Beitr Algebra Geom (2016) 57:1–36
DOI 10.1007/s13366-015-0245-7

ORIGINAL PAPER

Rigidity of circle packings with crosscuts

David Krieg · Elias Wegert

Received: 26 October 2014 / Accepted: 2 March 2015 / Published online: 1 May 2015

Abstract Circle packings with specified patterns of tangencies form a discrete counterpart of analytic functions. In this paper we study univalent packings (with a combinatorial closed disk as tangent graph) which are embedded in (or fill) a bounded, simply connected domain. We introduce the concept of crosscuts and investigate the rigidity of circle packings with respect to maximal crosscuts. The main result is a discrete version of an indentity theorem for analytic functions (in the spirit of Schwarz' Lemma), which has implications to uniqueness statements for discrete conformal mappings.

Keywords Circle packing · Crosscut · Prime ends · Conformal mapping · Schwarz's lemma · Apollonian packing

Mathematics Subject Classification 52C26 · 30C80 · 30D40

1 Introduction

The study of circle packings, as they are understood in this paper, was initiated by Paul Koebe as early as 1936 in the context of conformal mapping, but the real success of the topic began with William Thurston's talk at the celebration of the proof of the Bieberbach conjecture in 1985. The publication of Ken Stephenson's book (Stephenson (2005)) inspired further research and made the topic accessible to a wide audience.

D. Krieg was supported by Sächsisches Landesgraduiertenstipendium. E. Wegert was supported by the Deutsche Foschungsgemeinschaft, Grant We 1704/8-2.

D. Krieg · E. Wegert (✉)
Institute of Applied Analysis, TU Bergakademie Freiberg, Akademiestr. 6, 09596 Freiberg, Germany
e-mail: wegert@math.tu-freiberg.de

Since then many classical results in complex analysis found their discrete counterpart in circle packing.

In this paper we consider circle packings embedded in a bounded, simply connected domain. We introduce the concept of crosscuts for domain-filling circle packings, and study the rigidity of packings with respect to maximal crosscuts (for definitions see below). The main result is a discrete version of an identity theorem for analytic functions, which has implications to uniqueness results for boundary value problems for circle packings, and especially to discrete conformal mappings.

To be more specific, we recall that the tangency relations of a circle packing are encoded in a *2-dimensional simplicial complex* K, referred to as the combinatorics of the packing. In this paper it is assumed that K is a finite triangulation of a topological disk.

Circle packings are a mixture of flexibilty and rigidity. Counting the degrees of freedom for the centers and the radii, and comparing this with the number of conditions caused by the tangency relations, we see that the first number exceeds the latter by $m + 3$, where m is the number of boundary circles. In fact, the set of all circle packings for a fixed complex K forms a smooth manifold of dimension $m + 3$ (Bauer et al. 2012). So the question arises which sort of conditions are appropriate to eliminate the flexibility of a packing and make it rigid. Motivated by our work on nonlinear Riemann-Hilbert problems, we are interested in *boundary value problems* for circle packings. These problems involve m boundary conditions (one for each boundary circle) and three additional conditions, which can be imposed in different form on boundary circles and interior circles as well.

A standard boundary value problem of this kind consists in finding circle packings with (given combinatorics and) prescribed radii of its boundary circles. Somewhat surprisingly, this problem has always a *locally univalent* solution, and the solution is unique up to a rigid motion of the complete packing (see Stephenson 2005, Sect. 11.4, for details).

The existence of solutions is also known for a related more general problem, the *discrete Beurling problem*, where the radii of the boundary circles are prescribed as functions of their centers (see Wegert et al. 2012), but the question of uniqueness has not yet been answered satisfactorily.

Last but not least there are several approaches to *discrete conformal mapping* via circle packing which fall into this category (see Stephenson 2005, in particular Chap. 19 and 20, with many interesting comments on the history of this topic, also summarizing He and Schramm (1996), Rodin and Sullivan (1987) and Thurston (1985).

In our favorite setting of discrete conformal mapping, the domain packing $\mathcal{P}$ is a so-called *maximal packing*, which 'fills' the complex unit disk $\mathbb{D}$, while the range packing $\mathcal{P}'$ is required to 'fill' a bounded, simply connected domain G. That a packing 'fills a domain G' basically means that all its circles lie in the closure $\overline{G}$ of that domain and all its boundary circles intersect (touch) the boundary ∂G of G. For domains which are not Jordan this has to be complemented by a more subtle condition (see Definition 2).

In a series of papers, Oded Schramm proved several outstanding results about packings which fill a Jordan domain. His very general existence theorems do not only address packings of circles, but of much more general *packable sets* (for an explanation see Schramm 1990).

Surprisingly, much less is known about uniqueness. It is clear that uniqueness of a domain-filling (circle) packing can only be expected if one imposes additional conditions which eliminate the (three) remaining degrees of freedom. Whether this works depends on the type of normalization conditions and on the geometry of the domain. For example, in his uniqueness proofs, Schramm needs that the Jordan domain is (as he says) *decent* (see Schramm 1991).

This paper is devoted to the question which *additional conditions* are appropriate to make a *domain filling* circle packing *unique*. In analogy to the standard normalization of conformal mappings, it seems reasonable to fix the center of a distinguished circle (the so-called *alpha-circle*) at some point in G and to require that the center of a neighboring circle lies on a given ray emerging from that point. Keeping the first condition, we have chosen another setting for the second one. This condition, involving crosscuts, is non-standard, more flexible and allows one to address other uniqueness problems too.

In order to give the reader a flavor of the result, we first state an analogous theorem for analytic functions. Recall that a *crosscut* of a domain G in the complex plane $\mathbb{C}$ is an open Jordan arc J in G such that $\overline{J} = J \cup \{a, b\}$ with $a, b \in \partial G$ (see Pommerenke 1992). Slightly abusing terminology, we shall also denote $\overline{J}$ as a crosscut in G.

Theorem 1 (Identity Theorem for Analytic Functions) *Let J be a crosscut of a simply connected domain G, with G^- and G^+ denoting the (simply connected) components of $G \setminus J$. If $f : G \to G$ is analytic, $f(z_0) = z_0$ for some $z_0 \in G^+$, and $f(G^-) \subset G^-$, then $f(z) = z$ for all $z \in G$.*

Proof Let $g : G \to \mathbb{D}$ be a conformal mapping of G onto the unit disk $\mathbb{D}$ with $g(z_0) = 0$. Then g maps the crosscut J of G to a crosscut of $\mathbb{D}$ (see Pommerenke 1992, Prop. 2.14) and the composition $g \circ f \circ g^{-1}$ satisfies the assumptions of the lemma with $G := \mathbb{D}$ and $z_0 := 0$. Hence it suffices to consider this special case.

Let z_1 be a point on J with $|z_1| = \min_{z \in J} |z|$. Since J is a crosscut in $\mathbb{D}$, and $0 = z_0 \in G^+$, we have

$$0 < |z_1| \le \min \left\{ |z| : z \in \overline{G^-} \right\} < 1.$$

By continuity, $f(G^-) \subset G^-$ and $z_1 \in \overline{G^-}$ imply that $f(z_1) \in \overline{G^-}$, and hence $|f(z_1)| \ge |z_1|$. Invoking Schwarz' Lemma, we get $f(z) = cz$ in $\mathbb{D}$, where c is a unimodular constant. Finally, the only rotation of $\mathbb{D}$ which maps G^- into itself is the identity. □

Although Schwarz' Lemma has already been investigated in the framework of circle packing (see Rodin 1987; Pommerenke 1992, Chap. 13) the following interpretation of Theorem 1 is new. Though precise definitions will be deferred to the next section, we hope that Fig. 1 helps to get an intuitive understanding of the setting. The domain G^- is the one containing the brighter (yellow) disks.

Theorem 2 (Rigidity of Circle Packings with Crosscuts) *Assume that a univalent circle packing $\mathcal{P} = \{D_v\}$ for a complex K with vertex set V fills a bounded, simply*

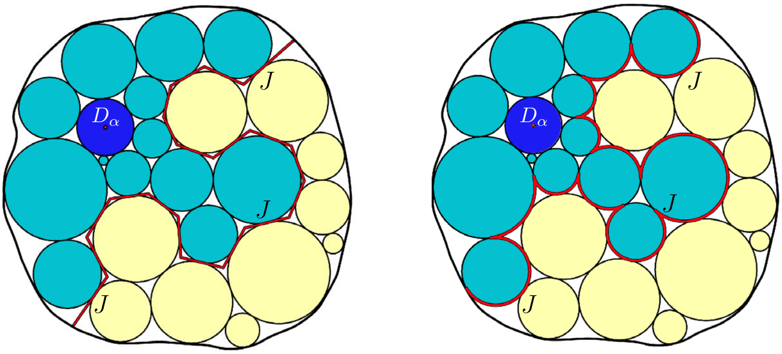

Fig. 1 A domain-filling packing $\mathcal{P}$ with a crosscut and a maximal crosscut

connected domain G. Let J be a (maximal) crosscut of $\mathcal{P}$ in G, such that G^- is a simply connected component of $G \backslash J$, and denote by V^- and V^+ the sets of vertices of K associated with circles in G^- and $G^+ := G \backslash \overline{G^-}$, respectively. Let D_α be an interior circle of $\mathcal{P}$ which is contained in G^+.

Assume further that a second univalent packing $\mathcal{P}' = \{D'_v\}$ for K is contained in G, such that D_α and D'_α have the same center, and $D'_v \subset G^-$ for all $v \in V^-$. Then $D'_v = D_v$ for all accessible *vertices $v \in V$.*

As follows from a simple topological argument, the condition $D'_v \subset G^-$ need only be required for those vertices $v \in V^-$ which are associated with circles D_v *touching the crosscut J*.

We point out that everything hinges on the assumption about the common center of the two alpha-circles. Since we do not assume that $\mathcal{P}'$ fills G, it is solely this condition which prevents $\mathcal{P}'$ from lying entirely in G^-.

The notion of accessible vertices will be explicated in Definition 1. Here we only note that *all vertices $v \in V$ are accessible* if and only if the complex K is *strongly connected*, which means K satisfies the following conditions (1) and (2):

(1) Every boundary vertex has an interior neighbor.
(2) The interior of K is connected.

Note that some authors of the circle packing community make the general assumption that the underlying complex K is strongly connected (see Stephenson 2005). For circle packings with this simpler combinatoric structure the theorem yields *complete rigidity* with respect to crosscuts, i.e., $D'_v = D_v$ for all $v \in V$.

Figure 2 illustrates some effects which can be observed for packings with general combinatorics. The picture on the left shows an Apollonian packing $\mathcal{P}$ with four generations. The highlighted line is a maximal crosscut, separating the disks in the "lower domain" from the disks in the "upper domain". The disk with the darkest color is the alpha-disk with fixed center. The accessible disks are those which can be connected with the alpha-disk by a chain of interior disks (see Definition 1).

The packing $\mathcal{P}'$, depicted in the middle, satisfies the assumptions of the theorem. In this example, only the accessible disks of $\mathcal{P}'$ (shown in darker colors) coincide with

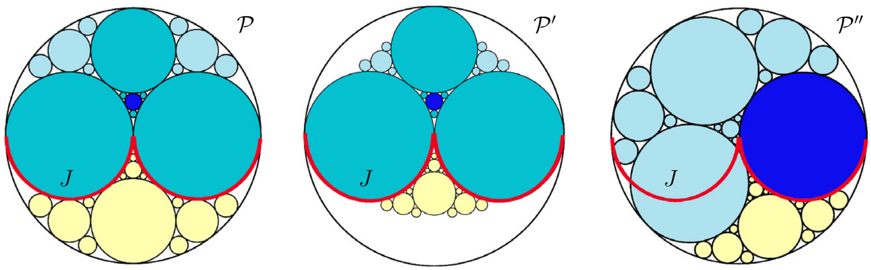

Fig. 2 Some examples illustrating assumptions and assertions of Theorem 2

their partners in $\mathcal{P}$. The non-accessible disks (shown in lighter colors) differ from the corresponding disks in $\mathcal{P}$.

The example on the right illustrates that the result need not hold if the alpha-disk is a boundary disk. The depicted packing $\mathcal{P}''$ satisfies all other assumptions (for the same crosscut), but, apart from the alpha disk, it is completely different from the packing $\mathcal{P}$ shown on the left-hand side.

The result has an intuitive interpretation when we think of circle packings as dynamic structures: Suppose that $\mathcal{P}$ fills G, and allow its circles to move (change position and size) in such a way, that they all remain in G, the center of the alpha-circle is fixed in G^+, and the circles in G^- are not allowed to leave G^-. Then only those circles which are not accessible can be moved, while the core part of the packing is *rigid*.

In order to illustrate the analogies with Theorem 1, we interpret the result in the framework of discrete analytic functions: The circle packing $\mathcal{P}$ filling G is the domain packing, the packing $\mathcal{P}'$ lies in G, so that $\mathcal{P} \to \mathcal{P}'$ defines a discrete analytic function from G into itself. Fixing the centers of the alpha-circles of both packings at the same point z_0 corresponds to the normalization $f(z_0) = z_0$. Finally, the condition $D'_v \subset G^-$ for all $v \in V^-$ expresses the invariance of the subdomain G^-.

Since the packing $\mathcal{P}$ represents the identity function on G, it is natural to suppose that $\mathcal{P}$ is univalent. Contrary to the continuous setting of Theorem 1, also $\mathcal{P}'$ was assumed to be univalent in Theorem 2. It is challenging to investigate what happens when this condition is dropped.

Terminological remark. For our purposes it would be better to work with *disk* packings instead of *circle* packings. Though we stay with the traditional notion, we shall often speak of the disks in a circle packing. In order to avoid cumbersome formulations, we also say that *a circle ∂D lies in a domain G* when this holds for the open disk D bounded by that circle. We already made use of this convention above.

2 Circle packings

In order to make the paper self-contained we recall basic concepts and notions of topology and circle packing (for details we refer to Henle 1979; Stephenson 2005).

Some geometry If A and B are subsets of the (complex) plane, we say that A *intersects* B if $A \cap B \neq \emptyset$. If A is a disk, then the phrase A *touches* B is in general used when $\overline{A} \cap \overline{B} \neq \emptyset$ and $A \cap B = \emptyset$. In this case we also say that the circle ∂A touches B. As usual, the symbol ∂ denotes the boundary operator.

By a *curve* γ we understand the image of a continuous mapping $\varphi : [a, b] \to \mathbb{C}$. The points $\varphi(a)$ and $\varphi(b)$ are said to be the *initial point* and the *terminal point* of γ, respectively; both are referred to as *endpoints* of γ. A *Jordan arc* and a *Jordan curve* are the homeomorphic images of a segment and a circle, respectively. By an *open Jordan arc* we mean a Jordan arc without its endpoints.

Let J be an *oriented* Jordan curve. For $p, q \in J$ with $p \neq q$ we denote by $J(p, q)$ the (oriented) open subarc of J with initial point p and terminal point q. If p, q, r are three pairwise different points on J, we say that *q lies between p and r on J* if $q \in J(p, r)$. Corresponding to whether q lies between p and r, or q lies between r and p, the *orientation of the triplet (p, q, r) with respect to J* is said to be *positive* or *negative*, respectively.

Let G be a bounded, simply connected domain in $\mathbb{C}$. A conformal mapping $g : \mathbb{D} \to G$ of $\mathbb{D}$ onto G has a continuous extension to $\overline{\mathbb{D}}$ if and only if ∂G is a *closed curve*, i.e., a continuous image of the unit circle $\mathbb{T}$ (see Pommerenke 1992, Theorem 2.1). This extension (which we again denote by g) is a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{G}$ if (and only if) G is a *Jordan domain*, i.e., ∂G is a Jordan curve (see Pommerenke 1992, Theorem 2.6).

