

Faithful tropicalization of Mumford curves of genus two

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Abstract In the present paper, we investigate the question if the skeleton of a Mumford curve of genus two can be tropicalized faithfully in dimension three, i.e. if there exists an embedding of the curve in projective three space such that the tropicalization maps the skeleton of the curve isometrically to its image. Baker, Payne and Rabinoff showed that the skeleton of every analytic curve can be tropicalized faithfully. However the dimension of the ambient space in their proof can be quite large. We will define a map from the skeleton to the tropicalization of the Jacobian, which is an isometry on the cycles. It allows us to find principal divisors and simultaneously to determine the retractions of their support on the skeleton which is necessary to calculate their tropicalization. It turns out that a Mumford curve of genus two whose cycles of its skeleton are either disjoint or share an edge of length at most half of the length of the cycles, can be tropicalized faithfully in dimension three.

Keywords Mumford curves · Tropical geometry · Berkovich spaces · Tropicalization

Mathematics Subject Classification 14T05 · 14H30 · 14H45

1 Introduction

This paper is inspired by recent work of (Baker et al. 2011, 2013) which compares the analytic and tropical geometry of curves. Let K be a complete, non-archimedean field with a non-trivial absolute value, and let X be a smooth, proper and connected

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curve over K . Then the subset $H_o(X^{\text{an}})$ of non-leaves in the Berkovich space X^{an} carries a natural metric which is induced by skeletons of semistable models. If X is embedded in a toric variety and meets the dense torus T , the resulting tropicalization $\text{Trop}(T \cap X)$ is a one-dimensional polyhedral complex. All its edges have rational slopes with respect to the cocharacter lattice of the torus. Hence $\text{Trop}(T \cap X)$ can be endowed with a natural metric locally given by the lattice length on each edge. In general the tropicalization map $(X \cap T)^{\text{an}} \rightarrow \text{Trop}(X \cap T)$ is not injective and does not respect the metrics on both sides.

Among the main results in [Baker et al. \(2011\)](#) are Theorems 6.20 and 6.22, which say that for every skeleton Γ in $H_o(X^{\text{an}})$ there exists a closed immersion of X in a quasiprojective toric variety such that X meets the dense torus T and Γ maps isometrically to its image under the induced tropicalization map on $X \cap T$. In the terminology of [Baker et al. \(2011\)](#), 6.15.2 we say that in this case Γ is *faithfully tropicalized*.

Given a skeleton of an analytic curve, it is an interesting question to determine the minimal dimension necessary to tropicalize it faithfully. Using the construction in the proof of Theorem 6.22 in [Baker et al. \(2011\)](#), we may assume that the skeleton has at least two vertices and there are at least $g + 1$ edges. To tropicalize the skeleton faithfully they define two functions for each edge as well as one function for each pair of vertices. These functions may not define an embedding thus we have to add more rational functions (at most four additional functions since every curve can be embedded in projective space of dimension three). This sums up to at least $2g + 6$ functions. Given a finite subset $D \subset X(K)$ for every point p in D one needs two functions to tropicalize the ray leading to p faithfully. Hence we obtain an embedding of dimension at least $2(g + |D|) + 5$ that leads to a faithful tropicalization of a skeleton.

However, in special cases one can do much better. For example, for Tate curves, this bound would be 7, whereas by [Baker et al. \(2011\)](#), Theorem 7.2 there exists a faithful tropicalization in two-dimensional projective space. Furthermore Chan and Sturmfels showed that for every Tate elliptic curve, there exists a tropicalization that is in so called honeycomb form, which is a faithful tropicalization ([Chan and Sturmfels 2013](#), Theorem 7).

In [Baker and Rabinoff \(2013\)](#), Theorem 8.2 Baker and Rabinoff defined for each curve X together with a skeleton Γ a rational morphism $X \dashrightarrow \mathbb{P}^3$ whose tropicalization, when restricted to Γ , is an isometry onto its image. However this morphism may not be an embedding.

In the present paper, we investigate this question for Mumford curves of genus two. There are two types of such curves: the cycles in the skeleton can share an edge or the cycles are disjoint. It turns out that for curves whose cycles are disjoint there exists an embedding in projective three-space such that the tropicalization is faithful (Theorem 5.3). This is also true for curves whose cycles share an edge if the cycles are at least twice as long as the shared edge. Let D be a very ample divisor of degree five such that its support retracts to given points on the skeleton. We define two divisors as follows: Choose three points from the support of D (three poles) and one additional K -rational point on the curve (one zero). From general theory we know that there exist two K -rational points (zeros) so that the resulting degree zero divisor is principal, i.e. we obtain a rational function on the curve with three poles. However we do not

know where the retractions of these new zeros lie on the skeleton. This is necessary to calculate the slopes of the tropicalization.

To solve this problem we will make use of the tropicalization of the Jacobian, which inherits a group structure from the Jacobian, and a map μ from the skeleton to the tropicalization of the Jacobian, which is an isometry on the cycles.

These will tell us where the new zeros of the function are retracting to (Lemma 5.1) and we are able to show that there exists a rational morphism from the curve to two-dimensional projective space tropicalizing the skeleton faithfully.

Next we check if the rays from the extended skeleton are also tropicalized faithfully. We will see that two rays may intersect. To correct this we have to define a third function in the linear system $L(D)$, linearly independent from the others. These three functions tropicalize the extended skeleton faithfully, and, since they are generators of $L(D)$, they also define an embedding of the curve in projective three-space.

2 Skeletons and tropicalizations of curves

Let K be a algebraically closed field which is complete with respect to a non-archimedean non-trivial absolute value $|\cdot|$. As usual, we put $K^\circ = \{x \in K : |x| \leq 1\}$ and $K^{\circ\circ} = \{x \in K : |x| < 1\}$ and we denote by $\tilde{K} = K^\circ/K^{\circ\circ}$ the residue field of K .

For every K -scheme X of finite type we denote by X^{an} the associated Berkovich analytic space, as defined in Berkovich (1990). If X is a smooth, projective and geometrically connected curve of positive genus, then X^{an} is a special quasipolyhedron in the sense of Berkovich (1990), Definition 4.1.1 and 4.1.5. Its Betti number is at most g by Berkovich (1990), Theorem 4.3.2.

Let X be a smooth, projective curve over K with semistable reduction, and fix a semistable model \mathfrak{X} of X . This gives rise to a *skeleton* $\mathcal{S}(\mathfrak{X})$ inside X^{an} which is a deformation retract, see Berkovich (1990), Chapter 4. Let $\tau : X^{\text{an}} \rightarrow \mathcal{S}(\mathfrak{X})$ be the continuous retraction map.

Let D be a finite subset of $X(K)$ and V be a finite set of type 2 points in X^{an} . Set $U = X \setminus D$. Then $U^{\text{an}} \setminus V$ is a *semistable decomposition* in the sense of Baker et al. (2013), Definition 3.1 if it is the disjoint union of

- infinitely many open balls,
- finitely many open annuli,
- finitely many punctured balls, each of them containing precisely one point of D .

The punctured balls can be seen as open annuli with infinite modulus. Their skeleton is a semi-infinite ray (see Baker et al. 2013, chapter 2.1).

For D as before set $W = \{\tau(x) : x \in D\} \cup \{\text{vertices of } \mathcal{S}(\mathfrak{X})\} \subset X^{\text{an}}$. This is a *semistable vertex set* of X^{an} in the sense of Baker et al. (2013), Definition 3.1. If two distinct points x, y of D lie in the same open ball B of $X^{\text{an}} \setminus W$ then there exists a smallest closed disc in B containing x and y . The Gauss point of this closed disc will be denoted by $x \vee y$. If we adjoin all such points $x \vee y$ to W then it is still a semistable vertex set and $U^{\text{an}} \setminus W$ is a semistable decomposition. This follows from the proof of Baker et al. (2013), Lemma 3.13 (3). The path from $x \vee y$ to $\mathcal{S}(\mathfrak{X})$ yields a new bounded edge. The skeleton associated to these data in Baker et al. (2013), Definition 3.3 is the union of $\mathcal{S}(\mathfrak{X})$ with the new bounded edges, the vertices W and the rays leading to

points in D . We denote it by $\mathcal{S}(\mathfrak{X}, D)$ and call it an *extended skeleton*. By Baker et al. (2013), Lemma 3.8 there exists a continuous retraction map $\tau : U^{\text{an}} \rightarrow \mathcal{S}(\mathfrak{X}, D)$ from the open curve to the extended skeleton.

A skeleton carries a natural metric, the shortest-path metric. The distance between two points is the smallest length of all paths connecting the points, where the *length* of an edge is defined to be the logarithmic modulus of the corresponding annulus (Baker et al. 2013, Definition 3.10 and chapter 3.12). This metric extends to rays giving the skeleton a degenerate metric.

We will use the following slope formula (Baker et al. 2013, Theorem 5.15), see also Thuillier’s thesis (Thuillier 2005, 3.3.15).

