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Generalized derivations and commutativity of prime Banach algebras

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Abstract In this article we study various conditions on a unital prime Banach algebra that ensure its commutativity. More specifically, we prove that a unital prime Banach algebra A with a nonzero continuous linear generalized derivation g associated with a nonzero linear continuous derivation d satisfying either $g((xy)^n) - d(x^n)d(y^n) \in Z(A)$ or $g((xy)^n) - d(y^n)d(x^n) \in Z(A)$, for sufficiently many x, y and an integer $n = n(x, y) > 1$ is commutative.

Keywords Prime Banach algebra · Generalized derivation · Left multiplier

Mathematics Subject Classification 16W25 · 16N60 · 46J10

1 Introduction

The symbol A shall denote a Banach algebra over the complex field $\mathbf{C}$ with unity e. The symbols $Z(A)$ and M will denote the center and a closed linear subspace of A. An algebra A is said to be prime if for any $x, y \in A$ such that $xAy = 0$, either $x = 0$ or $y = 0$. An additive mapping $d : A \to A$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ and $d(cx) = cd(x)$, for all $x, y \in A$ and for all $c \in \mathbf{C}$. An additive mapping $g : A \to A$ is said to be a generalized derivation associated with a derivation d if $g(xy) = g(x)y + xd(y)$ and $g(cx) = cg(x)$, for all $x, y \in A$

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and for all $c \in \mathbf{C}$. An additive mapping $T : A \rightarrow A$ is said to be a left multiplier if $T(xy) = T(x)y$ and $T(cx) = cT(x)$, for all $x, y \in A$ and for all $c \in \mathbf{C}$.

Yood (1991), has shown that if A is a unital Banach algebra and $n = n(x, y) > 1$ is a positive integer such that either $(xy)^n - x^n y^n \in M$ or $(xy)^n - y^n x^n \in M$ for sufficiently many x and y, then $[x, y] \in M$. Ali and Khan (2015) have shown that if A is a unital prime Banach algebra with nonzero continuous linear derivation $d : A \rightarrow A$ such that either $d((xy)^m) - x^m y^m$ or $d((xy)^m) - y^m x^m$ is in the center $Z(A)$ of A for an integer $m = m(x, y) > 1$ and sufficiently many x, y, then A is commutative. In this article we extend these results for a generalized derivation. Let $p(t) = \sum_{i=0}^{n} a_i t^i$ be a polynomial in real variable t with coefficients in A. As mentioned in Yood (1991), if $p(t) \in M$ for infinitely many real t then each $a_i \in M$.

For a derivation d on a prime ring R, Posner proved the following result:

Lemma 1 [Posner (1957), Theorem 2] *Let R be a prime ring and d be a nonzero derivation of R such that $[d(x), x]$ is in the center of R, for all $x \in R$. Then R is commutative.*

Many other authors have generalized Posner's result in several ways for rings and algebras (see Bell 1999; Brešar 2004; Herstein 1961; Vukman 1992; Yood 1984, 1990, where further references can be found). One of the generalizations of Posner's result by Lee and Lee is of our interest:

Lemma 2 [Lee and Lee (1983), Theorem 2] *Let d be a nonzero derivation on a prime ring R and U be a lie ideal of R such that $[x, d(U)] \subseteq Z(R)$. Then either $x \in Z(R)$ or $U \subseteq Z(R)$.*

Ashraf and Ali have given a relationship between the commutativity of a ring and its left multipliers (see Ashraf and Ali 2008). In this article we also find a connection between the commutativity of a prime Banach algebra and its left multipliers.

2 Main results

Theorem 1 *Let A be a unital prime Banach algebra and $g : A \rightarrow A$ be a nonzero continuous generalized derivation associated with nonzero continuous derivation d on A such that $g(e) \in Z(A)$ and $d(g(e)) \neq 0$. Suppose that there are open subsets G_1 and G_2 of A such that either $g((xy)^n) - d(x^n)d(y^n) \in Z(A)$ or $g((xy)^n) - d(y^n)d(x^n) \in Z(A)$ for each $x \in G_1$ and for each $y \in G_2$ and an integer $n = n(x, y) > 1$. Then A is commutative.*

