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ORIGINAL PAPER

The Steiner–Lehmus theorem and "triangles with congruent medians are isosceles" hold in weak geometries

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Abstract We prove that (a) a generalization of the Steiner–Lehmus theorem due to A. Henderson holds in Bachmann's standard ordered metric planes, (b) that a variant of Steiner–Lehmus holds in all metric planes, and (c) that the fact that a triangle with two congruent medians is isosceles holds in Hjelmslev planes without double incidences of characteristic $\neq 3$.

Keywords Steiner–Lehmus theorem · Bachmann's ordered metric planes · Hjelmslev groups · Equal medians

Mathematics Subject Classification 51F10 · 51F15 · 51F20 · 51G05

1 Introduction

The Steiner–Lehmus theorem, stating that a triangle with two congruent interior bisectors must be isosceles, has received, over the 170 years since it was first proved in

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1840, a wide variety of proofs. Some of those provided within the first hundred years have been surveyed in Simon (1906), p. 131–134, MacKay (1902, 1939), M'Bride (1943), Martini (1994): most use results of Euclidean geometry. Several proofs have been provided for foundational reasons, being valid in Hilbert's absolute geometry [the geometry axiomatized by the plane axioms of the groups I, II, and III of Hilbert (1977)], the first one being provided by Tarry (1895), the second one by Blichfeldt (1902a), the third one in Anonymous (1933), p. 125, attributed to Casey, and with the mention that H. G. Forder "points out that this proof is independent of the parallel postulate", and the fourth one—which, we are told, "excels most by being "absolute"" and "came in a letter from H. G. Forder"—in (Coxeter 1969, p. 460). The simplest proof among these is the one provided in Descube (1880), and repeated, without being aware of predecessors, in Tarry (1895) (and for congruent symmedians in Tarry 1895), and then in (Anonymous 1933, p. 124–125, Anonymous 1885; Hogg 1982), is valid not only in Hilbert's absolute planes, but in more general geometries as well. While Blichfeldt's, Casey's, and Forder's proofs rely on the free mobility property of the Hilbertian absolute plane (the segment and angle transport axioms), Descube's proof can be rephrased inside the geometry of a special class of Bachmann's ordered metric planes, in which no free mobility assumptions are made (and thus not all pairs of points need to have a midpoint, and not all angles need to be bisectable), but in which the foot of the perpendicular to the hypotenuse needs to lie between the endpoints of that hypotenuse, to be referred to as *standard ordered metric planes*.

On the other hand, the Steiner–Lehmus theorem has been generalized, in the Euclidean setting, by [Henderson (1937), p. 265, 272, Generalized Theorem (7)] [repeated, without being aware of Henderson (1937), in Nicula and Pohoaţă (2009),Russell (1961), Woyda (1973), Oxman (2012)] by replacing the requirement that two internal bisectors be congruent by the weaker one that two internal Cevians which intersect on the internal angle bisector of the third angle be congruent.

The purpose of this note is to present a proof, along the lines of Descube's proof, of Henderson's generalization of the Steiner–Lehmus theorem in an axiom system for standard ordered metric planes. Since the statement of the generalized Steiner–Lehmus theorem presupposes both notions of order—so one can meaningfully refer to "internal bisector" (without mentioning that the bisector is internal, the statement is false, see Henderson (1955), van Yzeren (1997), Hajja (2001), Kharazishvili (2012), and the remarkable generalization contained in [Montes and Recio (2014), Theorem 4.2], as well as Abu-Saymeh and Hajja (2010) for the generalized version we shall prove) —and metric notions—so that one can meaningfully refer to " angle bisector", "congruent" segments, and to an "isosceles triangle"—the setting of Bachmann's ordered metric planes represents the weakest absolute geometry in which the Steiner–Lehmus theorem or its generalization by Henderson can be expected to hold. However, we will show in Sect. 5 that there is a like-minded, purely metric statement, which holds in the absence of any notion of order.

The assumption that the ordered metric plane be *standard* is very likely not needed for the generalized Steiner–Lehmus theorem to hold, but it is indispensable for our proof.

We will also present in Sect. 4 a short proof inside the theory of Hjelmslev planes of the second of the "pair of theorems" considered in Blichfeldt (1902b), stating that a triangle with congruent medians must be isosceles.

2 The axiom system for standard ordered metric planes

For the reader's convenience, we list the axioms for ordered metric planes, in a language with one sort of individual variables, standing for *points*, and two predicates, a ternary one Z, with $Z(abc)$ to be read as "the point b lies strictly between a and c" (b is not allowed to be equal to a or to c) and a quaternary one $\equiv$, with $ab \equiv cd$ to be read as "ab is congruent to cd". To improve the readability of the axioms, we will use the two abbreviations λ and L, defined by

$$\lambda(abc) :\Leftrightarrow Z(abc) \vee Z(bca) \vee Z(cab),$$
$$L(abc) :\Leftrightarrow \lambda(abc) \vee a = b \vee b = c \vee c = a,$$

with $\lambda(abc)$ to be read as "a, b, and c are three different collinear points" and $L(abc)$ to be read as "a, b, and c are collinear points (not necessarily different)." Although we have only points as variables, we will occasionally refer to *lines*, with the following meaning: "point c *lies on the line determined by* a and b" is another way of saying $L(abc)$, and the line determined by a and b will be denoted by $\langle a, b\rangle$. The axiom system for ordered metric planes consists of the lower-dimension axiom $(\exists abc)\neg L(abc)$, which we will not need in our proof, as well as the following axioms:

A 1 $Z(abc) \rightarrow Z(cba)$,

A 2 $Z(abc) \rightarrow \neg Z(acb)$,

A 3 $\lambda(abc) \wedge (\lambda(abd) \vee b = d) \rightarrow (\lambda(cda) \vee c = d)$,

A 4 $(\forall abcde)(\exists f)\neg L(abc) \wedge Z(adb) \wedge \neg L(abe) \wedge c \neq e \wedge \neg\lambda(cde) \rightarrow [(Z(afc) \vee Z(bfc)) \wedge (\lambda(edf) \vee f = e)]$,

A 5 $ab \equiv pq \wedge ab \equiv rs \rightarrow pq \equiv rs$,

A 6 $ab \equiv cc \rightarrow a = b$,

A 7 $ab \equiv ba \wedge aa \equiv bb$,

A 8 $(\forall abca'b')(\exists^{=1}c')\,[\lambda(abc) \wedge ab \equiv a'b' \rightarrow \lambda(a'b'c') \wedge ac \equiv a'c' \wedge bc \equiv b'c']$,

A 9 $\neg L(abx) \wedge L(abc) \wedge L(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ac \equiv a'c' \wedge ax \equiv a'x' \wedge bx \equiv b'x' \rightarrow xc \equiv x'c'$,

A 10 $(\forall abx)(\exists^{=1}x')\,[\neg L(abx) \rightarrow x' \neq x \wedge ax \equiv ax' \wedge bx \equiv bx']$,

A 11 $(\forall abxx')(\exists y)\,[\neg L(abx) \wedge x' \neq x \wedge ax \equiv ax' \wedge bx \equiv bx' \rightarrow L(aby) \wedge Z(xyx')]$.