Theorem 2.1 (Slope formula) *Let f be a rational function on X and D be a finite subset of $X(K)$ containing $\text{supp}(\text{div}(f))$. Set $U = X \setminus D$, and consider the retraction map $\tau : U^{\text{an}} \rightarrow \mathcal{S}(\mathfrak{X}, D)$. Then the function $\log |f| : U^{\text{an}} \rightarrow \mathbb{R}$, mapping a point x to $\log |f(x)|$, has the following properties:*

- (i) $\log |f| = \log |f| \circ \tau$, i.e. $\log |f|$ factors over the skeleton.
- (ii) $\log |f|$ is piecewise affine-linear with integer slopes on each edge of the skeleton $\mathcal{S}(\mathfrak{X}, D)$.
- (iii) Let x be a point in the skeleton and let $d_v(x)$ denote the outgoing slope of $\log |f|$ along an edge v . Then $\sum_v d_v(x) = 0$ where only finitely many outgoing slopes are unequal to zero.
- (iv) If v is a vertex in \mathcal{S} and r a ray in v leading to a point $x \in \text{supp}(\text{div}(f))$, let d_v be the outgoing slope of $\log |f|$ along r . If x has multiplicity m in $\text{div}(f)$, then $d_v(x) = -m$.

If x is a point in the skeleton that is neither a vertex nor a retraction of a pole or zero then there are two edges of the skeleton emanating from x . Thus by (iii) the slopes along these edges have to be equal. Hence the slope of $\log |f|$ can only change at the retractions of the poles and zeros of f and the vertices of the skeleton.

Let U be a connected K -scheme together with a closed immersion $\varphi : U \hookrightarrow \mathbb{G}_m^n$ into a K -split torus. As a set, the associated tropical variety $\text{Trop}(U) = \text{Trop}_\varphi(U)$ is the image of U^{an} under the tropicalization map

$$(\mathbb{G}_m^n)^{\text{an}} = (\text{Spec} K[x_1^\pm, \dots, x_n^\pm])^{\text{an}} \longrightarrow \mathbb{R}^n, \quad p \longmapsto (\log |x_1(p)|, \dots, \log |x_n(p)|).$$

Note that $\text{Trop}_\varphi(U) = \text{Trop}_{\varphi_L}(U_L)$ for any base change by a non-archimedean complete extension field L/K , see Gubler (2013), Proposition 3.7. So if we start with a non-algebraically closed field we can perform a base change to apply the theory developed in this paper. The tropicalization $\text{Trop}(U)$ can be enriched with the structure of a balanced, weighted, integral polyhedral complex of pure dimension $d = \dim(U)$, see e.g. Maclagan and Sturmfels (2014), Theorem 3.3.6. $\text{Trop}(U)$ is called *faithful* if the restriction of the tropicalization map to the skeleton of the curve is an isometry. If we consider extended skeletons we call the tropicalization faithful if it is faithful on the skeleton, homeomorphic on the extended skeleton and the rays have slopes ± 1 . Let $\varphi : X^{\text{an}} \rightarrow (\mathbb{P}^n)^{\text{an}}$ be an embedding in projective space given by rational functions f_0, \dots, f_n and let $D = \bigcup_i \text{supp}(\text{div}(f_i))$ be the union of the support of

the corresponding principal divisors. We say that the curve is *tropicalized faithfully* with respect to φ if $\text{Trop}(X \setminus D)$ is a faithful tropicalization of the extended skeleton $S(\mathcal{X}, D)$.

3 Mumford curves

The present paper deals with analytic Mumford curves, for which the Betti number of X^{an} is equal to the genus. Mumford curves are smooth curves with totally degenerate stable reductions. In Mumford (1972) Mumford shows that these are precisely the curves admitting a non-archimedean Schottky uniformization. Let us recall some facts about these uniformizations.

Let Γ be a subgroup of $\text{PGL}(2, K)$. A point $x \in \mathbb{P}^1(K)$ is called a *limit point* of Γ , if there exists some $y \in \mathbb{P}^1(K)$ and an infinite subset $\{\gamma_n : n \geq 1\}$ of Γ such that $\gamma_n(y) \rightarrow x$. We denote by $\mathcal{L} = \mathcal{L}_\Gamma$ the set of all limit points. A subgroup Γ of $\text{PGL}(2, K)$ is called a *Schottky group* if it is finitely generated, free and discontinuous, where the last property means that $\mathcal{L} \neq \mathbb{P}^1(K)$ and that all orbit closures in $\mathbb{P}^1(K)$ are compact. Every element $\gamma \neq 1$ of a Schottky group is hyperbolic, i.e. it has two different fixed points with different absolute values.

Let Γ be a Schottky group, and write $\Omega = (\mathbb{P}^1_K)^{\text{an}} \setminus \mathcal{L}$, where $(\mathbb{P}^1_K)^{\text{an}}$ is the Berkovich analytic space associated to the projective line over K . Then Γ acts freely and properly discontinuously on Ω , and the quotient Ω/Γ is a proper analytic curve over K , and hence the analytification of an algebraic curve X . Every curve over K with such a Schottky uniformization is called a Mumford curve.

Recall that a closed disc of radius $r \geq 0$ with center $a \in K$ is the set $B := \{z \in K : |z - a| \leq r\}$ and the corresponding open disc is $B^\circ := \{z \in K : |z - a| < r\}$. Every point inside the disc can serve as its center. Note that topologically each disc is both open and closed.

It is shown in Gerritzen and van der Put (1980), section 1, paragraph 4, that every Schottky group has a *good fundamental domain*, i.e. for each Schottky group Γ of rank g there exist $2g$ pairwise disjoint closed discs $B_1, \dots, B_g, C_1, \dots, C_g$ with centers in K and generators $\gamma_1, \dots, \gamma_g$ of Γ such that

- $\mathcal{F} = \mathbb{P}^1(K) \setminus (\bigcup_i B_i \cup \bigcup_i C_i^\circ)$ is a fundamental domain for the action of Γ on Ω ,
- $\gamma_i(\mathbb{P}^1(K) \setminus B_i^\circ) = C_i$ and $\gamma_i^{-1}(\mathbb{P}^1(K) \setminus B_i) = C_i^\circ$ for all $i = 1, \dots, g$.

We denote the corresponding open discs by B_i°, C_i° etc.

Note that we always choose our coordinates on \mathbb{P}^1 so that 0 lies outside these discs.

We fix centers b_i of B_i and c_i of C_i . Let r_i be the radius of B_i and R_i the one of C_i . Since 0 is not contained in any disc, the radii of B_i and C_i satisfy $r_i < |b_i|$ and $R_i < |c_i|$. Furthermore, the absolute value of all points in B_i is equal to $|b_i|$ and the absolute value of all points in C_i is equal to $|c_i|$. Since B_i contains one fixed point of γ_i and C_i contains the other, and since the two fixed points of a hyperbolic element have different absolute values, we have $|b_i| \neq |c_i|$. Moreover, for every $a \in B_i$ and $b \notin B_i$ we have $|b - a| = |b - b_i|$. Similarly, for every $a \in C_i$ and $b \notin C_i$, we have $|b - a| = |b - c_i|$.

Now choose some $a \in \mathcal{F}$ and define for all $i = 1, \dots, g$ *theta functions*

$$u_i(z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma a}{z - \gamma \gamma_i a}.$$

By [Gerritzen and van der Put \(1980\)](#), section II.2, $u_i(z)$ is an analytic function without zeros on Ω which is independent of the choice of a . Moreover, the function $u_i(z)$ is an automorphic function with respect to Γ , i.e. the quotient $c_i(\gamma) = u_i(z)/u_i(\gamma z)$ is a constant in K independent of z by [Gerritzen and van der Put \(1980\)](#), chapter VI, paragraph 2. This constant is multiplicative in γ and satisfies $c_i(\gamma_j) = c_j(\gamma_i)$. We put $q_{ij} = c_i(\gamma_j)$. Then the matrix $(q_{ij})_{ij}$ is symmetric.

Proposition 3.1 *For every $z \notin \bigcup_{i=1}^g (B_i^\circ \cup C_i^\circ)$ we have $|u_i(z)| = \left| \frac{z - b_i}{z - c_i} \right|$.*

Proof Choose $a \in \mathbb{P}^1(K)$ inside the complement of the union of the discs B_i, C_i . Let γ be in Γ such that $\gamma \neq 1$ and $\gamma \neq \gamma_i$. Then $\gamma = \gamma_{i_m}^{s_m} \gamma_{i_{m-1}}^{s_{m-1}} \dots \gamma_{i_1}^{s_1}$ for $i_1, \dots, i_m \in \{1, \dots, g\}$ and signs $s_1, \dots, s_m \in \{-1, 1\}$. We assume that this expression is reduced, i.e. that γ_j and γ_j^{-1} are never adjacent factors. Let us show by induction on m that the points γa and $\gamma \gamma_i a$ are contained in the same disc. Since γ_j maps the complement of B_j to C_j° and γ_j^{-1} maps the complement of C_j to B_j° , the point $\gamma_i a$ is contained in C_i° which lies in the complement of all B_j and in the complement of those C_j with $j \neq i$. Hence our claim holds for $m = 1$.