In general, the conformal mapping g induces a one-to-one correspondence between the points on $\mathbb{T}$ and certain equivalence classes of open Jordan arcs γ in G with terminal point q on ∂G, so called *prime ends*. For the details we refer to Pommerenke (1992), Chap. 2, and Golusin (1957), Sect. 2.3.

If G contains a disk D which touches the boundary ∂G at some point $p \in \partial D \cap \partial G$, then every Jordan arc with starting point in D and terminal point p is contained in the same equivalence class. Hence there is a well-defined *prime end* of G *associated with* p *by* D.

Complexes The skeleton of a circle packing is a *simplicial 2-complex* K. Throughout this paper it is assumed that K is a *combinatorial closed disk*, i.e., it is finite, simply connected and has a nonempty boundary. Simply speaking of a complex, we always mean a complex of this class. Properties of complexes which are relevant in circle packing are summarized in Lemma 3.2 of Stephenson (2005).

We denote the sets of vertices, edges and faces of K by V, E, F, respectively. The edge adjacent to the vertices u and v is denoted by $e(u, v)$ or $\langle u, v\rangle$, where the first version stands for the *non-oriented edge*, while the second means the oriented edge from u to v. Similarly, a face of K with vertices u, v, w is written as $f(u, v, w)$ (non-oriented) or $\langle u, v, w\rangle$ (oriented), respectively. Two vertices u and v are said to be *neighbors* if they are connected by an edge $e(u, v)$ in E. For any vertex $v \in V$ we denote by $E(v)$ the set of edges adjacent to v. This set is endowed with a natural cyclic (counterclockwise) ordering, so that for $e_1, e_2 \in E(v)$ definitions like $\{e \in E(v) : e_1 < e \leq e_2\}$ make sense. Any edge e of K is adjacent to one or two faces. In the first case e is a *boundary edge*, otherwise it is an *interior edge* of K. *Boundary vertices* are those vertices of K which are adjacent to a boundary edge. The sets of boundary edges and boundary vertices are denoted ∂E and ∂V, respectively, the vertices in $V \backslash \partial V$ are

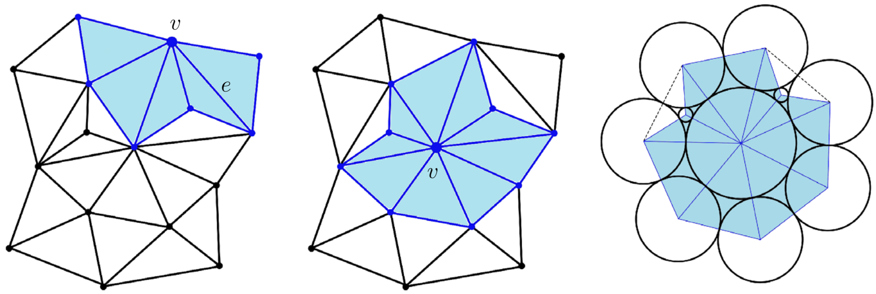

Fig. 3 The sub-complex of a (incomplete) flower and a corresponding packing

called *interior vertices*. We point out that a boundary vertex can be adjacent to many other boundary vertices, and that an edge which connects two boundary vertices need not be a boundary edge (as vertex v and edge e in Fig. 3, left). We let $B(v)$ denote the smallest sub-complex of K which contains a vertex v and all its neighbors. If v is an interior vertex $B(v)$ is said to be the *flower* of v (see Fig. 3, middle), if v is a boundary vertex we speak of an *incomplete flower* (see Fig. 3, left). Note that $B(v)$ need *not* contain all edges which connect neighbors of v (see Fig. 3, right).

Since K is a triangulation with non-void boundary, it must have at least three boundary vertices. The natural cyclic ordering of boundary edges, corresponding to the orientation of the boundary of the triangulated surface, induces a cyclic ordering of the boundary vertices. With respect to this ordering, any boundary vertex has a *precursor* and a *successor* which are well-defined.

Speaking of a *chain*, we mean a finite sequence $(c_1, \ldots, c_n)$ of vertices, edges or faces, such that neighboring elements c_j and c_{j+1} are adjacent to a common edge (if the c_j are vertices or faces) or a common vertex (if the c_j are edges), respectively.

We have illustrated some limitations of Theorem 2 in Figure 2. The reason for the observed effects is the relative independence of some substructures from the rest of the packing. This is described more precisely in the following definition.

Definition 1 Let K be a complex with a distinguished interior vertex, the *alpha-vertex* v_α. Then a vertex $v \in V$ is called *accessible* (from v_α) if there is a chain of vertices $(v, v_1, \ldots, v_n, v_\alpha)$ such that $v_1, \ldots, v_n$ are interior vertices. The set of all accessible vertices of K is denoted by V^*, the set of all edges $e(u, v) \in E$ with $u, v \in V^*$ by E^*, and the set of all faces $f(u, v, w) \in F$ with $u, v, w \in V^*$ by F^*. The *kernel* K^* of K is defined as the simplicial-2-complex arising from V^*, E^*, F^*, that is $K^*(V^*, E^*, F^*) \subset K(V, E, F)$.

Recall that a complex K is *strongly connected*, if the interior of K is connected, and every boundary vertex has an interior neighbor. The following lemma establishes a relation between this property and accessible vertices, and summarizes the crucial properties of the kernel. Since the statements are intuitive, we leave the details of the (somewhat tedious) proof to the reader.

Lemma 1 *Let K be a complex with a distinguished interior alpha-vertex v_α. Then the following assertions hold:*

(1) *The kernel* $K^*(V^*, E^*, F^*)$ *of* $K(V, E, F)$ *is a strongly connected complex with* $\partial V^* = \partial V \cap V^*$.
(2) *The complex* K *coincides with its kernel* K^* *(i.e., all vertices of* K *are accessible) if and only if* K *is strongly connected.*

Circle packings A collection $\mathcal{P}$ of *open* disks D_v is said to be a *circle packing* for the complex $K = K(V, E, F)$ if it satisfies the following conditions (1)–(3):

(1) Each vertex $v \in V$ has an associated disk $D_v \in \mathcal{P}$, such that $\mathcal{P} = \{D_v : v \in V\}$.
(2) If $\langle u, v\rangle \in E$ is an edge of K, then the disks D_u and D_v touch each other.
(3) If $\langle u, v, w\rangle \in F$ is a positively oriented face of K, then the centers of the disks D_u, D_v, D_w form a positively oriented triangle in the plane.

A circle packing is called *univalent*, if its disks are *non-overlapping*, $D_u \cap D_v = \emptyset$ for all $u, v \in V$ with $u \neq v$. In this paper all circle packings are assumed to be univalent.

Since the structure of the underlying complex K carries over to the associated packing $\mathcal{P}$, all related attributes can be applied to the disks D_v as well—so we shall speak of boundary disks, interior disks, neighboring disks, etc.

The *contact point* of two neighboring disks D_u, D_v is defined by $c(u, v) := \overline{D}_u \cap \overline{D}_v$. The *contact points of a packing* $\mathcal{P}$ for the complex $K(V, E, F)$ are the points $c(u, v)$ with $e(u, v) \in E$.

We denote by D the union of all disks in $\mathcal{P}$, $D := \bigcup_{v \in V} D_v$. If $\mathcal{P}$ is univalent and p and q are different points of ∂D, there is at most one disk D_v whose boundary ∂D_v contains p and q. If such a disk exists, we define $\delta(p, q)$ as the positively oriented open subarc of ∂D_v from p to q, and $\delta[p, q] := \overline{\delta(p, q)}$. In addition we set $\delta(p, p) := \emptyset$ and $\delta[p, p] := \{p\}$. Note that $\delta(p, q)$ and $\delta[q, p]$ are complementary subarcs of ∂D_v, provided that $p \neq q$.

If $\langle u, v, w\rangle$ is a face of K, the *interstice* $I(u, v, w)$ of $\mathcal{P}$ is the Jordan domain bounded by the arcs $\delta_u := \delta\big(c(u, v), c(u, w)\big)$, $\delta_v := \delta\big(c(v, w), c(v, u)\big)$ and $\delta_w := \delta\big(c(w, u), c(w, v)\big)$ (see Fig. 4, left).

Besides the union D of all disks in a packing $\mathcal{P}$ we need the *carrier* of $\mathcal{P}$, which is the compact set

$$D^* := \overline{D} \cup \bigcup_{f(u,v,w) \in F} I(u, v, w)$$

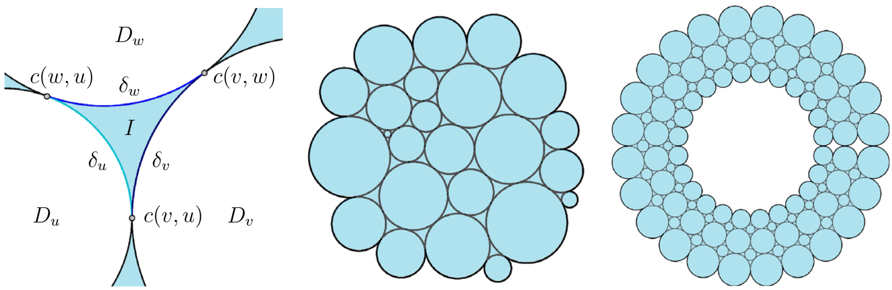

Fig. 4 Definition of the interstice $I := I(u, v, w)$ and the carrier D^* of two packings

(see Fig. 4, middle and right). Note that this definition is somewhat different from Stephenson's (cp. Stephenson 2005, p. 58). The carrier is essential in the next definition.

Definition 2 Let G be a bounded, simply connected domain. We say that a (univalent) circle packing $\mathcal{P}$ is *contained in* G (or *lies in* G) if the interior of D^* is a subset of G. A packing $\mathcal{P}$ contained in G is said to *fill* G if every boundary disk of $\mathcal{P}$ touches ∂G.

If G is a Jordan domain, $\mathcal{P}$ is contained in G if and only if any disk of $\mathcal{P}$ is a subset of G. For general domains the latter condition alone would be too week, since then it could happen that "spikes" of ∂G (think of G as a slit disk) penetrate into the packing, sneaking through between two boundary disks at their contact point. This is prevented by our definition; in particular it guarantees that $\partial G \cap I = \emptyset$ for every interstice I of $\mathcal{P}$.

What happens when ∂G meets a contact point of two boundary disks is explored in the following lemma (an explanation of associated prime ends is given at the beginning of this section).

Lemma 2 *Let G be a bounded, simply connected domain, and let $\mathcal{P}$ be a circle packing contained in G. Then every contact point $c(u, v) \in \partial G$ is associated with the same prime end by both D_u and D_v.*

Proof Let $c = c(u, v)$ be a contact point of $\mathcal{P}$ which lies on the boundary of G. Then there exists a vertex $w \in V$ such that $f(u, v, w)$ is a face in the complex of $\mathcal{P}$, and we denote by $I = I(u, v, w)$ the corresponding interstice.

For $\varepsilon > 0$, let B_ε be an open disk centered at c with radius ε and define

$$\widetilde{B}_\varepsilon := B_\varepsilon \cap \big(D_u \cup D_v \cup \overline{I}\big).$$

If ε is sufficiently small, $\widetilde{B}_\varepsilon \backslash \{c\}$ is a Jordan domain contained in G, and we have $D_u \cap B_\varepsilon \subset \widetilde{B}_\varepsilon$, $D_v \cap B_\varepsilon \subset \widetilde{B}_\varepsilon$ (see Fig. 5, left). As a Jordan domain $\widetilde{B}_\varepsilon \backslash \{c\}$ has a unique prime end c^* corresponding to its boundary point c, so the prime ends of G associated with c by the disks D_u and D_v, respectively, must coincide. □

A packing which fills the unit disk $\mathbb{D}$ is called *maximal*. A celebrated result, the Koebe-Andreev-Thurston-Theorem (which can be traced back to Koebe's paper 1936),

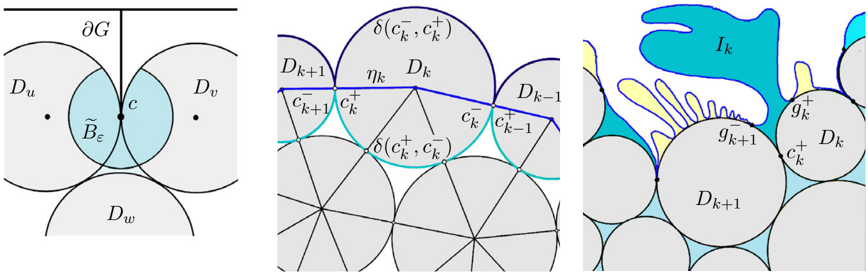

Fig. 5 Definitions of $\widetilde{B}_\varepsilon$, boundary arcs and boundary interstices

tells us that any complex K has an associated maximal packing, which is unique up to conformal automorphisms of $\mathbb{D}$. A far reaching generalization is the Uniformization Theorem of Beardon and Stephenson (1990, see also Chap. II in Stephenson 2005).

Recall that the boundary disks of a packing form a chain $D_1, \ldots, D_m$. Since this is a cyclic structure, we label it modulo m, in particular $D_0 := D_m$ and $D_{m+1} := D_1$. For $k \in \{1, \ldots, m\}$, we denote by η_k the closed segment which connects the centers of D_k and D_{k+1}. These *boundary segments* form a (polygonal) Jordan curve η.

If D_{k-1}, D_k and D_{k+1} are three consecutive boundary disks, the contact points $c_k^- := \overline{D}_{k-1} \cap \overline{D}_k$ and $c_k^+ := \overline{D}_k \cap \overline{D}_{k+1}$ split ∂D_k into two arcs. We call $\delta(c_k^-, c_k^+)$ the *exterior boundary arc* and $\delta(c_k^+, c_k^-)$ the *interior boundary arc* of D_k, respectively (see Fig. 5, middle).

Lemma 3 *Let D_k be a boundary disk of a circle packing $\mathcal{P}$. Then the exterior boundary arc of D_k contains no contact points of disks in $\mathcal{P}$.*

Note that, by definition, the contact points of the disks in a packing are prescribed by the combinatorics of $\mathcal{P}$.

Proof The polygonal line η which connects consecutive centers of the boundary disks is a Jordan curve which separates the exterior boundary arcs from the interior boundary arcs. The interior of η contains the closures $\overline{D}_v$ of all interior disks. Any contact point c of $\mathcal{P}$ is either a contact point of two boundary disks, or it lies on the boundary of an interior disk. In both cases c does not belong to any exterior boundary arc. □

To provide some more notation, let $\mathcal{P}$ be a circle packing which fills a bounded, simply connected domain G. By definition, every boundary disk D_k touches ∂G in a non-void (possibly uncountable) set G_k of points, and G_k must be contained in the closure $\delta[c_k^-, c_k^+]$ of the exterior boundary arc $\delta(c_k^-, c_k^+)$ of D_k. Let $\delta_k := \delta[g_k^-, g_k^+]$ be the smallest subarc (we allow the possibility that this 'arc' degenerates to a point) of $\delta[c_k^-, c_k^+]$ which contains G_k. Since G_k is a closed set, we have $g_k^-, g_k^+ \in G_k$.