Axioms A1–A 4 are axioms of ordered geometry, A 4 being the Pasch axiom. If we were to add the lower-dimension axiom and $(\forall ab)(\exists c)\, a \neq b \rightarrow Z(abc)$ to A 1–A 4, we'd get an axiom system for what Coxeter (1969) refers to as *ordered geometry*. That we do not need the axiom stating that the order is unending on all lines follows from the fact that, given two distinct points a and b, by (4.3) of Sörensen (1984), there is a perpendicular in b on the line $\langle a, b\rangle$, and the image c (which exists and is unique by A 10) of the reflection of a in that perpendicular line is, by A 11, such that $Z(abc)$.

Axioms A 5–A 7 are axioms K1–K3 of Sörensen (1984), A 8 is W1 of Sörensen (1984), A 9 is W2 of Sörensen (1984), A 10 is W3 of Sörensen (1984), and A 11 is the conjunction of axioms W4 and WA of Sörensen (1984). The reflection of point x in line $\langle a, b\rangle$, whose existence is ensured by A 10, will be denoted by $\sigma_{ab}(x)$, and the reflection of a in b, which exists as W5 of Sörensen (1984) holds in our axiom system, will be denoted by $\varrho_b(a)$.

That the lower-dimension axiom and A 1–A 11 form an axiom system for ordered (non-elliptic) metric planes was shown in Sörensen (1984). The theory of metric planes has been studied intensely in Bachmann (1973), where two axiom systems for it can be found. Two other axiom systems were put forward in Pambuccian (2003) (for the non-elliptic case only) and Pambuccian (2007).

The models of ordered metric planes have been described algebraically in the case with Euclidean metric (i. e., in planes in which there is a rectangle) in Bachmann (1949, 1948) and [Bachmann (1973), §19], and in the case with non-Euclidean metric in Pejas (1964).

We will be interested in a class of ordered metric planes in which no right angle can be enclosed within another right angle with the same vertex, or, expressed differently, in which the foot c of the altitude oc in a right triangle oab (with right angle at o) lies between a and b, i. e. in ordered metric planes that satisfy

A 12 $Z(aoa') \wedge oa \equiv oa' \wedge ba \equiv ba' \wedge o \neq b \wedge \lambda(abc) \wedge Z(bcb')cb \equiv cb' \wedge ob \equiv ob' \rightarrow Z(acb)$.

To shorten statements, we denote the foot of the perpendicular from a to line $\langle b, c\rangle$ by $F(bca)$.

That A 12 is equivalent to the statement **RR**, that no right angle can be enclosed within another right angle with the same vertex, can be seen by noticing that (i) if, assuming the hypothesis of A 12 we have, instead of its conclusion, that $Z(abc)$ holds, then the half-line $\overrightarrow{ox}$, with origin o, of the perpendicular in o to $\langle o, c\rangle$ that lies on the same side of $\langle o, b\rangle$ as a, must lie outside of triangle oac (if it lied inside it, $\overrightarrow{ox}$ would have to intersect its side ac in a point p, so we would have two perpendiculars from p to $\langle o, c\rangle$, namely $\langle p, c\rangle$ and $\langle p, o\rangle$), so the right angle $\angle aob$ is included inside the right angle $\angle xoc$, so $\neg A$ 12 $\Rightarrow$ $\neg$**RR**, and (ii) if $\langle o, a\rangle \perp \langle o, d\rangle$ and $\langle o, b\rangle \perp \langle o, c\rangle$ with $\overrightarrow{ob}$ and $\overrightarrow{oc}$ between $\overrightarrow{oa}$ and $\overrightarrow{od}$, with $\overrightarrow{oc}$ between $\overrightarrow{ob}$ and $\overrightarrow{od}$, then, with $e = F(odb)$, we must have, by the crossbar theorem, that $\overrightarrow{oc}$ intersects eb in a point p, so that $F(obe)$ cannot lie on the segment ob, for else the segments $eF(obe)$ and op would intersect, and from that intersection point there would be two perpendiculars to $\langle o, b\rangle$, namely $\langle e, F(obe)\rangle$ and $\langle o, c\rangle$, so that $\neg$**RR** $\Rightarrow$ $\neg$ A 12. A 12 was first considered as an axiom in [Bachmann (1951), p. 298], in the analysis of the interplay between Sperner's ordering functions and the orthogonality relation in affine planes. Ordering functions satisfying A 12 are referred to in [Bachmann (1951) p. 298] as “singled out by the orthogonality relation” (*durch die Orthogonalität ausgezeichnet*). An axiom more general than **RR**, stating that if an angle is enclosed within another angle with the same vertex, then the two angles cannot be congruent, has been first considered as axiom III,7 in [Bernays (1948), p. 31].

That ordered metric planes do not need to be standard, not even if the metric is Euclidean (i. e. if there is a rectangle in the plane), can be seen from the following example:

The point-set of the model is $\mathbb{Q} \times \mathbb{Q}$, with the usual betweenness relation (i. e. point $\mathbf{c}$ lies between points $\mathbf{a}$ and $\mathbf{b}$ if and only if $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$, with $0 < t < 1$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are in $\mathbb{Q} \times \mathbb{Q}$ and t is in $\mathbb{Q}$) and with segment congruence $\equiv$ given by $\mathbf{ab} \equiv \mathbf{cd}$ if and only if $\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{c} - \mathbf{d}\|$, where $\|\mathbf{x}\|$ stands for $x_1^2 - 2x_2^2$, with $\mathbf{x} = (x_1, x_2)$. Two lines $ux + vy + w = 0$ and $u'x + v'y + w = 0$ are orthogonal if and only if $-2uu' + vv' = 0$, and -2 is called the *orthogonality constant* of the Euclidean plane [see Bachmann (1949, 1948), Schröder (1985), Schnabel and Pambuccian (1985), Pambuccian (1994b), and Pambuccian (1994a) for more on Euclidean planes, and (Bachmann 1951, p. 300) for a general result regarding Sperner’s ordering function that satisfies A 12 in Euclidean planes]. If $\mathbf{o} = (0, 1)$, $\mathbf{a} = (1, 0)$, and $\mathbf{b} = (2, 0)$, then $\widehat{aob}$ is a right angle, and the foot $c = (0, 0)$ of the perpendicular from o to line ab does not lie between a and b.

