcal{O}_{\mathbb{X},p_j}$. Let $\{e_{j1},\ldots,e_{j\varkappa_j}\} \subseteq \mathcal{O}_{\mathbb{X},p_j}$ be such that whose residue classes form a K-basis of $K(p_j)$. For $k_j \in \{1,\ldots,\varkappa_j\}$, we set

$$f^*_{jk_j} := \widetilde{\imath}^{-1}((0,\ldots,0,e_{jk_j}s_jT_j^{\mu(e_{jk_j}s_j)},0,\ldots,0)).$$

Since $\mathbb{X}$ is a CB-scheme, there is for each $j \in \{1,\ldots,s\}$ an index $k_j \in \{1,\ldots,\varkappa_j\}$ such that $f^*_{jk_j} \in R_{r_{\mathbb{X}}}\setminus x_0R_{r_{\mathbb{X}}-1}$. W.l.o.g. we assume that $f^*_{j1} \in R_{r_{\mathbb{X}}}\setminus x_0R_{r_{\mathbb{X}}-1}$ for $j = 1,\ldots,s$. Let us fix an index $j \in \{1,\ldots,s\}$. Then we can define a K-linear map $\overline{\varphi}_j : R_{r_{\mathbb{X}}} \to K$ such that $\overline{\varphi}_j(x_0R_{r_{\mathbb{X}}-1}) = 0$ and $\overline{\varphi}_j(f^*_{j1}) \neq 0$. By Kreuzer (1994, Lemma 1.5), we may lift $\overline{\varphi}_j$ to obtain a $K[x_0]$-linear map $\varphi_j : R \to K[x_0]$ of degree $-r_{\mathbb{X}}$, i.e., φ_j is an element of $\underline{\mathrm{Hom}}_{K[x_0]}(R,K[x_0])_{-r_{\mathbb{X}}}$ such that $\varphi_j|_{R_{r_{\mathbb{X}}}} = \overline{\varphi}_j$, especially, $\varphi_j(f^*_{j1}) \neq 0$.

Given a homogeneous element $f \in (I_{\mathbb{Y}_j/\mathbb{X}})_{r_\mathbb{X}} \setminus \{0\}$, we proceed to show that $f \cdot \varphi_j \neq 0$. Obviously, we have $I_{\mathbb{Y}_j/\mathbb{X}} = \langle f \rangle^{\mathrm{sat}} = \langle f_{j1}^* \rangle^{\mathrm{sat}}$. This implies that $x_0^k f_{j1}^* \in \langle f \rangle$ for some $k \geq 0$, so we may write $x_0^k f_{j1}^* = fh$ for some $h \in R_k \setminus \{0\}$. Consequently, $(f \cdot \varphi_j)(h) = \varphi_j(hf) = \varphi_j(x_0^k f_{j1}^*) = x_0^k \varphi_j(f_{j1}^*) \neq 0$. From this we conclude $f \cdot \varphi_j \neq 0$ for all $f \in (I_{\mathbb{Y}_j/\mathbb{X}})_{r_\mathbb{X}} \setminus \{0\}$.

Since $\underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \cong \mathfrak{C}^\sigma_\mathbb{X} = \Phi(\omega_R(1))$, where Φ is the monomorphism of graded R-modules in Definition 3.5, we find $g_j^* = \Phi(\varphi_j) \in (\mathfrak{C}^\sigma_\mathbb{X})_{-r_\mathbb{X}}$ such that $f \cdot g_j^* \neq 0$ for all $f \in (I_{\mathbb{Y}_j/\mathbb{X}})_{r_\mathbb{X}} \setminus \{0\}$. By Proposition 3.7, we have $x_0^{2r_\mathbb{X}} \in \delta^\sigma_\mathbb{X}$, so we may write $g_j^* = x_0^{-2r_\mathbb{X}} \widetilde{g}_j^* \in (\mathfrak{C}^\sigma_\mathbb{X})_{-r_\mathbb{X}}$ with $\widetilde{g}_j^* \in R_{r_\mathbb{X}} \setminus \{0\}$. Then, for $k_j = 1, \ldots, \varkappa_j$, we get $f_{jk_j}^* \cdot \widetilde{g}_j^* \neq 0$, and so $e_{jk_j} s_j \cdot (\widetilde{g}_j^*)_{p_j} \neq 0$ in $\mathcal{O}_{\mathbb{X},p_j}$. Since $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X},p_j})$, we must have $(\widetilde{g}_j^*)_{p_j} \notin \mathfrak{m}_{\mathbb{X},p_j}$. Therefore, for every $j \in \{1, \ldots, s\}$, we have constructed a non-zero element $g_j^* \in (\mathfrak{C}^\sigma_\mathbb{X})_{-r_\mathbb{X}}$ such that $g_j^* = x_0^{-2r_\mathbb{X}} \widetilde{g}_j^*$ with $\widetilde{g}_j^* \in R_{r_\mathbb{X}}$ and $(\widetilde{g}_j^*)_{p_j} \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{\mathbb{X},p_j}$.

Conversely, we assume for contradiction that the scheme $\mathbb{X}$ is not a CB-scheme, i.e., $\deg_\mathbb{X}(p_j) < r_\mathbb{X}$ for some $j \in \{1, \ldots, s\}$. For such an j, we let $g_j^* \in (\mathfrak{C}^\sigma_\mathbb{X})_{-r_\mathbb{X}}$ such that $g_j^* = x_0^{-2r_\mathbb{X}} \widetilde{g}_j^*$ with $\widetilde{g}_j^* \in R_{r_\mathbb{X}}$ and $(\widetilde{g}_j^*)_{p_j} \notin \mathfrak{m}_{\mathbb{X},p_j}$, and let $\varphi_j = \Phi^{-1}(g_j^*)$. Clearly, $f \cdot \varphi_j \neq 0$ for all $f \in (I_{\mathbb{Y}_j/\mathbb{X}})_{r_\mathbb{X}} \setminus \{0\}$. We set $f_{j1} := x_0^{r_\mathbb{X} - \deg(f_{j1}^*)} f_{j1}^* \in (I_{\mathbb{Y}_j/\mathbb{X}})_{r_\mathbb{X}} \setminus \{0\}$. Let $i \geq 0$, and let $h \in R_i$ be a non-zero homogeneous element. If $hf_{j1} = 0$, then $(f_{j1} \cdot \varphi)(h) = 0$. Suppose that $hf_{j1} \neq 0$. In this case we write $hf_{j1} = \sum_{k_j=1}^{\varkappa_j} c_{jk_j} x_0^{r_\mathbb{X}+i-\deg(f_{jk_j}^*)} f_{jk_j}^*$ for some $c_{j1}, \ldots, c_{j\varkappa_j} \in K$. By assumption, we have $\deg(f_{jk_j}^*) < r_\mathbb{X}$, and so $\varphi(f_{jk_j}^*) = 0$ for all $k_j = 1, \ldots, \varkappa_j$. Thus

$$
\begin{aligned}
(f_{j1} \cdot \varphi)(h) = \varphi(hf_{j1}) &= \varphi\left(\sum_{k_j=1}^{\varkappa_j} c_{jk_j} x_0^{r_\mathbb{X}+i-\deg(f_{jk_j}^*)} f_{jk_j}^* \right) \\
&= \sum_{k_j=1}^{\varkappa_j} c_{jk_j} x_0^{r_\mathbb{X}+i-\deg(f_{jk_j}^*)} \varphi(f_{jk_j}^*) = 0.
\end{aligned}
$$

Hence we obtain $f_{j1} \cdot \varphi = 0$, a contradiction. □

This characterization is related to Kreuzer (1994, Theorem 2.4) as follows.

Remark 4.13 In the setting of Proposition 4.12, if there is a homogeneous element $g \in (\mathfrak{C}^\sigma_\mathbb{X})_{-r_\mathbb{X}}$ such that $\mathrm{Ann}_R(g) = \langle 0 \rangle$ then $\mathbb{X}$ is a CB-scheme. The converse holds true if the field K is infinite.

Our next example shows that the converse of the preceding remark may fail if the base field is finite.

Example 4.14 Let $\mathbb{X} \subseteq \mathbb{P}^n_{\mathbb{F}_2}$ be the set consisting of three points $p_1 = (1 : 1 : 0)$, $p_2 = (1 : 0 : 1)$, and $p_3 = (1 : 1 : 1)$. We have $\mathrm{HF}_\mathbb{X} :\ 1\ 3\ 3 \ldots$ and $r_\mathbb{X} = 1$. It is not difficult to check that $\mathbb{X}$ is a CB-scheme. A calculation gives us $(\mathfrak{C}_\mathbb{X})_{-1} = \langle g_1, g_2 \rangle_{\mathbb{F}_2}$, where $g_1 = x_0^{-2} x_1$ and $g_2 = x_0^{-2} x_2$. If $g \in (\mathfrak{C}_\mathbb{X})_{-1}$, then g is one of three forms:

g_1, g_2, and $g_1 + g_2$. We see that $x_0 + x_1 \in \mathrm{Ann}_R(g_1)$, $x_0 + x_2 \in \mathrm{Ann}_R(g_2)$, and $x_0 + x_1 + x_2 \in \mathrm{Ann}_R(g_1 + g_2)$. Thus $(\mathfrak{C}_{\mathbb{X}})_{-1}$ cannot contain an element g such that $\mathrm{Ann}_R(g) = \langle 0 \rangle$.

The most widely known class of 0-dimensional schemes which are CB-schemes are arithmetically Gorenstein schemes. The following characterization of arithmetically Gorenstein schemes is a generalization of Kreuzer (1992, Theorem 1.1) where the base field K was assumed to be algebraically closed.