In order to define the *boundary interstice* I_k between two consecutive boundary disks D_k and D_{k+1} (see Fig. 5, right) we distinguish two cases. If $g_k^+ = c_k^+$, we set $I_k := \emptyset$. Otherwise we let δ be the union of the arcs $\delta(g_k^+, c_k^+]$ (a subarc of ∂D_k) and $\delta[c_k^+, g_{k+1}^-)$ (a subarc of ∂D_{k+1}). The open Jordan arc δ is contained in G with different endpoints on ∂G, hence it is a crosscut. The set $G \backslash \delta$ consists of two simply connected components G_1 and G_2. One of these components contains all disks of $\mathcal{P}$, the other one is (by definition) the boundary interstice I_k.

Lemma 4 *$I_k \cap D = \emptyset$ for all $k = 1, \ldots, m$.*

Proof Let $k \in \{1, \ldots, m\}$ be fixed. If $I_k = \emptyset$ the assertion is trivially fulfilled. Let $I_k \neq \emptyset$ and let δ be the crosscut defined above, so that $G \backslash \delta$ consists of exactly two simply connected domains $G_1 = I_k$ and G_2.

Clearly every disk of $\mathcal{P}$ is contained either in G_1 or G_2. We assume that there is a disk D_u in G_1 (remember $D_k \subset G_2$). Because K is connected there is a chain C of vertices $\{u, \ldots, v\}$, where v is the vertex associated with D_k. Because $D_u \subset G_1$ and $D_k \subset G_2$ there have to be two consecutive vertices w_1, w_2 in C, so that D_{w_1}

is contained in G_1 and D_{w_2} in G_2. The contact point $c(w_1, w_2)$ must lie on $\partial G_1 \backslash \delta$, because there are no contact points of $\mathcal{P}$ on δ according to Lemma 3.

Let w_3 be a vertex, so that $f(w_1, w_2, w_3)$ is a face of K. The interstice $I := I(w_1, w_2, w_3)$ is contained either in G_1 or G_2, because it is disjoint from ∂G. Moreover both arcs $\partial D_{w_1} \cap \partial I$ and $\partial D_{w_2} \cap \partial I$ (up to their endpoints) lie in the same domain as I, without being contained in the boundary of G. This implies, that both disks D_{w_1} and D_{w_2} are contained either in G_1 or G_2, a contradiction. Hence, $I_k \cap D = \emptyset$ for all $k = 1, \ldots, m$. □

Last but not least we state a result about glueing simply connected domains along a common boundary arc. The proof is left as an exercise (see Pommerenke 1992).

Lemma 5 *Let G_1 and G_2 be simply connected domains with locally connected boundaries. If G_1 and G_2 touch each other along a Jordan arc J with endpoints a, b, i.e., $G_1 \cap G_2 = \emptyset$ and $\overline{G}_1 \cap \overline{G}_2 = J$, then $\big(G_1 \cup J \cup G_2\big) \backslash \{a, b\}$ is a simply connected domain and its boundary is locally connected.*

3 Crosscuts

Before we introduce crosscuts of a (univalent) circle packing which fills a domain G, we define crosscuts of its complex.

Definition 3 A (combinatoric) *crosscut of a complex* K is a sequence $L = (e_0, e_1, \ldots, e_l)$ of edges in K with the following properties (1)–(4):

(1) The edges are pairwise different, if $0 \le j < k \le l$ then $e_j \ne e_k$.
(2) For $1 \le j \le l$ the edges e_{j-1} and e_j are adjacent to a common face of K.
(3) Three consecutive edges are not adjacent to the same face of K.
(4) The edges e_0 and e_l are boundary edges.

It is easy to see that only the first and the last edge of a crosscut can be boundary edges of K. Because $e_0 \ne e_l$ we have $l \ge 1$. When one edge of a face f belongs to

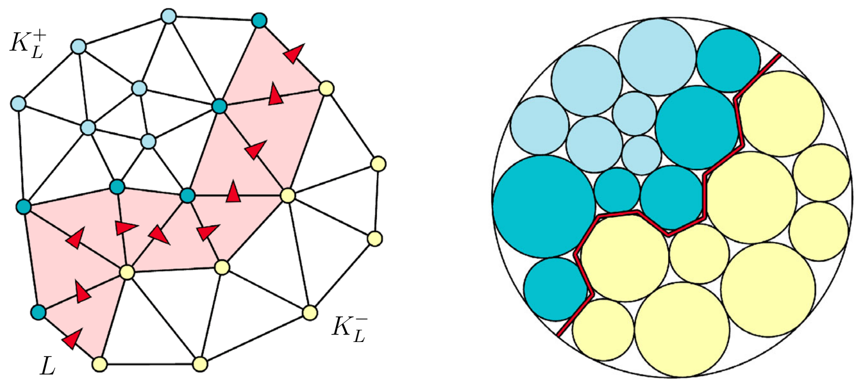

Fig. 6 A crosscut L of K, the vertex sets V_L^-, V_L^+, U_L^+, and a corresponding packing

L, then L must contain exactly two edges of f, and these are subsequent members of L. So a crosscut can also be represented by a sequence $(f_1, \ldots, f_l)$ of faces, where e_{j-1} and e_j are adjacent to f_j. Since the three edges of a face are not allowed to be consecutive members of L, all faces f_j must be pairwise different.

After removing the edges of a crosscut L from K, the remaining graph consists of two edge-connected components K_L^- and K_L^+. We assume that K_L^- 'lies to the right' and K_L^+ 'lies to the left', respectively, when we move along the edges $e_0, e_1, \ldots, e_l$ in this order. The vertex sets of K_L^- and K_L^+ are denoted by V_L^- and V_L^+, respectively, and we call them the *lower* and the *upper vertices* of K with respect to L. The set U_L^+ is constituted by all vertices v in V_L^+ which are adjacent to an edge in L. These vertices and the corresponding disks are said to be the *upper neighbors* of L. An analogous definition is made for the set U_L^- of *lower neighbors* of L (see Fig. 6).

Given a (combinatoric) crosscut L of a complex K and a circle packing $\mathcal{P}$ for K which fills a domain G, we define several related (geometric) crosscuts J of $\mathcal{P}$ in G. To begin with, we associate with every edge $e_j = e(u, v)$ in L the contact point $x_j := \overline{D}_u \cap \overline{D}_v$ of the disks $D_u, D_v \in \mathcal{P}$. The common tangent to D_u and D_v at x_j is denoted τ_j. The set $X := \{x_0, \ldots, x_l\}$ of all contact points associated with edges of L has a natural ordering, induced by the ordering of edges in the crosscut. Since the indexing of the elements fits with this ordering, we write $x_j < x_k$ if $j < k$.

The *polygonal crosscut* J_L^0 is built from the common tangents τ_i of disks at their contact points x_i as follows. Let $i \in \{1, \ldots, l\}$ and assume that x_{i-1} and x_i are consecutive contact points of the pairs D_u, D_v and D_v, D_w, respectively. Then the three circles $\partial D_u, \partial D_v, \partial D_w$ bound an interstice $I := I(u, v, w)$. The tangents τ_{i-1} and τ_i intersect each other at a point s_i in I, and the union of the closed segments $[s_i, s_{i+1}]$ for $i = 1, \ldots, l-1$ is a Jordan arc in G (see Fig. 7).

In order to complete this arc to a crosscut in G we look at the boundary disks D_k and D_{k+1} which touch each other at x_0. If x_0 is not a boundary point of G we define s_0 as the endpoint of the largest segment (x_0, s_0) on the tangent τ_0 which is contained in I_k. Since there is no disk of $\mathcal{P}$ intersecting I_k (Lemma 4) we see that $[x_0, s_0) \subset G$ is disjoint from $\mathcal{P}$ and $s_0 \in \partial G$. If x_0 is a boundary point of G we set $s_0 := x_0$.

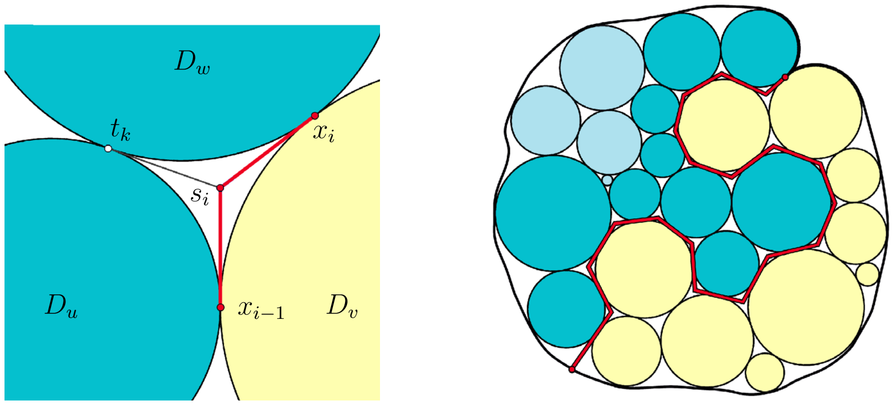

Fig. 7 Local construction and global view of a polygonal crosscut

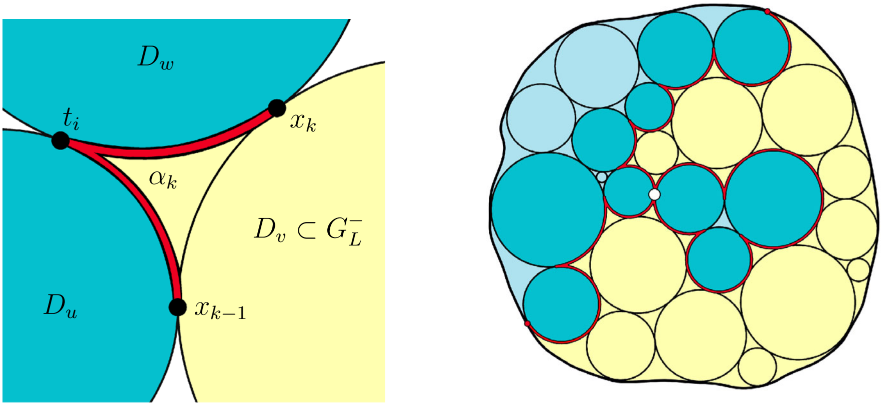

Fig. 8 Construction of a maximal crosscut (which is not a Jordan arc)

A similar construction is made for the point s_{l+1} as ("the first") intersection point of the tangent τ_l with ∂G. Here $x_0 \neq x_l$ ensures that $[s_0, s_1]$ and $[s_l, s_{l+1}]$ live in two different boundary interstices. Although this does not exclude $s_0 = s_{l+1}$, it guarantees that s_0 and s_{l+1} are endpoints of the segments $[s_1, s_0]$ and $[s_l, s_{l+1}]$, belonging to *different prime ends* s_0^* and s_{l+1}^*, respectively.

Finally, the union of the closed segments $[s_k, s_{k+1}]$ for $k = 0, \ldots, l$ forms the desired polygonal crosscut $J_L^0 := \bigcup_{k=0}^{l} [s_k, s_{k+1}]$ in G. It can easily be verified that J_L^0 is a (topologically closed) Jordan arc which meets $\overline{D}$ at the contact points x_k – more precisely we have $X \subset J_L^0 \cap \overline{D} \subset X \cup \{s_0, s_{l+1}\}$. The open set $G \backslash J_L^0$ has two simply connected components G_0^+ and G_0^-, containing the disks associated with V_L^+ and V_L^-, respectively.

It is clear that, for a fixed combinatorial crosscut L of K, the statement of Theorem 2 depends on the choice of the geometric crosscut J: the assertion becomes the stronger, the larger the domain G_J^- is. Unfortunately, there exists (in general) no crosscut J which maximizes G_J^-, since the boundary of the largest domain G_J^- need not be a Jordan curve. We therefore extend the concept of crosscuts somewhat, defining the *maximal crosscut* J_L^+ in $\mathcal{P}$ as follows.

Recall that U_L^+ is the vertex set of upper neighbors of L. If x_{k-1} and x_k are contact points of the disks D_u, D_v and D_v, D_w, respectively, then either $v \in U_L^+$ or $u, w \in U_L^+$. The interstice $I(u, v, w)$ is bounded by three (topologically closed) circular arcs α_u, α_v and α_w, respectively. If $v \in U_L^+$ we connect x_{k-1} with x_k by the arc $a_k := \alpha_v$, in the second case we connect these points by the concatenation $a_k := \alpha_u \cup \alpha_w$ (see Fig. 8). In addition we connect x_0 and x_l with ∂G by arcs $a_0 := \delta(g_j^+, x_0)$ and $a_{l+1} := \delta(x_l, g_k^-)$ of those circles ∂D_j and ∂D_k which are upper neighbors of L and contain x_0 and x_l, respectively. The union $J_L^+ := \bigcup_{k=0}^{l+1} a_k$ of these arcs is a curve which we call the *maximal crosscut* in $\mathcal{P}$ with respect to L.

The maximal crosscut J_L^+ is composed from a finite number of circular (topologically closed) arcs ω_i which are linked at the *turning points* t_i of J_L^+, and every contact point x_k lies exactly on one arc ω_i (see Fig. 8). If J_L^+ is not a Jordan arc, $G \backslash J_L^+$ may consist of several connected components (see Fig. 8, right), one of them containing all disks associated with vertices v in V_L^-. We call this component G_L^- the *maximal lower domain* for L with respect to $\mathcal{P}$, and we set $G_L^+ := G\backslash\overline{G_L^-}$. For the sake of brevity we define $\omega := J_L^+$ and $\Omega := G_L^-$.

Since the curve ω can have multiple points (see Fig. 8, right) there is no natural ordering of the *points* on ω. However, considering ω as part of the boundary of Ω, we can introduce an ordering of the *terminal points* $q \in \omega$ of open Jordan arcs $\gamma(p, q)$ in Ω. In order to describe this procedure we need the following result.

Lemma 6 *For any combinatorial crosscut L the maximal lower domain $\Omega = G_L^-$ is simply connected and has a locally connected boundary.*

Proof Let G_0^- be the lower domain with respect to the polygonal crosscut J_0 in $\mathcal{P}$. Then $G\backslash J_L^0$ consists of two simply connected domains G_0^- and G_0^+, respectively. The maximal lower domain G_L^- is constructed by glueing a finite number of simply connected domains along straight line segments to G_0^-. Hence the assertion follows from Lemma 5. □

The assertion of Lemma 6 guarantees that any (fixed) conformal mapping $g : \mathbb{D} \to \Omega$ has a continuous extension to $\overline{\mathbb{D}}$, which we again denote by g (see Pommerenke 1992, Theorem 2.1). With respect to this mapping, we let $\sigma_i \subset \mathbb{T}$ denote the preimage of the circular arcs ω_i with $i = 1, \ldots, n$. Then $\sigma := \bigcup_{i=1}^n \sigma_i$ is the preimage of the maximal crosscut ω.

By the Prime End Theorem, the mapping g induces a bijection g^* between $\mathbb{T}$ the set of prime ends of Ω. We denote by $\omega^* := g^*(\sigma)$ the set of prime ends associated with Ω, and, for $i = 1, \ldots, n$, we let $\omega_i^* := g^*(\sigma_i)$ be the subsets of ω^* corresponding to the arcs σ_i.

Note that the preimages σ_i of the circular arcs ω_i are topologically closed subarcs of $\mathbb{T}$, and that the preimage $\mathbb{T}\backslash\sigma$ of $\partial\Omega\backslash\omega$ is not empty. Therefore σ_i and σ_j, and thus ω_i^* and ω_j^*, are disjoint if $|i - j| > 1$, while their intersection contains exactly one element if $|i - j| = 1$.