However, all ordered Euclidean planes with free mobility (i. e. those that can be coordinatized by Pythagorean fields) must be standard (see Bachmann 1973, p. 217), as must be all absolute planes in Hilbert’s sense (see Schwabhäuser et al. 2011, I.11.47).

3 Proof of the Steiner–Lehmus theorem in standard ordered metric planes

Before starting the proof (a variant of Descube’s proof) of the Steiner–Lehmus theorem, we will first state it inside our language, and then list, without proof, the proofs being straightforward, a series of results true in standard ordered metric planes.

With the abbreviation $M(abc)$ standing for $Z(abc) \wedge ba \equiv bc$, to be read as “$b$ is the midpoint of the segment ac”, the generalized Steiner–Lehmus theorem can be stated as

$$\begin{aligned} \neg L(abc) \wedge Z(amc) \wedge Z(anb) \wedge ad \equiv ab \wedge (Z(adc) \vee Z(acd) \vee d = c) \\ \wedge sb \equiv sd \wedge Z(bsm) \wedge Z(csn) \wedge bm \equiv cn \\ \wedge M(mpn) \wedge M(boc) \rightarrow ab \equiv ac. \end{aligned} \tag{1}$$

Notice that we assume the existence of two midpoints: the midpoint p of the segment mn and the midpoint o of the segment bc. These midpoints do exist whenever the triangle abc is isosceles, but do not have to exist in general.

Notation Whenever the segment bc has a midpoint, we define $ab < ac$ to mean that the perpendicular bisector of bc intersects the open segment ac. We say that the lines $\langle a, b\rangle$ and $\langle c, d\rangle$ are perpendicular ($\langle a, b\rangle \perp \langle c, d\rangle$) if there are two different points x and x' on $\langle c, d\rangle$ such that $ax \equiv ax'$ and $bx \equiv bx'$. We denote by $\angle xoy$ the angle formed by the rays $\overrightarrow{ox}$ and $\overrightarrow{oy}$. We say that "$\angle abc$ is acute" if $\neg L(abc)$ and if $\overrightarrow{bc}$ lies between $\overrightarrow{ba}$ and $\overrightarrow{bb'}$, where $\langle b, b'\rangle \perp \langle b, a\rangle$, with b' on the same side of $\langle a, b\rangle$ as c. A point p lies in the *interior* of $\angle xoy$ if $\overrightarrow{op}$ intersects the open segment xy. We say that a segment ab can be *transported* from c on the ray $\overrightarrow{cd}$, if there is a point x on $\overrightarrow{cd}$ (to be referred as the second endpoint of the transported segment) such that $ab \equiv cx$.

The facts that we will need for (1)'s proof, which will be stated without proof, their proofs being either well known or straightforward, are:

F 1 *If $\neg L(abu)$, $ba \equiv bu$, $m \neq b$, and $ma \equiv mu$, then, for every point p with $L(bmp)$, we have $px \equiv py$, where $x = F(bap)$ and $y = F(bup)$ (points on the angle bisector of an angle are equidistant from the legs of the angle). Also, $bx \equiv by$.*

F 2 *If $\angle bac$ is acute, then $\angle cab$ is acute as well, and any angle with the same vertex a and inside $\angle bac$ is also acute. If the triangles abc and $a'b'c'$ are congruent (i. e. $ab \equiv a'b'$, $bc \equiv b'c'$, and $ca \equiv c'a'$), and $\angle bac$ is acute, then so is $\angle b'a'c'$.*

F 3 *If a and a' have a midpoint, $Z(abc)$, $Z(a'b'c')$, $ac \equiv a'c'$, $ab \equiv a'b'$, then $bc \equiv b'c'$ (a special case of Euclid's Common Notion III "If equals are subtracted from equals, then the remainders are equal.")*

F 4 *The betweenness relation is preserved under orthogonal projection, i. e. if $Z(oa'b')$, $L(oab)$, and $\langle a, a'\rangle$ and $\langle b, b'\rangle$ are perpendicular on $\langle a, b\rangle$, then $Z(oab)$.*

F 5 *The base angles of an isosceles triangles are acute, i. e. if $\neg L(abc)$, $ab \equiv ac$, $\langle b, b'\rangle \perp \langle b, c\rangle$ and $\langle c, c'\rangle \perp \langle c, b\rangle$, and b' and c' lie on the same side of $\langle b, c\rangle$ as a, then $\overrightarrow{ba}$ lies between $\overrightarrow{bc}$ and $\overrightarrow{bb'}$ and $\overrightarrow{ca}$ lies between $\overrightarrow{cb}$ and $\overrightarrow{cc'}$.*

F 6 *If $ab \equiv a'b'$, $ac \equiv a'c'$, $Z(abc)$, and $Z(a'b'c') \vee Z(a'c'b')$, then $Z(a'b'c')$.*

F 7 *If $\angle bac$ has an interior angle bisector, then any segment xy on $\langle a, b\rangle$ can be transported on any halfline that is included in $\langle a, c\rangle$, i. e. for all x, y with $x \neq y$, $L(abx)$ and $L(aby)$, and any u, v with $L(acu)$ and $L(acv)$ and $u \neq v$, there exists a z with $uz \equiv xy$ and $Z(vuz)$.*

F 8 *If $ab \equiv a'b'$, with $a \neq b$, and if the lines $\langle a, b\rangle$ and $\langle a', b'\rangle$ intersect, then the two lines have an angle bisector (as both segments ab and $a'b'$ can be transported along the lines $\langle a, b\rangle$ and $\langle a', b'\rangle$ respectively (via A8) to their intersection point).*

F 9 *If $\neg L(abc)$, $ab \equiv a'b'$, $ac \equiv a'c'$, $cb \equiv c'b'$, $L(adc)$, $ad \equiv a'd'$, $dc \equiv d'c'$, then $dF(abd) \equiv d'F(a'b'd')$.*