Theorem 4.15 *A 0-dimensional subscheme $\mathbb{X}$ of $\mathbb{P}^n_K$ is arithmetically Gorenstein if and only if it is a locally Gorenstein Cayley–Bacharach scheme and its Hilbert function is symmetric, i.e., we have $HF_{\mathbb{X}}(i) + HF_{\mathbb{X}}(r_{\mathbb{X}} - i - 1) = deg(\mathbb{X})$ for all $i \in \mathbb{Z}$.*

Proof If $\mathbb{X}$ is arithmetically Gorenstein, then it is also locally Gorenstein. By Goto and Watanabe (1978, Proposition 2.1.3), R is a Gorenstein ring if and only if $\omega_R \cong R(d)$ for some $d \in \mathbb{Z}$. Since $\mathfrak{C}^{\sigma}_{\mathbb{X}} \cong \omega_R(1)$ and $\mathrm{HF}_{\mathfrak{C}^{\sigma}_{\mathbb{X}}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-i-1)$ for all $i \in \mathbb{Z}$, this is equivalent to $\mathfrak{C}^{\sigma}_{\mathbb{X}} \cong R(r_{\mathbb{X}})$. Consequently, the Hilbert function $\mathrm{HF}_{\mathbb{X}}$ is symmetric. Moreover, there is an element $g \in (\mathfrak{C}^{\sigma}_{\mathbb{X}})_{-r_{\mathbb{X}}}$ such that $\mathfrak{C}^{\sigma}_{\mathbb{X}} = \langle g \rangle_R$ and $\mathrm{Ann}_R(g) = \langle 0 \rangle$. Hence $\mathbb{X}$ is a CB-scheme by Proposition 4.12.

Conversely, suppose that $\mathbb{X}$ is a locally Gorenstein CB-scheme and its Hilbert function is symmetric. We have $\mathrm{HF}_{\mathfrak{C}^{\sigma}_{\mathbb{X}}}(-r_{\mathbb{X}}) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1) = 1$. It follows that the K-vector space $(\mathfrak{C}^{\sigma}_{\mathbb{X}})_{-r_{\mathbb{X}}}$ is generated by one element $g = x_0^{-2r_{\mathbb{X}}}\widetilde{g}$, where $\widetilde{g} \in R_{r_{\mathbb{X}}} \setminus \{0\}$. Since $\mathbb{X}$ is a CB-scheme, Proposition 4.12 implies that the element $\widetilde{g}_{p_j}$ is a unit of $\mathcal{O}_{\mathbb{X},p_j}$ for every $j \in \{1, \ldots, s\}$. Thus $\widetilde{g}$ is a non-zerodivisor of R (cf. Kreuzer 1994, Lemma 1.1]), and hence $\mathrm{Ann}_R(g) = \langle 0 \rangle$. Because $\mathrm{HF}_{\mathbb{X}}$ is symmetric, we must have $\mathfrak{C}^{\sigma}_{\mathbb{X}} = \langle g \rangle_R \cong R(r_{\mathbb{X}})$. Therefore Goto and Watanabe (1978, Proposition 2.1.3) yields that $\mathbb{X}$ is arithmetically Gorenstein, as wanted. □

5 Characterizations of zero-dimensional complete intersections

In this section we discuss some characterizations of 0-dimensional complete intersection schemes $\mathbb{X}$ in $\mathbb{P}^n_K$ using their Kähler and Dedekind differents. Before we begin, let us examine the relations between several versions of the definition of a complete intersection.

Recall that a local ring $(S, \mathfrak{m})$ is called a **complete intersection** if it is Noetherian and its $\mathfrak{m}$-adic completion $\widehat{S}$ is a quotient of a regular local ring A by an ideal generated by an A-regular sequence. It is well known (cf. Bruns and Herzog 1993, Theorem 2.3.3) that if S is a Noetherian local ring and $S = A/I$ with a regular local ring A, then S is a complete intersection if and only if I is generated by an A-regular sequence. For more properties of complete intersection rings, we refer to Bruns and Herzog (1993, Section 2.3).

Definition 5.1 Given a ring S and an algebra T/S, we say that T/S is **locally a complete intersection** if for all $\mathfrak{P} \in \mathrm{Spec}(T)$ the algebra $T_{\mathfrak{P}}/S_{\mathfrak{p}}$ with $\mathfrak{p} = \mathfrak{P} \cap S$ is flat and the local ring $T_{\mathfrak{P}}/\mathfrak{p}T_{\mathfrak{P}}$ is a complete intersection.

At this point we can describe 0-dimensional complete intersection schemes in the following ways.

Proposition 5.2 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme. Then the following statements are equivalent.*

(a) *The scheme $\mathbb{X}$ is a complete intersection.*
(b) *The algebra $R/K[x_0]$ is locally a complete intersection.*
(c) *The local ring $\overline{R} = R/\langle x_0\rangle$ is a complete intersection.*

Proof Let $\{F_1, \dots, F_r\}$ be a minimal homogeneous system of generators of $I_{\mathbb{X}}$, where $r \geq n$. If $\mathbb{X}$ is a complete intersection, then $r = n$ and $\{F_1, \dots, F_n\}$ is a P-regular sequence. Thus the algebra $R/K[x_0]$ is locally a complete intersection by Kunz (1986, C.7) and we have "(a)⇒(b)". "(b)⇒(c)" follows from the observation that $\langle x_0\rangle_{K[x_0]} = \mathfrak{m} \cap K[x_0]$ and $R_{\mathfrak{m}}/\langle x_0\rangle R_{\mathfrak{m}} = (R/\langle x_0\rangle)_{\overline{\mathfrak{m}}} = \overline{R}_{\overline{\mathfrak{m}}} = \overline{R}$, where $\overline{\mathfrak{m}} = \mathfrak{m}/\langle x_0\rangle$ is the maximal ideal of $\overline{R}$.

It remains to prove "(c)⇒(a)". Let $\mathfrak{M}$ denote the homogeneous maximal ideal $\langle X_0, \dots, X_n\rangle_P$ of P. Observe that if $\{F_1, \dots, F_r, X_0\}$ is a minimal homogeneous system of generators of $I_{\mathbb{X}} + \langle X_0\rangle$ then we have

$$\overline{R} = \overline{R}_{\overline{\mathfrak{m}}} \cong (P/I_{\mathbb{X}} + \langle X_0\rangle)_{\overline{\mathfrak{m}}} \cong P_{\mathfrak{M}}/(\langle F_1, \dots, F_r, X_0\rangle)_{\mathfrak{M}}.$$

Since $\overline{R}$ is a complete intersection, the set $\{F_1, \dots, F_r, X_0\}$ is a $P_{\mathfrak{M}}$-regular sequence (see Bruns and Herzog 1993, Theorem 2.1.2). Now Kunz (1986, Lemma C.28) implies that $\{F_1, \dots, F_r, X_0\}$ is a P-regular sequence, and hence $r = n$ or $\mathbb{X}$ is a complete intersection. Therefore it suffices to show that $\{F_1, \dots, F_r, X_0\}$ is a minimal homogeneous system of generators of $I_{\mathbb{X}} + \langle X_0\rangle$. Clearly, we have $X_0 \notin I_{\mathbb{X}}$. If there is an index $i \in \{1, \dots, r\}$ such that $F_i \in \langle F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_r, X_0\rangle$, then we get a representation $F_i = \sum_{j\neq i} G_j F_j + G X_0$ where $G_j \in P$ is a homogeneous polynomial of degree $\deg(F_i) - \deg(F_j)$ for $j \neq i$ and $G \in P$ is a homogeneous polynomial of degree $\deg(F_i) - 1$ (cf. Kreuzer and Robbiano 2000, Corollary 1.7.11). This implies $G X_0 = F_i - \sum_{j\neq i} G_j F_j \in I_{\mathbb{X}}$, and so $G \in I_{\mathbb{X}}$ (as x_0 is a non-zerodivisor of R). Thus there are homogeneous polynomials $H_1, \dots, H_r \in P$ such that $G = \sum_{j=1}^r H_j F_j$ and $\deg(H_j) = \deg(G) - \deg(F_j)$. Note that $H_i = 0$ (as $\deg(G) < \deg(F_i)$). Hence we have $F_i = \sum_{j\neq i}(G_j + H_j X_0)F_j$, in contradiction to the minimality of $\{F_1, \dots, F_r\}$. □

The following characterization of 0-dimensional complete intersections in Proposition 5.4 generalizes a result of G. Scheja and U. Storch (see Scheja and Storch 1975, p. 187) where the characteristic of the base field was assumed to be zero, but higher dimension was allowed. Our proof follows essentially the argument given in Scheja and Storch (1975), but offers some simplifications and clarifications. The following lemma makes some arguments in Zariski and Samuel (1958, Ch. 4, Thm. 34) explicit.

Lemma 5.3 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme, let $\{F_1, \dots, F_n\}$ be a P-regular sequence in $I_{\mathbb{X}}$, and let $\mathfrak{M} = \langle X_0, \dots, X_n\rangle$. Then the colon ideal $\langle X_0, F_1, \dots, F_n\rangle : \mathfrak{M}$ is the smallest ideal in P that properly contains $\langle X_0, F_1, \dots, F_n\rangle$.*

Proof By localizing at $\mathfrak{M}$ and applying Kunz (1986, C.27), it suffices to prove that the ideal $\mathfrak{q} : \mathfrak{M}_{\mathfrak{M}}$ is the smallest ideal in $P_{\mathfrak{M}}$ that properly contains $\mathfrak{q}$, where

$\mathfrak{q} = \langle X_0, F_1, \ldots, F_n \rangle_{P_\mathfrak{M}}$. Since $P_\mathfrak{M}$ is a regular local ring and $\{X_0, F_1, \ldots, F_n\}$ is a regular sequence, the residue class ring $P_\mathfrak{M}/\mathfrak{q}$ is a complete intersection, and so it is a Gorenstein ring. In particular, we have $\dim_K((\mathfrak{q} : \mathfrak{M}_\mathfrak{M})/\mathfrak{q}) = 1$.