Further we see that the arcs $\sigma_1, \sigma_2, \ldots, \sigma_n$ (in this order) are arranged in clockwise direction on $\mathbb{T}$. It is therefore just natural to order the *points* on the arc σ (and hence on each subarc σ_i) also in *clockwise* direction. The mapping g^* transplants this ordering from σ to the set ω^* of prime ends. If $\gamma_1^* = g^*(s_1)$ and $\gamma_2^* = g^*(s_2)$ are two prime ends of ω^*, the notion $\gamma_1^* \le \gamma_2^*$ refers to the ordering $s_1 \le s_2$ of the associated points on σ.

Remark Every ω_i without its endpoints is an open Jordan arc, so there is a one-to-one correspondence between the interior points of ω_i and σ_i. Let γ in Ω be an open Jordan arc with terminal point q on ω, then the associated unique prime end γ^* in ω^* must lie in ω_i^*, whenever q is an interior point of ω_i. Only if q is an endpoint of ω_i there is a chance that the prime end γ^* is not contained in ω_i^*, because now γ^* depends on how γ approaches q.

4 Loners

So far we have studied properties of a single circle packing $\mathcal{P}$ which fills G. In the next step we consider pairs $(\mathcal{P}, \mathcal{P}')$ of packings which are subject to the assumptions of Theorem 2.

Definition 4 A pair $(\mathcal{P}, \mathcal{P}')$ of univalent circle packings for the complex K is said to be *admissible* (for the crosscut L of K in G with alpha-vertex v_α) if it satisfies the following conditions:

(1) The packing $\mathcal{P}$ fills the bounded, simply connected domain G, and the packing $\mathcal{P}'$ is contained in G (see Definition 2).
(2) For all vertices $v \in U_L^-$ (the lower neighbors of L) the disks D_v' are contained in G_L^- (the maximal lower domain of G for L with respect to $\mathcal{P}$).
(3) The centers of the alpha-disks of $\mathcal{P}$ and $\mathcal{P}'$ coincide and lie in $G_L^+ := G\backslash G_L^-$.

Though it would be more precise to speak of an admissible sixtuple $(K, L, G, \mathcal{P}, \mathcal{P}', v_\alpha)$, we shall use the term "admissible" generously, for instance saying that "L is an admissible crosscut for $(\mathcal{P}, \mathcal{P}')$".

Recall that U_L^+ denotes the vertex set of those disks in $\mathcal{P}$ which lie in G_L^+ and touch the crosscut ("upper neighbors of L"). In the next step we are going to explore the interplay of the disks D_v' in $\mathcal{P}'$ and D_w in $\mathcal{P}$ for $v, w \in U_L^+$.

Definition 5 Let $(\mathcal{P}, \mathcal{P}')$ be an admissible pair of circle packings for the complex K with crosscut L. A vertex v in U_L^+ is called a *loner*, if $D_v' \cap D_w = \emptyset$ for all $w \in U_L^+$ with $w \neq v$.

The concept of loners was introduced by Schramm (1991) in a similar but somewhat different context. The main characteristic of a loner is the following.

Lemma 7 *Let v in U_L^+ be a loner of the admissible pair $(\mathcal{P}, \mathcal{P}')$ with complex K and crosscut L. Then $D_v' \cap (G_L^+\backslash D_v) = \emptyset$.*

Proof Let $u \in U_L^-$ and $w \in U_L^+$ be neighbors of v, and let p and q be the contact points of the disks D_v' with D_u' and D_v with D_w, respectively. Clearly $p \neq q$, otherwise D_u' had to intersect D_v or D_w, a contradiction to condition (2) of the admissible pair $(\mathcal{P}, \mathcal{P}')$.

Assume that p is a boundary point of D_v. Then ∂D_v and $\partial D_v'$ have a common tangent at p, otherwise D_u' had to intersect D_v, a contradiction to condition (2) of the admissible pair $(\mathcal{P}, \mathcal{P}')$. It follows that either $\overline{D_v'}\backslash\{p\} \subset D_v$ or $D_v' = D_v$ or $\overline{D_v}\backslash\{p\} \subset D_v'$. The latter implies that $q \in D_v'$, hence $D_v' \cap D_w \neq \emptyset$, which is impossible since v is a loner. The other two cases imply the statement we want to prove.

Assume that p is not a boundary point of D_v. Suppose that the assertion of Lemma 7 were false, i.e., there is some point r in D_v' which is also contained in $G_L^+\backslash D_v$. Because p lies in the maximal lower domain G_L^-, and r lies in the upper domain G_L^+, the boundary of D_v' must intersect the maximal crosscut J_L^+. Since the vertex v is a loner, every such intersection point must lie in ∂D_v. If $\partial D_v' \cap \partial D_v$ consists of exactly one point

r_1, then the boundary of D'_v is the union of $\delta[p, r_1]$ and $\delta[r_2, p]$, hence $D'_v \cap G_L^+ = \emptyset$, a contradiction to $r \in D'_v$. If there is a second point $r_2 \in \partial D'_v \cap \partial D_v$ with $r_1 \neq r_2$, then we have $\partial D'_v \cap D_v = \delta(r_2, r_1)$, hence r must be contained in D_v, a contradiction to $r \in G_L^+ \backslash D_v$. □

In Sect. 6 the property of loners described in Lemma 7 will allow us to move the crosscut L through the packing, reducing in every step the number of disks in G_L^+. The next result is crucial for the applicability of this procedure.

Lemma 8 (Existence of loners) *Every admissible pair* $(\mathcal{P}, \mathcal{P}')$ *of circle packings with crosscut L has a loner.*

The proof is divided into several steps; the first part uses the *geometry* of disks, then we employ some *topology*, and finally everything is reduced to pure *combinatorics*. We start with some preparations.

Recall the definition of the contact points x_k: If $L = (e_0, \dots, e_l)$ and $e_k = \langle u, v \rangle$, for some $k \in \{0, \dots, l\}$, then $x_k := \overline{D_u} \cap \overline{D_v}$. Using the same notation, the corresponding contact points of disks in $\mathcal{P}'$ are given by $y_k := \overline{D'_u} \cap \overline{D'_v}$, where $Y := \{y_0, \dots, y_l\}$ is the set of all such contact points.

The contact points x_k form an ordered set on the maximal crosscut $\omega := J_L^+$, which is the upper boundary of the maximal lower domain $\Omega := G_L^-$. Since every x_k lies on exactly one arc ω_i, the set X of contact points splits into classes $X_i := \{x_k \in X : x_k \in \omega_i\}$, $i = 1, \dots, n$. The set Y of the contact points of $\mathcal{P}'$ is divided accordingly, $Y_i := \{y_k \in Y : x_k \in \omega_i\}$ (the x_k is no typo here). Like X, the set Y is endowed with a natural ordering, we write $y_j < y_k$ if $j < k$.

Our next aim is to construct a Jordan arc α which is contained in $\overline{\Omega}$ and carries the contact points y_k in their natural order.

Lemma 9 *If* $(\mathcal{P}, \mathcal{P}')$ *is an admissible pair, then there exist oriented Jordan arcs* α_k *from* y_{k-1} *to* y_k *such that* $\alpha := \cup_{k=1,\dots l} \alpha_k$ *is a Jordan arc in* $\overline{\Omega}$ *and* $\alpha \cap \omega \subset Y$.

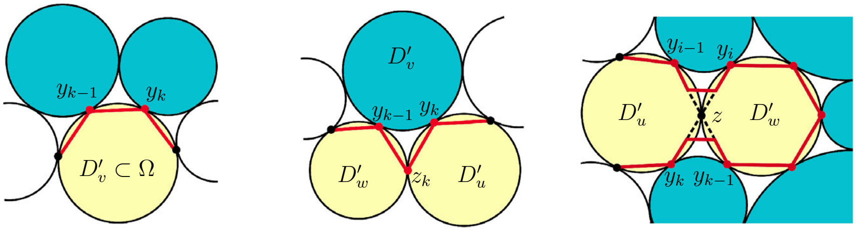

Fig. 9 Construction of the Jordan arc α in Case 1 (*left*) and Case 2 (*middle*, *right*)

Proof Let $k \in \{1, \ldots, l\}$. In order to determine the arc α_k of α which connects y_{k-1} with y_k we remark that both points lie on the boundary of one and the same disk $D'_v \in \mathcal{P}'$. We distinguish two cases:

Case 1 If $v \in V_L^-$, then the disk D'_v is contained in Ω, and we choose the segment $\alpha_k := [y_{k-1}, y_k]$ (see Fig. 9, left).

Case 2 If $v \in V_L^+$, then e_{k-1}, e_k and a third edge $\langle u, w\rangle$ of K form a face of K, and the (neighboring) disks D'_u and D'_w are both contained in Ω. So we let $z_k := \overline{D'_u} \cap \overline{D'_w}$ and connect y_{k-1} with y_k by $[y_{k-1}, z_k] \cup [z_k, y_k] \subset \overline{\Omega}$ (see Fig. 9, middle).

It is clear that all *open* segments (y_{k-1}, y_k), (y_{k-1}, z_k), (z_k, y_k) for $k = 1, \ldots, l$ are pairwise disjoint, and that $y_k \neq z_j$. However, it is possible that two endpoints z_k and z_j coincide for $j \neq k$, in which case the concatenation of the arcs α_k is not a Jordan arc.

If this happens, the point $z := z_j = z_k$ is the contact point of two disks D'_u and D'_w with $u, w \in V_L^-$. A little thought shows that then z can neither lie on the boundary of G nor on ω, and hence it must be an interior point of Ω. This allows one to resolve the double point of α at z without destroying its other properties (see Fig. 9, right). □

In the next step we transform the existence of loners to a topological problem. Technically this is much simpler when α and ω are disjoint. We consider this 'regular case' in Sect. 4.1. The 'critical case', where intersections of α and ω are admitted, will be treated in Sect. 4.2.

4.1 The regular case

Here we assume that $\alpha \cap \omega = \emptyset$, which implies that all contact points y_k $(k = 0, \ldots, l)$ lie in the lower domain Ω.

We fix $i \in \{1, \ldots, n\}$ and denote by y_i^- and y_i^+ the smallest and the largest member of Y_i with respect to the natural ordering of Y, respectively. Both points (which may coincide), as well as all elements of Y_i, lie on the same circle $\partial D'_v$, associated with a vertex $v = v(i) \in V$.

Let δ'_i be the negatively oriented topologically closed subarc of $\partial D'_v$ from y_i^- to y_i^+. We consider the largest subarcs ν_i and π_i of δ'_i which are contained in $\overline{\Omega}\backslash\omega$ and have initial points y_i^- (for η_i) and y_i^+ (for π_i), respectively (see Fig. 10).

Lemma 10 *If there exists no loner, then the terminal points ν_i^+ and π_i^+ of ν_i and π_i, respectively, lie on ω for $i = 1, \ldots, n$.*

Proof If one of the arcs ν_i or π_i does not intersect ω, then both coincide with δ'_i. In this case, the disk $D'_{v(i)}$ is separated from G_L^+ by the union of the arcs α and δ'_i, which implies that $D'_{v(i)}$ cannot intersect any disk D_w with $w \in U_L^+$, so that $v(i)$ is a loner. □

Since (with the exception of their endpoints) the circular arcs ν_i $(i = 2, \ldots, n)$ and π_i $(i = 1, \ldots, n-1)$ lie in Ω and have terminal points ν_i^+ and π_i^+ on ω, they define prime ends ν_i^* and π_i^* in ω^*.

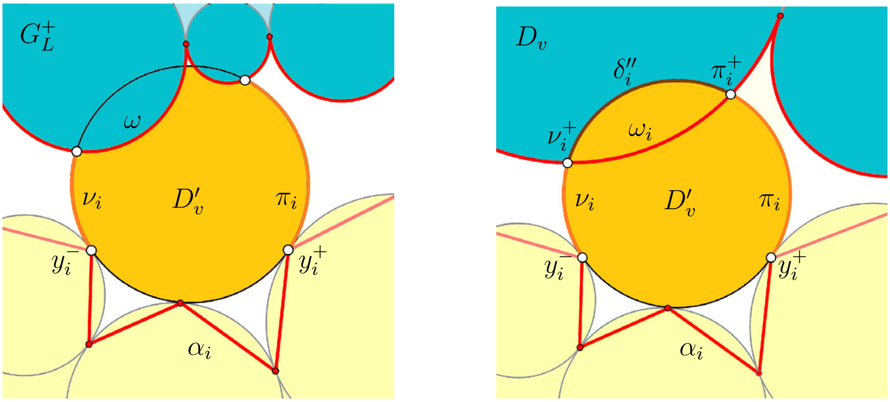

Fig. 10 The arcs ν_i and π_i and their intersection with the boundary of G_L^+

Because the arcs ν_1 and π_n need not lie in Ω, a modified definition is needed for the prime ends ν_1^* and π_n^*. To do so, we replace ν_1 and π_n by slightly perturbed circular arcs ν_1^ε and π_n^ε, respectively, which have the same endpoints as ν_1 and π_n, respectively, and lie in Ω (with the exception of their endpoints). Then ν_1^* and π_n^* are defined as the prime ends associated with the terminal points of ν_1^ε and π_n^ε, respectively. Clearly such arcs ν_1^ε and π_n^ε exist, and for all sufficiently small ε they define the same prime ends $\nu_1^*, \pi_n^* \in \omega^*$, respectively.

Since the set of prime ends ω^* is endowed with a natural ordering, we can compare the prime ends ν_i^* and π_i^*.

Lemma 11 *If $(\mathcal{P}, \mathcal{P}')$ has no loner, the prime ends ν_i^* and π_i^* form an interlacing sequence with respect to the prime end ordering of ω^*,*

$$\nu_1^* \le \pi_1^* \le \nu_2^* \le \pi_2^* \le \cdots \le \nu_n^* \le \pi_n^*.$$

Proof Let $y_- := y_0$ and z_- be the initial and terminal points of ν_1, while $y_+ := y_l$ and z_+ are the initial and terminal points of π_n, respectively. We have $z_-, z_+ \in \omega$ due to Lemma 10. Further, let ω_0^* be the set of all prime ends γ^* of ω^* with $\nu_1^* \le \gamma^* \le \pi_n^*$, and denote the set of all corresponding points on ω by ω_0. The set ω_0 is a curve or a single point. Together with the Jordan arcs ν_1, α and π_n it forms the boundary of a simply connected domain $\Omega_0 \subset \Omega$ with locally connected boundary. Let Ω_0^* be the set of all prime ends associated with points on $\partial\Omega_0$. Because $\Omega_0 \backslash \omega_0$ is an open Jordan arc, the points y_-, y_+ are associated with uniquely determined prime ends y_-^*, y_+^* of Ω_0.

Contrary to this, the points z_-, z_+ may be associated with several prime ends of Ω_0. In order to explain which one we choose, let again $\nu_1^\varepsilon, \pi_n^\varepsilon$ be small perturbations (as explained above) of ν_1, π_n, respectively, so that both arcs are crosscuts in Ω_0. We define z_-^* and z_+^* as the prime ends in ω^* associated with the terminal points z_- and z_+ of $\nu_1^\varepsilon, \pi_n^\varepsilon$, respectively.

We have $n > 1$, because otherwise a loner would exist. It follows that $y_- \ne y_+$, so $y_-^* \ne y_+^*$. From $\alpha \cap \omega = \emptyset$ we get $z_-, z_+ \notin \{y_-, y_+\}$, hence $z_-^*, z_+^* \notin \{y_-^*, y_+^*\}$.