If γ is an element of length $m > 1$ in Γ , consider $\gamma' = \gamma_{i_{m-1}}^{s_{m-1}} \dots \gamma_{i_1}^{s_1}$. Then $\gamma' \neq 1$. If $\gamma' = \gamma_i^{-1}$, then $\gamma = \gamma_j^{-1} \gamma_i^{-1}$ or $\gamma = \gamma_j \gamma_i^{-1}$ such that $\gamma_j \neq \gamma_i$. In the first case, the points $\gamma_i^{-1} a$ and a lie in the complement of C_j , hence they are mapped to the disc B_j° by γ_j^{-1} , and our claim holds. It also holds in the second case, since the points $\gamma_i^{-1} a$ and a lie in the complement of B_j and are mapped to the disc C_j° by γ_j . Therefore we may assume that γ' is neither 1 nor γ_i . We conclude by induction hypothesis that $\gamma' a$ and $\gamma' \gamma_i a$ are contained in either $B_{i_{m-1}}^\circ$ or $C_{i_{m-1}}^\circ$, depending on s_{m-1} . Since $\gamma_{i_m}^{s_m} \neq \gamma_{i_{m-1}}^{-s_{m-1}}$, the element $\gamma_{i_m}^{s_m}$ maps $C_{i_{m-1}}$ to $B_{i_m}^\circ$ if $s_m = -1$ or $B_{i_{m-1}}$ to $C_{i_m}^\circ$ if $s_m = 1$ and our claim follows.

Hence for all γ not equal to 1 or γ_i^{-1} we have $|z - \gamma \gamma_i a| = |z - \gamma a|$, since z lies outside of the $2g$ open discs. This implies that

$$|u_i(z)| = \left| \prod_{\gamma \in \Gamma} \frac{z - \gamma a}{z - \gamma \gamma_i a} \right| = \frac{|z - \gamma_i^{-1} a|}{|z - \gamma_i a|} = \frac{|z - b_i|}{|z - c_i|},$$

since $\gamma_i^{-1} a$ lies in B_i and $\gamma_i a$ lies in C_i . □

Let us recall some facts about the Berkovich projective line $(\mathbb{P}^1)^{\text{an}} = (\mathbb{A}^1)^{\text{an}} \cup \{\infty\}$. The points in the affine line $(\mathbb{A}^1)^{\text{an}}$ correspond to multiplicative seminorms on the polynomial ring $K[T]$ extending the absolute value on K . These seminorms can be described explicitly and are classified into four types, see [Berkovich \(1990\)](#), 1.4.4. Every point a in $\mathbb{A}^1(K)$ induces a point of type 1, i.e. the multiplicative seminorm

$f \mapsto |f(a)|$. For every point $a \in \mathbb{A}^1(K)$ and every positive real number r , there is a corresponding multiplicative seminorm $\zeta_{a,r}(f)$ mapping $f = \sum_n c_n(x - a)^n$ to

$$\zeta_{a,r}(f) = \max_n \{|c_n|r^n\}.$$

It can be described as a supremum norm over the disc in K around a with radius r . The point $\zeta_{a,r}$ in $(\mathbb{A}^1)^{\text{an}}$ is called of type 2, if the radius r lies in the value group $|K^*|$. Otherwise it is called of type 3. Note that $\zeta_{a,r} = \zeta_{b,s}$ if and only if $|a - b| \leq r = s$. Points of type 4 are realized as limits of series of points of type 2 and 3. They do only appear over fields that are not spherically complete.

The Berkovich projective line is an \mathbb{R} -tree in the sense of Baker and Rumely (2010), Appendix B. It is uniquely arcwise connected. We denote the unique path between two elements x, y of $(\mathbb{P}^1)^{\text{an}}$ by $[x, y]$. For $a, b \in \mathbb{A}^1(K)$ and $r < s$ the path $[\zeta_{a,r}, \zeta_{a,s}]$ consists of all points $\zeta_{a,t}$ such that $t \in [r, s]$. To determine $[\zeta_{a,r}, \zeta_{b,s}]$ in general, put $R = \max\{r, s, |a - b|\}$ and note that $\zeta_{a,R} = \zeta_{b,R}$. Then $[\zeta_{a,r}, \zeta_{b,s}] = [\zeta_{a,r}, \zeta_{a,R}] \cup [\zeta_{b,R}, \zeta_{b,s}]$.

The complement of the subset of points of type 1 in $(\mathbb{A}^1)^{\text{an}}$ can be endowed with a path distance metric, see Baker and Rumely (2010), section 2.7. For points $x = \zeta_{a,r}$ and $y = \zeta_{b,s}$ as above, it is equal to

$$\rho(x, y) = 2 \log R - \log r - \log s.$$

Set $\zeta_i^+ = \zeta_{b_i, r_i}$ and $\zeta_i^- = \zeta_{c_i, R_i}$. Note that $\gamma_i(\zeta_i^-) = \zeta_i^+$. By the explicit description above, is easy to see that the path $[\zeta_i^-, \zeta_i^+]$ is contained in the fundamental domain \mathcal{F} .

The Berkovich curve X^{an} contains the skeleton $\mathcal{S}(X^{\text{an}})$ of the semistable model of X constructed by Mumford (1972), Theorem 3.3. The graph $\mathcal{S}(X^{\text{an}})$ is equal to the image under the natural map $\pi : \Omega \rightarrow X^{\text{an}}$ of the convex hull of all the points ζ_i^+, ζ_i^- in $(\mathbb{P}^1)^{\text{an}}$. Note that this convex hull is contained in $(\Omega)^{\text{an}}$. The skeleton $\mathcal{S}(X^{\text{an}})$ can be endowed with a natural metric induced by the path distance metric on the projective line. It contains the g cycles $Z_i = \pi([\zeta_i^-, \zeta_i^+])$.

- Proposition 3.2** (i) *The cycle Z_i has length $-\log |q_{ii}|$.*
 (ii) *The intersection $Z_i \cap Z_j$ has length $|\log |q_{ij}||$.*

Proof (i) After exchanging γ_i with γ_i^{-1} if necessary, we may assume that $|b_i| < |c_i|$. The length of Z_i is equal to the path distance $\rho(\zeta_i^-, \zeta_i^+)$. Since $r_i, R_i < |c_i| = |c_i - b_i|$, we find

$$\rho(\zeta_i^-, \zeta_i^+) = 2 \log |c_i| - \log r_i - \log R_i.$$

On the other hand, $|q_{ii}| = |u_i(z)/u_i(\gamma_i z)|$ for any point $z \in \Omega$. Let us choose a rational point $z \in \partial B_i$. Then $\gamma_i z$ lies in ∂C_i . Therefore we can apply Proposition 3.1 to z and $\gamma_i z$ to deduce

$$-\log |q_{ii}| = -\log \frac{|z - b_i||\gamma_i z - c_i|}{|z - c_i||\gamma_i z - b_i|}.$$

Since by the choice of z we have $|z - b_i| = r_i$ and $|\gamma_i z - c_i| = R_i$ and $|z - c_i| = |c_i|$ as well as $|\gamma_i z - b_i| = |\gamma_i z| = |c_i|$, we deduce that

$$-\log |q_{ii}| = 2 \log |c_i| - \log r_i - \log R_i.$$

- (ii) After exchanging γ_i and γ_i^{-1} if necessary, we may again assume that $|b_i| < |c_i|$. Similarly, we may assume that $|b_j| < |c_j|$. After exchanging i and j if necessary we also have $|b_i| \leq |b_j|$.

Choose $z \in \partial B_j$. Then $\gamma_j z \in \partial C_j$. Hence we can apply Proposition 3.1 to z and $\gamma_j z$ to deduce

$$-\log |q_{ij}| = -\log \left| \frac{u_i(z)}{u_i(\gamma_j z)} \right| = -\log \frac{|z - b_i| |\gamma_j z - c_i|}{|z - c_i| |\gamma_j z - b_i|}.$$

By the choice of z we have $|z - b_i| = |b_j - b_i|$ and $|z - c_i| = |b_j - c_i|$ as well as $|\gamma_j z - b_i| = |c_j - b_i|$ and $|\gamma_j z - c_i| = |c_j - c_i|$. Hence

$$-\log |q_{ij}| = \log \frac{|b_j - c_i| |c_j - b_i|}{|b_j - b_i| |c_j - c_i|}. \tag{1}$$

Recall that $[\zeta_i^+, \zeta_i^-] = [\zeta_i^+, \zeta_{b_i, |c_i|}] \cup [\zeta_{c_i, |c_i|}, \zeta_i^-]$. The first part $[\zeta_i^+, \zeta_{b_i, |c_i|}]$ intersects $[\zeta_j^+, \zeta_j^-]$ in all points $\zeta_{b_i, r}$ satisfying $|b_i - b_j| \leq r \leq \min\{|c_i|, |c_j|\}$. The second part $[\zeta_{c_i, |c_i|}, \zeta_i^-]$ intersects $[\zeta_j^+, \zeta_j^-]$ in all points $\zeta_{c_i, r}$ such that either $|c_j - c_i| \leq r \leq \min\{|c_i|, |c_j|\}$ or $|b_j - c_i| \leq r \leq \min\{|c_i|, |c_j|\}$.