F 10 *If p_1 and p_2 are two distinct points on the same side of the line $\langle a, b\rangle$, and $p_1F(abp_1) \equiv p_2F(abp_2)$, then the lines $\langle p_1, p_2\rangle$ and $\langle a, b\rangle$ do not meet (given that the perpendicular from the point of intersection of segments $p_1F(abp_2)$ and $p_2F(abp_1)$ to $\langle a, b\rangle$ is perpendicular to $\langle p_1, p_2\rangle$ as well).*

We will also need the following lemmas:

Lemma 1 $\neg L(bsc) \wedge Z(bxs) \wedge Z(cxm') \wedge Z(csn) \wedge cn \equiv cm' \rightarrow bm' < bn$

Proof Since $cn \equiv cm'$, $d = F(nm'c)$ is the midpoint of the segment $m'n$, and thus $\langle d, c\rangle$ is the perpendicular bisector of the segment $m'n$. By the crossbar theorem the ray $\overrightarrow{cd}$ intersects the segment xs in a point p. By the Pasch axiom applied to Δbsn and secant $\langle d, p\rangle$, we get that the latter must intersect the segment bn, thus $bm' < bn$.

Lemma 2 $M(boc) \wedge \neg L(abc) \wedge Z(ab'c) \wedge ab \equiv ab' \rightarrow ab < ac$.

Proof Let $o' = F(bb'a)$, i. e. the midpoint of segment bb'. By the Pasch axiom, $\langle a, o'\rangle$ intersects the side bc of $\triangle bcb'$ in a point p with $Z(ao'p)$ and $Z(bpc)$. Point p cannot coincide with o, as $\langle o, o'\rangle$ and $\langle c, b'\rangle$ do not intersect (given that $\langle o, o'\rangle$ and $\langle c, b'\rangle$ have a common perpendicular (by Bachmann 1973, §4,2, Satz 2), and it also cannot be such that $Z(cpo)$, for in that case the Pasch axiom with Δpac and secant $\langle o, o'\rangle$ would ask the latter to intersect segment ac, contradicting the fact that $\langle o, o'\rangle$ and $\langle a, c\rangle$ do not intersect. Thus $Z(opb)$ must hold. Also by the Pasch axiom the perpendicular bisector of segment bc must intersect one of the segments ac or ab. Suppose it intersects segment ab in s. By A12, with $f = F(pbo')$, we have $Z(pfb)$. Since we have $Z(po'a)$ and $Z(asb)$, for $g = F(pba)$ and for o, which is $F(pbs)$, we have, by F4, $Z(pfg)$ and $Z(gob)$. From these two betweenness relations and $Z(pfb)$ we get that $Z(opb)$ cannot hold, a contradiction. Hence the perpendicular bisector of bc must intersect ac.

Lemma 3 $M(bab') \wedge o \neq a \wedge Z(abc) \wedge ob \equiv ob' \wedge od \equiv ob \wedge (Z(odc) \vee Z(ocd)) \rightarrow Z(odc)$.

Proof Suppose we have $Z(ocd)$. Let x be the intersection point of the perpendicular in b on $\langle b, a\rangle$ with side oc of Δaoc (which it must intersect by the Pasch axiom and the fact that it cannot intersect $\langle o, a\rangle$), and let y be the point of intersection of the perpendicular in b on $\langle b, d\rangle$ with segment xd (by **RR**, i. e. by A12, there must be such an intersection point). Let $m = F(bdo)$. Since we have $Z(dmb)$, we should, by F4, also have $Z(doy)$, which cannot be the case, since we have $Z(dyx)$ and $Z(dxo)$, thus $Z(dyo)$. Hence we must have $Z(odc)$.

Lemma 4 *If* $\neg L(oac)$*, ray* $\overrightarrow{ob}$ *lies between* $\overrightarrow{oa}$ *and* $\overrightarrow{oc}$*, then one of the halflines determined by* o *on* $\langle o, d\rangle$*, the line for which* $\sigma_{oc}\sigma_{ob}\sigma_{oa} = \sigma_{od}$ *(which exists, since metric planes satisfy the three reflection theorem for concurrent lines), also lies between* $\overrightarrow{oa}$ *and* $\overrightarrow{oc}$.

Proof Let $a' = \sigma_{ob}(a)$. If a' lies on the side determined by $\langle o, c\rangle$ opposite to the one in which a lies, then segment aa' must intersect $\overrightarrow{oc}$ in a point z and thus we have, with $x = a$ and $y = F(oba)$, that x, y, and z are three points on the rays $\overrightarrow{oa}$, $\overrightarrow{ob}$, and $\overrightarrow{oc}$ respectively, such that $Z(xyz)$ and $y = F(xzo)$. Three such points can be found even in case a' lies on the same side determined by $\langle o, c\rangle$ as a. For, in that case, it must be that $c' = \sigma_{ob}(c)$ is on the side determined by $\langle o, a\rangle$ opposite to the one in which c lies. To see this, notice that, by the crossbar theorem, $\overrightarrow{oa'}$ must intersect the segment $cF(obc)$ in a point p, so we have $Z(F(obc)pc)$. Since σ_{ob} preserves betweenness and $p' = \sigma_{ob}(p)$ is on $\overrightarrow{oa}$, we have $Z(F(obc)p'c')$, i. e. segment cc' intersects $\langle o, a\rangle$. So, in this case, we set $x = p'$, $y = F(obc)$, $z = c$, to have three points on the rays $\overrightarrow{oa}$, $\overrightarrow{ob}$, and $\overrightarrow{oc}$ respectively, such that $Z(xyz)$ and $y = F(xzo)$.

Let now $z' = \sigma_{oc}\sigma_{ob}\sigma_{oa}(z) = \sigma_{oa}\sigma_{ob}(z)$. Note that, with $d = F(zz'o)$, we have $\sigma_{oc}\sigma_{ob}\sigma_{oa} = \sigma_{od}$, so if we prove that z' lies inside the angle $\angle aoc$, we are done, since d, as the midpoint of zz', must lie inside the angle $\angle aoc$ as well.