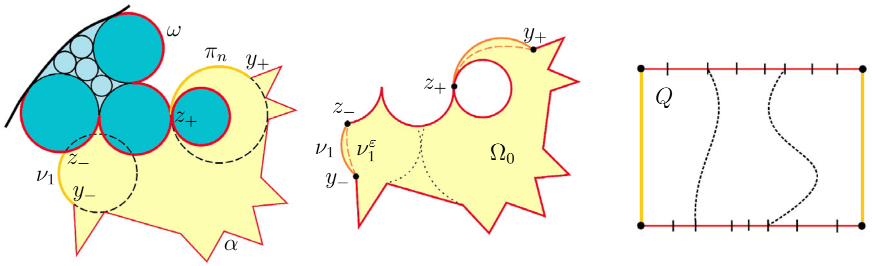

Fig. 11 Construction of Ω_0 and Q from ω, α and ν_1, π_n

If $z_-^* = z_+^* =: z^*$, we directly get $\omega^* \cap \Omega_0^* = z^*$. This implies $\nu_1^* = \pi_1^* = \nu_2^* = \cdots = \pi_n^* = z^*$, so the lemma holds. (We consider this case here, though Lemma 12 shows, that it cannot occur).

If $z_-^* \neq z_+^*$, the prime ends y_-^*, y_+^*, z_-^* and z_+^* are pairwise distinct and with respect to the (cyclic) ordering of Ω_0 we have $y_-^* < y_+^* < z_-^* < z_+^* < y_-^*$. Therefore Ω_0 can be mapped conformally onto a rectangle Q (with appropriately chosen aspect ratio) such that y_-^*, y_+^*, z_-^* and z_+^* correspond to the four corners of Q (see Pommerenke 1992), which is depicted in Fig. 11.

Any of the arcs ν_i ($i = 2, \ldots, n$) and π_i ($i = 1, \ldots, n-1$) is mapped onto a crosscut of Q which connects two opposite sides of this rectangle. Since these Jordan arcs cannot cross each other in the interior of Q, the ordering of their initial points on one side of Q is transplanted to the ordering of their terminal points on the opposite side of Q. Translated back to Ω_0, this implies that the ordering of the prime ends ν_i^* and π_i^* is the same as the ordering of the initial points y_i^- and y_i^+ of ν_i and π_i, respectively, along the Jordan curve α. By construction, the latter points form an interlacing sequence. □

Lemma 12 *If both prime ends ν_i^* and π_i^* belong to ω_i^*, then the corresponding vertex $v(i)$ is a loner.*

Proof Let $v := v(i)$. It follows from $\nu_i^*, \pi_i^* \in \omega_i^*$ that $\nu_i^+, \pi_i^+ \in \omega_i \subset \partial D_v$. If $\pi_i^+ \neq \nu_i^+$, the positively oriented open subarc δ_i'' of P_v' from π_i^+ to ν_i^+ lies in D_v. If $\pi_i^+ = \nu_i^+$, we set $\delta_i'' := \emptyset$. In both cases the union of $\alpha_i, \pi_i, \delta_i''$ and ν_i is a Jordan curve which does not intersect the disks D_u with $u \in U_L^+$ and $u \neq v$. So either D_v' is disjoint to all such disks D_u, or one of the disks D_u is contained in D_v'. In the latter case the prime ends ν_i^* and π_i^* cannot both belong to the same set ω_i^*. □

Proof of Lemma 8 After these preparations we are ready to harvest the fruits: Assume that $(\mathcal{P}, \mathcal{P}')$ has no loner. Then, by Lemma 10, the endpoint ν_i^+ of the arc ν_i must lie on ω and hence ν_i is associated with a prime end $\nu_i^* \in \omega^*$. If $\nu_i^* \in \omega_k^*$, we choose the smallest such k and set $l(i) := k$. Similarly, we denote by $r(i)$ the smallest number k for which $\pi_i^* \in \omega_k^*$.

Lemma 11 tells us that $l(i) \leq r(i) \leq l(i+1)$. In conjunction with Lemma 12 we conclude that the first condition implies $r(i) \geq l(i) + 1$. Starting with $l(1) \geq 1$, we

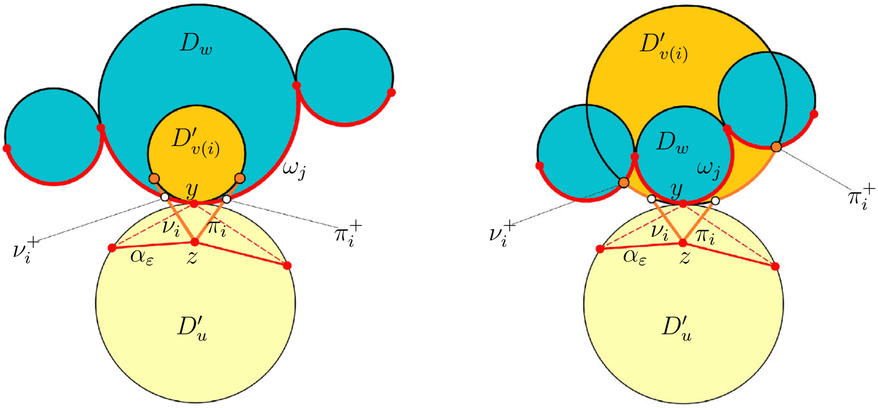

Fig. 12 Modification of α and definition of the arcs ν_i and π_i for critical contact points y

get inductively that $r(i) \geq i+1$ for $i = 1, \ldots, n$, ending up with the contradiction $r(n) \geq n+1$. This proves Lemma 8 in the regular case. $\square$

4.2 The critical case

The second case, where we admit that $\alpha \cap \omega \neq \emptyset$, will be reduced to the regular case by an appropriate deformation of the Jordan arc α.

Definition 6 A contact point $y \in Y$ is called *regular* if $y \notin \omega$, otherwise it is said to be *critical*.

If $y \in Y$ is a critical contact point, then $y \in \alpha \cap \omega \neq \emptyset$, and hence $y \in \omega_j$ for some j. Since $y = \partial D'_u \cap \partial D'_v$ with some $u \in U_L^-$ and $v = v(i) \in U_L^+$, we see that y cannot be an endpoint of ω_j (turning point of ω) – otherwise D'_u would not be contained in Ω. Moreover, the circles $\partial D'_u$, $\partial D'_v$, and ω_j must be mutually tangent at y. The arc ω_j is a subset of the circle ∂D_w (with $w = v(j) \in U_L^+$). Hence either $D'_v \subset D_w$ (with $D'_v = D_w$ admitted) or D_w is a proper subset of D'_v.

In the next step we modify the Jordan arc α in a neighborhood of y and redefine the arcs ν_i and π_i (connecting y with ω) introduced in the regular case.

Let ε be a sufficiently small positive number. Denote by z the ε-shift of y in the direction of the center of D'_u. Append to D'_v an equilateral open triangular domain T with one vertex at z, two vertices on $\partial D'_v$, and symmetry axis through y and z (see Fig. 12).

For $y \notin \{y_0, y_l\}$ let ν_i (and π_i) be the largest negatively (positively) oriented subarc of $\partial(D'_v \cup T)$ which has initial point z and is contained in Ω. For $y \in \{y_0, y_l\}$ (and only then) it can happen that y is a boundary point of G. Therefore we define $\nu_i := [z, y]$ in the case $y = y_0$, and $\pi_i := [y, z]$ in the case $y = y_l$. The case $y_0 = y_l$ can never occur, because $l \geq 1$.

Denote by ν_i^+ and π_i^+ the terminal points of ν_i and π_i. Clearly, $\nu_i^+, \pi_i^+ \in \omega$, so let $\nu_i^*, \pi_i^* \in \omega^*$ be their associated prime ends.

We see, that the statement of Lemma 10 holds in the critical case, too. Moreover, for the critical case, Lemma 11 can be proved in exactly the same way as for the regular case, we just have to apply the adapted definitions of ν_i^* and π_i^*. All what is missing is the following "critical" version of Lemma 12.

Lemma 13 *Assume that $\partial D'_v$ with $v = v(i) \in U_L^+$ contains a critical contact point $y \in Y \cap \omega$. Then v is a loner if and only if ν_i^* and π_i^* belong to ω_i^*.*

Proof We use the notations introduced above, with $\varepsilon > 0$ fixed and sufficiently small. We distinguish two cases.
Case 1 Let $D'_v \subset D_w$ (see Fig. 12, left). Then v is a loner if and only if $w = v$, and this holds, if and only if $j = i$ and $\nu_i^*, \pi_i^* \in \omega_i^*$.
Case 2 Let $D_w \subset D'_v$ and $D_w \neq D'_v$ (see Fig. 12, right). Then D'_v intersects at least two "upper" disks (namely D_w and one of its neighbors), so that v is not a loner. According to our construction, we have $\nu_i^* \leq y^* \leq \pi_i^*$ (where $y^* \in \omega_j^*$ is the prime end corresponding to y and $w = v(j)$), but both equalities are never fulfilled at the same time, and $\nu_i^*, \pi_i^* \notin \omega_j^*$ for $w = v(j)$. Therefore $\nu_i^* \in \omega_m^*$ and $\pi_i^* \in \omega_n^*$ with $m \leq j \leq n$, but $m < n$, so the prime ends ν_i^* and π_i^* cannot both belong to the same class ω_i^*. □

Remark If D'_v has several critical contact points $y \in Y \cap \omega_j$ with the same arc ω_j, then D'_v must be tangent to D_w with $w = v(j)$ at two different points. This implies that $D'_v = D_w$, which explains why the criterion is independent of the choice of y.

After replacing all critical contact points y_k by the shifted points z_k, and modifying the construction of the curve α accordingly, Lemma 8 can be proved completely the same way as in the regular case.

In Sect. 5 we need the following generalization of Lemma 8. We point out that $v(i) = v(j)$ is allowed in assertion (i).

Lemma 14 *Let $D_{v(i)} = D'_{v(i)}$ and $D_{v(j)} = D'_{v(j)}$ with $1 \leq i \leq j \leq n$. Then, in each of the following cases* (1)–(3)*, there exists a loner $v(k)$ which is different from $v(i)$ and $v(j)$, such that k satisfies the following conditions:*

(1) *if $1 \leq i < j - 1 \leq n - 1$, then $i < k < j$,*
(2) *if $i > 1$, then $1 \leq k < i$,*
(3) *if $j < n$, then $j < k \leq n$.*

Proof The proof differs only slightly from the proof of Lemma 8. For example, in order to prove (1) we need only replace the first inequality $l(1) \geq 1$ by $l(i+1) \geq i+1$ (which follows from $D_{v(i)} = D'_{v(i)}$) and, assuming that no loner $v(k)$ with $i < k < j$ exists, proceed inductively for $k = i + 1, \ldots, j$ until we arrive at $r(j) \geq j + 1$. The last condition contradicts $D_{v(j)} = D'_{v(j)}$.

If $v(k) = v(i)$ or $v(k) = v(j)$, we repeat the procedure, replacing i (in the first case) or j (in the second case) by k, respectively. Iterating this a number of times, if necessary, we eventually find a loner $v(k)$ which is different from $v(i)$ and $v(j)$, because for all $m = 2, 3, \ldots, n-1$ we have $v(m-1) \neq v(m)$ and $v(m) \neq v(m+1)$. □

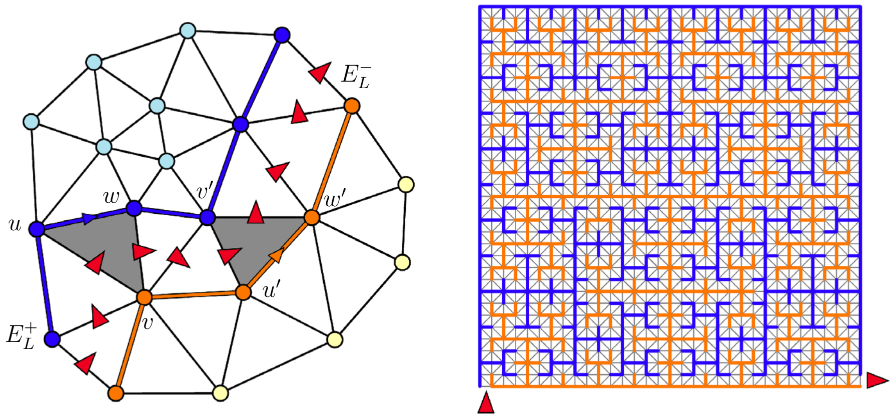

Fig. 13 The upper and the lower accompanying edges of a crosscut

5 Structure of upper neighbors

In this section we analyze the structure of the set of upper neighbors U_L^+ and its subset of loners in more detail.

Two consecutive (non-oriented) edges e_{j-1} and e_j of $L = (e_0, \ldots, e_l)$ can be represented as $e_{j-1} = e(u, v)$ and $e_j = e(v, w)$. The third edge of the face $f(u, v, w)$ is considered as oriented from u to w, and we set $e_j^0 := \langle u, w \rangle$. The set of edges e_j^0 splits into two classes. We define E_L^- as the set of those e_j^0 where the face $\langle u, v, w \rangle$ is oriented clockwise, whereas E_L^+ consists of those edges with counter-clockwise orientation of $\langle u, v, w \rangle$, respectively. After renumbering the elements of E_L^- and E_L^+, without changing their order, we get two sequences of oriented edges $E_L^- = \{e_1^-, \ldots, e_p^-\}$ and $E_L^+ = \{e_1^+, \ldots, e_q^+\}$ (with $p + q = l$), which are called the *sequences of lower* and *upper accompanying edges* of the crosscut L, respectively.

Here are some basic properties of E_L^-, E_L^+, which follow quite easy from the definition of L (proofs are left as exercises). The *oriented* edges in $E_L^- \cup E_L^+$ are pairwise disjoint; the corresponding non-oriented edges can appear at most twice, and either both in E_L^- or both in E_L^+. Two consecutive edges $e_{j-1}^\pm$ and $e_j^\pm$ are linked at a common vertex. The vertex set of all edges in E_L^+ is precisely the set U_L^+ of upper neighbors of L.

Figure 13 shows two examples. The involved crosscut on the right models the fourth generation of the Hilbert curve. With the exception of boundary edges, all edges in E_L^- (lighter color) and in E_L^+ (darker color) appear with both orientations (not shown in the picture).

When we arrange the elements of U_L^+ in the order they are met along the edge path E_L^+ we get the *sequence* S_L^+ *of upper accompanying vertices*. A similar definition is made for the *sequence* S_L^- *of lower accompanying vertices*. The geometry of circle packings causes some combinatorial obstructions for these sequences.

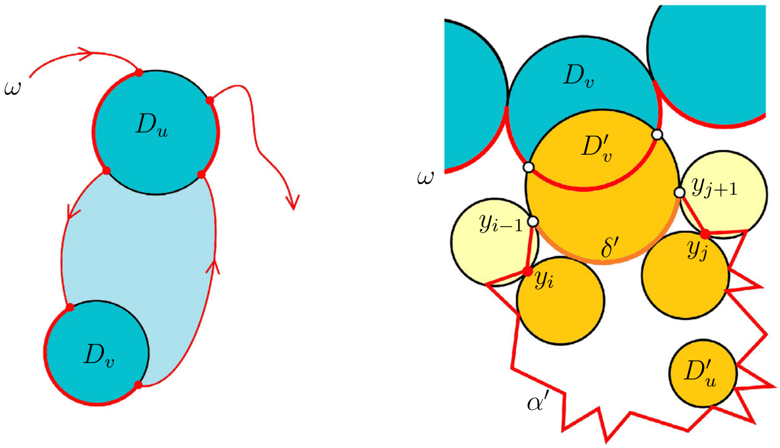

Fig. 14 Illustrations to Lemmas 15 and 17

Lemma 15 *The sequence S_L^+ of upper accompanying vertices cannot contain the pattern $(\dots, u, \dots, v, \dots, u, \dots, v, \dots)$ with $u \neq v$.*

Proof If the sequence S_L^+ contains the pattern $(\dots, u, \dots, v, \dots, u, \dots)$, the oriented curve ω has three subarcs $\omega_i, \omega_j, \omega_k$ with $i < j < k$ such that $\omega_i, \omega_k \subset \partial D_u$ and $\omega_j \subset \partial D_v$. But then ω cannot contain a subarc of $\partial D_v \backslash \omega_j$ (see Fig. 14, left), which would be necessary to append another v to the sequence. □

Definition 7 A vertex $v \in U_L^+$ which appears only once in the sequence S_L^+ is called *simple*, the other elements in U_L^+ are said to be *multiple* vertices.