Proof Set

$$f(x, y, n) := g((xy)^n) - d(x^n)d(y^n)$$

and

$$h(x, y, n) := g((xy)^n) - d(y^n)d(x^n).$$

Let $x \in G_1$ be an arbitrarily fixed element. For each positive integer n consider the set $U_n = \{y \in A \mid f(x, y, n) \notin Z(A), h(x, y, n) \notin Z(A)\}$. We show that U_n is open by showing that its complement U_n^c is closed. Let (s_k) be a sequence in U_n^c such that $\lim s_k = s$. Since $s_k \in U_n^c$ we have either

$$h(x, s_k, n) = g((xs_k)^n) - d(x^n)d(s_k^n) \in Z(A) \tag{1}$$

or

$$f(x, s_k, n) = g((xs_k)^n) - d(s_k^n)d(x^n) \in Z(A). \tag{2}$$

Since g and d are continuous it follows that

$$g((xs)^n) - d(x^n)d(s^n) \in Z(A) \text{ or } g((xs)^n) - d(s^n)d(x^n) \in Z(A).$$

This shows that $s \in U_n^c$ and so U_n is open.

By the Baire category theorem, if each U_n is dense then their intersection is also dense, which contradicts the existence of G_2. Therefore there exists an integer $i = i(x) > 1$ such that U_i is not dense and a nonempty open set G_3 in the complement of U_i such that either $g((xy)^i) - d(x^i)d(y^i) \in Z(A)$ or $g((xy)^i) - d(y^i)d(x^i) \in Z(A)$, for all $y \in G_3$. Take $z \in G_3$ and $w \in A$. So for sufficiently small real t, $(z + tw) \in G_3$ and either

$$g((x(z + tw))^i) - d(x^i)d((z + tw)^i) \in Z(A) \tag{3}$$

or

$$g((x(z + tw))^i) - d((z + tw)^i)d(x^i) \in Z(A). \tag{4}$$

Thus for infinitely many t either (3) or (4), say (3) must hold. Since $g((x(z+tw))^i) - d(x^i)d((z + tw)^i)$ is polynomial in t which is in $Z(A)$, each of its coefficients must be in $Z(A)$. The coefficient of t^i is $g((xw)^i) - d(x^i)d(w^i)$ which is in $Z(A)$. We have therefore shown that, given $x \in G_1$, there exists a positive integer i depending on x such that for each $w \in A$ either

$$g((xw)^i) - d(x^i)d(w^i) \in Z(A)$$

or

$$g((xw)^i) - d(w^i)d(x^i) \in Z(A).$$

Next we show that for each $y \in A$ there is an integer $j = j(y) > 1$ such that for each $v \in A$ either

$$g((vy)^j) - d(v^j)d(y^j) \in Z(A)$$

or

$$g((vy)^j) - d(y^j)d(v^j) \in Z(A).$$

Fix an arbitrary $y \in A$ and for each $k > 1$, define

$$W_k = \{z \in A \mid f(z, y, k) \notin Z(A), h(z, y, k) \notin Z(A)\}.$$

As shown above each W_k is open, so by the Baire category theorem, if each W_k is dense then its intersection is also dense which contradicts the existence of G_1. Hence there exists an integer $j = j(y) > 1$ and a nonempty open subset G_4 in the complement of W_j. If $z \in G_4$ and $v \in A$ then for sufficiently small real t, $(z + tv) \in G_4$ and either

$$g(((z + tv)y)^j) - d((z + tv)^j)d(y^j) \in Z(A) \tag{5}$$

or

$$g(((z + tv)y)^j) - d(y^j)d((z + tv)^j) \in Z(A). \tag{6}$$

With the same argument as earlier we see that for each $v \in A$, either

$$g((vy)^j) - d(v^j)d(y^j) \in Z(A)$$

or

$$g((vy)^j) - d(y^j)d(v^j) \in Z(A).$$

Now let S_k, $k > 1$ be the set of all $y \in A$ such that for each $w \in A$ either

$$g((wy)^k) - d(w^k)d(y^k) \in Z(A)$$

or

$$g((wy)^k) - d(y^k)d(w^k) \in Z(A).$$

The union of S_k is A. It is obvious to see that each S_k is closed. Again, by the Baire category theorem, some S_n must contain a nonempty open subset G_5. Take $z \in G_5$ and $x \in A$. For all sufficiently small real t and each $w \in A$ either

$$g((w(z + tx))^n) - d(w^n)d((z + tx)^n) \in Z(A)$$

or

$$g((w(z + tx))^n) - d((z + tx)^n)d(w^n) \in Z(A).$$

By earlier arguments, for all $x, w \in A$, we have either

$$g((wx)^n) - d(w^n)d(x^n) \in Z(A)$$

or

$$g((wx)^n) - d(x^n)d(w^n) \in Z(A).$$

Since A is unital, for all real t and for all $x, y \in A$ we have either

$$g(((e + tx)y)^n) - d((e + tx)^n)d(y^n) \in Z(A) \tag{7}$$

or

$$g(((e+tx)y)^n) - d(y^n)d((e+tx)^n) \in Z(A). \tag{8}$$

Now, by collecting the coefficient of t in the above expressions, we have either

$$g(xy^n + Q) - nd(x)d(y^n) \in Z(A) \quad \textit{for all} \quad x, y \in A \tag{9}$$

or

$$g(xy^n + Q) - nd(y^n)d(x) \in Z(A) \quad \textit{for all} \quad x, y \in A, \tag{10}$$

where $Q = \sum_{k=1}^{n-1} y^k x y^{n-k}$.