Now we can check our claim case by case.

Case 1 $|c_i| < |b_j|$. This implies $|b_i| < |c_i| < |b_j| < |c_j|$. In this case $Z_i \cap Z_j$ is empty and by formula (1) we have $-\log |q_{ij}| = 0$.

Case 2 $|c_i| = |b_j|$. This implies $|b_i| < |c_i| = |b_j| < |c_j|$. Here, $Z_i \cap Z_j$ is the image of $[\zeta_{c_i, |b_j - c_i|}, \zeta_{c_i, |c_i|}]$, which has length $\log(|c_i| |b_j - c_i|^{-1})$, which coincides with $\log |q_{ij}| = |-\log |q_{ij}||$ by formula (1).

Case 3 $|c_i| > |b_j|$ and $|c_i| \neq |c_j|$. Put $m = \min\{|c_i|, |c_j|\}$. Then $Z_i \cap Z_j$ is the image of $[\zeta_{b_i, |b_i - b_j|}, \zeta_{b_i, m}]$, which has length $\log(m |b_i - b_j|^{-1})$. This coincides with $-\log |q_{ij}|$ by formula (1).

Case 4 $|c_i| > |b_j|$ and $|c_i| = |c_j|$. In this case, $Z_i \cap Z_j$ is the image of $[\zeta_{b_i, |b_i - b_j|}, \zeta_{b_i, |c_i|}] \cup [\zeta_{c_i, |c_i|}, \zeta_{c_i, |c_i - c_j|}]$, which has length

$$\log \frac{|c_i| |c_i|}{|b_i - b_j| |c_i - c_j|} = -\log |q_{ij}|.$$

□

Note that part (ii) of the preceding statement follows also from van der Put (1992), proof of Theorem 6.4 (2).

We can use the functions u_i to give an explicit description of the embedding of X into its Jacobian. We give here a short summary, for details see Gerritzen and van

der Put (1980), chapter II (5.1), chapter IV (3.9) and Manin (1974), chapter 3. Fix $a \in \Omega(K)$ and consider the analytic morphism

$$u : \Omega \longrightarrow (\mathbb{G}_m^g)^{\text{an}}, \quad z \mapsto \left(\frac{u_1(z)}{u_1(a)}, \dots, \frac{u_g(z)}{u_g(a)} \right).$$

Let Λ be the multiplicative lattice in $(\mathbb{G}_m^g)^{\text{an}}$ generated by all $\lambda_i = (q_{i1}, \dots, q_{ig})$. Then the quotient $(\mathbb{G}_m^g)^{\text{an}}/\Lambda$ is isomorphic to the analytification J^{an} of the Jacobian J of X , and the following uniformization diagram is commutative, where j is the embedding of X into its Jacobian associated to the point $\pi(a) \in X(K)$.

$$\begin{array}{ccc} \Omega^{\text{an}} & \xrightarrow{u} & (\mathbb{G}_m^g)^{\text{an}} \\ \downarrow & & \downarrow \\ X^{\text{an}} & \xrightarrow{j^{\text{an}}} & J^{\text{an}} \end{array}$$

There exists a birational morphism $j^{(g)} : X^{(g)} \rightarrow J$ where $X^{(g)}$ denotes the g -th symmetric power of the curve sending (z_1, \dots, z_g) to $j(z_1) \cdots j(z_g)$ (see Milne (1986), Theorem 5.1).

There is a natural tropicalization map

$$\text{trop} : (\mathbb{G}_m^g)^{\text{an}} \longrightarrow \mathbb{R}^g,$$

which maps a multiplicative seminorm x on the Laurent polynomial ring $K[T_1^{\pm 1}, \dots, T_g^{\pm 1}]$ to $(\log |T_1(x)|, \dots, \log |T_g(x)|)$, where $|T_i(x)|$ denotes the value of x on T_i (Baker et al. 2013, 2.1).

It induces a tropicalization map $\text{trop} : J^{\text{an}} \longrightarrow \mathbb{R}^g/\log |\Lambda|$ by passing to quotients, where $\log |\Lambda|$ denotes the tropicalization of the lattice Λ , which is generated by the vectors $(\log |q_{i1}|, \dots, \log |q_{ig}|)$ (see Baker and Rabinoff 2013, 6.5). We will use this tropicalization map on the Jacobian to construct faithful tropicalizations.

4 Mumford curves of genus two

We study the situation of Mumford curves of genus $g = 2$. In this case, the skeleton $\mathcal{S}(X^{\text{an}})$ contains two distinguished cycles Z_1 and Z_2 . These cycles are either disjoint, meet in a point or meet in an edge (Fig. 1). We know from the proof of Proposition 3.2 that in the first two cases $|q_{12}| = |q_{21}| = 1$, and that in the third case the length of the shared edge is $|\log |q_{12}||$.

Lemma 4.1 *For a Schottky group Γ of rank two there exists an element $\gamma \in \text{PGL}(2, K)$ such that $\gamma\Gamma\gamma^{-1}$ has a good fundamental domain satisfying $|b_1| < |c_1| < |c_2|$ and $|b_1| < |b_2| < |c_2|$. If the skeleton of the corresponding Mumford curve has a shared edge, we may also assume $|b_2| < |c_1|$.*

Proof Let γ_1, γ_2 be generators of Γ and B_1, B_2, C_1, C_2 discs in \mathbb{P}^1 providing a good fundamental domain. For $i = 1, 2$ let b_i be the fixed point of Γ in B_i , and let c_i be

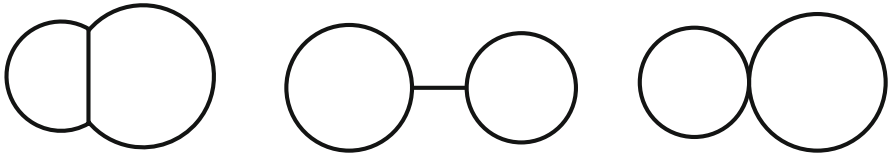


Fig. 1 The three possible types of skeletons in genus two, from the left to the right skeleton with shared edge, connecting edge and connecting point

the fixed point of Γ in C_i . Recall that all elements in B_i have absolute value $|b_i|$, all elements in C_i have absolute value $|c_i|$, and the discs B_1, B_2, C_1, C_2 are pairwise disjoint. We have

$$|c_1 - b_1| \leq \max\{|c_1 - c_2|, |c_2 - b_1|\}.$$

If $|c_1 - c_2| > |c_2 - b_1|$ we exchange the generators γ_1 and γ_1^{-1} , and hence the pair of fixed points. Therefore we may assume that $|c_1 - c_2| \leq |c_2 - b_1|$, which implies $|c_1 - b_1| \leq |c_2 - b_1|$. Similarly, after exchanging γ_2 and γ_2^{-1} if necessary, we may assume that $|b_1 - b_2| \leq |c_2 - b_1|$, which implies that $|b_2 - c_2| \leq |c_2 - b_1|$. Since

$$|b_1 - b_2| \leq \max\{|b_1 - c_2|, |c_2 - b_2|\}$$

we have $|b_1 - b_2| \leq |c_2 - b_1|$ and similarly $|c_1 - c_2| \leq |c_2 - b_1|$. Thus

$$|c_2 - b_1| \geq \max\{|c_1 - b_1|, |c_2 - b_2|, |c_2 - c_1|, |b_2 - b_1|\}.$$

We find points $p, q \in K, p \neq q$ lying outside the closed discs B_1 and C_2 such that

$$\begin{aligned} |b_1 - p| &< \min\{|b_1 - c_1|, |b_1 - b_2|, |c_2 - b_1|\} \text{ and} \\ |c_2 - q| &< \min\{|c_2 - b_1|, |c_2 - c_1|, |c_2 - b_2|\}. \end{aligned}$$

Such points do exist since the discs are pairwise disjoint. This implies that

$$|c_1 - p| = |c_1 - b_1|, |b_2 - p| = |b_2 - b_1| \text{ and } |c_2 - p| = |c_2 - b_1|,$$

as well as

$$|b_1 - q| = |b_1 - c_2|, |c_1 - q| = |c_1 - c_2| \text{ and } |b_2 - q| = |b_2 - c_2|.$$

Hence p and q lie in the fundamental domain \mathcal{F} .