If v is a multiple vertex in U_L^+, there are sequences $M := \{e_i^+, e_{i+1}^+, \dots, e_j^+\} \subset E_L^+$ of accompanying edges such that v is the initial vertex of e_i^+, as well as the terminal vertex of e_j^+ with $i < j$. Any such sequence is called a *loop* for v. We say that a loop M *meets a vertex* $u \in U_L^+$, if u is adjacent to an edge in M and $u \neq v$. The *set of vertices* met by M is denoted by V_M. A loop M also generates a *sequence of vertices* $U_M = \{v, v_1, \dots, v_m, v\}$ when we arrange the elements of V_M in the order they are met along the edge path M.

Lemma 16 *Every loop M of a multiple vertex v meets a simple vertex u.*

Proof We consider the sequence $U_M = \{v, v_1, \dots, v_m, v\}$ of vertices in V_M, arranged in the order as they are met by the edge path M. Let w denote the element of this sequence with the earliest second appearance (this does *not* mean the first element which appears twice). Since w cannot appear twice in direct succession, there exists a vertex u in between the first two symbols w.

In order to show that u is a simple vertex, we remark that U_M is a subsequence of the sequence S_L^+ of upper accompanying vertices. By definition of w, there cannot be a second u in S_L^+ between the two symbols w next to u, and by Lemma 15, the sequence S_L^+ cannot contain a second u outside these two w's. □

Since loners are vertices in U_L^+, it makes sense to speak of simple and multiple loners.

Lemma 17 *Let v be a multiple loner with $D'_v \neq D_v$. If $u \neq v$ is a vertex which is met by a loop of v, then u is a loner and $D'_u \cap D_u = \emptyset$.*

Proof Let M be a loop of v with $U_M = \{v, v_1, \ldots, v_m, v\}$. Let i be the smallest index, so that y_i is a contact point of v_1, and let j be the largest index, so that y_j is a contact point of v_m. According to the ordering of Y and U_M (as subsequences of S_L^+), y_{i-1} and y_{j+1} are contact points of D'_v. Let $u \in \{v_1, \ldots, v_m\}$ with $u \neq v$.

The disk D'_u is enclosed by the union of the subarc $\delta' := \delta[y_{i-1}, y_{j+1}]$ of D'_v and the subarc $\alpha' \subset \alpha$ which connects the points y_{i-1} and y_{j+1} on α (see Fig. 14). Since v is a loner with $D'_v \neq D_v$, it is clear that $y_{i-1}, y_{j+1} \notin D_v$, and hence either $D'_v \cap D_v = \emptyset$ or $\partial D'_v \cap \partial D_v$ consists of one or two points. In both cases δ' does not intersect D_v. Therefore the union $\alpha' \cup \delta'$ is contained in $\overline{\Omega}$, hence u is a loner. In particular $D'_u \cap D_u = \emptyset$, which proves the last assertion. □

Combining Lemma 8, Lemma 14 (applied recursively), Lemma 16 and Lemma 17 (applied recursively), the essence of this section can be summarized in the following lemma.

Lemma 18 *Let $(\mathcal{P}, \mathcal{P}')$ be an admissible pair of circle packings with crosscut L.*

1. *The pair $(\mathcal{P}, \mathcal{P}')$ contains a simple loner $v \in U_L^+$.*
2. *Every loop of a multiple loner v meets a simple loner u, and if $D'_v \neq D_v$ then $D'_u \neq D_u$.*

6 Proof of the main theorem

After all these preparations we are eventually in a position to prove Theorem 2. To begin with, we use the concept of loners and combinatorial surgery to modify the crosscut L. In every step of this procedure the number of vertices in V_L^+ is reduced. At the end we get a special combinatorial structure which is called a slit. Roughly speaking, this is a chain of vertices connecting the alpha-vertex with a boundary vertex. We shall prove that the disks of both packings coincide along a slit.

Then a subdivision procedure generates a sequence of slits, such that any accessible boundary vertex appears among their end points. So we get $D'_v = D_v$ for all accessible $v \in \partial V$, and finally a well-known theorem tells us that $D'_v = D_v$ for all accessible $v \in V$.

6.1 Combinatoric reduction

Let L be a combinatoric crosscut of the complex K. In this section we describe how a simple vertex $v \in U_L^+$ can be "shifted" from V_L^+ to V_L^- such that we get a new crosscut L' with $\left|V_{L'}^+\right| < \left|V_L^+\right|$. Depending on the properties of v we distinguish three cases.

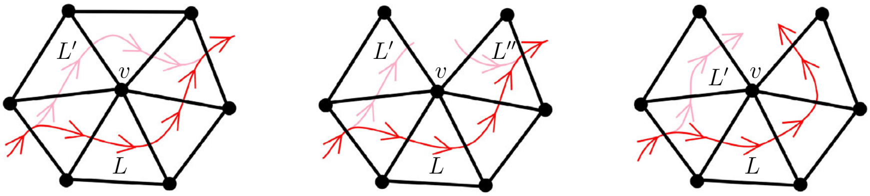

Fig. 15 Modification of the crosscut L in Case 1 (*left*), Case 2 (*middle*) and Case 3(*right*)

Case 1 Let $v \in U_L^+$ be a simple interior vertex.
Case 2 Let $v \in U_L^+$ be a simple boundary vertex, and assume that neither the initial nor the terminal edge of L are adjacent to v.
Case 3 Let $v \in U_L^+$ be a simple boundary vertex, and assume that either the initial or the terminal edge of L are adjacent to v.

Remark The case where the initial *and* the terminal edge of L are adjacent to v cannot appear. Indeed, otherwise either v is a multiple vertex (which is not considered) or all edges adjacent to v must belong to L. The latter implies that v is the only vertex in V_L^+, which is not allowed.

Reduction of Type 1 In order to modify the crosscut $L = (e_0, e_1, \ldots, e_l)$ in Case 1, we consider the flower $B = B(v)$ of v. Since v is simple, the set of edges adjacent to v consists of a subsequence $S = (e_i, \ldots, e_j)$ (with $0 \le i \le j \le l$) of L and a complementary sequence, which we denote by $S' = (e'_1, \ldots, e'_k)$ (with $k \ge 1$). Replacing in L the sequence S by S', we get a new edge sequence

$$L' = \left(e_0, \ldots, e_{i-1}, e'_1, \ldots, e'_k, e_{j+1}, \ldots, e_l\right).$$

The reader can easily convince herself (see Fig. 15, left), that the sequence L' is a crosscut for K with $\left|V_{L'}^+\right| < \left|V_L^+\right|$.

Reduction of Type 2 In Case 2 the flower of v is incomplete. Nevertheless, the edges in L which are adjacent to v form again a sequence of consecutive edges in this incomplete flower, because v is simple. However, the local modification of L in a neighborhood of v described above does not result in a crosscut L', since the complementary sequence $S' = S'_1 \cup S'_2$ consists of exactly two connected components $S'_1 = (e'_1, \ldots, e'_k)$ and $S'_2 = (e''_1, \ldots, e''_m)$ (see Fig. 15, middle). Replacing in L the sequence S by S'_1 or S'_2, we get a new edge sequence L' or L'', respectively, with

$$L' = \left(e_0, \ldots, e_{i-1}, e'_1, \ldots, e'_k\right), \qquad L'' = \left(e''_1, \ldots, e''_m, e_{j+1}, \ldots, e_l\right).$$

Both sequences L' and L'' are new crosscuts of K, but only one (L', say) contains v_α among its upper vertices, so we choose this one as the new crosscut. Clearly $\left|V_{L'}^+\right| < \left|V_L^+\right|$.

Reduction of Type 3 If either the initial or the terminal edge of L are adjacent to v, then the Type 1 reduction applied to the incomplete flower of v results in an admissible crosscut L', which has one vertex (namely v) less in $V_{L'}^+$ than in V_L^+ (see Fig. 15, right).

Remark No matter which type of reduction we used, the sets U_L^- and $U_{L'}^-$ of lower neighbors before and after the reduction, respectively, always fulfill $U_{L'}^- \setminus U_L^- = \{v\}$.

In order to not lose the normalization, we will only reduce vertices different from v_α. This leads to a situation where none of the above reductions can be applied, namely when v_α is the only simple vertex in U_L^+. This special case will be explored in Sect. 6.2.

6.2 Slits

The next definition and the following lemma describe the situation when all but exactly one vertex of U_L^+ are multiple.

Definition 8 A combinatoric *slit* of the complex $K(V, E, F)$ is a sequence $S = (v_1, v_2, \ldots, v_s)$ of vertices in V which satisfies the following conditions (1)–(4):

(1) The vertices of S are pairwise different, $v_j \neq v_k$ if $1 \leq j < k \leq s$.
(2) For $j = 1, \ldots, s-1$, the edges $e_j := e(v_j, v_{j+1})$ belong to E.
(3) For $j = 1, \ldots, s$, the vertices v_{j-1} and v_{j+1} are the only neighbors of v_j in K which belong to S (where $v_0 := \emptyset$ and $v_{s+1} := \emptyset$).
(4) The vertex v_1 is a boundary vertex, and v_j are interior vertices for $j = 2, \ldots, s$.

The vertices v_1 and v_s are referred to as the *initial vertex* and the *terminal vertex* of S, respectively. The sequence $E_S := (e_1, \ldots, e_{s-1})$ [see (2)] is said to be the *edge sequence* of S. Note that all e_j are interior edges.

Lemma 19 *Assume that the interior vertex v is the only simple vertex in U_L^+. Then the sequence of upper accompanying vertices S_L^+ has the symmetric form $(v_1, \ldots, v_{s-1}, v, v_{s-1}, \ldots, v_1)$ and $S = (v_1, \ldots, v_{s-1}, v)$ is a slit.*

Proof By definition of a multiple vertex, any vertex in U_L^+ except v must appear at least twice in the sequence S_L^+. If there are vertices which show up twice *at a position left of* v, we choose one, say u, whose appearances have minimal distance in the sequence $S_L^+ = (\ldots, u, \ldots, u, \ldots, v, \ldots)$. Since neighboring vertices of S_L^+ must be different, there exists $w \neq u$ such that $S_L^+ = (\ldots, u, \ldots, w, \ldots, u, \ldots, v, \ldots)$. Because v is assumed to be simple and w is a multiple vertex, we have $w \neq v$ and w must appear again at another place in S_L^+. By Lemma 15 this can only happen in between the two occurrences of u, which is in conflict with the minimal distance property of u.

Similarly, the assumption that there exists a vertex which appears in S_L^+ twice at a position right of v leads to a contradiction. Hence, with the only exception of v, any vertex of U_L appears in S_L^+ exactly once on either side of v. Applying Lemma 15 again, we see that the ordering of the vertices left of v must be reverse to the ordering on the right of v, so that S_L^+ has the symmetric form claimed in the lemma.

Moreover we have shown that $v_1, \ldots, v_{s-1}, v$ are pairwise different, which is condition (1) of Definition 8. The second condition (2) is trivial.

In order to verify condition (4), it remains to show that v_j is an interior vertex for $j = 2, \ldots, s-1$, because v_1 is obviously a boundary vertex, while $v_s := v$ is an

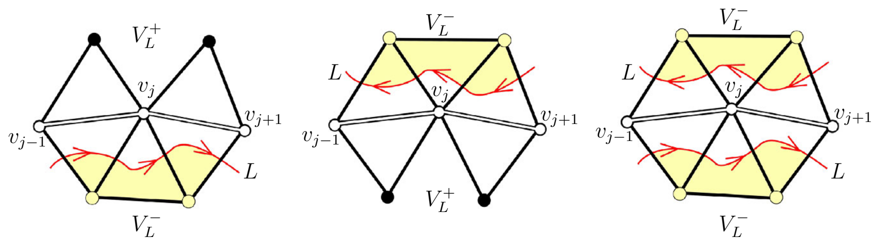

Fig. 16 Illustrations for the proof of Lemma 19

interior vertex, by assumption. Assume v_j is a boundary vertex. The flower of v_j is incomplete and it is clear that v_{j-1} and v_{j+1} are neighbors of v_j. On the one hand, since (v_{j-1}, v_j, v_{j+1}) is a subsequence of S_L^+, the crosscut L must look locally like shown in Fig. 16 left. On the other hand, the subsequence (v_{j+1}, v_j, v_{j-1}) of S_L^+ forces L to look locally like depicted in the middle of Fig. 16, a contradiction. Hence v_j must be an interior vertex and its flower must look qualitatively like shown in Fig. 16 right.

To verify condition (3) let $j \in \{2, \ldots, s-1\}$ be fixed. Looking at the behavior of the crosscut L in the flower of v_j, it becomes clear that any edge $e(v_{j-1}, v_{j+1})$ (with the convention $v_s := v$) belonging to E must be contained in L twice, a contradiction. Furthermore, all other neighbors of v_j belong to V_L^- and hence not to $V_L^+ \supset S_L^+$. A similar result can be derived by looking at the local behavior of L in the flower of v and the incomplete flower of v_1, now using the subsequences (v_{s-1}, v_s, v_{s-1}) and $(v_1, v_2, \ldots, v_2, v_1)$ of S_L^+, respectively. □

The following lemma explains why we are interested in slits.

Lemma 20 *Let $(\mathcal{P}, \mathcal{P}')$ be an admissible pair of circle packings for the complex K with crosscut L and alpha-vertex v_α. Then there exists a slit $S = (v_1, \ldots, v_s, v_\alpha) \subset V_L^+$ with terminal vertex v_α such that $D'_v = D_v$ for all $v \in S$.*

Proof To begin with, we invoke Lemma 18, which tells us that the pair $(\mathcal{P}, \mathcal{P}')$ has a simple loner v_λ. The idea is to use the reduction procedures of the last section to shift v_λ from V_L^+ to V_L^- which results in a new crosscut L'.

As we remarked earlier, the one and only lower neighbor of L' which has not already been a lower neighbor of L is the simple loner v_λ. Therefore Lemma 7 guarantees that L' is admissible for $(\mathcal{P}, \mathcal{P}')$. In order to find the appropriate type of reduction we distinguish the following cases:

Case 1 There exists a simple interior loner v_λ different from the alpha-vertex v_α.
Case 2 There exists a simple boundary loner v_λ.
Case 3 The only simple loner v_λ is the alpha-vertex v_α.

In Case 1 we apply the reduction of Type 1, while in Case 2 either the reduction of Type 2 or Type 3 can be applied, respectively, depending on whether v_λ is adjacent to the initial or the terminal edge of L, or not. In any case we get a new combinatoric crosscut L' of K. Applying the reduction in Cases 1 and 2 recursively as long as possible, the number of vertices in V_L^+ decays in every step at least by one, so that we eventually arrive at Case 3.