Again if we start with $g((y(e+tx))^n)$ in place of $g(((e+tx)y)^n)$, we have either

$$g(y^n x + Q) - nd(y^n)d(x) \in Z(A) \quad \textit{for all} \quad x, y \in A \tag{11}$$

or

$$g(y^n x + Q) - nd(x)d(y^n) \in Z(A) \quad \textit{for all} \quad x, y \in A. \tag{12}$$

At least one of the pairs of equations {(9) (11)}, {(10) (12)},{(9) (12)} and {(10) (11)} must hold. Subtracting these pairs we have either

$$g[x, y^n] - n[d(x), d(y^n)] \in Z(A) \quad \textit{for all} \quad x, y \in A \tag{13}$$

or

$$g[x, y^n] + n[d(x), d(y^n)] \in Z(A) \quad \textit{for all} \quad x, y \in A \tag{14}$$

or

$$g[x, y^n] \in Z(A) \quad \textit{for all} \quad x, y \in A. \tag{15}$$

Now if $g[x, y^n] \in Z(A)$ then by replacing y by $(e + ty)$ we have $g[x, y] \in Z(A)$ for all $x, y \in A$. Replace x by ex we get $g(e)[x, y] + d[x, y] \in Z(A)$, or equivalently,

$$[g(e)[x, y] + d[x, y], z] = 0, \quad \textit{for all} \quad x, y, z \in A. \tag{16}$$

This can be written as $[g(e)[x, y]+[x, dy], z]+[[dx, y], z] = 0$. Replace y by $[y, w]$ we get $[g(e)[x, [y, w]] + [x, d[y, w]], z] + [[dx, [y, w]], z] = 0$. Now use (16) to get $[[dx, [y, w]], z] = 0$; hence, $[dx, [y, w]] \in Z(A)$, for all $x, y, w \in A$. In the light Lemma 2 we have either $[y, w] \in Z(A)$ or $A \subseteq Z(A)$. In both the cases A is commutative.

Now if (13) holds then by replacing y by $(e + ty)$ we have $g[x, y] - n[d(x), d(y)] \in Z(A)$ for all $x, y \in A$. In this expression replace x by $xg(e)$ in order to obtain

$(g[x, y] - n[d(x), d(y)])g(e) + [x, y]d(g(e)) - n[xd(g(e)), d(y)] \in Z(A)$ for all $x, y \in A$. Consequently, since $Z(A)$ is a linear subspace of A and $(g[x, y] - n[d(x), d(y)])g(e) \in Z(A)$, it follows that $[x, y]d(g(e)) - n[xd(g(e)), d(y)] \in Z(A)$ for all $x, y \in A$. Hence if we set $y = x$ and observe that $g(e) \in Z(A)$ implies $d(g(e)) \in Z(A)$, we obtain $[x, d(x)]d(g(e)) \in Z(A)$ for all $x \in A$. So, in particular, $[[x, d(x)]d(g(e)), z] = 0$ for all $x, z \in A$. Replacing z by zy now yields $[[x, d(x)], z]yd(g(e)) = 0$ for all $x, y, z \in A$. Thus, since A is prime and $d(g(e)) \neq 0$, it follows that $[x, d(x)] \in Z(A)$ for all $x \in A$. Hence, by Lemma 1 we may infer that A is commutative. Similarly it can be shown that if (14) holds then A is commutative. □

Theorem 2 *Let A be a unital prime Banach algebra and $g : A \to A$ be a nonzero continuous generalized derivation associated with a nonzero continuous derivation d on A such that $g(e) \in Z(A)$. Suppose that there are open subsets G_1 and G_2 of A such that either $g((xy)^n) - x^n y^n \in Z(A)$ or $g((xy)^n) - y^n x^n \in Z(A)$ for each $x \in G_1$ and for each $y \in G_2$ and an integer $n = n(x, y) > 1$. Then A is commutative.*