Now let J be an ideal properly containing $\mathfrak{q}$. Then we have $J \cap (\mathfrak{q} : \mathfrak{M}_\mathfrak{M}) \nsubseteq \mathfrak{q}$. Suppose that k is the smallest exponent such that $J \cdot \mathfrak{M}_\mathfrak{M}^k \subseteq \mathfrak{q}$. Clearly, $k \geq 1$ and $J \cdot \mathfrak{M}_\mathfrak{M}^{k-1} \subseteq (\mathfrak{q} : \mathfrak{M}_\mathfrak{M}) \cap J$ and $J \cdot \mathfrak{M}_\mathfrak{M}^{k-1} \nsubseteq \mathfrak{q}$. Because $\dim_K((\mathfrak{q} : \mathfrak{M}_\mathfrak{M})/\mathfrak{q}) = 1$, there are no ideals between $\mathfrak{q}$ and $\mathfrak{q} : \mathfrak{M}_\mathfrak{M}$. Moreover, the intersection of J with $\mathfrak{q} : \mathfrak{M}_\mathfrak{M}$ is different from $\mathfrak{q}$, and therefore J contains $\mathfrak{q} : \mathfrak{M}_\mathfrak{M}$. □

Now we are ready to characterize 0-dimensional complete intersections using the non-vanishing of the reduced Kähler different.

Proposition 5.4 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme, let $\{F_1, \ldots, F_r\}$ be a minimal homogeneous system of generators of the vanishing ideal $I_\mathbb{X}$, and suppose char$(K) > \max\{deg(\mathbb{X}), deg(F_1), \ldots, deg(F_r)\}$. Then $\mathbb{X}$ is a complete intersection if and only if $\overline{\vartheta}_\mathbb{X}$ is non-zero.*

Proof If $\mathbb{X}$ is a complete intersection, then $\vartheta_\mathbb{X} = \delta_\mathbb{X}$ is a principal ideal generated by the Jacobian determinant $h = \det(\frac{\partial F_j}{\partial x_i})$. According to Kunz (1986, F.20), there is a trace map $\sigma : R \to K[x_0]$ associated with the presentation $R = P/\langle F_1, \ldots, F_n \rangle$. By Kunz (1986, F.23), the canonical trace map $\mathrm{Tr}_{R/K[x_0]} : R \to K[x_0]$ satisfies $\mathrm{Tr}_{R/K[x_0]} = h \cdot \sigma$. Let $d = \deg(\mathbb{X})$, let $\{t_1, \ldots, t_d\}$ be the $K[x_0]$-basis of R as in Remark 3.6(b), and let $\{t'_1, \ldots, t'_d\}$ be the dual $K[x_0]$-basis of R to $\{t_1, \ldots, t_d\}$ w.r.t. σ (see Kunz 1986, F.11). For $k = 1, \ldots, d$, we see that

$$\left(\left(\sum_{j=1}^{d} \mathrm{Tr}_{R/K[x_0]}(t_j)t'_j\right) \cdot \sigma\right)(t_k) = \sigma\left(\sum_{j=1}^{d} \mathrm{Tr}_{R/K[x_0]}(t_j)t'_j t_k\right) = \sum_{j=1}^{d} \mathrm{Tr}_{R/K[x_0]}(t_j)\sigma(t'_j t_k)$$
$$= \sum_{j=1}^{d} \mathrm{Tr}_{R/K[x_0]}(t_j)\delta_{jk} = \mathrm{Tr}_{R/K[x_0]}(t_k).$$

This implies $h \cdot \sigma = \mathrm{Tr}_{R/K[x_0]} = (\sum_{j=1}^{d} \mathrm{Tr}_{R/K[x_0]}(t_j)t'_j) \cdot \sigma$. Hence we get the equality $h = \sum_{j=1}^{d} \mathrm{Tr}_{R/K[x_0]}(t_j)t'_j$. We may assume $t_1 = 1$. Then $\mathrm{Tr}_{R/K[x_0]}(t_1) = d \neq \mathrm{char}(K)$, and so $h = dt'_1 + \sum_{j=2}^{d} \mathrm{Tr}_{R/K[x_0]}(t_j)t'_j$. Consequently, we have $\vartheta_\mathbb{X} = \langle h \rangle_R \nsubseteq \langle x_0 \rangle_R$. In particular, $\overline{\vartheta}_\mathbb{X}$ is non-zero.

Conversely, suppose that $\mathbb{X}$ is not a complete intersection, i.e., that $r > n$. For every subset $\{F_{i_1}, \ldots, F_{i_n}\}$ of $\{F_1, \ldots, F_r\}$ consisting of n elements, we want to show that $\frac{\partial(F_{i_1}, \ldots, F_{i_n})}{\partial(X_1, \ldots, X_n)} \in \langle X_0 \rangle + I_\mathbb{X} =: J$. It suffices to do this for $\{F_1, \ldots, F_n\}$.

W.l.o.g., let $F_1, \ldots, F_t$ be a P-regular sequence of maximal length in $\langle F_1, \ldots, F_n \rangle$. In the case $t = n$, Euler's rule yields

$$\deg(F_j)F_j = \sum_{i=0}^{n} \frac{\partial F_j}{\partial X_i} X_i$$

for $j = 1, \ldots, n$. Then Lemma 5.3 implies

$$\frac{\partial(X_0, F_1, \ldots, F_n)}{\partial(X_0, \ldots, X_n)} = \frac{\partial(F_1, \ldots, F_n)}{\partial(X_1, \ldots, X_n)} \in \langle X_0, F_1, \ldots, F_n \rangle : \mathfrak{M} \subseteq J.$$

Now we consider the case $t < n$. It is well-known that all associated prime ideals of $\langle X_0, F_1, \ldots, F_t \rangle$ have the same height $t+1$. So, we may assume that X_{t+1} is not contained in any associated prime ideal of $\langle X_0, F_1, \ldots, F_t \rangle$. Let G_{t+1} be a homogeneous polynomial in the intersection of the associated prime ideals of $\langle X_0, F_1, \ldots, F_t \rangle$ which does not contain F_{t+1}. For $k > 1$, the elements $X_0, F_1, \ldots, F_t, F_{t+1} + G_{t+1}^k X_{t+1}^{k+1}$ form a P-regular sequence. Repeating this process, we can construct a P-regular sequence $X_0, F_1, \ldots, F_t, F_{t+1} + G_{t+1}^k X_{t+1}^{k+1}, \ldots, F_n + G_n^k X_n^{k+1}$ eventually.

We set $J_k = \langle X_0, F_1, \ldots, F_t, F_{t+1} + G_{t+1}^k X_{t+1}^{k+1}, \ldots, F_n + G_n^k X_n^{k+1} \rangle$. If $G_n^k X_n^k \in J_k$, then

$$G_n^k X_n^k = X_0 H_0 + F_1 H_1 + \cdots + (F_n + G_n^k X_n^{k+1}) H_n$$

for some $H_0, \ldots, H_n \in P$. Since $\{X_0, F_1, \ldots, F_t, F_n\}$ is not a P-regular sequence, there exist $H, H_0', \ldots, H_t' \in P$ such that $H \notin \langle X_0, F_1, \ldots, F_t \rangle$ and

$$H F_n = X_0 H_0' + F_1 H_1' + \cdots F_t H_t'.$$

Hence we get

$$\begin{aligned} H(F_n + G_n^k X_n^{k+1}) &= (X_n H_0 H + H_0') X_0 + (X_n H_1 H + H_1') F_1 + \cdots \\ &\quad + (X_n H_t H + H_t') F_t + H X_n H_{t+1} (F_{t+1} + G_{t+1}^k X_{t+1}^{k+1}) + \cdots \\ &\quad + X_n H_n H (F_n + G_n^k X_n^{k+1}). \end{aligned}$$

This contradicts the fact that $X_0, F_1, \ldots, F_t, F_{t+1} + G_{t+1}^k X_{t+1}^{k+1}, \ldots, F_n + G_n^k X_n^{k+1}$ form a P-regular sequence. Thus we have $G_n^k X_n^k \notin J_k$.

Consequently, we get $J_k : \mathfrak{M} \subseteq J_k + \langle G_n^k X_n^k \rangle$ by Lemma 5.3. Let $d_j = \deg(F_j)$ for $j = 1, \ldots, n$ and let

$$\mathcal{M} = \begin{pmatrix} 1 & \frac{\partial F_1}{\partial X_0} & \cdots & \frac{\partial F_t}{\partial X_0} & \frac{\partial F_{t+1}}{d_{t+1} \partial X_0} & \cdots & \frac{\partial F_n}{d_n \partial X_0} \\ 0 & \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_t}{\partial X_1} & \frac{\partial F_{t+1}}{d_{t+1} \partial X_1} & \cdots & \frac{\partial F_n}{d_n \partial X_1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{\partial F_1}{\partial X_{t+1}} & \cdots & \frac{\partial F_t}{\partial X_{t+1}} & \frac{\partial F_{t+1}}{d_{t+1} \partial X_{t+1}} + G_{t+1}^k X_{t+1}^k & \cdots & \frac{\partial F_n}{d_n \partial X_{t+1}} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{\partial F_1}{\partial X_n} & \cdots & \frac{\partial F_t}{\partial X_n} & \frac{\partial F_{t+1}}{d_{t+1} \partial X_n} & \cdots & \frac{\partial F_n}{d_n \partial X_n} + G_n^k X_n^k \end{pmatrix}.$$

We shall show that $\det(\mathcal{M}) \in J_k : \mathfrak{M}$, and hence $\det(\mathcal{M}) \in J_k + \langle G_n^k X_n^k \rangle$. Let $\mathcal{M}_i$ denote the i-th row of the matrix $\mathcal{M}$ for $i = 1, \ldots, n+1$, and let

$$\mathcal{V} := (X_0, d_1 F_1, \ldots, d_t F_t, F_{t+1} + G_{t+1}^k X_{t+1}^{k+1}, \ldots, F_n + G_n^k X_n^{k+1}).$$