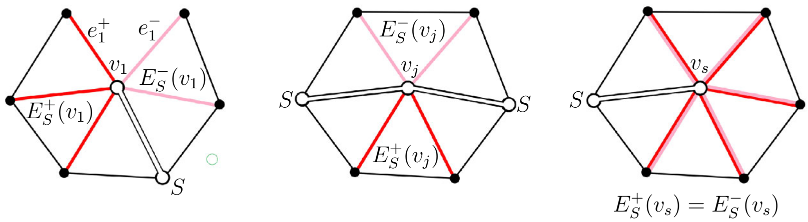

Fig. 17 *Left* and *right* neighboring edges of vertices $v = v_1, v_j, v_s$ in a slit S

Since the disks D'_α and D_α have the same centers, we either have one of the strict inclusions $D'_\alpha \subset D_\alpha$, $D_\alpha \subset D'_\alpha$ or $D'_\alpha = D_\alpha$. The first case cannot occur, since otherwise all neighboring disks of D'_α would intersect D_α, a contradiction for those disks associated with a vertex in U_L^-. The second case clearly implies that v_α is an intruder. So the alpha-vertex v_α is a loner if and only if $D'_\alpha = D_\alpha$. This implies, by Lemma 14, that there exists another loner v_μ. Since v_α is the only simple loner, v_μ must be a multiple loner. If $D'_\mu \neq D_\mu$, then according to Lemma 18 (1), the vertex set V_M of any loop M of v_μ contains a simple loner, i.e., M meets v_α. Because $D'_\alpha = D_\alpha$, assertion (2) of this lemma tells us that $D'_\mu = D_\mu$.

Applying Lemmas 14 and 18 repeatedly in this manner, we see that all vertices in $U_L^+ \backslash \{v_\alpha\}$ must be multiple loners and hence that $D'_v = D_v$ for all $v \in U_L^+$. Furthermore v_α is the only simple vertex in U_L^+, so, by Lemma 19, we just constructed a slit $S \subset V_L^+$ with terminal vertex v_α. ☐

In the next step we are going to construct crosscuts from slits. To begin with, we introduce some more notations.

Let $S = (v_1, \ldots, v_s)$ be a slit. For any vertex v in S we define the subsets $E_S^-(v)$ and $E_S^+(v)$ of $E(v)$ as follows. For $v = v_1$, the (boundary) vertex v_1 has two adjacent boundary edges e_1^- and e_1^+ in $E(v_1)$, such that e_1^- is the predecessor of e_1^+ in the chain of boundary edges. The meaning of the inequalities in the following definitions is explained in the second paragraph on complexes in Section 2. For the initial vertex v_1 we set (see Fig. 17, left)

$$E_S^-(v_1) := \left\{e \in E(v_1) : e(v_1, v_2) < e \le e_1^-\right\},$$
$$E_S^+(v_1) := \left\{e \in E(v_1) : e_1^+ \le e < e(v_1, v_2)\right\}.$$

If $v = v_j$, with $j = 2, \ldots s-1$, we define (Fig. 17, middle)

$$E_S^-(v_j) := \left\{e \in E(v_j) : e(v_j, v_{j+1}) < e < e(v_{j-1}, v_j)\right\},$$
$$E_S^+(v_j) := \left\{e \in E(v_j) : e(v_{j-1}, v_j) < e < e(v_j, v_{j+1})\right\},$$

and for the terminal vertex v_s of S we let (see Fig. 17, right)

$$E_S^-(v_s) = E_S^+(v_s) := \{e \in E(v_s) : e(v_{s-1}, v_s) < e < e(v_{s-1}, v_s)\}.$$

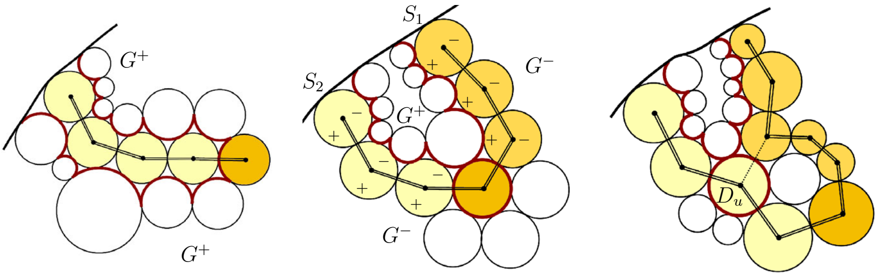

Fig. 18 Constructing crosscuts from one slit (*left*) and two slits (*middle*, *right*)

The edges in

$$E_S^- := \cup_{j=1}^{s-1} E_S^-(v_j) \quad \text{and} \quad E_S^+ := \cup_{j=1}^{s-1} E_S^+(v_j)$$

are called the *left* and the *right neighbors of* S, respectively. Note that condition (3) in Definition 8 guarantees that every edge e which is a neighbor of a slit S has *exactly one* adjacent vertex in S.

Lemma 21 *If* $S = (v_1, \ldots, v_s, v)$ *is a slit in* K*, then there exists a combinatoric crosscut* L *such that* $v \in S_L^+$*, and* $S_L^- = (v_1, \ldots, v_{s-1}, v_s, v_{s-1}, \ldots, v_1)$ *is the sequence of* lower accompanying vertices *of* L.

Proof Walking along the slit S from v_1 to v_s and back to v_1, we build the crosscut L from the concatenation of the edge sequences

$$E_S^-(v_1), \ldots, E_S^-(v_s), e(v_s, v), E_S^+(v_s), \ldots, E_S^+(v_1).$$

It is easy to see that all edges in L are pairwise different, so that L satisfies condition (1) of Definition 3. Condition (2) can easily be verified and (4) is obvious. In order to prove (3) we assume that three edges of L would form a face of K. Since these edges are neighbors of S, exactly one vertex of every edge must belong to S, which is impossible.

The construction also guarantees that the sequence S_L^- of lower accompanying edges of L has the desired form and that v belongs to S_L^+ (see, for example, Fig. 18, left). □

A crosscut L can also be constructed from joining two slits S_1 and S_2 with a common terminal vertex v. This procedure is somewhat more complicated, in particular when the "right side" of S_1 is close to the "left side" of S_2. In those cases we cannot join the cuts at their common terminal vertex v, since then the resulting edge sequence L would contain some edges more than once. Instead we modify the procedure by linking S_1 and S_2 at some appropriately chosen vertex u in S_2 or S_1 which has a neighbor in S_1 or S_2, respectively. Figure 18 (middle, right) illustrates the result, showing an associated circle packing and the related maximal crosscuts.

Lemma 22 *Let $S_1 = (v_1, \dots, v_t, v)$ and $S_2 = (w_1, \dots, w_s, v)$ be slits in K with $S_1 \cap S_2 = \{v\}$. Assume further that $E^+_{S_1}(v_1) \cap E^-_{S_2}(w_1) = \emptyset$. Then there exists a combinatoric crosscut L and a vertex $u \in (S_1 \cup S_2) \cap U^+_L$ such that*

$$S^-_L = \big(w_1, w_2, \dots, w_\sigma, u_1, \dots, u_k, v_\tau, v_{\tau-1}, \dots, v_1\big), \qquad 1 \le \tau \le t,\ 1 \le \sigma \le s, \tag{1}$$

where $\big(w_\sigma, u_1, \dots, u_k, v_\tau\big)$ is a (positively oriented) chain of neighbors of u.

Note that the condition $E^+_{S_1}(v_1) \cap E^-_{S_2}(w_1) = \emptyset$ does not exclude that v_1 and w_1 share an edge. Loosely speaking, it means that there is no such edge connecting the "plus side" of S_1 and the "minus side" of S_2;

Proof We set $v_{t+1} := v$ and $w_{s+1} := v$. Let i be the smallest number in $\{1, ..., t+1\}$ for which $E^+_{S_1}(v_i)$ contains an edge $e(v_i, w)$ with $w \in S_2$. Then let j be the smallest number in $\{1, ..., s+1\}$ for which $E^-_{S_2}(w_j)$ contains an edge $e(w_j, v_i)$. If $i \ne 1$ and $j \ne s+1$ we set $\tau := i-1$, $\sigma := j$ and $u := v_i$. If $i \ne 1$ but $j = s+1$, then $i = t$ must hold (otherwise v would have more then one neighbor in S_1), and we set $\tau := t$, $\sigma := s$ and $u := v$. If $i = 1$ we set $\tau := 1$, $\sigma := j-1$ and $u := w_j$. In the last case we have $j > 1$, since otherwise $i = j = 1$ would contradict the assumption $E^+_{S_1}(v_1) \cap E^-_{S_2}(w_1) = \emptyset$.

In every case $1 \le \tau \le t$ and $1 \le \sigma \le s$ hold, and u is well defined. We now build L as the concatenation of the edge sequences

$$E^-_{S_2}(w_1), \dots, E^-_{S_2}(w_\sigma), E^*(u), E^+_{S_1}(v_\tau), \dots, E^+_{S_1}(v_1),$$

where $E^*(u) = \big(e(u, w_\sigma), e(u, u_1), \dots, e(u, u_k), e(u, v_\tau)\big)$ is the negatively oriented chain of edges in the set $\{e' \in E(v) : e(u, w_\sigma) \le e' \le e(u, v_\tau)\}$.

Because S_1, S_2 are slits, all edges in the "$E^+_{S_1}$-part" and in the "$E^-_{S_2}$-part" of L are pairwise different. Furthermore, it cannot happen that such an edge is contained in both parts (according to the definition of u), or that it belongs to $E^*(u)$ (by definition of $E^*(u)$). Hence, L satisfies condition (1) of the crosscut definition.

Condition (2) can easily be verified and (4) is trivial. In order to prove (3) we assume that three edges of L form a face of K. By definition of u, the sequence $(w_1, w_2, \dots, w_\sigma, u, v_\tau, \dots, v_2, v_1)$ divides K into two parts K_1, K_2. All edges of the "$E^+_{S_1}$-part" and of the "$E^-_{S_2}$-part" have exactly one vertex lying in $S^0_1 \cup S^0_2$ and one in K_1, so three of them can never form a face of K. All edges of $E^*(u) \backslash \{e(u, v_\tau), e(u, w_\sigma)\}$ have exactly one vertex lying in $S^0_1 \cup S^0_2$ and one in K_2, so again three of them can never form a face of K. The only remaining edges are $e(u, v_\tau), e(u, w_\sigma)$, but two edges cannot form a face, and a combination of edges from more than one of the three distinguished edge types can clearly never form a face. Hence, L is a crosscut with $u \in (S_1 \cup S_2) \cap U^+_L$, and S^-_L has the form (1). □

The operation described in the proof is well defined by the slits S_1 and S_2, and will be referred to as *reflected concatenation* $S_1 \ominus S_2$ of S_1 with S_2. It delivers a crosscut L, a vertex u, and the reduced slits S^0_1, S^0_2. Note that the reflected concatenation is not commutative.

6.3 Subdivision by disk chains

Let v_β be an arbitrary accessible boundary vertex. In this section we describe an approach which allows us to apply Lemma 20 recursively, until we find a slit S with initial vertex v_β such that $D'_v = D_v$ for all $v \in S$, so especially $D'_{v_\beta} = D_{v_\beta}$. During this procedure we construct a sequence of crosscuts L_j such that $V^+_{L_j}$ contains v_β and the number of elements in $V^+_{L_j}$ is strictly decreasing for increasing j. This procedure will be crucial for proving the following lemma, and finally Theorem 2.

Lemma 23 *Let* $(\mathcal{P}, \mathcal{P}')$ *be an admissible pair with complex* K*, interior alpha vertex* v_α *and crosscut* L*. Then* $D'_v = D_v$ *for all accessible boundary vertices* $v \in \partial V^*$*.*

Proof To begin with, let $S_0 = (v_1, \ldots, v_s, v_\alpha)$ be a slit according to Lemma 20. Let v_β be an accessible boundary vertex. If $v_1 = v_\beta$ then $D'_\beta = D_\beta$ and we are done. So let us assume that $v_\beta \notin S_0$.

By Lemma 21 there exists a crosscut L_1 such that $S^-_{L_1} = (v_1, \ldots, v_{s-1}, v_s, v_{s-1}, \ldots, v_1)$ and $v_\alpha \in S^+_{L_1}$. Applying Lemma 20 again, but now with respect to the crosscut L_1, we get another slit $S_1 = (w_1, \ldots, w_t, v_\alpha) \subset V^+_{L_1}$, such that $D'_v = D_v$ for all $v \in S_1$. If $w_1 = v_\beta$ then $D'_\beta = D_\beta$ and we are done. So suppose that $v_\beta \notin S_1$.

The three boundary vertices v_1, w_1 and v_β are pairwise different, and we assume, without loss of generality, that they are oriented such that $w_1 < v_\beta < v_1$. This ensures that $E^+_{S_1}(v_1) \cap E^-_{S_0}(w_1) = \emptyset$, because otherwise v_β could be either accessible or a boundary vertex, but not both. Since, except v_α, all vertices of S_0 belong to $V^-_{L_1}$, we have $S_0 \cap S_1 = \{v_\alpha\}$. Consequently, by Lemma 22, the reflected concatenation $S_0 \ominus S_1$ of S_0 with S_1 is well defined. It delivers a crosscut L_2, a vertex v_{α_2}, and reduced slits $S^-_2 \subset S_0$, $S^+_2 \subset S_1$ with common terminal vertex v_{α_2}. Since $E^+_{S_1}(v_1) \cap E^-_{S_2}(w_1) = \emptyset$ (see above), by Lemma 22 the vertex v_{α_2} belongs to S_1 or S_2 and the set $U^-_{L_2}$ of lower neighbors of L_2 consists solely of elements of $S_0 \cup S_1$ and of (lower) neighbors of v_{α_2}. Since $D'_v = D_v$ for all $v \in S_0 \cup S_1$, this implies that L_2 is an admissible crosscut for $(\mathcal{P}, \mathcal{P}')$. Moreover, the order of S_0 and S_1 in the reflected concatenation has been chosen such that v_β belongs to $V^+_{L_2}$.

The general step of the procedure is as follows. Assume that we already have an admissible crosscut L_j, the alpha vertex v_{α_j}, and the reduced slits S^-_j and S^+_j, such that $v_\beta \in V^+_{L_j}$ (see Fig. 19, left). Denoting by v^-_j and v^+_j the initial vertices of S^-_j and S^+_j, respectively, we may assume that $v^-_j < v_\beta < v^+_j$, which will again be essential to ensure the special condition of Lemma 22.

Applying Lemma 20, we get a new slit $S_j \subset V^+_{L_j}$, such that S^-_j, S_j and S^+_j are pairwise disjoint, except at their common terminal vertex v_{α_j}, and $D'_v = D_v$ for all $v \in S_j$ (see Fig. 19, middle).

If $v_\beta \in S_j$ we are done. Otherwise we either have $v^-_j < v_\beta < v_j$ or $v_j < v_\beta < v^+_j$. In the first case we build the reflected concatenation $S^-_j \ominus S_j$, in the second case we form $S_j \ominus S^+_j$. The result is a new crosscut L_{j+1}, a corresponding alpha-vertex $v_{\alpha_{j+1}}$, and reduced slits S^-_{j+1}, S^+_{j+1} (see Fig. 19, right).

