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Another equivalent of the Lotschnittaxiom

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Abstract We prove that Bachmann's *Lotschnittaxiom*, which states that perpendiculars raised on the two legs of a right angle always meet, is equivalent, with respect to Hilbert's plane absolute geometry, to the statement **P**: "For any point P, line l, with P not incident with l, and any line g, there exists a point G on g for which the distance to P exceeds the distance to l".

Keywords Absolute geometry · Lotschnittaxiom

Mathematics Subject Classification 51F05

1 Introduction

Bachmann (1964) introduced an important axiom in the foundation of geometry, stating that "Every quadrilateral with three right angles closes" (or "If a, b, c and d are lines such that a is orthogonal to b, b is orthogonal to c, and c is orthogonal to d, then a and d must intersect.") He called it the *Lotschnittaxiom* (to be referred to as $\mathbf{A}_1$) and provided two equivalent statements for it, the equivalence holding over Hilbert's plane absolute geometry $\mathcal{A}$ (whose axioms are the plane axioms of groups I, II, and III of Hilbert's *Grundlagen der Geometrie*, being equivalent to the axioms A1–A9 in Schwabhäuser et al. (1983), the models of which are referred to as *Hilbert planes*). One of those equivalent statements is $\mathbf{A_2}$, which states that: "If l and m are two lines which form a half-right angle (i.e., such that the reflection of l in m is orthogonal to

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l), then every line perpendicular to l must intersect m." The other statement that is equivalent to $\mathbf{A}_1$ is one indicative of the fact that the *Lotschnittaxiom* is weaker than the Euclidean parallel postulate. It states that "Through any point in the interior of a right angle there is a line which intersects the sides of that angle" ($\mathbf{A}_3$). A like-minded statement equivalent with respect to $\mathcal{A}$ to the Euclidean parallel postulate was first stated by J. F. Lorenz in 1791: "Through every point in the interior of any given angle there is a line intersecting the sides of that angle." An order-free variant of $\mathbf{A}_3$, which is equivalent to $\mathbf{A}_3$ with respect to $\mathcal{A}$, was stated as axiom (s) in (Knüppel (1977), p. 6): "If a and b are two orthogonal lines, then any line c must intersect one of a or b."

The first time a statement equivalent to $\mathbf{A}_1$ was presented was long before 1964. On February 3, 1806 by Lagrange used the following axiom, whose equivalence to $\mathbf{A}_1$ was shown in Pambuccian (2009): "If a and b are two parallels from P to g, then the reflection of a in b is parallel to g as well." As shown in Pambuccian (1994), $\mathbf{A}_1$ is also equivalent, with respect to $\mathcal{A}$, to the *universal* statement "The altitude to the base in an isosceles triangle with base angles of 45° is less than the base."

The *Lotschnittaxiom* is essential for a geometric understanding of Pejas's (Pejas 1961) algebraic characterization of the models of $\mathcal{A}$. A Hilbert plane $\mathfrak{H}$ can be embedded in a projective-metric plane over a Pythagorean ordered field K, in such a way that no point in it lies on the line [0, 0, 1] (written in homogeneous coordinates), so all its points can be written as $(a, b, 1)$. The set $M = \{a \in K : (a, 0, 1) \in \mathfrak{H}\}$ is called the set of abscissae of $\mathfrak{H}$. If M is a submodule of K (i.e. if M is a subgroup of the additive group of K), then $\mathfrak{H}$ is called *modular*. According to Bachmann (1964, Satz 2), a Hilbert plane embedded in a projective-metric plane is modular precisely if it satisfies $\mathbf{A}_1$.

More general absolute geometries satisfying the *Lotschnittaxiom* have been investigated in Dress (1966) and Knüppel (1977).

The aim of this note is to show that the *Lotschnittaxiom* is equivalent to yet another statement of geometric interest. In Euclidean geometry, given a non-incident pair of point P and line l, the locus of all points Q equidistant from P and l is a parabola. The locus of all the points Q for which the distance to l is greater than the distance to P is the open convex region determined by that parabola. That region does not contain any (complete) line. Which absolute geometries, i.e. models of $\mathcal{A}$, have this property? We can state this property as

P *For any non-incident pair (P, l) and any line h, there is a point Q on h whose distance to l is less than or equal to its distance to P.*

The somewhat surprising answer is that those are precisely the absolute geometries satisfying the *Lotschnittaxiom*.

Theorem 1 $\mathcal{A} \vdash \mathbf{A}_2 \leftrightarrow \mathbf{P}$.

2 Proving the equivalence

Suppose $\mathfrak{M}$ is a model of $\mathcal{A}$ in which $\mathbf{A}_2$ holds, and let (P, l) be a non-incident pair consisting of a point P and a line l. Let O denote the foot of the perpendicular from P to l, and let m denote the line OP. Let b and b' denote the angle bisectors of the right angle formed by the lines m and l.