Proof Proceeding as in Theorem 1, we get either

$$g[x, y] \in Z(A) \quad \text{for all} \quad x, y \in A \tag{17}$$

or

$$g[x, y] - n[x, y] \in Z(A) \quad \text{for all} \quad x, y \in A \tag{18}$$

or

$$g[x, y] + n[x, y] \in Z(A) \quad \text{for all} \quad x, y \in A. \tag{19}$$

Suppose (17) holds. Replace x by ex we get $g(e)[x, y] + d[x, y] \in Z(A)$, or equivalently,

$$[g(e)[x, y] + d[x, y], z] = 0, \quad \text{for all} \quad x, y, z \in A. \tag{20}$$

This can be written as $[g(e)[x, y] + [x, dy], z] + [[dx, y], z] = 0$. Replace y by $[y, w]$ we get $[g(e)[x, [y, w]] + [x, d[y, w]], z] + [[dx, [y, w]], z] = 0$. Use (20) to get $[[dx, [y, w]], z] = 0$ or $[dx, [y, w]] \in Z(A)$, for all $x, y, w \in Z(A)$. In the light Lemma 2 we have either $[y, w] \in Z(A)$ or $A \subseteq Z(A)$. In both the cases A is commutative.

Now consider $g[x, y] - n[x, y] \in Z(A)$, for all $x, y \in A$. Replace x by ex we get $g(e)[x, y] + d[x, y] - n[x, y] \in Z(A)$ or $[(g(e) - n)[x, y] + d[x, y], z] = 0$ for all $x, y, z \in A$, which is similar to Eq. (20). Thus in this case it can also be shown that A is commutative. Similarly, we can prove that if $g[x, y] + n[x, y] \in Z(A)$ then A is commutative. □

Theorem 3 *Let A be a unital prime Banach algebra and $g : A \to A$ be a nonzero continuous generalized derivation associated with a nonzero continuous derivation d*

on A such that $g(e) \in Z(A)$. Suppose that there are open subsets G_1 and G_2 of A such that either $g((xy)^n - x^n y^n) \in Z(A)$ or $g((xy)^n - y^n x^n) \in Z(A)$ for each $x \in G_1$ and for each $y \in G_2$ and an integer $n = n(x, y) > 1$. Then A is commutative.

Proof Proceeding as in Theorem 1, we obtain either

$$g[x, y] \in Z(A) \quad \text{for all} \quad x, y \in A \tag{21}$$

or

$$g([x, y] - n[x, y]) \in Z(A) \quad \text{for all} \quad x, y \in A \tag{22}$$

or

$$g([x, y] + n[x, y]) \in Z(A) \quad \text{for all} \quad x, y \in A. \tag{23}$$

Equations (22) and (23) can be written as $(1-n)g[x, y] \in Z(A)$ and $(1+n)g[x, y] \in Z(A)$, respectively. Thus the equations (21), (22) and (23) all reduce to $g[x, y] \in Z(A)$. The result then follows using the argument in the proof of Theorem 2. □

Theorem 4 *Let A be a unital prime Banach algebra and $T : A \to A$ be a continuous left multiplier on A such that $T(x) \neq \pm nx$, for all nonzero $x \in A$ and integers $n \geq 0$. Suppose that there are open subsets G_1 and G_2 of A such that either $T((xy)^n) - x^n y^n \in Z(A)$ or $T((xy)^n) - y^n x^n \in Z(A)$ for each $x \in G_1$ and for each $y \in G_2$ and an integer $n = n(x, y) > 1$. Then A is commutative.*

Proof Proceeding as in Theorem 1, we get either

$$T[x, y] \in Z(A) \quad \text{for all} \quad x, y \in A \tag{24}$$

or

$$T[x, y] - n[x, y] \in Z(A) \quad \text{for all} \quad x, y \in A \tag{25}$$

or

$$T[x, y] + n[x, y] \in Z(A) \quad \text{for all} \quad x, y \in A. \tag{26}$$

If $T[x, y] \in Z(A)$, for all $x, y \in A$, then $[T[x, y], z] = 0$, for all $x, y, z \in A$. Replacing y by yx and then z by rz we have $T[x, y]r[x, z] = 0$, for all $r, x, y, z \in A$. Hence, $T[x, y]r[x, y] = 0$, for all $r, x, y \in A$, and so, by hypothesis on T and the fact that A is prime, we have $[x, y] = 0$, for all $x, y \in A$. If $T[x, y] \pm n[x, y] \in Z(A)$, for all $x, y \in A$, then, as before $(T[x, y] \pm n[x, y])r[x, z] = 0$, for all $r, x, y, z \in A$. Since $n > 1$ it follows that $T[x, y] \neq \pm n[x, y]$, for all $x, y \in A$, and so, $[x, z] = 0$, for all $x, z \in A$. This completes the proof.