We conjugate Γ by the element $\gamma(z) = \frac{z-p}{z-q}$ in $\text{PGL}(2, K)$. Then we obtain

$$|\gamma(b_1)| = \left| \frac{b_1 - p}{b_1 - q} \right| < \left| \frac{c_1 - p}{c_1 - q} \right| = |\gamma(c_1)|$$

since $|b_1 - p| < |c_1 - p|$ and $|c_1 - q| = |c_2 - c_1| \leq |c_2 - b_1| = |b_1 - q|$.

Moreover, we have

$$|\gamma(c_1)| = \left| \frac{c_1 - p}{c_1 - q} \right| < \left| \frac{c_2 - p}{c_2 - q} \right| = |\gamma(c_2)|$$

since $|c_2 - q| < |c_1 - q|$ and $|c_1 - p| = |c_1 - b_1| \leq |c_2 - b_1| = |c_2 - p|$.

This shows $|\gamma(b_1)| < |\gamma(c_1)| < |\gamma(c_2)|$. We also find

$$|\gamma(b_1)| = \left| \frac{b_1 - p}{b_1 - q} \right| < \left| \frac{b_2 - p}{b_2 - q} \right| = |\gamma(b_2)|$$

since $|b_1 - p| < |b_2 - p|$ and $|b_1 - q| = |c_2 - b_1| \geq |b_2 - c_2| = |b_2 - q|$. Similarly,

$$|\gamma(b_2)| = \left| \frac{b_2 - p}{b_2 - q} \right| < \left| \frac{c_2 - p}{c_2 - q} \right| = |\gamma(c_2)|$$

since $|b_2 - p| = |b_2 - b_1| \leq |c_2 - b_1| = |c_2 - p|$ and $|b_2 - q| = |b_2 - c_2| > |c_2 - q|$. Thus we also get $|\gamma(b_1)| < |\gamma(b_2)| < |\gamma(c_2)|$.

Hence $\gamma\Gamma\gamma^{-1}$ has a good fundamental domain satisfying the first claim of the lemma.

In order to prove the second claim, we consider the case that the skeleton has a shared edge. By the first part of the proof we may assume that $|b_1| < |c_1| < |c_2|$ and $|b_1| < |b_2| < |c_2|$ hold. Recall from the proof of Proposition 3.2 that the inequality $|b_2| > |c_1|$ implies that the cycles are disjoint. Hence we have $|b_2| \leq |c_1|$. If the inequality is strict, our claim follows. Therefore we may assume that $|c_1| = |b_2|$. Note that in the case of a shared edge we have $|c_1 - b_2| < |c_1| = |b_2|$. This follows from the discussion of Case 2 in the proof of Proposition 3.2.

There exists a point $p \in K$ outside the disc C_1 such that

$$|c_1 - p| < \min\{|c_2 - c_1|, |b_1 - c_1|, |b_2 - c_1|\}$$

and a point $q \in K$, $q \neq p$ outside C_2 such that

$$|c_2 - q| < \min\{|c_2 - c_1|, |c_2 - b_2|, |c_2 - b_1|\}.$$

A similar argument as in the proof of the first claim shows that p and q lie in the fundamental domain \mathcal{F} . Let $\gamma(z) = \frac{z-p}{z-q}$. We find

$$|\gamma(c_1)| = \left| \frac{c_1 - p}{c_1 - q} \right| = \frac{|c_1 - p|}{|c_2|} < \frac{|b_2 - p|}{|c_2|} = \left| \frac{b_2 - p}{b_2 - q} \right| = |\gamma(b_2)|$$

since $|c_1 - p| < |b_2 - c_1| = |b_2 - p|$ and $|c_1 - q| = |c_1 - c_2| = |c_2|$ as well as $|b_2 - q| = |b_2 - c_2| = |c_2|$. Similarly, we find

$$|\gamma(b_2)| = \left| \frac{b_2 - p}{b_2 - q} \right| = \frac{|b_2 - c_1|}{|c_2|} < \frac{|b_1 - c_1|}{|b_1 - q|} = \left| \frac{b_1 - p}{b_1 - q} \right| = |\gamma(b_1)|$$

since $|b_2 - p| = |b_2 - c_1| < |c_1| = |b_1 - c_1| = |b_1 - p|$ and $|b_2 - q| = |b_2 - c_2| = |c_2| = |c_2 - b_1| = |b_1 - q|$. Moreover, we have

$$|\gamma(b_1)| = \left| \frac{b_1 - p}{b_1 - q} \right| = \frac{|c_1|}{|c_2|} < \frac{|c_2|}{|c_2 - q|} = \left| \frac{c_2 - p}{c_2 - q} \right| = |\gamma(c_2)|$$

since $|b_1 - p| = |b_1 - c_1| = |c_1| < |c_2|$ and $|c_2 - q| < |c_2|$. Thus $|\gamma(c_1)| < |\gamma(b_2)| < |\gamma(b_1)| < |\gamma(c_2)|$. Now exchange γ_1 and γ_1^{-1} which exchanges c_1 and b_1 . This concludes the proof. \square

Let X be a Mumford curve of genus 2 over K obtained by a Schottky uniformization with a Schottky group Γ . Since conjugation of Γ by an element in $\text{PGL}(2, K)$ corresponds to a K -rational automorphism of the curve, we may assume by Lemma 4.1 that the set of generators of Γ satisfy the inequalities $|b_1| < |c_1| < |c_2|$ and $|b_1| < |b_2| < |c_2|$ with the additional inequality $|b_2| < |c_1|$ in the case of a shared edge.

Every K -rational point in X (and hence every K -rational point of Ω) induces an embedding $j : X \hookrightarrow J$ into the Jacobian. Recall from the last section that the uniformization of J^{an} by a two-dimensional torus gives rise to a tropicalization map $\text{trop} : J^{\text{an}} \rightarrow \mathbb{R}^2/\log|\Lambda|$. We consider the composition

$$\mu : \mathcal{S}(X^{\text{an}}) \hookrightarrow X^{\text{an}} \xrightarrow{j^{\text{an}}} J^{\text{an}} \xrightarrow{\text{trop}} \mathbb{R}^2/\log|\Lambda|$$

Note that μ is in general not injective.

Theorem 4.2 *Denote by e_1, e_2 the standard basis of \mathbb{R}^2 . There exists a point $a \in \Omega(K)$ such that the map μ defined with the embedding $j : X \hookrightarrow J$ associated to a has the following image in a fundamental parallelepiped in $\mathbb{R}^2/\log|\Lambda|$:*

- (1) *In the case of disjoint cycles or cycles meeting in a point we have*

$$\text{im}(\mu) = [0, -\log|q_{11}|) e_1 \cup [0, -\log|q_{22}|) e_2.$$

- (2) *In the case of cycles sharing an edge, we put $l = |\log|q_{12}|| = |\log|q_{21}||$ and $v = (l, l)$. Then we have*

$$\text{im}(\mu) = [0, 1) v \cup v + [0, \log|q_{11}| - l) e_1 \cup v + [0, \log|q_{22}| - l) e_2.$$

- (3) *In the case of cycles sharing an edge, we have furthermore $\mu(Z_1 \setminus Z_2) \subset v + [0, \log|q_{11}| - l) e_1$ and $\mu(Z_2 \setminus Z_1) \subset v + [0, \log|q_{22}| - l) e_2$.*
- (4) *μ maps the two cycles Z_1, Z_2 in $\mathcal{S}(X^{\text{an}})$ isometrically to their images $\mu(Z_1)$ resp. $\mu(Z_2)$ where we endow the fundamental parallelepiped with the metric induced from the maximum metric on \mathbb{R}^2 . See Fig. 2 for an illustration.*

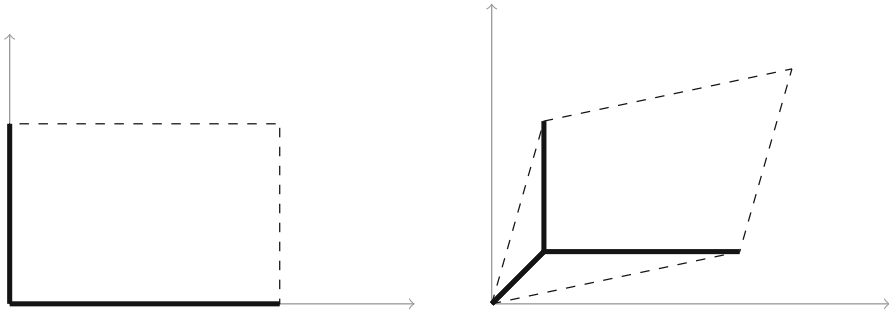


Fig. 2 The fundamental parallelepiped and the image of μ . Depicted on the *left* is the case of a connecting edge, on the *right* the case of a shared edge