We use Euler's rule and calculate

$$\sum_{j=0}^{n} X_j \cdot \mathcal{M}_{j+1} = \Bigg(X_0,\ \sum_{j=0}^{n} \tfrac{\partial F_1}{\partial X_j} X_j, \ldots,\ \sum_{j=0}^{n} \tfrac{\partial F_t}{\partial X_j} X_j,\ \tfrac{1}{d_{t+1}} \sum_{j=0}^{n} \tfrac{\partial F_{t+1}}{\partial X_j} X_j + G_{t+1}^k X_{t+1}^{k+1}, \ldots,\ \tfrac{1}{d_n} \sum_{j=0}^{n} \tfrac{\partial F_n}{\partial X_j} X_j + G_n^k X_n^{k+1} \Bigg)$$
$$= (X_0, d_1 F_1, \ldots, d_t F_t, F_{t+1} + G_{t+1}^k X_{t+1}^{k+1}, \ldots, F_n + G_n^k X_n^{k+1}).$$

Thus we obtain $\mathcal{V} = \sum_{j=0}^{n} X_j \cdot \mathcal{M}_{j+1}$. Therefore we have

$$X_{i-1} \cdot \det(\mathcal{M}) = \det \begin{pmatrix} \mathcal{M}_1 \\ \vdots \\ \mathcal{M}_{i-1} \\ X_{i-1} \cdot \mathcal{M}_i \\ \mathcal{M}_{i+1} \\ \vdots \\ \mathcal{M}_n \end{pmatrix} = \det \begin{pmatrix} \mathcal{M}_1 \\ \vdots \\ \mathcal{M}_{i-1} \\ \mathcal{V} \\ \mathcal{M}_{i+1} \\ \vdots \\ \mathcal{M}_n \end{pmatrix} \in J_k$$

for $i = 1, \ldots, n+1$, and hence we get $\det(\mathcal{M}) \in J_k : \mathfrak{M} \subseteq J_k + \langle G_n^k X_n^k \rangle$. Furthermore, we have $\det(\mathcal{M}) - \frac{1}{d_{t+1} \cdots d_n} \cdot \frac{\partial(F_1, \ldots, F_n)}{\partial(X_1, \ldots, X_n)} \in \langle G_{t+1}^k X_{t+1}^k, \ldots, G_n^k X_n^k \rangle$, and so $\frac{\partial(F_1, \ldots, F_n)}{\partial(X_1, \ldots, X_n)} \in J_k + \langle G_{t+1}^k X_{t+1}^k, \ldots, G_n^k X_n^k \rangle \subseteq J + \langle G_{t+1}^k X_{t+1}^k, \ldots, G_n^k X_n^k \rangle$ for all $k > 1$. Therefore, by the Krull Intersection Theorem (cf. Eisenbud 1995, Corollary 5.4), we obtain $\frac{\partial(F_1, \ldots, F_n)}{\partial(X_1, \ldots, X_n)} \in J$, as desired. □

The characterization given in the preceding proposition is nice in the sense that it only uses the reduced Kähler different. But we are also looking for a refined version which lets us distinguish between complete intersections and arithmetically Gorenstein schemes. If we know already that $\mathbb{X}$ is arithmetically Gorenstein, then we can use the following characterization.

Proposition 5.5 *Let $\mathbb{X} \subseteq \mathbb{P}_K^n$ be a 0-dimensional smooth scheme which is arithmetically Gorenstein. Then the following conditions are equivalent.*

(a) *The scheme $\mathbb{X}$ is a complete intersection.*
(b) *The Hilbert function of $\vartheta_{\mathbb{X}}$ satisfies $HF_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$.*
(c) *We have $\vartheta_{\mathbb{X}} = \delta_{\mathbb{X}}$.*

Proof (a)⇒(b): This follows from Corollary 2.7.

(b)⇒(c): Since $\mathbb{X}$ is arithmetically Gorenstein, as in the proof of Theorem 4.15 we find an element $g \in (\mathfrak{C}_{\mathbb{X}})_{-r_{\mathbb{X}}}$ such that $\mathfrak{C}_{\mathbb{X}} = \langle g \rangle_R$ and $\mathrm{Ann}_R(g) = \langle 0 \rangle$. Note that g is a unit of $Q^h(R)$ and $h = g^{-1} \in (\delta_{\mathbb{X}})_{r_{\mathbb{X}}}$. For $f \in (\delta_{\mathbb{X}})_i$ with $i < r_{\mathbb{X}}$, we see that $f \cdot g \in R_{i - r_{\mathbb{X}}} = \langle 0 \rangle$, and hence $f = 0$. If $f \in (\delta_{\mathbb{X}})_i$ with $i \geq r_{\mathbb{X}}$, then $f_1 = f \cdot g \in R_{i - r_{\mathbb{X}}}$. This implies $(f - f_1 h) \cdot g = 0$, and consequently $f = f_1 h$. So, we get $\delta_{\mathbb{X}} = \langle h \rangle_R$.

Furthermore, we have

$$\langle 0\rangle \neq (\vartheta_{\mathbb{X}})_{r_{\mathbb{X}}} \subseteq (\delta_{\mathbb{X}})_{r_{\mathbb{X}}} = \langle h\rangle_K,$$

and so $h \in \vartheta_{\mathbb{X}}$. Therefore we obtain $\vartheta_{\mathbb{X}} = \delta_{\mathbb{X}}$.

(c)⇒(a): Suppose that $\vartheta_{\mathbb{X}} = \delta_{\mathbb{X}}$. Since $\mathbb{X}$ is arithmetically Gorenstein, we argue as above to get $\vartheta_{\mathbb{X}} = \langle h\rangle_R$ for some non-zerodivisor $h \in R_{r_{\mathbb{X}}}$. In particular, $\vartheta_{\mathbb{X}}$ is an invertible ideal. Moreover, if we have $\Omega^1_{Q(R)/K[x_0]} = \langle 0\rangle$, where $Q(R)$ is the full ring of quotients of R, then it follows from Kunz (1986, Theorem 10.14) that the algebra $R/K[x_0]$ is locally a complete intersection, and hence $\mathbb{X}$ is a complete intersection by Proposition 5.2. So, it suffices to prove that $\Omega^1_{Q(R)/K[x_0]} = \langle 0\rangle$. According to Corollary 3.4, the algebra $Q(R)/K(x_0)$ is étale, free of rank $\deg(\mathbb{X})$. Thus Kunz (1986, Proposition 6.8) yields $\Omega^1_{Q(R)/K(x_0)} = \langle 0\rangle$. Additionally, it is not difficult to see that $\mathrm{Ker}(K(x_0) \otimes_{K[x_0]} K(x_0) \xrightarrow{\mu} K(x_0)) = \langle f\otimes 1 - 1\otimes f \mid f \in K(x_0)\rangle_{K(x_0)\otimes_{K[x_0]}K(x_0)} = \langle 0\rangle$, so $\Omega^1_{K(x_0)/K[x_0]} = \langle 0\rangle$. On the other hand, we have $Q(R) \cong K(x_0)\otimes_K \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}$ (as $\mathbb{X}$ is smooth). This implies

$$Q(R) \cong K(x_0) \otimes_{K[x_0]} \left(K[x_0] \otimes_K \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j} \right) \cong K(x_0) \otimes_{K[x_0]} \widetilde{R}$$

where $\widetilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}[T_j]$ and $T_1,\dots,T_s$ are indeterminates. By Kunz (1986, Formulas 4.4), we obtain $\Omega^1_{Q(R)/K(x_0)} \cong K(x_0)\otimes_{K[x_0]} \Omega^1_{\widetilde{R}/K[x_0]}$ and

$$\begin{aligned}\Omega^1_{Q(R)/K[x_0]} &\cong K(x_0)\otimes_{K[x_0]} \Omega^1_{\widetilde{R}/K[x_0]} \oplus \widetilde{R}\otimes_{K[x_0]} \Omega^1_{K(x_0)/K[x_0]}\\ &\cong \Omega^1_{Q(R)/K(x_0)} \oplus \widetilde{R}\otimes_{K[x_0]} \Omega^1_{K(x_0)/K[x_0]} = \langle 0\rangle.\end{aligned}$$

This completes the proof. □

Now we present our main result of this section. It answers a question posed in Griffiths and Harris (1978) and Davis and Maroscia (1984): **CB-scheme + (?) = Complete intersection?** in the case of smooth 0-dimensional subscheme $\mathbb{X}$ of $\mathbb{P}^n_K$. In other words, we replace the assumption that $\mathbb{X}$ is arithmetically Gorenstein by the weaker assumption that $\mathbb{X}$ is a CB-scheme and show again that the non-vanishing of a single homogeneous component of the Kähler different characterizes 0-dimensional complete intersections.

Theorem 5.6 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a smooth 0-dimensional scheme. Then $\mathbb{X}$ is a complete intersection if and only if it is a CB-scheme and $HF_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$.*

Proof Suppose that the scheme $\mathbb{X}$ is a complete intersection. Then $\mathbb{X}$ is arithmetically Gorenstein. It follows from Goto and Watanabe (1978, Proposition 2.1.3) that $\mathfrak{C}_{\mathbb{X}} \cong R(r_{\mathbb{X}})$, and so there exists a homogeneous element $g \in (\mathfrak{C}_{\mathbb{X}})_{-r_{\mathbb{X}}}$ such that $\mathfrak{C}_{\mathbb{X}} = \langle g\rangle_R$ and $\mathrm{Ann}_R(g) = \langle 0\rangle$. By Proposition 4.12, the scheme $\mathbb{X}$ is a CB-scheme. Moreover, it follows from Corollary 2.7 that $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$.