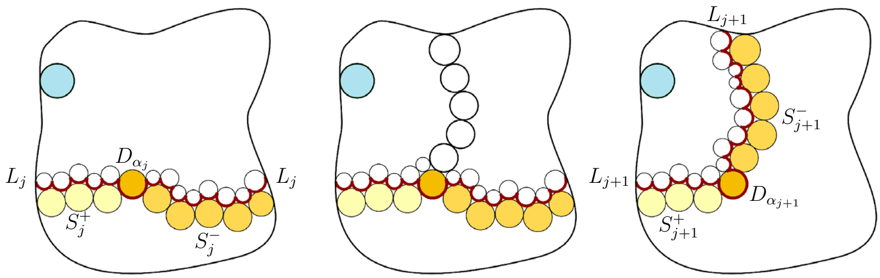

Fig. 19 Construction of the crosscut L_{j+1} from L_j

It follows directly from the construction of the reflected concatenation that $v_{\alpha_{j+1}}, v_\beta \in V^+_{L_{j+1}}$. Moreover, $v_{\alpha_{j+1}} \in S_j^-$, and hence $D'_{\alpha_{j+1}} = D_{\alpha_{j+1}}$. To see that L_{j+1} is admissible for the pair $(\mathcal{P}, \mathcal{P}')$ it remains to prove that $D'_v \subset G^-_{L_{j+1}}$ for all $v \in U^-_{L_{j+1}}$.

By Lemma 22 the set $U^-_{L_{j+1}}$ of lower neighbors of L_{j+1} consists solely of elements of $S_j^- \cup S_j^+$ and of (lower) neighbors of $v_{\alpha_{j+1}}$. Since $D'_v = D_v$ for all $v \in S_j^- \cup S_j^+ \cup \{v_{\alpha_{j+1}}\}$, and $D_v \subset G^-_{L_{j+1}}$ for all $v \in U^-_{L_{j+1}}$, the assertion follows.

The number of elements in $V^+_{L_j}$ is strictly decreasing in every step, and hence the procedure must come to end. This can only happen if $v_\beta \in S_{j^*}$ for some $j^* \in \mathbb{N}$. Because $D'_v = D_v$ for all $v \in S_j$ with $j \le j^*$, we have shown $D'_{v_\beta} = D_{v_\beta}$. □

Now the proof of Theroem 2 is near to its end. By Lemma 1 the kernel K^* is a strongly connected complex with vertex set V^*. Since we have shown that $D'_v = D_v$ for all boundary vertices $v \in \partial V^*$ of K^*, and every boundary vertex of K^* is also a boundary vertex of K (that is $\partial V^* = V^* \cap \partial V$), Theorem 11.6 in Stephenson (2005) (on the uniqueness of a locally univalent packing with presribed combinatorics and given radii of boundary disks) tells us that $D'_v = D_v$ for all $v \in V^*$, which is the assertion of Theorem 2.

7 Concluding remarks

All proofs in this paper work with (simple) geometric or combinatoric arguments, alone in the very last step we had recourse to a theorem established in the literature. For purists we mention that even this could have been avoided, at the expense of adding a few pages to this rather longish text.

Theorem 2 can be interpreted as a uniqueness result for (the range packing of) discrete conformal mappings. Here is a simple version:

Theorem 3 *Suppose that two univalent packings $\mathcal{P}$ and $\mathcal{P}'$ for K fill G. If D'_α and D_α have the same center, and if $D'_\beta \subset D_\beta$ for some boundary vertex v_β, then $D'_v = D_v$ for all vertices $v \in V^*$.*

The proof follows immediately from Theorem 2 applied to the maximal crosscut which separates the disk D_β from the rest of the packing $\mathcal{P}$ (see the leftmost image of Fig. 20).

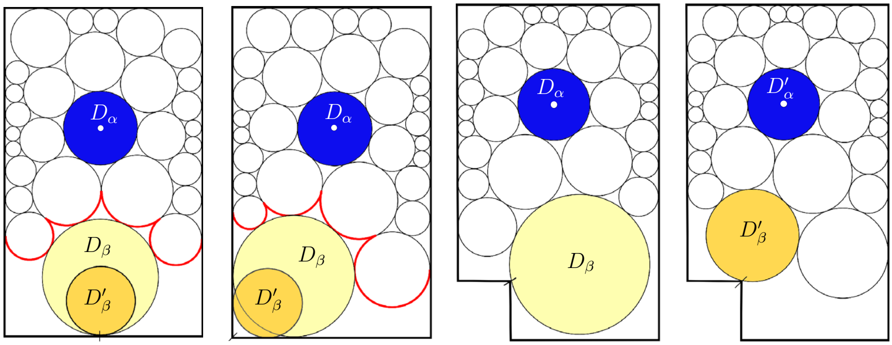

Fig. 20 Applications of Theorem 2 to discrete conformal mapping

The condition $D'_\beta \subset D_\beta$ can even be relaxed, it suffices to require that D'_β lies in the lower domain G_- with respect to this crosscut (see the second image of Fig. 20). Note that both figures show the packing $\mathcal{P}$ and a single disk D'_β of $\mathcal{P}'$ in G_-.

We point out that the condition $D'_\beta \subset G_-$ is always satisfied (possibly after exchanging the roles of $\mathcal{P}$ and $\mathcal{P}'$), if the packings are normalized so that D'_β and D_β *touch the boundary* ∂G *in a generalized sense* at the same *regular point* (or, more generally, at the same *regular prime end*). Without explaining these concepts here (see Wegert and Krieg 2014), we mention that a point which lies on a smooth subarc of ∂G is always regular, while a point at a re-entrant corner fails to be regular. The two pictures on the right of Fig. 20 illustrate that uniqueness of domain-filling circle packings may be violated in that case. Both displayed packings $\mathcal{P}$ and $\mathcal{P}'$ fill a Jordan domain G, D_α and D'_α have the same center, and D_β and D'_β touch ∂G at the same point. While this type of normalization implies uniqueness of classical conformal mappings, the corresponding circle packings $\mathcal{P}$ and $\mathcal{P}'$ are completely different.

We further mention that for domain-filling circle packings $\mathcal{P}$ and $\mathcal{P}'$ the assertions of Theorems 2 and 3 can be strengthened to $D'_v = D_v$ for all $v \in V$, using the results of our forthcoming paper (Krieg and Wegert 2015).

In the general setting of Theorem 2, a complete description of which disks are uniquely determined by a crosscut seems not to be known. Figure 21 shows some examples. The accessible disks are depicted in darker colors, the alpha-disk is the darkest one. By Theorem 2 these disks are uniquely determined (rigid) by the crosscut, but the rigid part also comprises the non-accessible disks shown in brighter color.

The example on the right is of special interest: a short crosscut separates only one non-accessible disk D_β from the alpha-disk. Here the theorem yields rigidity for the dark (blue) disks, while it says nothing about the disks depicted in lighter colors. This is somewhat counterintuitive, since the bright disks separate the dark disks from the crosscut, so that the latter seem to have no relation to the cut at all. However, a little thought shows that in fact all colored disks in the upper domain are rigid. It is a challenging problem to precisely describe the set of all rigid disks in a circle packing with crosscut.

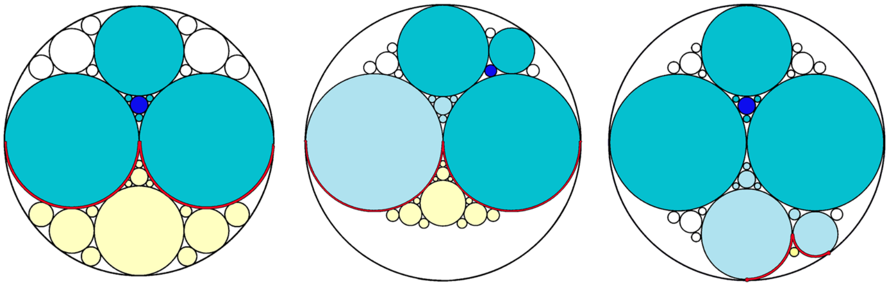

Fig. 21 Rigid configurations of disks in a packing with crosscut

Isn't it wonderful that simple circles can form such fascinating structures?

Glossary

$\ominus$, $S_1 \ominus S_2$	reflected concatenation of slit S_1 with slit S_2
$\langle u, v, w\rangle$	oriented face of K with vertices u,v and w
$\langle u, v\rangle$	oriented edge of K from vertex u to vertex v
α_i, α	special Jordan arcs connecting y_i^- and y_i^+, and their concatenation
$B(v)$	the flower of the vertex v, a subcomplex of K
$c(u, v)$	contact point of the disks D_u and D_v, $c(u, v) = \overline{D}_u \cap \overline{D}_v$
c_k^-, c_k^+	contact points of boundary disk D_k with D_{k-1} and D_{k+1}, respectively
D	union of all disks in $\mathcal{P}$
D^*	carrier of $\mathcal{P}$
D_k, D_k'	boundary disks in $\mathcal{P}$ and $\mathcal{P}'$, respectively
D_v, D_v'	disks in $\mathcal{P}$ and $\mathcal{P}'$, respectively
∂	boundary operator, applied to various objects
$\delta(p, q)$	positively oriented open circular arc from p to q on ∂D
$\delta[p, q]$	positively oriented closed circular arc from p to q on ∂D
$\delta(c_k^-, c_k^+)$	exterior boundary arc of D_k
$\delta(c_k^+, c_k^-)$	interior boundary arc of D_k
δ_k	smallest subarc of $\delta[c_k^-, c_k^+]$ which contains G_k
E_S	the edge sequence of the slit S
E	the set of edges of the complex K
∂E	boundary edges of the complex K
$E(v)$	the (cyclically ordered) sequence of edges adjacent to $v \in V$
$E_L^{\pm}(v)$	sequences of upper and lower accompanying edges of the crosscut L
$E_S^{\pm}(v)$	sequences of edges adjacent to a vertex v in a slit S
$E_S^{\pm}$	sequences of left and right neighbor edges of slit S, respectively
$e(u, v)$	non-oriented edge between vertices u and v
e_j	edges in a crosscut, $L = (e_0, e_1, \dots, e_l)$
e_j^-, e_j^+	lower and upper accompanying edges of the crosscut L, respectively
η_k, η	segments connecting the centers of D_k and D_{k+1} and their concatenation
F	set of faces of the complex K

$f(u, v, w)$	non-oriented face with vertices u, v and w
G	a bounded simply connected domain in $\mathbb{C}$
G_L^-, G_L^+	lower and upper domains of G with maximal crosscut J_L^+, $G_L^- = \Omega$
G_k	set of contact points of D_k with ∂G, $G_k := \overline{D}_k \cap \partial G$
g_k^-, g_k^+	first and the last contact point of D_k with ∂G
I_k	boundary interstice between D_k and D_{k+1}
$I(u, v, w)$	interstice between the disks D_u, D_v and D_w
J_L^0	polygonal (geometric) crosscut in G for (combinatoric) crosscut L in K
J_L^+	maximal 'crosscut', the upper boundary of the lower domain G_L^-, $J_L^+ = \omega$
K	simplicial 2-complex, combinatorial disk, finite triangulation, $K(V, E, F)$
K^*	kernel of K, largest sub-complex of K with vertex set V^*
L	combinatorial crosscut, sequence of edges in K
$l(i)$	smallest label k of prime end set ω_k^* associated with ν_i
M, $M(\mu)$	loop of a multiple loner v_μ, a sequence of edges
ν_i, π_i	negatively and positively oriented arcs on ∂D from y_i^-, y_i^+ to ω, respectively
ν_i^+, π_i^+	terminal points of the arcs ν_i, π_i, respectively
ν_i^*, π_i^*	prime ends of Ω associated with ν_i, π_i, respectively
Ω	lower subdomain of G with respect to a maximal crosscut, $\Omega = G_L^-$
ω	upper boundary of lower domain Ω, concatenation of the ω_i, maximal crosscut
ω^*	prime ends of Ω associated with ω
ω_i	circular subarcs of ω in between its turning points
ω_i^*	classes of prime ends associated with the arcs ω_i
$\mathcal{P}$	a univalent circle packing for K filling G
$\mathcal{P}'$	a univalent circle packing for K in G
$r(i)$	largest label k of prime end set ω_k^* associated with π_i
S	combinatoric slit, a sequence of vertices
S_L^-, S_L^+	sequences of lower and upper accompanying vertices of L, respectively
t_i	turning points of the upper boundary ω, cusps of Ω
U_L^-, U_L^+	sets of lower and upper neighbors of L, respectively, $U_L^- \subset V_L^-$, $U_L^+ \subset V_L^+$
U_M	sequence of the vertices in V_M for a loop M
V	vertex set of the complex K
V^*	the set of all accessible vertices of K
∂V	boundary vertices of the complex K
V_L^-, V_L^+	lower and upper vertices of K with crosscut L, respectively, subsets of V
V_M	set of all vertices met by a loop M
v_α	alpha vertex of K, a distinguished interior vertex
$v(i)$	vertex of the disk which contains the circular arc ω_i, $v(i) \in U_L^+$
x_k, X	contact points of upper with lower disks in $\mathcal{P}$, the set of all x_k

X_i	sets of contact points x_k on ω_i, $X_i \subset X$
y_-, y_+	initial point and terminal point of α, respectively
y_k, Y	contact points of upper with lower disks in $\mathcal{P}'$, the set of all y_k
y_i^-, y_i^+	minimal and maximal element of Y_i, respectively
Y_i	sets of contact points y_k with $x_k \in \omega_i$, $Y_i \subset Y$
z_-, z_+	terminal points of ν_1 and π_n, respectively
z_k	shifted contact points when y_k is critical

Acknowledgments We would like to thank the referee for reading this long manuscript with great care and making valuable comments and suggestions. We also thank Beate Uhl and the Springer Correction Team for their constructive collaboration in resolving some issues concerning page referencing.

References

Bauer, D., Stephenson, K., Wegert, E.: Circle packings as differentiable manifolds. Contrib. Algebra Geom. **53**, 399–420 (2012)
Beardon, A.F., Stephenson, K.: The uniformization theorem for circle packings. Indiana Univ. Math. J. **39**, 1383–1425 (1990)
Golusin, G.M.: Geometrische Funktionentheorie. Dt. Verl. d. Wissenschaften, Berlin (1957)
He, Z.-X., Schramm, O.: On the convergence of circle packings to the Riemann map. Invent. Math. **125**, 285–305 (1996)
Henle, M.: A combinatorial introduction to topology. Dover Publ. (1979)
Koebe, P.: Kontaktprobleme der konformen Abbildung. Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. 88, Leipzig 1936: 141–164
Krieg, D., Wegert, E.: Domain-filling circle packings (2015) (in preparation)
Pommerenke, Ch.: Boundary behaviour of conformal maps. Springer, Berlin (1992)
Rodin, B.: Schwarz's lemma for circle packings. Invent. Math. **89**, 271–289 (1987)
Rodin, B., Sullivan, D.: The convergence of circle packings to the Riemann Mapping. J. Differ. Geom. **89**, 349–360 (1987)
Schramm, O.: Combinatorically prescribed packings and applications to conformal and quasiconformal maps. Ph.D. thesis., Princeton (1990)
Schramm, O.: Existence and uniqueness of packings with specified combinatorics. Israel J. of Math. **73**, 321–341 (1991)
Stephenson, K.: Introduction to circle packing. Cambridge Univ. Press, Cambridge (2005)
Thurston, W.: The finite Riemann mapping theorem. In: Invited talk in the International Symposium at Purdue University on the occasion of the proof of the Bieberbach conjecture (1985)
Wegert, E., Krieg, D.: Incircles of trilaterals. Contrib. Algebra Geom. **55**, 277–287 (2014)
Wegert, E., Roth, O., Kraus, D.: On Beurling's boundary value problem in circle packing. Complex Var. Elliptic Equ. **57**, 397–410 (2012)