Proof Recall that by Proposition 3.1 $|u_i(z)| = \frac{|z-b_i|}{|z-c_i|}$ on the fundamental domain. Hence we find for all $x = \zeta_{b_1,r}$ with $r_1 \leq r \leq |c_1|$:

$$(|u_1(x)|, |u_2(x)|) = \left(\frac{r}{|c_1|}, \frac{\max\{r, |b_1 - b_2|\}}{|c_2|} \right)$$

For all $x = \zeta_{c_1,r}$ with $R_1 \leq r \leq |c_1|$ we have

$$(|u_1(x)|, |u_2(x)|) = \left(\frac{|c_1|}{r}, \frac{\max\{r, |c_1 - b_2|\}}{\max\{r, |c_1 - c_2|\}} \right)$$

Let us first discuss the case where the cycles are disjoint or share a point. By Proposition 3.2 this corresponds to the case $|b_1| < |c_1| < |b_2| < |c_2|$ or to the case $|b_1| < |c_1| = |b_2| < |c_2|$ with $|b_2 - c_1| = |c_1|$. Choose a point $a \in \Omega(K)$ such that $|a| = |c_1| = |c_1 - a|$ and $|b_2| = |b_2 - a|$. Then $(|u_1(a)|, |u_2(a)|) = (1, |b_2|/|c_2|)$. If we use $\pi(a)$ to define the embedding j of X into its Jacobian, the associated map μ is given by $\mu(x) = (\log |u_1(x)|/|u_1(a)|, \log |u_2(x)|/|u_2(a)|) \pmod{\log |\Lambda|}$ on the fundamental domain. Hence the cycle Z_1 is mapped to

$$\{(\log r - \log |c_1|, 0) : r \in [r_1, |c_1|]\} \cup \{(\log |c_1| - \log r, 0) : r \in [R_1, |c_1|]\}.$$

Recall that in this case $\log |q_{12}| = 0$, so that we can add $(-\log |q_{11}|, 0) = (2 \log |c_1| - \log r_1 - \log R_1, 0) \in \log |\Lambda|$ to the first subset. Therefore the image of Z_1 is $[0, -\log |q_{11}|] e_1$. The second cycle Z_2 can be treated in the same way. This proves claim (1).

Now let us discuss the cases of cycles sharing an edge. In this case we have $|b_1| < |b_2| < |c_1| < |c_2|$. Choose a point $a \in \Omega(K)$ such that $|a| = |b_2| = |b_2 - a|$. Then $(|u_1(a)|, |u_2(a)|) = (|b_2|/|c_1|, |b_2|/|c_2|)$. If we use $\pi(a)$ to define the embedding j of X into its Jacobian, the associated map μ is given by $\mu(x) = (\log |u_1(x)|/|u_1(a)|, \log |u_2(x)|/|u_2(a)|) \pmod{\log |\Lambda|}$ on the fundamental domain. Recall from the proof of Proposition 3.2 that the intersection $Z_1 \cap Z_2$ corresponds to the line segment $[\zeta_{b_1,|b_2|}, \zeta_{b_1,|c_1|}]$. Hence $Z_1 \cap Z_2$ is mapped to the line segment

between $(0, 0)$ and $(\log |c_1| - \log |b_2|, \log |c_1| - \log |b_2|)$ with slope one. Formula (1) in part (ii) of the proof of Proposition 3.2 implies that $-\log |q_{12}| = \log |c_1| - \log |b_2|$. Therefore the shared edge is mapped to $[0,1] \ v$. Let us now look at $Z_1 \setminus Z_2$. It corresponds to the union of the paths $[\zeta_{b_1,r_1}, \zeta_{b_1,|b_2|}]$ and $[\zeta_{c_1,R_1}, \zeta_{c_1,|c_1|}]$. On the first path $[\zeta_{b_1,r_1}, \zeta_{b_1,|b_2|}]$ the map μ is given by

$$\zeta_{b_1,r} \mapsto \left(\log \frac{r}{|b_2|}, 0 \right).$$

Adding $(\log |q_{11}|, \log |q_{12}|) = (-2 \log |c_1| + \log r_1 + \log R_1, -\log |c_1| + \log |b_2|) \in \log |\Delta|$ we find that this is the line segment between $(\log \frac{|c_1|^2}{R_1|b_2|}, l)$ and the lattice point $(\log |q_{11}|, l)$. On the second path $[\zeta_{c_1,R_1}, \zeta_{c_1,|c_1|}]$ the map μ is given by

$$\zeta_{c_1,r} \mapsto \left(\log \frac{|c_1|^2}{r|b_2|}, \log \frac{|c_1|}{|b_2|} \right).$$

Its image is the line segment between v and $(\log \frac{|c_1|^2}{R_1|b_2|}, l)$. The cycle Z_2 can be treated similarly. Hence claims (2) and (3) hold. To show claim (4), write $\mu(x) = (\mu_1(x), \mu_2(x))$. Recall that on the first cycle μ_1 is of the form $\zeta_{b,r} \mapsto \log r - \log |b|$. In the case of a connecting edge μ_2 is equal to zero, which implies that μ is an isometry on this cycle. In the shared edge case there is also a change in μ_2 , but μ_1 will increase in the same way, so the maximum will be achieved in μ_1 . A similar argument holds for the second cycle. □

Lemma 4.3 *Again let $v = (-\log |q_{12}|, -\log |q_{12}|)$. If $x = 2v + \alpha e_1 + \beta e_2$ with $0 < \alpha < -\log |q_{11}| + \log |q_{12}|$ and $0 < \beta < -\log |q_{22}| + \log |q_{12}|$ then there are exactly two points S and T in the skeleton such that $\mu(S) + \mu(T) = x$, namely $S = \mu^{-1}(v + \alpha e_1)$ and $T = \mu^{-1}(v + \beta e_2)$.*

Proof If $\mu(S) = v + \alpha e_1$ and $\mu(T) = v + \beta e_2$ then clearly $\mu(S) + \mu(T) = x$. So let us suppose there exist two other points U and V in the skeleton such that $\mu(U) + \mu(V) = x$. We have to handle several cases:

- (1) $\mu(U) = v + \alpha' e_1$ and $\mu(V) = v + \beta' e_2$ with $0 < \alpha' < -\log |q_{11}| + \log |q_{12}|$ and $0 < \beta' < -\log |q_{22}| + \log |q_{12}|$. Then $2v + \alpha e_1 + \beta e_2 = 2v + \alpha' e_1 + \beta' e_2$ so $v + \alpha e_1 + \beta e_2 = v + \alpha' e_1 + \beta' e_2$. Since these points lie in the fundamental domain of the tropicalization of the Jacobian we obtain $\alpha' = \alpha$ and $\beta = \beta'$, thus U and V coincide with S and T .
- (2) $\mu(U) = v + \alpha' e_1$ and $\mu(V) = v + \alpha'' e_1$ with $0 < \alpha', \alpha'' < -\log |q_{11}| + \log |q_{12}|$. Then $\mu(U) + \mu(V)$ lies inside the image of the interior of the first cycle, but x does not. Thus such points U and V cannot exist. The same argument holds if both points lie on the other cycle.
- (3) $\mu(U) = v + \alpha' e_1$ and $\mu(V) = \gamma v$ with $0 < \alpha' < -\log |q_{11}| + \log |q_{12}|$ and $\gamma \in [0,1]$. Then $2v + \alpha e_1 + \beta e_2 = (1 + \gamma)v + \alpha' e_1$ thus $v + \alpha e_1 + \beta e_2 = \gamma v + \alpha' e_1$. To satisfy this equality $\gamma v + \alpha' e_1$ cannot lie in the fundamental domain of the

tropicalization of the Jacobian. Then $(1 + \gamma)v + \alpha'e_1 - \log |q_{22}|e_2$ lies in the fundamental domain, but this point is not equal to $x = 2v + \alpha e_1 + \beta e_2$.

- (4) $\mu(U) = \gamma v$ and $\mu(V) = \gamma'v$ with $\gamma, \gamma' \in [0,1]$. Then $2v + \alpha e_1 + \beta e_2 = (\gamma' + \gamma)v$ thus $v + \alpha e_1 + \beta e_2 = \delta v$, $\delta \in [-1, 1]$. This is not possible if $\delta \geq 0$. So suppose $\delta < 0$. We replace δv by an equivalent point:

$$\begin{aligned} \delta v &= 2v + (-\log |q_{11}| + \log |q_{12}|)e_1 + (-\log |q_{22}| + \log |q_{12}|)e_2 + \delta v \\ &= (1 + \delta)v - \log |q_{11}|e_1 - \log |q_{22}|e_2 \end{aligned}$$

Thus $\delta v - \log |q_{11}|e_1 - \log |q_{22}|e_2 = \alpha e_1 + \beta e_2$ and we obtain $\alpha = -\log |q_{11}| - \delta \log |q_{12}|$. This contradicts $\alpha < -\log |q_{11}| + \log |q_{12}|$. The same argument holds for β thus the points U and V cannot exist. \square

5 Faithful tropicalization in genus two

In this section we will always assume that the cycle lengths are at least twice as long as the length of the shared edge, i.e. $|q_{11}| < |q_{12}|^2$ and $|q_{22}| < |q_{12}|^2$.