Now we prove the converse. Since $\vartheta_{\mathbb{X}} \subseteq \delta_{\mathbb{X}}$ and $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$, we have $\mathrm{HF}_{\delta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$. We let h be a non-zero element in $(\delta_{\mathbb{X}})_{r_{\mathbb{X}}}$. Then there is a point $p_j \in \mathrm{Supp}(\mathbb{X})$ such that $h_{p_j} \neq 0$ in $\mathcal{O}_{\mathbb{X},p_j}$. Because $\mathbb{X}$ is a CB-scheme, Proposition 4.12 implies that there is an element $g_j^* \in (\mathfrak{C}_{\mathbb{X}})_{-r_{\mathbb{X}}} \setminus \{0\}$ such that $g_j^* = x_0^{-2r_{\mathbb{X}}} \widetilde{g}_j^*$ with $\widetilde{g}_j^* \in R_{r_{\mathbb{X}}}$ and $(\widetilde{g}_j^*)_{p_j} \neq 0$. In $\mathcal{O}_{\mathbb{X},p_j}$, we have $h_{p_j} \cdot (\widetilde{g}_j^*)_{p_j} \neq 0$. This implies $h \cdot \widetilde{g}_j^* \neq 0$, and then $h \cdot g_j^* \neq 0$ (as x_0 is a non-zerodivisor of R). Thus there is an element $c \in K \setminus \{0\}$ such that $c = h \cdot g_j^* \in R_0 = K$. In particular, h is a non-zerodivisor of R and $\mathrm{Ann}_R(g_j^*) = \langle 0 \rangle$. W.l.o.g. we may assume that $c = 1$.

Next we prove that $\mathfrak{C}_{\mathbb{X}} = \langle g_j^* \rangle_R$. Let $i \geq 0$ and let $g \in (\mathfrak{C}_{\mathbb{X}})_{i-r_{\mathbb{X}}}$. Then we have $g \cdot h \in R_i$. Set $f = g \cdot h \in R_i$. We have $g \cdot h = f = fh \cdot g_j^*$, and so $(x_0^{2r_{\mathbb{X}}} \cdot g - x_0^{2r_{\mathbb{X}}} f \cdot g_j^*)h = 0$. Since $x_0^{2r_{\mathbb{X}}} \in \delta_{\mathbb{X}}$ and $\mathrm{Ann}_R(h) = \langle 0 \rangle$, we have $x_0^{2r_{\mathbb{X}}} \cdot g - x_0^{2r_{\mathbb{X}}} f \cdot g_j^* = x_0^{2r_{\mathbb{X}}} \cdot (g - fg_j^*) = 0$. The fact that x_0 is a non-zerodivisor on $\mathfrak{C}_{\mathbb{X}}$ implies $g = fg_j^*$. Thus we get $\mathfrak{C}_{\mathbb{X}} = \langle g_j^* \rangle_R$, as claimed.

Consequently, we have $\mathfrak{C}_{\mathbb{X}} \cong R(r_{\mathbb{X}})$, since $\mathrm{Ann}_R(g_j^*) = \langle 0 \rangle$. Hence the scheme $\mathbb{X}$ is arithmetically Gorenstein by Goto and Watanabe (1978, Proposition 2.1.3). Therefore an application of Proposition 5.5 yields that $\mathbb{X}$ is a complete intersection, as we wanted to show. □

The following corollary is an immediate consequence of Theorem 5.6. This corollary can be deduced from Kunz (1986, Theorem 9.5), and its corresponding local version is given in Lenstra (1993, Proposition 1).

Corollary 5.7 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional smooth scheme. Then $\mathbb{X}$ is a complete intersection if and only if $\vartheta_{\mathbb{X}}$ is a principal ideal of R generated by a non-zero homogeneous element of degree $r_{\mathbb{X}}$.*

Proof Suppose that $\vartheta_{\mathbb{X}} = \langle h \rangle_R$ for some element $h \in R_{r_{\mathbb{X}}} \setminus \{0\}$. Since $\mathrm{HP}_{\vartheta_{\mathbb{X}}} = \deg(\mathbb{X})$ by Corollary 3.11, the element h must be a non-zerodivisor of R. Clearly, $\mathrm{HF}_{\mathfrak{C}_{\mathbb{X}}}(-r_{\mathbb{X}}) = \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1) \geq 1$, and so there exists a non-zero element $g \in (\mathfrak{C}_{\mathbb{X}})_{-r_{\mathbb{X}}}$. We can write $g = x_0^{-2r_{\mathbb{X}}} \widetilde{g}$ for some $\widetilde{g} \in R_{r_{\mathbb{X}}}$. Since x_0 and h are non-zerodivisors of R, we have

$$0 \neq h \cdot g = x_0^{-2r_{\mathbb{X}}} h\widetilde{g} \in (\vartheta_{\mathbb{X}})_{r_{\mathbb{X}}} (\mathfrak{C}_{\mathbb{X}})_{-r_{\mathbb{X}}} \subseteq R_0.$$

It follows that $h \cdot g \in K \setminus \{0\}$, and hence $\mathrm{Ann}_R(g) = \langle 0 \rangle$. Thus Proposition 4.12 yields that $\mathbb{X}$ is a CB-scheme, and therefore $\mathbb{X}$ is a complete intersection by Theorem 5.6. The other implication follows from Corollary 2.7. □

We conclude this section with the following criterion for 0-dimensional arithmetically Gorenstein schemes.

Proposition 5.8 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional locally Gorenstein scheme. Then the following conditions are equivalent.*

(a) *The scheme $\mathbb{X}$ is arithmetically Gorenstein.*
(b) *There exists an element $h \in R_{r_{\mathbb{X}}} \setminus \{0\}$ such that $\delta_{\mathbb{X}} = \langle h \rangle_R$.*

(c) *There exists an element* $h \in (\delta_{\mathbb{X}})_{r_{\mathbb{X}}}$ *with* $Ann_R(h) = \langle 0 \rangle$.
(d) *The scheme* $\mathbb{X}$ *is a CB-scheme and* $HF_{\delta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$.

If these conditions are satisfied, then the Hilbert function of the Dedekind different satisfies $HF_{\delta_{\mathbb{X}}}(i) = HF_{\mathbb{X}}(i - r_{\mathbb{X}})$ *for all* $i \in \mathbb{Z}$, *and* $ri(\delta_{\mathbb{X}}) = 2r_{\mathbb{X}}$.

Proof (a)⇒(b): This follows from the proof of "(b)⇒(c)" of Proposition 5.5.

(b)⇒(c): Assume that $\delta_{\mathbb{X}} = \langle h \rangle_R$ with $h \in R_{r_{\mathbb{X}}} \setminus \{0\}$. According to Proposition 3.7, there is an element $f \in R_{r_{\mathbb{X}}}$ such that $fh = x_0^{2r_{\mathbb{X}}} \in \delta_{\mathbb{X}}$, and therefore $\mathrm{Ann}_R(h) = \langle 0 \rangle$.

(c)⇒(d): Suppose that there is $h \in (\delta_{\mathbb{X}})_{r_{\mathbb{X}}}$ with $\mathrm{Ann}_R(h) = \langle 0 \rangle$. Obviously, we have $\mathrm{HF}_{\delta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$. Since $\mathrm{HF}_{\mathfrak{C}_{\mathbb{X}}}(-r_{\mathbb{X}}) \geq 1$, there is a non-zero homogeneous element $g \in (\mathfrak{C}_{\mathbb{X}})_{-r_{\mathbb{X}}}$. Note that $\delta_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}} \subseteq R$. So, we can argue similarly as in the proof of Corollary 5.7 to get $\mathrm{Ann}_R(g) = \langle 0 \rangle$. Hence $\mathbb{X}$ is a CB-scheme by Proposition 4.12.

(d)⇒(a): This follows easily from the proof of Theorem 5.6.

The additional claim follows from the fact that $\delta_{\mathbb{X}} \cong R(-r_{\mathbb{X}})$. □

6 Characterizations using higher Kähler differents

Previously, we mainly considered the Kähler different $\vartheta_{\mathbb{X}} = F_0(\Omega^1_{R/K[x_0]})$ of the algebra $R/K[x_0]$ to study complete intersections, arithmetically Gorenstein schemes, etc. But what about the algebra R/K? Since R is 1-dimensional, it is natural to consider the higher Kähler different $\vartheta^{(1)}_{\mathbb{X}} = F^1(\Omega^1_{R/K})$ in this case. After exhibiting relations between this higher Kähler different and the differents studied earlier and collecting some results about its Hilbert function, we shall show that it, too, can be used to characterize 0-dimensional complete intersections in a nice way.

As in the previous sections, we let $\mathbb{X}$ be a 0-dimensional subscheme of $\mathbb{P}^n_K$ such that $\mathrm{Supp}(\mathbb{X}) \cap \mathcal{Z}(X_0) = \emptyset$. Recall that the first Kähler different $\vartheta^{(1)}_{\mathbb{X}}$ of the algebra R/K is the first Fitting ideal of $\Omega^1_{R/K}$ and that we called it the **higher Kähler different** of $\mathbb{X}$. Using Kunz (1986, Proposition 4.19), we can compute $\vartheta^{(1)}_{\mathbb{X}}$ as follows.

Proposition 6.1 *Let* $\{F_1, \ldots, F_r\}$ *be a homogeneous set of generators of* $I_{\mathbb{X}}$. *For every subset* $S = \{i_1, \ldots, i_n\}$ *of* $\{1, \ldots, r\}$ *and every* $j \in \{0, \ldots, n\}$, *we define* $\Delta_{S,j} := \frac{\partial(F_{i_1}, \ldots, F_{i_n})}{\partial(x_0, \ldots, \hat{x}_j, \ldots, x_n)}$. *Then we have*

$$\vartheta^{(1)}_{\mathbb{X}} = \langle \Delta_{S,j} \mid S \subseteq \{1, \ldots, r\},\ \#S = n,\ j \in \{0, \ldots, n\} \rangle.$$

The following lemma provides useful relations between the Kähler differents of $R/K[x_0]$ and of R/K.

Lemma 6.2 *Let* $\mathbb{X} \subseteq \mathbb{P}^n_K$ *be a* 0*-dimensional scheme. Then we have*

$$x_0 \vartheta^{(1)}_{\mathbb{X}} = \vartheta_{\mathbb{X}}\mathfrak{m} \subseteq \vartheta_{\mathbb{X}} \subseteq \vartheta^{(1)}_{\mathbb{X}}.$$

Proof Analogous to Kreuzer et al. (2015, Lemma 2.2). □

By applying the lemma, we can give bounds for the Hilbert polynomial and the regularity index of the Kähler different $\vartheta_{\mathbb{X}}^{(1)}$, as the following proposition shows.