Theorem 5 *Let A be a unital prime Banach algebra and $T : A \to A$ be a continuous left multiplier on A such that $T(x) \neq \pm nx$, for all $n \geq 0$. Suppose that there are open subsets G_1 and G_2 of A such that either $T((xy)^n - x^n y^n) \in Z(A)$ or $T((xy)^n - y^n x^n) \in Z(A)$ for each $x \in G_1$ and for each $y \in G_2$ and an integer $n = n(x, y) > 1$. Then A is commutative.*

Proof The proof is similar to the proof of Theorem 4.

Theorem 6 *Let A be a unital prime Banach algebra and $T : A \to A$ be a nonzero continuous left multiplier on A such that $T(e) \in Z(A)$ and $nT(e) \neq \pm e$, for all integers $n \geq 1$. Suppose that there are open subsets G_1 and G_2 of A such that either $T((xy)^n) - T(x^n)T(y^n) \in Z(A)$ or $T((xy)^n) - T(y^n)T(x^n) \in Z(A)$ for each $x \in G_1$ and for each $y \in G_2$ and an integer $n = n(x, y) > 1$. Then A is commutative.*

Proof Proceeding as in Theorem 1, we obtain either

$$T[x, y] \in Z(A) \quad \text{for all} \quad x, y \in A \tag{27}$$

or

$$T[x, y] - n[T(x), T(y)] \in Z(A) \quad \text{for all} \quad x, y \in A \tag{28}$$

or

$$T[x, y] + n[T(x), T(y)] \in Z(A) \quad \text{for all} \quad x, y \in A. \tag{29}$$

Since $T(e) \in Z(A)$, it follows from the fact that A is prime that $T(x) = 0$ implies $x = 0$. Now, if $T[x, y] \in Z(A)$, for all $x, y \in A$, then A is commutative as in the proof of Theorem 4. On the other hand, if $T[x, y] \pm n[T(x), T(y)] \in Z(A)$, for all $x, y \in A$, then as in the proof of Theorem 4, we have

$$(T[x, y] \pm n[T(x), T(y)])r[x, z] = 0, \quad \text{for all} \quad r, x, y, z \in A.$$

Hence, $T[x, y](e \pm nT(e))r[x, y] = 0$, for all $r, x, y \in A$. If $T[x, y](e \pm nT(e)) = 0$ then $T[x, y]r(e \pm nT(e)) = 0$, for all $r \in A$ and so, $T[x, y] = 0$ and hence $[x, y] = 0$. So, $[x, y] = 0$, for all $x, y \in A$. The result now follows. □

3 Open questions

The authors would like to open the following questions for further studies:

Question 1 Can the hypothesis that $g(e) \in Z(A)$ be removed from the assumptions in Theorem 1, Theorem 2 and Theorem 3?

Question 2 Can the hypothesis that $T(x) \neq \pm nx$ be removed from the assumptions in Theorem 4 and Theorem 5?

Question 3 Can the hypotheses $T(e) \in Z(A)$ and $nT(e) \neq \pm e$, for $n \geq 1$ be removed from the assumptions in Theorem 6?

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References

Ali, S., Khan, A.N.: On commutativity of banach algebras with derivations. Bull. Aust. Math. Soc. **91**(03), 419–425 (2015)

Ashraf, M., Ali, S.: On left multipliers and the commutativity of prime rings. Demonstratio Math. **41**(4), 763–771 (2008)

Bell, H.E.: On the commutativity of prime rings with derivation. Quaestiones Mathematicae **22**(3), 329–335 (1999)

Brešar, M.: Commuting maps: a survey. Taiwanese J. Math. **8**(3), 361 (2004)

Herstein, I.N.: Power maps in rings. Michigan Math. J. **8**(1), 29–32 (1961)

Lee, P.H., Lee, T.K.: Lie ideals of prime rings with derivations. Bull. Inst. Math. Acad. Sinica (NS) **11**(1), 75–80 (1983)

Posner, E.C.: Derivations in prime rings. Proc. Am. Math. Soc. **8**(6), 1093–1100 (1957)

Vukman, J.: On derivations in prime rings and banach algebras. Proc. Am. Math. Soc. **116**(4), 877–884 (1992)

Yood, B.: Continuous homomorphisms and derivations on banach algebras. Contemp. Math **32**, 279–284 (1984)

Yood, B.: Commutativity theorems for banach algebras. Michigan Math. J. **37**(2), 203–210 (1990)

Yood, B.: On commutativity of unital banach algebras. Bull. London Math. Soc. **23**(3), 278–280 (1991)