Lemma 5.1 *Let $A = -\log |q_{11}| + 2 \log |q_{12}|$, $B = -\log |q_{22}| + 2 \log |q_{12}|$ and $v = (-\log |q_{12}|, -\log |q_{12}|)$. Then there exist points $P_1, P_2, P_3, P_4, S_1, S_2, S_3, T_1, T_2, T_3 \in X(K)$ such that*

- $\mu \circ \tau(P_1) = v + \frac{1}{4}Ae_1$
- $\mu \circ \tau(P_2) = v + \frac{1}{2}Ae_1$
- $\mu \circ \tau(P_3) = v + \frac{1}{4}Be_2$
- $\mu \circ \tau(P_4) = v + \frac{1}{2}Be_2$
- $\mu \circ \tau(S_1) = \mu \circ \tau(T_3) = v + (\frac{3}{4}B - \log |q_{12}|)e_2$
- $\mu \circ \tau(T_1) = \mu \circ \tau(S_3) = v + (\frac{3}{4}A - \log |q_{12}|)e_1$
- $\mu \circ \tau(S_2) = v - \frac{1}{2} \log |q_{22}|e_2$
- $\mu \circ \tau(T_2) = v - \frac{1}{2} \log |q_{11}|e_1$
- $\frac{j(P_1)j(P_2)j(P_3)}{j(S_1)j(S_2)j(S_3)}, \frac{j(P_1)j(P_3)j(P_4)}{j(T_1)j(T_2)j(T_3)}$ are trivial in J .

Proof All points on the skeleton stated in the lemma are of type 2 thus there exist K -rational points P_1, P_2, P_3, P_4, S_1 and T_1 as postulated. Since the map $j^{(2)}$ from $X^{(2)}$ to J is surjective there are points S_2, S_3 and T_2, T_3 in $X(K)$ with

$$j(S_2)j(S_3) = \frac{j(P_1)j(P_2)j(P_3)}{j(S_1)} \quad \text{and} \quad j(T_2)j(T_3) = \frac{j(P_1)j(P_3)j(P_4)}{j(T_1)}$$

thus $\frac{j(P_1)j(P_2)j(P_3)}{j(S_1)j(S_2)j(S_3)}$ and $\frac{j(P_1)j(P_3)j(P_4)}{j(T_1)j(T_2)j(T_3)}$ are trivial in J . To determine the position of the retractions of S_2 and S_3 on the skeleton we use that

$$\begin{aligned} \mu \circ \tau(S_2) + \mu \circ \tau(S_3) &= \mu \circ \tau(P_1) + \mu \circ \tau(P_2) \\ &\quad + \mu \circ \tau(P_3) - \mu \circ \tau(S_1) \pmod{\log |\Lambda|}. \end{aligned}$$

Thus

$$\begin{aligned} \mu \circ \tau(S_2) + \mu \circ \tau(S_3) &= 2v + \frac{3}{4}Ae_1 - \left(\frac{1}{2}B + \log |q_{12}|\right) e_2 \pmod{\log |\Lambda|} \\ &= 2v + \frac{3}{4}Ae_1 + \frac{1}{2} \log |q_{22}| e_2 \pmod{\log |\Lambda|} \\ &= 3v + \frac{3}{4}Ae_1 + \left(-\log |q_{22}| + \log |q_{12}| + \frac{1}{2} \log |q_{22}|\right) e_2 \\ &\quad \pmod{\log |\Lambda|} \end{aligned}$$

where we replaced the point $\frac{1}{2} \log |q_{22}| e_2$ by an equivalent one. Splitting one of the vectors $v = -\log |q_{12}| e_1 - \log |q_{12}| e_2$ yields

$$\mu \circ \tau(S_2) + \mu \circ \tau(S_3) = 2v + \left(\frac{3}{4}A - \log |q_{12}|\right) e_1 - \frac{1}{2} \log |q_{22}| e_2.$$

Hence by Lemma 4.3

$$\mu \circ \tau(S_2) = v - \frac{1}{2} \log |q_{22}| e_2 \text{ and } \mu \circ \tau(S_3) = v + \left(\frac{3}{4}A - \log |q_{12}|\right) e_1.$$

$\mu \circ \tau(T_2)$ and $\mu \circ \tau(T_3)$ can be determined by a similar calculation. \square

Theorem 5.2 *Let $P_1, P_2, P_3, P_4 \in X(K)$ be given as in the previous lemma and let $D = P_1 + P_2 + P_3 + P_4$ be the corresponding effective divisor. Then there exist functions f, g in the linear system $L(D)$ such that the rational morphism $(1, f, g) : X \dashrightarrow \mathbb{P}^2$ tropicalizes the skeleton faithfully.*

Proof By the previous lemma there exist points $S_1, S_2, S_3, T_1, T_2, T_3 \in X(K)$ such that $S_1 + S_2 + S_3 - P_1 - P_2 - P_3 = \text{div}(f)$ and $T_1 + T_2 + T_3 - P_2 - P_3 - P_4 = \text{div}(g)$ are principal divisors. Both functions f and g are elements in the linear system $L(D)$. Since μ is an isometry on the cycles by Theorem 4.2 the points $P_1, P_2, P_3, P_4, S_1, S_2, S_3, T_1, T_2, T_3$ retract to the skeleton as shown in Fig. 3 where we label the edges on the first cycle by a, b, c, d, e, λ . Note that λ and c are not present in the connecting edge case. We orient all edges so that b, c, d, e, λ, a is the path $\tau(P_1) \rightarrow \tau(P_2) \rightarrow \tau(T_2) \rightarrow \tau(T_1) \rightarrow \tau(P_1)$. Similarly we label the edges on the second cycle by $\alpha, \beta, \gamma, \delta$ and λ , where again λ and γ are not present in the case of a connecting edge. We orient the cycle so that $\beta, \gamma, \delta, \lambda, \alpha$ is the path $\tau(P_3) \rightarrow \tau(P_4) \rightarrow \tau(S_2) \rightarrow \tau(S_1) \rightarrow \tau(P_3)$. In the connecting edge case the edge χ will be oriented from the first to the second cycle. \square

By $|a|$ we mean the length of edge a , and we use similar notation for the other edges. By isometry of μ we obtain

$$|c| + |d| = \frac{3}{4}A - \log |q_{12}| - \frac{1}{2}A = \frac{1}{4}A - \log |q_{12}| = |a| + |\lambda| \quad (2)$$

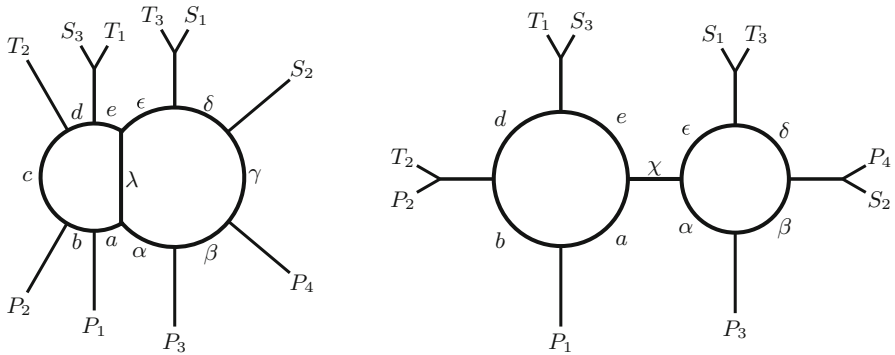


Fig. 3 The extended skeletons

and

$$\begin{aligned}
 |\beta| + |\gamma| - |\epsilon| &= -\frac{1}{2} \log |q_{22}| - \frac{1}{4} B - (-\log |q_{22}| + \log |q_{12}|) + \left(\frac{3}{4} B - \log |q_{12}|\right) \\
 &= \frac{1}{2} \log |q_{22}| - 2 \log |q_{12}| + \frac{1}{2} B = -\log |q_{12}| = |\lambda|
 \end{aligned}$$

where $|c| = |\gamma| = |\lambda| = 0$ in the connecting edge case.