Proposition 6.3 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme, and let $\mathbb{X}_{\mathrm{sm}}$ be the set of smooth points of $\mathbb{X}$ in Supp$(\mathbb{X}) = \{p_1, \ldots, p_s\}$. Then we have*

$$\sum_{p_j \in \mathbb{X}_{\mathrm{sm}}} \dim_K(\mathcal{O}_{\mathbb{X},p_j}) \le HP_{\vartheta_{\mathbb{X}}^{(1)}} = HP_{\vartheta_{\mathbb{X}}} \le deg(\mathbb{X}) - (s - \#\mathbb{X}_{\mathrm{sm}})$$

and $ri(\vartheta_{\mathbb{X}}) - 1 \le ri(\vartheta_{\mathbb{X}}^{(1)}) \le ri(\vartheta_{\mathbb{X}})$.

Proof By Lemma 6.2, we have $x_0\vartheta_{\mathbb{X}}^{(1)} \subseteq \vartheta_{\mathbb{X}} \subseteq \vartheta_{\mathbb{X}}^{(1)}$. This implies the equalities $\mathrm{HP}_{x_0\vartheta_{\mathbb{X}}^{(1)}} = \mathrm{HP}_{\vartheta_{\mathbb{X}}^{(1)}} = \mathrm{HP}_{\vartheta_{\mathbb{X}}}$, since x_0 is a non-zerodivisor of R. Hence Proposition 2.15 yields the bounds for $\mathrm{HP}_{\vartheta_{\mathbb{X}}^{(1)}}$. Now we prove the claimed inequalities between regularity indices. Obviously, we have $\mathrm{ri}(\vartheta_{\mathbb{X}}^{(1)}) \le \mathrm{ri}(\vartheta_{\mathbb{X}})$. It follows from the inclusion $x_0\vartheta_{\mathbb{X}}^{(1)} \subseteq \vartheta_{\mathbb{X}}$ that $\mathrm{HF}_{\vartheta_{\mathbb{X}}^{(1)}}(i) \le \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1)$ for every $i \in \mathbb{Z}$. Consequently, we get $\mathrm{ri}(\vartheta_{\mathbb{X}}) - 1 \le \mathrm{ri}(\vartheta_{\mathbb{X}}^{(1)}) \le \mathrm{ri}(\vartheta_{\mathbb{X}})$. □

If $\mathbb{X} \subseteq \mathbb{P}^n_K$ is a 0-dimensional smooth scheme, then the Kähler different $\vartheta_{\mathbb{X}}^{(1)}$ satisfies $\mathrm{HP}_{\vartheta_{\mathbb{X}}^{(1)}} = \deg(\mathbb{X})$, and we have $\mathrm{ri}(\vartheta_{\mathbb{X}}^{(1)}) \le \mathrm{ri}(\vartheta_{\mathbb{X}}) \le r_{\mathbb{X}}(n+1)$ by Corollary 3.11 and the preceding proposition. Furthermore, we have the following particular values of the Hilbert function of $\vartheta_{\mathbb{X}}^{(1)}$.

Corollary 6.4 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a reduced 0-dimensional complete intersection.*

(a) *Assume that $\mathbb{X}$ contains no smooth point in its support. Then we have $HF_{\vartheta_{\mathbb{X}}^{(1)}}(i) = 0$ for all $i \in \mathbb{Z}$.*
(b) *Assume that $\mathbb{X}$ contains at least one smooth point in its support. Let $\mathbb{Y} \subseteq \mathbb{X}$ be the subscheme defined by $I_{\mathbb{Y}} = \bigcap_{p_j \in Supp(\mathbb{X}):\, p_j \text{ smooth}} \mathfrak{P}_j$. Then, for all $i \in \mathbb{Z}$, we have*

$$HF_{\vartheta_{\mathbb{X}}^{(1)}}(i) = \begin{cases} 0 & \text{if } i < r_{\mathbb{X}}, \\ HF_{\mathbb{Y}}(i+1-r_{\mathbb{X}}) & \text{if } i \ge r_{\mathbb{X}}. \end{cases}$$

In particular, we have $ri(\vartheta_{\mathbb{X}}^{(1)}) = ri(\vartheta_{\mathbb{X}}) - 1 = r_{\mathbb{X}} + r_{\mathbb{Y}} - 1$ in this case.

Proof Since (a) follows immediately from the proposition, we prove (b). Let $I_{\mathbb{X}} = \langle F_1, \ldots, F_n \rangle$ and set $\Delta_0 = \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)}$. Then $\vartheta_{\mathbb{X}} = \langle \Delta_0 \rangle_R$. It follows from Lemma 6.2 that $x_0\vartheta_{\mathbb{X}}^{(1)} = \Delta_0\mathfrak{m}$. Since x_0 is a non-zerodivisor of R, the Hilbert function of $\vartheta_{\mathbb{X}}^{(1)}$ satisfies $\mathrm{HF}_{\vartheta_{\mathbb{X}}^{(1)}}(i) = \mathrm{HF}_{\Delta_0\mathfrak{m}}(i+1)$ for all $i \in \mathbb{Z}$. If $i < r_{\mathbb{X}}$, then $0 \le \mathrm{HF}_{\vartheta_{\mathbb{X}}^{(1)}}(i) \le \mathrm{HF}_{\mathfrak{m}}(i+1-r_{\mathbb{X}}) = 0$, and so $\mathrm{HF}_{\vartheta_{\mathbb{X}}^{(1)}}(i) = 0$. For $i \ge r_{\mathbb{X}}$, we see that $\mathrm{HF}_{\vartheta_{\mathbb{X}}^{(1)}}(i) = \mathrm{HF}_{\Delta_0\mathfrak{m}}(i+1) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1)$. Furthermore, Proposition 2.6 yields that $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i) = \mathrm{HF}_{\mathbb{Y}}(i - r_{\mathbb{X}})$ for all $i \in \mathbb{Z}$. This implies $\mathrm{HF}_{\vartheta_{\mathbb{X}}^{(1)}}(i) = \mathrm{HF}_{\mathbb{Y}}(i+1-r_{\mathbb{X}})$ for all $i \ge r_{\mathbb{X}}$. Hence the claim follows. □

Corollary 6.5 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional smooth scheme which is a complete intersection. Then we have*

$$HF_{\vartheta^{(1)}_{\mathbb{X}}}(i) = \begin{cases} 0 & \text{if } i < r_{\mathbb{X}}, \\ HF_{\mathbb{X}}(i+1-r_{\mathbb{X}}) & \text{if } i \geq r_{\mathbb{X}}. \end{cases}$$

In particular, we have $ri(\vartheta^{(1)}_{\mathbb{X}}) = ri(\vartheta_{\mathbb{X}}) - 1 = 2r_{\mathbb{X}} - 1$.

Now we present a criterion for a smooth 0-dimensional scheme $\mathbb{X}$ to be a complete intersection which uses one value of the Hilbert function of the Kähler different $\vartheta^{(1)}_{\mathbb{X}}$.

Theorem 6.6 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional smooth scheme. Then the following conditions are equivalent.*

(a) *The scheme $\mathbb{X}$ is a complete intersection.*
(b) *The scheme $\mathbb{X}$ is a CB-scheme and $HF_{\vartheta^{(1)}_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$.*
(c) *We have $x_0\vartheta^{(1)}_{\mathbb{X}} \cdot \mathfrak{C}_{\mathbb{X}} = \mathfrak{m}$.*

Proof (a)⇔(b): According to Lemma 6.2, we know that $\mathrm{HF}_{\vartheta^{(1)}_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$ if and only if $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(r_{\mathbb{X}}) \neq 0$. Hence the claim follows from Theorem 5.6.

(a)⇒(c): If $\mathbb{X}$ is a complete intersection and $\{F_1, \ldots, F_n\}$ is a homogeneous regular sequence generating $I_{\mathbb{X}}$, then we let $\Delta_0 := \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)}$. By Corollary 2.7, we get $\vartheta_{\mathbb{X}} = \langle \Delta_0 \rangle_R$, $\mathfrak{C}_{\mathbb{X}} = \langle \Delta_0^{-1} \rangle_R$, and Δ_0 is a non-zerodivisor of R. We also have $x_0\vartheta^{(1)}_{\mathbb{X}} = \Delta_0\mathfrak{m}$ by Lemma 6.2. Then, multiplying by $\mathfrak{C}_{\mathbb{X}}$, we obtain $x_0\vartheta^{(1)}_{\mathbb{X}} \cdot \mathfrak{C}_{\mathbb{X}} = \mathfrak{m}$.

(c)⇒(a): Suppose that $x_0\vartheta^{(1)}_{\mathbb{X}} \cdot \mathfrak{C}_{\mathbb{X}} = \mathfrak{m}$. It follows from the equality $x_0\vartheta^{(1)}_{\mathbb{X}} = \vartheta_{\mathbb{X}}\mathfrak{m}$ that $\vartheta_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}}\mathfrak{m} = \mathfrak{m}$. Furthermore, since $\vartheta_{\mathbb{X}}$ is a subideal of $\delta_{\mathbb{X}}$, this implies

$$\vartheta_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}} \subseteq \delta_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}} \subseteq R.$$

If $\vartheta_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}} \subsetneq R$, then $\vartheta_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}}$ is a homogeneous ideal of R contained in $\mathfrak{m}$ and

$$\langle 0 \rangle \subseteq \mathfrak{m} = \vartheta_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}}\mathfrak{m} \subseteq \langle 0 \rangle + \mathfrak{m}^2.$$

By Nakayama's lemma (cf. Kreuzer and Robbiano 2000, Proposition 1.7.15), we have $\mathfrak{m} = \langle 0 \rangle$, which is impossible. Thus we must have $\vartheta_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}} = R$. Consequently, $\mathfrak{C}_{\mathbb{X}}$ is invertible and $\vartheta_{\mathbb{X}} = \delta_{\mathbb{X}}$. So, the scheme $\mathbb{X}$ is arithmetically Gorenstein and $\vartheta_{\mathbb{X}} = \delta_{\mathbb{X}}$. Therefore Proposition 5.5 yields that $\mathbb{X}$ is a complete intersection. □

In the last part of this section we characterize 0-dimensional complete intersections using the image of the canonical map $\Psi : \Omega^1_{R/K} \longrightarrow \Omega^1_{Q^h(R)/K}$. Given a 0-dimensional scheme $\mathbb{X} \subseteq \mathbb{P}^n_K$ with $\mathrm{Supp}(\mathbb{X}) = \{p_1, \ldots, p_s\}$, we have the isomorphism $Q^h(R) \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]$ (see Proposition 3.1). Our next lemma gives us a smoothness criterion for $\mathbb{X}$ in terms of the module of Kähler differentials $\Omega^1_{Q^h(R)/K}$ of the K-algebra $Q^h(R)$.