With $u = \sigma_{ob}(z)$, we notice that we must have one of $Z(yxu)$ or $Z(yux)$ or $u = x$, and that, in case $u = x$, we are done, for then $\langle o, d\rangle = \langle o, b\rangle$. Given that $\sigma_{oc}\sigma_{ob}\sigma_{oa} = \sigma_{oa}\sigma_{ob}\sigma_{oc}$, we may assume, w. l. o. g., that that $Z(yxu)$. Suppose now that z' does not lie inside the angle $\angle aoc$. With $v = F(oau)$ and $w = F(oay)$, we must have, by F4, $Z(wxv)$. By A12 we also have $Z(owx)$, thus also $Z(oxv)$. Since the line $\langle u, v\rangle$ intersects the extensions of two sides of Δoxz, it cannot, by the Pasch axiom, intersect the segment oz, so if the segment uz' intersects line $\langle o, z\rangle$, then it can intersect it only in a point q with $Z(ozq)$ (and $Z(uqz')$). In that case, by the Pasch axiom, the secant $\langle o, d\rangle$ must intersect the side qz' of $\Delta zqz'$ in a point r. Given $Z(ozq)$ and $Z(zdz')$, we have $Z(odr)$. The perpendicular in r on $\langle q, z'\rangle$ must intersect, by the Pasch axiom, one of the sides oq or oz' of $\Delta oqz'$. It cannot intersect oq, for then, it would also have to intersect, by the Pasch axiom, side vq of Δoqv, and from that intersection point there would be two perpendiculars to $\langle q, z'\rangle$. So it must intersect segment oz' in s. By the Pasch axiom applied to $\Delta doz'$ and secant $\langle r, s\rangle$, we conclude that there is a point f with $Z(rfs)$ and $Z(dfz')$. By A12 we have $Z(rF(rfd)f)$, and the Pasch axiom applied to $\Delta rz'f$ with secant $\langle d, F(rfd)\rangle$ gives a point of intersection of the latter with segment $z'r$, a contradiction, as from that point one has dropped two distinct perpendiculars to $\langle r, s\rangle$. Thus, z' has to lie inside the angle $\angle aoc$, and we are done. □

Theorem 1 *The generalized Steiner–Lehmus theorem, (1), holds in Bachmann's standard ordered metric planes.*

Proof Let $b' = \varrho_p(b)$. Then, since ϱ_p is an isometry, $nb \equiv mb'$ and $bm \equiv b'n$. Since $bm \equiv cn$, we have $nb' \equiv nc$ as well.

We will first show that $Z(aF(abs)b)$ and $Z(aF(acs)c)$ must hold. The perpendicular raised in s on $\langle a, s\rangle$ must intersect, by the Pasch axiom, one of the sides ac or an of Δacn (including the ends c and n of segments ac and an). Thus, it must intersect at least one of the sides ab and ac of Δabc. If it intersects both sides (including the ends b and c of the segments ab and ac), then, by A12, we get the desired conclusion, namely that $Z(aF(abs)b)$ and $Z(aF(acs)c)$. Suppose the perpendicular raised in s on $\langle a, s\rangle$ intersects one of the two, say ac, but *not* the closed segment ab. Since the point q, obtained by reflecting in $\langle a, s\rangle$ the intersection of the perpendicular raised in s on $\langle a, s\rangle$ with ac, lies on both $\langle a, s\rangle$ and on $\langle a, b\rangle$, we must have $Z(abq)$ (since we assumed that we do not have $Z(aqb)$) and $\langle s, a\rangle \perp \langle s, q\rangle$. We want to show that we still need to have $Z(aF(abs)b)$ in this case as well. Suppose that were not the case, and we'd have $Z(F(abs)ba)$. We thus have $Z(F(abs)bn)$ and $Z(cmF(acs))$. With $s' = \varrho_p(s)$, we have, given that point-reflections are isometries, $s'n \equiv sm$, $s'b' \equiv sb$. Recall that we also have $cn \equiv b'n$. With $u = F(b'cn)$, $w = \sigma_{nu}(s)$, we notice that $Z(nwb')$, $ns \equiv nw$, $sc \equiv wb'$ (since lines $\langle n, c\rangle$ and $\langle n, b'\rangle$ are symmetric with respect to $\langle n, u\rangle$, and symmetry in lines preserves both congruence and betweenness).

Let v be the point (whose existence is ensured by A8) for which (by F6) $Z(nvb')$, $ns' \equiv b'v$, and $b's' \equiv nv$. We thus have $bs \equiv nv$, $ns \equiv nw$, $sm \equiv b'v$, $sc \equiv wb'$. By Lemma 3 (with $(s, F(abs), b, n)$ and $(s, F(acs), m, c)$ for (o, a, b, c)), using F8 (i. e. bearing in mind that sb can be transported from n on $\overrightarrow{ns}$, and that sm can be transported from c on $\overrightarrow{cs}$, and that the second points resulting from the transport are on the open segments ns and cs), we deduce that $Z(nvw)$ and $Z(b'vw)$, which is impossible. This proves that both $Z(aF(abs)b)$ and $Z(aF(acs)c)$ must hold.

If o coincides with $F(bcs)$, then $sb \equiv sc$ and thus, since $bm \equiv cn$, also $sm \equiv sn$ (by F3). Since $\sigma_{so}(b) = c$, and σ_{so} is an isometry and preserves betweenness, we must, by the uniqueness requirement in A8, have $\sigma_{so}(m) = n$, and thus σ_{so} maps line $\langle b, n\rangle$ onto line $\langle m, c\rangle$. Since these two lines intersect in a, point a must lie on the axis of reflection, and thus $ab \equiv ac$.

Suppose $o \neq F(bcs)$. W. l. o. g. we may assume that $Z(boF(bcs))$. By the Pasch axiom, the perpendicular bisector of the segment bc must intersect one of the sides sb and sc of $\triangle sbc$. Given $Z(boF(bcs))$ and F4, it must intersect side sb in a point x (and thus does not intersect the side sc). By the Pasch axiom, line $\langle x, o\rangle$ must intersect one of the sides ab and ac of $\triangle abc$. Line $\langle x, o\rangle$ cannot intersect ac, for else, by the Pasch axiom applied to $\triangle asc$ and secant $\langle x, o\rangle$, it would have to intersect one of the sides sa and sc. Since we have already seen that $\langle x, o\rangle$ cannot intersect segment sc, $\langle x, o\rangle$ would have to intersect the segment sa in a point z. We will show that this leads to a contradiction. Let $z_1 = F(abz)$ and $z_2 = F(acz)$. We have shown that $Z(aF(abs)b)$ and $Z(aF(acs)c)$, so, given $Z(sza)$, we can apply F4, to obtain that we have $Z(az_1b)$, $Z(az_2c)$. We also have $az_1 \equiv az_2$ (by F1) and $zb \equiv zc$ (as z is a point on the perpendicular bisector of segment bc). Since σ_{az} maps line $\langle a, b\rangle$ onto line $\langle a, c\rangle$, we have that $zb \equiv z\sigma_{az}(b)$, and thus $zc \equiv z\sigma_{az}(b)$, and $L(ac\sigma_{az}(b))$. Since $\sigma_{az}(b) \neq c$ (else, we'd have $ab \equiv ac$, so $o = F(bcs)$), we must have $\sigma_{az}(b) = \varrho_{z_2}(c)$, a contradiction, as σ_{az} preserves the betweenness relation, and we have $Z(az_1b)$, and thus should have $Z(az_2\sigma_{az}(b))$. Thus $\langle x, o\rangle$ must intersect ab in a point g.