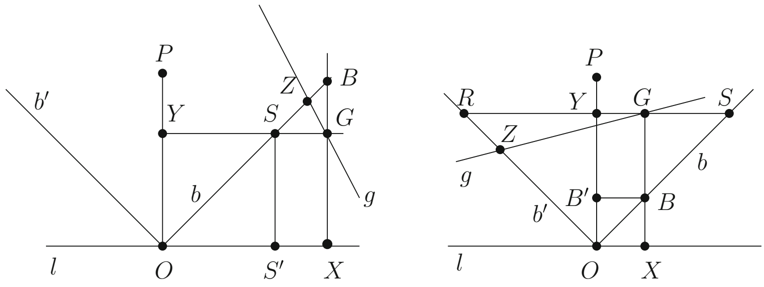


Fig. 1 $\mathbf{A_2} \to \mathbf{P}$

Suppose now that g is a line, such that all points Q on g have the property that their distance to l exceeds their distance to P. We want to show that g and b (or g and b') must intersect (see Fig. 1).

Let G be an arbitrary point on g that is not on m (there must be such points, as g cannot be m). If G is incident with b or b', then there is nothing left to prove. Suppose G lies in the quadrant determined by l and m in which a half-line of b lies (the proof proceeds analogously were G to lie in the quadrant determined by l and m in which a half-line of b' lies). Let X and Y denote the feet of the perpendiculars from G to l and to m respectively. Notice that no point of g can lie on l, so that G is not on l. By the Crossbar Theorem [see Greenberg (2008), p. 116] and the Pasch axiom, the half-line of b that lies in the same quadrant determined by l and m in which G lies intersects one of the sides GX or GY of the quadrilateral $OXGY$. By $\mathbf{A}_2$, the line GX, being a perpendicular to l, intersects b in some point B, and the line GY, being perpendicular to OP, intersects b in some point S. Thus either B belongs to the *segment* GX or S belongs to the *segment* GY.

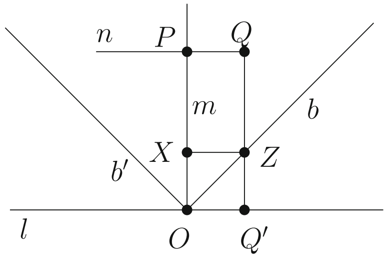
If S belongs to the segment GY, then let S' be the foot of the perpendicular from S to l. Since orthogonal projection preserves betweenness and S lies between G and Y, point S' must lie between O and X, thus S must lie between O and B, and finally G must lie between X and B. Applying the Pasch axiom to triangle OXB and line g, and bearing in mind that g cannot intersect l, we conclude that g must intersect b.

If B belongs to the segment GX, then let B' be the foot of the perpendicular from B to OP. Since B lies between X and G and orthogonal projection preserves betweenness, B' lies between O and Y, thus B lies between O and S, and finally G lies between Y and S. Let R be the reflection of S in OP. Notice that R lies on b'. Since G lies between Y and S, it lies between S and R as well. Unless g is YS, in which case we are done, as g and b intersect, we can apply the Pasch axiom to triangle ROS with secant g to conclude that g intersects one of the sides SO or RO, thus b or b'.

Let Z be the point of intersection of g and b. Let X' and Y' denote the feet of the perpendiculars from Z to l and OP respectively. Then ZY' is congruent to ZX', and $ZP \geq ZY'$, so $ZP \geq ZX'$, a contradiction, as Z was on g. Thus $\mathbf{A_2} \to \mathbf{P}$.

Suppose now $\mathfrak{M}$ is a model of $\mathcal{A}$ in which $\mathbf{P}$ holds. Let l be a line, O a point on it, m the perpendicular in O on l, and b and b' be the two angle bisectors of the angle formed by l and m. Let P be an arbitrary point on m. We will show that n, the perpendicular

Fig. 2 $\mathbf{P} \rightarrow \mathbf{A_2}$



in P on m intersects b and b', and thus that $\mathbf{A_2}$ must hold (see Fig. 2). By $\mathbf{P}$, there is a point Q on n such that $QQ' < QP$, where Q' is the foot of the perpendicular from Q to l. Now Q lies either in the quadrant determined by l and m in which a ray from b lies or in the quadrant in which a ray from b' lies. We can assume w. l. o. g. that it lies in the quadrant visited by b. The line b divides the plane into two half-planes: one containing P and one containing Q'. If Q were in the same half-plane in which P lies, then the segment QQ' would have to intersect b in a point Z. Being on the bisector, Z has the property that ZQ' is congruent to ZX, where X is the foot of the perpendicular from Z to m. Thus $QQ' = QZ + ZQ' = ZQ + ZX > QX > QP$, the first inequality being the triangle inequality and the second one stemming from the hypotenuse being greater than the side, contradicting the fact that $QQ' < QP$. Thus P and Q must lie on different sides of b, so the segment PQ must intersect b, thus n must intersect b. Thus $\mathbf{P} \rightarrow \mathbf{A_2}$.

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