Lemma 6.7 *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme with Supp$(\mathbb{X}) = \{p_1, \ldots, p_s\}$. Then $\mathbb{X}$ is smooth if and only if $\Omega^1_{Q^h(R)/K} \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]dT_j$.*

Proof On account of Kunz (1986, Corollary 4.8), we have

$$\Omega^1_{Q^h(R)/K} \cong \prod_{j=1}^{s} \Omega^1_{\mathcal{O}_{\mathbb{X},p_j}[T_j,T_j^{-1}]/K}.$$

Also, Kunz (1986, Formula 4.4(b)) implies

$$\begin{aligned}\Omega^1_{\mathcal{O}_{\mathbb{X},p_j}[T_j,T_j^{-1}]/K} &= \Omega^1_{\mathcal{O}_{\mathbb{X},p_j}\otimes_K K[T_j,T_j^{-1}]/K}\\ &= \mathcal{O}_{\mathbb{X},p_j} \otimes_K \Omega^1_{K[T_j,T_j^{-1}]/K} \oplus K[T_j, T_j^{-1}] \otimes_K \Omega^1_{\mathcal{O}_{\mathbb{X},p_j}/K}\\ &= \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]dT_j \oplus K[T_j, T_j^{-1}] \otimes_K \Omega^1_{\mathcal{O}_{\mathbb{X},p_j}/K}.\end{aligned}$$

It follows that $\Omega^1_{\mathcal{O}_{\mathbb{X},p_j}[T_j,T_j^{-1}]/K} = \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]dT_j$ if and only if $\Omega^1_{\mathcal{O}_{\mathbb{X},p_j}/K} = 0$. This is equivalent to the condition that p_j is a smooth point of $\mathbb{X}$ (see Kunz 1986, Theorem 7.14). Therefore $\Omega^1_{Q^h(R)/K} \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]dT_j$ if and only if the scheme $\mathbb{X}$ is smooth. □

For the remainder of this section, we assume that the scheme $\mathbb{X}$ is smooth. Using the lemma and the isomorphism $Q^h(R) \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]$, we see that $\Omega^1_{Q^h(R)/K} = Q^h(R)\, dx_0$ is a free $Q^h(R)$-module of rank one and that $dx_i = \frac{x_i}{x_0} dx_0$ for all $i = 0, \ldots, n$. Furthermore, letting $L_0 = K[x_0, x_0^{-1}]$, the canonical trace $\mathrm{Tr} = \mathrm{Tr}_{Q^h(R)/L_0} : Q^h(R) \longrightarrow L_0$ is a $Q^h(R)$-basis of $\underline{\mathrm{Hom}}_{L_0}(Q^h(R), L_0)$ and induces a homogeneous L_0-linear map of degree zero

$$\mathrm{Tr}^{\Omega} : \Omega^1_{Q^h(R)/K} \longrightarrow \Omega^1_{L_0/K}$$

such that $\mathrm{Tr}^{\Omega}(f\, dx_0) = \mathrm{Tr}(f)\, dx_0$ for $f \in Q^h(R)$.

Definition 6.8 In the setting defined above, the set

$$\Omega_{\mathbb{X}} = \{\omega \in \Omega^1_{Q^h(R)/K} \mid \mathrm{Tr}^{\Omega}(R\,\omega) \subseteq \Omega^1_{K[x_0]/K}\}$$

is clearly a graded R-module. It is called the **module of regular differential forms** of R/K (or of $\mathbb{X}$).

This module was introduced by Kunz (1975, 1978) and later generalized and extended by Kunz and Waldi (1988). In our setting, it has the following properties.

Proposition 6.9 *Let $\mathbb{X}$ be a smooth 0-dimensional subscheme of $\mathbb{P}^n_K$.*

(a) *The module of regular differential forms of $\mathbb{X}$ satisfies $\Omega_{\mathbb{X}} = \mathfrak{C}_{\mathbb{X}}\, dx_0$.*

(b) *The image of the canonical map* $\Psi : \Omega^1_{R/K} \longrightarrow \Omega^1_{Q^h(R)/K}$ *is contained in* $\Omega_{\mathbb{X}}$. *Thus it induces an R-linear map* $c_{\mathbb{X}} : \Omega^1_{R/K} \longrightarrow \Omega_{\mathbb{X}}$ *which is called the* **fundamental class** *of* $\mathbb{X}$.

Proof Let $f \in Q^h(R)$ be a non-zero homogeneous element. We see that $w = f dx_0 \in \Omega_{\mathbb{X}}$ if and only if $\mathrm{Tr}^{\Omega}(gf dx_0) = \mathrm{Tr}(gf)dx_0 \in K[x_0]dx_0$ for all $g \in R$. This is equivalent to $(f \cdot \mathrm{Tr})(g) \in K[x_0]$ for all $g \in R$, i.e., to $f \cdot \mathrm{Tr} \in \omega_R(1)$. Under the injection Φ given in Definition 3.5, this is also equivalent to $f \in \mathfrak{C}_{\mathbb{X}}$, and consequently claim (a) follows.

Next we prove claim (b). In $\Omega^1_{Q^h(R)/K}$, we have $dx_i = \frac{x_i}{x_0}dx_0$ for $i = 0, \ldots, n$. Since $\Omega^1_{R/K} = Rdx_0 + \cdots + Rdx_n$, we have $\Psi(\Omega^1_{R/K}) = \frac{\mathfrak{m}}{x_0}dx_0$. Moreover, we see that $\mathrm{HF}_{\mathfrak{C}_{\mathbb{X}}}(0) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-1) = \deg(\mathbb{X})$, and so we have $x_0(\mathfrak{C}_{\mathbb{X}})_0 = x_0(Q^h(R))_0 = (Q^h(R))_1$. Thus we get $\mathfrak{m} \subseteq x_0\mathfrak{C}_{\mathbb{X}}$. Therefore, by part (a), we obtain $\Psi(\Omega^1_{R/K}) = \frac{\mathfrak{m}}{x_0}dx_0 \subseteq \mathfrak{C}_{\mathbb{X}}dx_0 = \Omega_{\mathbb{X}}$, as we wanted to show. □

In the preceding proof we showed that $c_{\mathbb{X}}(\Omega^1_{R/K}) = \frac{\mathfrak{m}}{x_0}dx_0$. The kernel of the fundamental class can be described as follows.

Proposition 6.10 *We have* $Ker(c_{\mathbb{X}}) = \left\{ \sum_{i=0}^n g_i dx_i \in \Omega^1_{R/K} \mid \sum_{i=0}^n g_i x_i = 0 \right\}$.

Proof For an element $w = \sum_{i=0}^n g_i dx_i \in \Omega^1_{R/K}$, $c_{\mathbb{X}}(w) = (\frac{1}{x_0}\sum_{i=0}^n g_i x_i)dx_0$. So, $c_{\mathbb{X}}(w) = 0$ if and only if $\frac{1}{x_0}\sum_{i=0}^n g_i x_i = 0$. Since x_0 is a non-zerodivisor for $Q^h(R)$, this is equivalent to $\sum_{i=0}^n g_i x_i = 0$. Hence the conclusion follows. □

Next we use the image of the fundamental class to characterize 0-dimensional complete intersections.

Theorem 6.11 *Let* $\mathbb{X} \subseteq \mathbb{P}^n_K$ *be a* 0*-dimensional smooth scheme. Then* $\mathbb{X}$ *is a complete intersection if and only if* $c_{\mathbb{X}}(\Omega^1_{R/K}) = \vartheta^{(1)}_{\mathbb{X}}\Omega_{\mathbb{X}}$.

Proof Assume that $\mathbb{X}$ is a complete intersection. Let $\{F_1, \ldots, F_n\}$ be a homogeneous regular sequence generating $I_{\mathbb{X}}$. We set $\Delta_j := \frac{\partial(F_1,\ldots,F_n)}{\partial(x_0,\ldots,\widehat{x_j},\ldots,x_n)}$ for $j = 0, \ldots, n$. In $\Omega^1_{R/K}$, there are relations

$$\frac{\partial F_i}{\partial x_0}dx_0 + \frac{\partial F_i}{\partial x_1}dx_1 + \cdots + \frac{\partial F_i}{\partial x_n}dx_n = 0 \quad \text{for } i = 1, \ldots, n.$$

By Cramer's Rule, we have $\Delta_0 dx_j = (-1)^{n+1-j}\Delta_j dx_0$ for $j = 1, \ldots, n$. Thus we deduce from $\Omega^1_{R/K} = Rdx_0 + \cdots + Rdx_n$ and $\mathfrak{C}_{\mathbb{X}} = \langle \Delta_0^{-1} \rangle_R$ that

$$\begin{aligned} c_{\mathbb{X}}(\Omega^1_{R/K}) &= \left\langle \frac{\Delta_0}{\Delta_0}, \frac{\Delta_1}{\Delta_0}, \ldots, \frac{\Delta_n}{\Delta_0} \right\rangle_R dx_0 \\ &= \vartheta^{(1)}_{\mathbb{X}} \langle \Delta_0^{-1} \rangle_R dx_0 = \vartheta^{(1)}_{\mathbb{X}}\Omega_{\mathbb{X}}. \end{aligned}$$

Conversely, suppose $c_{\mathbb{X}}(\Omega^1_{R/K}) = \vartheta^{(1)}_{\mathbb{X}}\Omega_{\mathbb{X}}$. We know that $c_{\mathbb{X}}(\Omega^1_{R/K}) = \frac{\mathfrak{m}}{x_0}dx_0$. Thus we get the equality $\mathfrak{m}dx_0 = x_0\vartheta^{(1)}_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}}dx_0$. Since $\mathrm{Ann}_R(dx_0) = \langle 0 \rangle$, we obtain $\mathfrak{m} = x_0\vartheta^{(1)}_{\mathbb{X}}\mathfrak{C}_{\mathbb{X}}$. Therefore the conclusion follows from Theorem 6.6. □

From the proof of this theorem we also get a characterization of arithmetically Gorenstein schemes which is based on the following observations.