Let $m' = \sigma_{ox}(m)$. Since $c = \sigma_{ox}(b)$ and σ_{ox} is an isometry, we have $cm \equiv bm'$, as well as $bm \equiv cm'$, and thus, given the hypothesis that $bm \equiv cn$, we have $cm' \equiv cn$, and since x is a fixed point of σ_{ox} and $Z(bxm)$, we have $Z(cxm')$, and thus the hypothesis of Lemma 1 holds, and thus so must the conclusion, i. e., $bm' < bn$. By F7, cm can be transported from n on the ray $\overrightarrow{nb}$ to get m_1 with $nm_1 \equiv cm$ and $Z(bm_1n)$ (given $bm' < bn$ and Lemma 2). Since reflections in points are isometries and preserve betweenness, for $m_2 = \varrho_p(m_1)$ we have $mm_2 \equiv nm_1$ (thus $mm_2 \equiv cm$) and $Z(mm_2b')$, so, by Lemma 2 (which can be applied as the segment cb' does have u as midpoint), we have $mc < mb'$.

Let h be the intersection point of the perpendicular bisector $\langle u, n\rangle$ of $b'c$ with segment mb'. Let $a' = \varrho_p(a)$ and $a_1 = \sigma_{nh}(a')$. We have $na_1 \equiv na'$ and $na' \equiv am$ (since symmetries in both lines and points are isometries), thus $na_1 \equiv ma$. Let $m'' = \sigma_{as}(m)$ and $b_1 = \sigma_{as}(b)$. Given $ac < ab$, we must have $Z(acb_1)$ (by Lemma 2), and thus, by the Pasch axiom applied to $\triangle anc$ and secant $\langle b_1, s\rangle$, the latter must intersect side na, and the point of intersection must be m'', so $Z(am''n)$. Since $ma \equiv m''a$, we also have $na_1 \equiv m''a$. We also have $b'a' \equiv ba$, $b'a' \equiv ca_1$ (since symmetries in both points and lines are isometries), so $ba \equiv ca_1$, and, since $ba \equiv b_1a$, also $ca_1 \equiv b_1a$.

We turn our attention to the congruent triangles ca_1n and b_1am''. The $\angle aca_1$ being bisectable (by F8), let $\overrightarrow{cc_1}$ be its internal bisector (i. e. $\sigma_{cc_1}\sigma_{ca}\sigma_{cc_1} = \sigma_{ca_1}$ and $\overrightarrow{cc_1}$ lies between $\overrightarrow{ca}$ and $\overrightarrow{ca_1}$). Let $\varphi = \sigma_{ca}\sigma_{cc_1}$. By F2, $\angle\varphi(a_1)c\varphi(n)$ is acute (as triangles ca_1n and $c\varphi(a_1)\varphi(n)$ are congruent and $\angle a_1cn$ is acute (by F2, $\triangle ca_1n$ and $\triangle ab_1m''$ being congruent, and $\angle ab_1m''$ being included in the base angle $\angle ab_1b$ of isosceles $\triangle ab_1b$, thus acute, by F5). Given that $Z(a'mb')$ (preservation of the betweenness relation $Z(anb)$ under the reflection in point p), $\langle a', b'\rangle$ passes through h, and thus so must the $\langle c, a_1\rangle$, the reflection in $\langle n, u\rangle$ of $\langle a', b'\rangle$ passes through h, and thus so must the $\langle c, a_1\rangle$, given that h is fixed under σ_{nu}. Since $Z(b'hm)$ and $Z(cha_1)$, the points a_1 and b' must lie on the same side of $\langle c, a\rangle$. Points b' and b are known to lie on different sides of line $\langle c, a\rangle$ (since $\langle a, b\rangle$ and $\langle m, b'\rangle$ do not intersect, which is not possible if b' and b were to lie on the same side of $\langle c, a\rangle$). Thus $\overrightarrow{ca}$ is between $\overrightarrow{ca_1}$ and $\overrightarrow{cn}$. Notice that $\sigma_{c\varphi(n)} = \sigma_{\varphi(c)\varphi(n)} = \varphi\sigma_{cn}\varphi^{-1} = (\sigma_{ca}\sigma_{cc_1})\sigma_{cn}(\sigma_{cc_1}\sigma_{ca}) = \sigma_{ca}(\sigma_{cc_1}\sigma_{ca}\sigma_{cc_1})\sigma_{cn} = \sigma_{ca}\sigma_{ca_1}\sigma_{cn}$. If we denote by $\langle c, i\rangle$ the fourth reflection line that is the line of reflection for $\sigma_{ca_1}\sigma_{ca}\sigma_{cn}$, i. e., $\sigma_{ca_1}\sigma_{ca}\sigma_{cn} = \sigma_{ci}$, then by Lemma 4, i can be chosen such that $\overrightarrow{ci}$ lies between $\overrightarrow{ca_1}$ and $\overrightarrow{cn}$. Since $\sigma_{ca}\sigma_{ca_1} = \sigma_{cn}\sigma_{ci}$, we have $\sigma_{c\varphi(n)} = \sigma_{ca}\sigma_{ca_1}\sigma_{cn} = \sigma_{cn}\sigma_{ci}\sigma_{cn}$, thus $\langle c, \sigma_{cn}(i)\rangle = \langle c, \varphi(n)\rangle$. Since $\angle\varphi(a_1)c\varphi(n)$ is acute, this implies that $\overrightarrow{c\varphi(n)}$ is between $\overrightarrow{cn}$ and $\overrightarrow{cb_1}$. Thus $\overrightarrow{c\varphi(n)}$ must intersect segment b_1s in a point p_1. Let p_2 be a point on $\overrightarrow{cp_1}$ with $cp_2 \equiv b_1p_1$ (such a point exists by F8). Notice that $p_1 \neq p_2$, given that one of the base angles of $\triangle p_1b_1c$, $\angle b_1cp_1$ is not acute (since its supplement, $\angle p_1ca$ is acute), so $\triangle p_1b_1c$ cannot be isosceles by F5. On ray $\overrightarrow{cp_1}$ there are thus two points, p_1 and p_2, whose distance to line $\langle b_1, a\rangle$ is the same (by F9), contradicting F10.