We start by looking at $\log |f|$. It suffices to calculate the slopes along a and α denoted by m_a and m_α resp. since the other slopes can be determined using the slope formula (Theorem 2.1), e.g. the slope along edge b is $m_a - 1$. Since $\log |f|$ is piecewise linear by walking around the first cycle we get

$$m_\lambda |\lambda| + m_a |a| + (m_a - 1) |b| + (m_a - 2) (|c| + |d|) + (m_a - 1) |e| = 0.$$

Rearranging and making use of Eq. (2) leads us to

$$m_\lambda |\lambda| + m_a (-\log |q_{11}| - |\lambda|) = -\log |q_{11}|.$$

Thus $(m_\lambda - m_a) |\lambda| = (1 - m_a) \log |q_{11}|$. If $|\lambda| = 0$, i.e. if we are in the connecting edge case, then $m_a = 1$. If not then $m_\lambda - m_a = m_\alpha$ by the slope formula. Thus

$$m_\alpha |\lambda| = (1 - m_a) \log |q_{11}|. \tag{3}$$

Similarly since

$$m_\lambda |\lambda| + m_\alpha |\alpha| + (m_\alpha - 1) (|\beta| + |\gamma|) + m_\alpha |\delta| + (m_\alpha + 1) |\epsilon| = 0$$

and $|\beta| + |\gamma| = |\epsilon| + |\lambda|$ on the second cycle we obtain

$$m_\lambda |\lambda| + m_\alpha (-\log |q_{22}| - |\lambda|) = |\lambda|.$$

Thus

$$-m_\alpha \log |q_{22}| = (m_\alpha - m_\lambda + 1)|\lambda|.$$

If $|\lambda| = 0$ then $m_\alpha = 0$. If not then $m_\alpha - m_\lambda = -m_a$ thus

$$-m_\alpha \log |q_{22}| = (1 - m_a)|\lambda|.$$

Inserting $m_\alpha = -(1 - m_a) \frac{|\lambda|}{\log |q_{22}|}$ into (3) gives

$$-(1 - m_a) \frac{|\lambda|^2}{\log |q_{22}|} = (1 - m_a) \log |q_{11}|.$$

Since $|\lambda| = -\log |q_{12}|$ and $0 < \log |q_{12}|^2 < \log |q_{11}| \log |q_{22}|$ this implies $m_a = 1$ and hence $m_\alpha = 0$. The calculation for $\log |g|$ is similar to that of $\log |f|$ treating the second cycle first. Thus we obtain the following slope vectors for $(\log |f|, \log |g|)$:

$$m_a = (1, 0), \quad m_b = (0, -1), \quad m_c = (-1, -1), \quad m_d = (-1, 0), \quad m_e = (0, 1)$$

on the first cycle and

$$m_\alpha = (0, 1), \quad m_\beta = (-1, 0), \quad m_\gamma = (-1, -1), \quad m_\delta = (0, -1), \quad m_\epsilon = (1, 0)$$

on the second one. If the skeleton has a shared edge then by the slope formula (Theorem 2.1) the corresponding slope vector is $m_\lambda = (1, 1)$. If the skeleton has a connecting edge we obtain $m_\lambda = (-1, 1)$. Hence the tropicalization maps the skeleton of the curve isometrically to its image, i.e. the tropicalization is faithful. See Fig. 4 for an illustration.

Theorem 5.3 *Let $D' = D + S_3$. Then there exists a function $h \in L(D')$ such that $(1, f, g, h) : X \hookrightarrow \mathbb{P}^3$ is a closed immersion tropicalizing the extended skeleton faithfully.*

Proof Since $\deg(D') = 5$ the divisor is very ample. By Riemann–Roch $\dim L(D') = 3$ thus a basis of the linear system defines an embedding into three-dimensional projective space. The functions 1, f and g are linearly independent in $L(D')$ so they can be completed to a basis by a function having a pole at S_3 . We already know that $\log |f|$ and $\log |g|$ tropicalize the skeleton faithfully.

First note that there can be up to two additional bounded edges on the extended skeleton with respect to D' : The rays to S_1 and T_3 (resp. T_1 and S_3) may meet outside the skeleton. The slopes along these edges can be determined using the slope formula thus the slope vectors are $(-1, -1)$ in both cases (oriented away from the skeleton). Hence these edges are mapped isometrically to their image in the tropicalization.

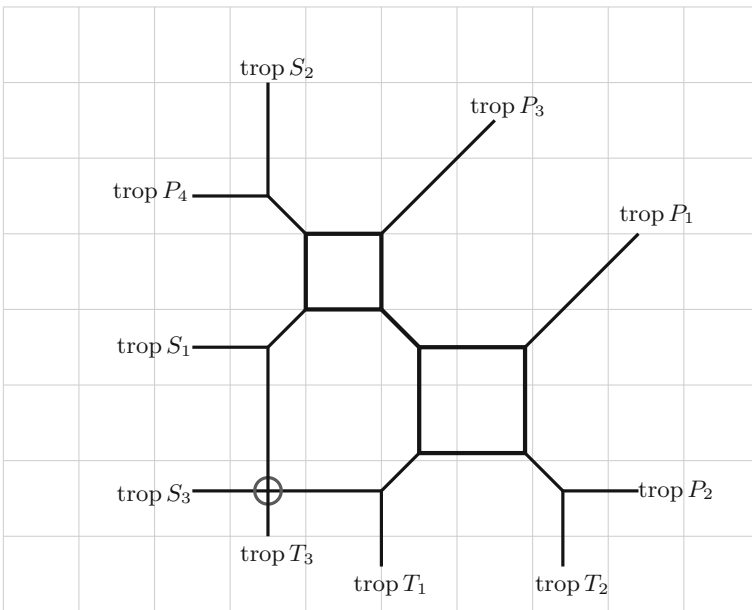
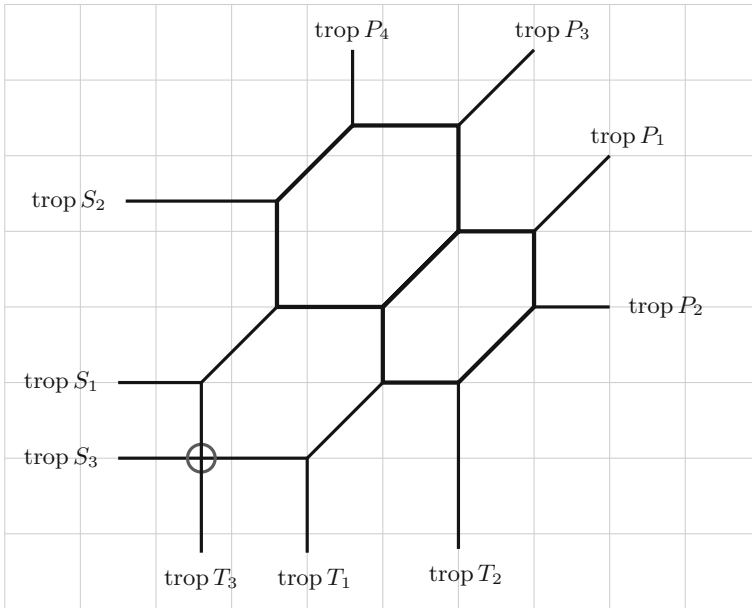


Fig. 4 Tropicalization with respect to $\log |f|$ and $\log |g|$ of Theorem 5.2 in the case of a shared edge resp. connecting edge where the possible intersection points are marked by a circle

In the case of a connecting edge there are possibly two more bounded edges between the point where the rays leading to S_2 and P_4 meet and the skeleton resp. between the point where the rays leading to P_2 and T_2 meet and the skeleton. Again using the slope

formula we see that the slope vectors are $(-1, 1)$ resp. $(1, -1)$ so also these edges are going to be tropicalized faithfully.

The slope vector of the ray leading to P_1 is denoted by m_{P_1} ; we use similar notation for other rays. By convention they are oriented away from the skeleton and are equal to the multiplicities of the poles resp. zeros of each function (Theorem 2.1). Thus we obtain

$$m_{P_1} = (1, 1), \quad m_{P_2} = (1, 0), \quad m_{T_2} = (0, -1), \quad m_{T_1} = (0, -1), \quad m_{S_3} = (-1, 0)$$

on the first cycle and

$$m_{P_3} = (1, 1), \quad m_{P_4} = (0, 1), \quad m_{S_2} = (-1, 0), \quad m_{S_1} = (-1, 0), \quad m_{T_3} = (0, -1)$$

on the second cycle.

Without loss of generality choose coordinates in \mathbb{R}^2 such that via the tropicalization map one vertex of the connecting resp. shared edge lies in the origin. Let the plane be divided into two halfspaces by the line $\langle(1, 1)\rangle$. Then each cycle lies in one of the two halfspaces. Except for the rays leading to S_3 resp. T_3 all rays of one cycle lie in the same halfspace. This means these rays cannot intersect with rays from the other cycle. However the rays leading to S_3 and T_3 may well intersect, see Fig. 4. By arguments similar to that in the Proof of Lemma 5.1 there exists a rational function h having poles at S_3 , P_1 and P_3 as well as a zero in T_2 which implies that none of the other zeros will retract to $\tau(S_3)$. Since h has a pole at S_3 the ray leading to this pole (and also a possible additional bounded edge) will be lifted into the third dimension as the ray leading to T_3 will not.

Thus $\log|h|$ separates the remaining rays, and the tropicalization of the extended skeleton with respect to $\log|f|$, $\log|g|$ and $\log|h|$ is faithful. \square

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