Remark 6.12 Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional smooth scheme.

(a) There is an exact sequence of graded R-modules

$$0 \longrightarrow \mathrm{Ker}(c_{\mathbb{X}}) \longrightarrow \Omega^1_{R/K} \longrightarrow \Omega_{\mathbb{X}} \longrightarrow \Omega_{\mathbb{X}}/c_{\mathbb{X}}(\Omega^1_{R/K}) \longrightarrow 0.$$

The R-module $J_{\mathbb{X}} = \Omega_{\mathbb{X}}/\Psi(\Omega^1_{R/K})$ is also known as the **Jacobian module** of $\mathbb{X}$.

(b) Notice that $c_{\mathbb{X}}(\Omega^1_{R/K}) \cong \mathfrak{m}$ and $\Omega_{\mathbb{X}} \cong \mathfrak{C}_{\mathbb{X}}(1)$. Thus the Hilbert function of $J_{\mathbb{X}}$ is given by

$$\mathrm{HF}_{J_{\mathbb{X}}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-i) - \mathrm{HF}_{\mathfrak{m}}(i)$$

for all $i \in \mathbb{Z}$. Hence $J_{\mathbb{X}}$ is a finite dimensional K-vector space with

$$\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1)\deg(\mathbb{X}) + 1 + \sum_{i=0}^{r_{\mathbb{X}}-1} (\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(i) \\ - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1)).$$

Finally, we characterize arithmetically Gorenstein schemes using the dimension of their Jacobian module as follows.

Corollary 6.13 *Let K be an infinite field, and let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional smooth scheme. Then the following conditions are equivalent.*

(a) *The scheme $\mathbb{X}$ is arithmetically Gorenstein.*
(b) *The scheme $\mathbb{X}$ is a CB-scheme and $\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1)deg(\mathbb{X}) + 1$.*

Proof If $\mathbb{X}$ is an arithmetically Gorenstein scheme, Theorem 4.15 implies $\mathrm{HF}_{\mathbb{X}}(i) + \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1) = \deg(\mathbb{X})$ for $i = 0, \dots, r_{\mathbb{X}} - 1$. Hence Remark 6.12(b) yields the equality $\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1)\deg(\mathbb{X}) + 1$. Moreover, Proposition 5.8 shows that $\mathbb{X}$ is a CB-scheme.

Conversely, assume that $\mathbb{X}$ is a CB-scheme and $\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1)\deg(\mathbb{X}) + 1$. Since K is infinite, Remark 4.13 tells us that there is an element $g \in (\mathfrak{C}_{\mathbb{X}})_{-r_{\mathbb{X}}}$ such that $\mathrm{Ann}_R(g) = \langle 0 \rangle$. Thus $R(-r_{\mathbb{X}}) \cong R \cdot g \subseteq \mathfrak{C}_{\mathbb{X}}$. This implies

$$\mathrm{HF}_{\mathbb{X}}(i) \le \mathrm{HF}_{\mathfrak{C}_{\mathbb{X}}}(i - r_{\mathbb{X}}) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1)$$

for all $i \in \mathbb{Z}$. Since $\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1)\deg(\mathbb{X}) + 1$, we deduce $\sum_{i=0}^{r_{\mathbb{X}}-1}(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(i) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1)) = 0$. Hence we must have $\mathrm{HF}_{\mathbb{X}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1)$ for all $i \in \mathbb{Z}$. Observe that $\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1) = \mathrm{HF}_{\mathbb{X}}(0) = 1$ and $\mathrm{HF}_{\mathfrak{C}_{\mathbb{X}}}(i - r_{\mathbb{X}}) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1) = \mathrm{HF}_{\mathbb{X}}(i) = \mathrm{HF}_{\langle g \rangle_R}(i - r_{\mathbb{X}})$. Thus we get $\mathfrak{C}_{\mathbb{X}} = \langle g \rangle_R$, and therefore $\mathbb{X}$ is arithmetically Gorenstein. □

Acknowledgements This paper is partially based on the L. N. Long's dissertation (Long 2015). M. Kreuzer and L. N. Long thank Jürgen Herzog and Ernst Kunz for their encouragement to elaborate the results presented here. We are also grateful to Uwe Storch for pointing out the characterization in Scheja and Storch (1975). Moreover, we thank the Mathematics Department of IISc Bangalore (India), and in particular Dilip Patil, for their kind hospitality during part of the preparation of this paper. Last, but not least, we are extremely thankful to the referee for his very detailed and enlightening comments.

References

The ApCoCoA Team. ApCoCoA: approximate computations in commutative algebra (2007). http://www.apcocoa.org

Bruns, W., Herzog, J.: Cohen–Macaulay Rings Cambridge Stud. Adv. Math., vol. 39. Cambridge University Press, Cambridge (1993)

Bourbaki, N.: Commutative Algebra, Chap. 1–7. Springer, Berlin (1989)

Davis, E.D., Geramita, A.V., Orecchia, F.: Gorenstein algebras and the Cayley–Bacharach theorem. Proc. Am. Math. Soc. **93**, 593–597 (1985)

Davis, E., Maroscia, P.: Complete intersections in $\mathbb{P}^2$: Cayley–Bacharach characterizations. In: Complete Intersections- Acireale 1983. In: Lecture Notes in Mathematics, vol. 1092, pp. 253–269 (1984)

Eisenbud, D.: Commutative algebra with a view toward algebraic geometry. In: Grad. Texts in Math., vol. 150. Springer, New York (1995)

Griffiths, P., Harris, J.: Residues and zero-cycles on algebraic varieties. Ann. Math. **108**, 461–505 (1978)

Geramita, A.V., Kreuzer, M., Robbiano, L.: Cayley–Bacharach schemes and their canonical modules. Trans. Am. Math. Soc. **339**, 163–189 (1993)

Geramita, A.V., Maroscia, P.: The ideal of forms vanishing at a finite set of points in $\mathbb{P}^n$. J. Algebra **90**, 528–555 (1984)

Geramita, A.V., Orecchia, F.: On the Cohen–Macaulay type of s-lines in $\mathbb{A}^{n+1}$. J. Algebra **70**, 116–140 (1981)

Goto, S., Watanabe, K.: On graded rings I. J. Math. Soc. Jpn. **30**, 179–213 (1978)

Herzog, J., Kunz, E., Waldi, R.: Der kanonische Modul eines Cohen–Macaulay Rings. Lect. Notes in Math., vol. 238. Springer, Heidelberg (1971)

Kreuzer, M., Linh, T.N.K., Long, L.N.: Kähler differentials and Kähler differents for fat point schemes. J. Pure Appl. Algebra **219**, 4479–4509 (2015)

Kreuzer, M.: On 0-dimensional complete intersections. Math. Ann. **292**, 43–58 (1992)

Kreuzer, M.: On the canonical module of a 0-dimensional scheme. Can. J. Math. **141**, 357–379 (1994)

M. Kreuzer: Beiträge zur Theorie der nulldimensionalen Unterschemata projektiver Räume, Regensburger Math. Schr. 26, Universität Regensburg (1998)

Kreuzer, M.: *On the canonical ideal of a set of points*, Boll. U.M.I. (8) **1-B**, 221–261 (2000)

Kreuzer, M., Migliore, J., Nagel, U., Peterson, C.: Determinantal schemes and Buchsbaum–Rim sheaves. J. Pure Appl. Algebra **150**, 155–174 (2000)

Kreuzer, M., Robbiano, L.: Commutational Commutative Algebra 1. Springer, Heidelberg (2000)

Kreuzer, M., Robbiano, L.: Commutational Commutative Algebra 2. Springer, Heidelberg (2005)

Kunz, E.: Introduction to commutative algebra and algebraic geometry. Birkhäuser, Boston (1985)

Kunz, E.: Kähler differentials. In: Adv. Lectures Math. Wieweg Verlag, Braunschweig (1986)

Kunz, E.: Holomorphe Differentialformen auf algebraischen Varietäten mit Singularitäten I. Manuscripta Math. **15**, 91–108 (1975)

Kunz, E.: Differentialformen auf algebraischen Varitäten mit Singularitäten II. Abh. Math. Sem. Univ. Hamburg **47**, 42–70 (1978)

Kunz, E., Waldi, R.: Regular differential forms. Contemporary Mathematics, vol. 79. Am. Math. Soc., Providence (1988)

Long, Le Ngoc: Various Differents for 0-Dimensional Schemes and Appplications. Universtät Passau, Passau (2015)

Lenstra, H.W.: Complete intersections and Gorenstein rings. In: Conference on Elliptic Curves and Modular Forms, Hong Kong, pp. 99–109 (1993)

Morandi, P.: Field and Galois Theory. Springer, New York (1996)

Scheja, G., Storch, U.: Über Spurfunktionen bei vollständigen Durchschnitten. J. Reine Angew. Math. **278**(279), 174–190 (1975)

Wiebe, H.: Über homologische Invarianten lokaler Ringe. Math. Ann. **179**, 257–274 (1969)
Wiles, A.: Modular elliptic curves and Fermat's Last Theorem. Ann. Math. **141**, 443–551 (1995)
Zariski, O., Samuel, P.: Commutative Algebra, vol. 1. van Nostrand, Princeton (1958)