4 A triangle with two congruent medians is isosceles

We will turn to the proof of the second result proved in Blichfeldt (1902b) to be true in Hilbert's absolute planes, i. e.

Theorem 2 *A triangle with two congruent medians is isosceles.*

We will show that this theorem is true in a purely metric setting (without introducing a relation of order). The axiom system for this theory can be expressed in first order logic, as done in Pambuccian (2009). Here we will present it in its group-theoretical formulation of [Bachmann (1989), p. 20].

Basic assumption Let G be a group which is generated by an invariant set S of involutory elements.

Notation The elements of G will be denoted by lowercase Greek letters, its identity by 1, those of S will be denoted by lowercase Latin letters. The set of involutory elements of S^2 will be denoted by P and their elements by uppercase letters $A, B, \ldots$. The 'stroke relation' $\alpha \mid \beta$ is an abbreviation for the statement that α, β and $\alpha\beta$ are involutory elements. The statement $\alpha, \beta \mid \delta$ is an abbreviation of $\alpha \mid \delta$ and $\beta \mid \delta$. We denote $\alpha^{-1}\sigma\alpha$ by σ^α.

(G, S, P) is called a *Hjelmslev group without double incidences* if it satisfies the following axioms:

F 1 *For* A, b *there exists* c *with* $A, b \mid c$.

F 2 *If* $A, B \mid c, d$ *then* $A = B$ *or* $c = d$.

F 3 *If* $a, b, c \mid e$ *then* $abc \in S$.

F 4 *If* $a, b, c \mid E$ *then* $abc \in S$.

F 5 *There exist* a, b *with* $a \mid b$.

The elements of S can be thought of as reflections in lines (and can be thought of as lines), those of P as reflections in points (and can be thought of as points), thus $a|b$ can be read as "the lines a and b are orthogonal", $A|b$ as "A is incident with b". Thus, the axioms state that: through any point to any line there is a perpendicular, two points have at most one joining line, the three reflection theorem for three lines incident with a point or having a common perpendicular, stating that the composition of three reflections in lines which are either incident with a point or have a common perpendicular is a reflection in a line, and the existence of a point. In contrast to Bachmann's metric planes (Bachmann 1973) there may be points which have no joining line.

A notion of congruence for segments can be introduced in the following way:

Definition 1 AB and CD are called *congruent* ($AB \equiv CD$) if there is a motion α (i. e. $\alpha \in G$) with $A^\alpha = C$ and $B^\alpha = D$ or with $A^\alpha = D$ and $B^\alpha = C$.

Our theorem on triangles with congruent medians can be stated in this setting as:

(*) Let A, B, C be three non-collinear points and $C^U = B$ and $B^W = A$ and $b \mid A, C$ and $n \mid C, W$. If $AU \equiv CW$ then there exists a line v through B with $A^v = C$, i.e. triangle ABC is isosceles.

This statement does not hold in Bachmann's metric planes which are elliptic or of characteristic 3 (such as the Euclidean plane over $GF(3)$; see (Bachmann 1989, 7.4). A Hjelmslev group (G, S, P) (and with it a Bachmann plane) is called *non-elliptic* if $S \cap P = \emptyset$, and of *characteristic* $\neq 3$ if $(AB)^3 \neq 1$ for all $A \neq B$.

Theorem 3 *Let* (G, S, P) *be a Hjelmslev group without double incidences which is non-elliptic and of characteristic* $\neq 3$. *Let* A, B, C *be three non-collinear points and* $C^U = B$ *and* $B^W = A$ *and* $b \mid A, C$ *and* $n \mid C, W$. *If* $AU \equiv CW$ *then there exists a line* v *through* B *with* $A^v = C$.

For the proof of Theorem 3 we need the following:

Lemma 5 *Let* C, W *be points and* s, n *lines with* $C, W|n$ *and* $W|s$ *and* $C \nmid s$. *Then there exists at most one point* $V \neq W$ *with* $V|s$ *and* $CW \equiv CV$.

Proof Let $C, W \mid n$ and $V, W \mid s$ with $C \nmid s$. If $CW \equiv CV$ then there exists a motion α with $C^\alpha = C$ and $W^\alpha = V$ or with $C^\alpha = V$ and $W^\alpha = C$.

Suppose $C^\alpha = C$ and $W^\alpha = V$ (case 1). Then α is a line through C or a rotation which leaves C fixed. Since in the latter case $n\alpha$ is a line through C (see Bachmann 1989, Section 3.4), we can assume that α is a line g which leaves C fixed. Let h be

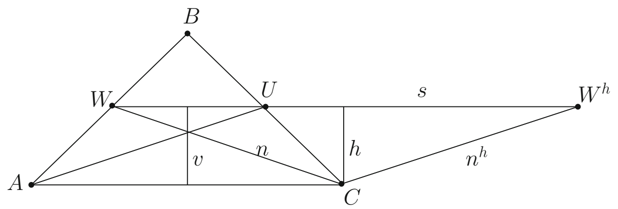

Fig. 1 A triangle with two congruent medians must be isosceles

the line with $h|W, g$. Then $V, W|h, s$ and according to H2 it is $h = s$. Hence g is the unique perpendicular with $g|C, s$ and V is the unique point with $V = W^g$ and $CW \equiv C^g W^g \equiv CV$.

Suppose now $C^\alpha = V$ and $W^\alpha = C$ (case 2). Since α or $n\alpha$ is a glide reflection (according to Bachmann 1989, Proposition3.2) we can assume without loss of generality that α is glide reflection i.e. $\alpha \in PS$.

Let M be the midpoint of C and W and let N be the midpoint of C and V (which exist according to (Bachmann 1989, Proposition 2.33). Let a be the axis of the glide reflection α. According to (Bachmann 1989, Proposition 2.32) we have $a|M, N$.

Since $\alpha \in PS$ there exists a line b with $\alpha = bN$ and $b|a$ (see Bachmann 1989, Section 2.3). Hence $V = C^\alpha = C^{bN}$ and $C^b = V^N = C$ (since N is the midpoint of V and C). Thus we get $b|C, a$. In an analogous way there exists a line d with $\alpha = Md$ and $d|a$. Hence $C = W^\alpha = W^{Md} = C^d$ and $d|C, a$. Since in a non-elliptic Hjelmslev group there is at most one perpendicular from C to a we get $b = d$.

Hence $\alpha = Mb = bN$ and $M^b = N$, i.e. b is the midline of M and N. Thus $W^b = (C^M)^b = M^b C^b M^b = NCN = C^N = V$, i.e. b is also the midline of W and V. Hence b is the unique perpendicular from C to s (the joining line of V, W) and V the unique point with $V = W^b$.

We now turn to the proof of Theorem 3.

Proof Let U and W be the midpoints of the sides BC and BA of triangle ABC and $AU \equiv CW$. Let n and b denote the lines $\langle C, W\rangle$ and $\langle A, C\rangle$ respectively. Since $A^{WU} = C$ there exists a midpoint V of A, C (see Bachmann 1989, Proposition 2.33) and a midline $v = Vb$ of A, C. Moreover according to (Bachmann 1989, Proposition 2.48) there is a joining line s of U, W which is orthogonal to v, i.e. $v|b, s$. Since $v, W, U|s$ the element $vWU = h$ is a line with $h|s$ and $h|C$ (since $C^{vWU} = A^{WU} = B^U = C$), i.e. h is the perpendicular from C to s.

Hence W, W^h are points on s with $CW \equiv CW^h$. Since $AU \equiv CW$ it is $A^v U^v \equiv CW$ and hence $CU^v \equiv CW$ with $U^v|s$ (since $U, v|s$). According to Lemma 5 there are at most two points P on s with $CW \equiv CP$, namely W and W^h. Hence $U^v = W$ or $U^v = W^h$.

If $U^v = W^h$ then $U^v = W^{vWU} = W^{UWv}$ and hence $U = W^{UW}$. This implies $(UW)^3 = 1$ which is a contradiction to our assumption that (G, S, P) is of characteristic $\neq 3$.

Hence $U^v = W$ and $B^v = (C^U)^v = U^v C^v U^v = WAW = B$ which shows that v is a line through B with $A^v = C$.

5 An absolute order-free version of the Steiner–Lehmus theorem

In its original version, stating that a triangle with two congruent internal bisectors must be congruent, the Steiner–Lehmus theorem requires the notion of betweenness, to ensure that the two angle bisectors are *internal*.

However, we will show that it is possible to state and prove an order-free absolute version of the Steiner–Lehmus theorem, one stated inside the theory of *metric planes*, from which all we need are the axioms H2–H4 and "For all A, B, with $A \neq B$, there exists c with $A, B \mid c$". Metric planes will be again considered in group-theoretical terms, with (G, S, P) as in Sect. 4. The elements of G will be again referred to as *motions*.

To this end, we first notice that, for the angle bisectors of a triangle ABC, we have the following facts that can be proved to hold in metric planes:

(a) If there is an angle bisector w through A, then there is exactly another angle bisector v through A, which is the perpendicular in A on w.
(b) Every triangle ABC has precisely six angle bisectors (through each of the points A, B, and C, there are precisely two perpendicular angle bisectors
(c) If an angle bisector through A intersects an angle bisector through B in a point M, then the line joining M and C is an angle bisector through C.

With $\equiv$ defined as in Definition 1, we have

Lemma 6 *If $AM \equiv BM$ then there exists a motion α with $A^\alpha = B$ and $M^\alpha = M$.*

Proof Suppose there is a motion α with $A^\alpha = M$ and $M^\alpha = B$. Given that M is the image of A under a motion, A and M must have, by [Bachmann (1973), §3, Satz 28] a midpoint N. Thus $A^{N\alpha} = M^\alpha = B$ and $M^{N\alpha} = A^\alpha = M$.

Here is now the order-free, absolute version of the Steiner–Lehmus theorem:

Theorem 4 *Let ABC be a triangle with sides a, b, and c, and u, v, and w are angle bisectors through A, respectively B, respectively C. Then we have:*

(a) If u, v, and w have a point M in common and $AM \equiv BM$, the triangle ABC is isosceles.
(b) If u, v, and w are the sides of a triangle with vertices U, V, and W, and triangle UVW is isosceles, then so is triangle ABC.

Proof (a) : By Lemma 1, we can assume that there is a motion $\alpha \in G$ with $M^\alpha = M$ and $A^\alpha = B$. Since $A, M|u$, the motion $u\alpha$ must also satisfy $M^{u\alpha} = M$ and $A^{u\alpha} = B$. According to (Bachmann 1989, §3.1), we must have $\alpha \in S$ or $u\alpha \in S$. We conclude that there exists a line h with $h|M$ and $A^h = B$. Given the uniqueness of the joining line of two points (i. e., given H2), we have $h|c$ and $u^h = v$ (the latter holds since $A, M|u$ and $B, M|v$). Since $u, h, v|M$, we have $uhv \in S$ and $b^{uhv} = c^{hv} = c^v = a$.
We conclude that $uhv = uhu^h = uh(huh) = h$ is an angle bisector of a and b. By Bachmann (1973), we have $h|C$, and since $A^h = B$, h is a symmetry axis of triangle ABC, i. e. the latter is isosceles.

(b) :Let $u, v|W$; $u, w|V$; $v, w|U$, and let $UW \equiv VW$. As in (a), one can prove that there is a line m with $m|W$, $U^m = V$, and $m|w$. Line m joins the point W of intersection of the angle bisectors u and v with C, and thus is (according to Bachmann 1973, §4,7, Satz 11) an angle bisector through C. The reflection in m thus switches the lines a and b, as well as the lines u and v. Thus it switches the intersection points A (of b and u) and B (of a and v). This means that m is symmetry axis of triangle ABC, i. e., the latter is isosceles.

This formulation of the Steiner–Lehmus theorem also shows that there is, indeed, a version of the Steiner–Lehmus theorem that is invariant under what is called an 'extraversion' in Conway and Ryba (2014).

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