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ORIGINAL PAPER

On centrally-extended maps on rings

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Abstract Let R be a ring with center Z. A map D of R (resp. T of R) is called a centrally-extended derivation (resp. a centrally-extended endomorphism) if for each $x, y \in R$, $D(x+y) - D(x) - D(y) \in Z$ and $D(xy) - D(x)y - xD(y) \in Z$ (resp. $T(x+y) - T(x) - T(y) \in Z$ and $T(xy) - T(x)T(y) \in Z$). We discuss existence of such maps which are not derivations or endomorphisms, we study their effect on Z, and we give some commutativity results.

Keywords Derivations · Epimorphisms · Centrally-extended derivations · Centrally-extended epimorphisms · Commutativity theorems

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1 Introduction

Let R be a ring, not necessarily with 1, with center $Z = Z(R)$. Let N be the set of nilpotent elements of R, and call R reduced if $N = \{0\}$. For each $x, y \in R$, denote by $[x, y]$ the commutator $xy - yx$. If $S \subseteq R$, define $f : R \to R$ to be centralizing on S if $[x, f(x)] \in Z$ for all $x \in S$; and define f to be strong commutativity-preserving on S if $[x, y] = [f(x), f(y)]$ for all $x, y \in S$.

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Define a map $D\colon R \to R$ to be a centrally-extended derivation (CE-derivation) if for each $x, y \in R$, $D(x+y) - D(x) - D(y) \in Z$ and $D(xy) - D(x)y - xD(y) \in Z$. Define a map $T\colon R \to R$ to be a centrally-extended endomorphism (CE-endomorphism) if for each $x, y \in R$, $T(x+y) - T(x) - T(y) \in Z$ and $T(xy) - T(x)T(y) \in Z$; and if T is also surjective, call it a CE-epimorphism. We present examples of CE-derivations and CE-endomorphisms and investigate when they are ordinary derivations or endomorphisms, we study their effect on Z, and we note applications to commutativity theorems.

2 Examples and existence theorems

Clearly, every derivation (resp. endomorphism) is a CE-derivation (resp. CE-endomorphism). If R is commutative, every map $f\colon R \to R$ is both a CE-derivation and a CE-endomorphism; hence we cannot get interesting results in this case.

Example 2.1 Let R be any ring with $Z \neq \{0\}$. Choose $a \in Z\backslash\{0\}$ and define $T(x) = a$ for all $x \in R$. Then $T(x+y) - T(x) - T(y) = -a \in Z$ and $T(xy) - T(x)T(y) = a - a^2 \in Z$, hence T is a CE-endomorphism. Since T is not additive, it is not an endomorphism.

Example 2.2 Let R be a ring with a nonzero central ideal I, and let f be any function from R into I.

(a) Let t be any endomorphism of R and define $T(x) = t(x) + f(x)$ for all $x \in R$. Then $T(x+y) - T(x) - T(y) = f(x+y) - f(x) - f(y) \in Z$ and $T(xy) - T(x)T(y) = f(xy) - f(x)t(y) - f(y)t(x) - f(x)f(y) \in Z$ for all $x, y \in R$, so T is a CE-endomorphism. If f is a nonzero constant function, T is not an endomorphism; and if we also take t to be the identity map, T is a CE-epimorphism which is not an epimorphism.

(b) With I and f as above, let d be a derivation on R and define $D(x) = d(x) + f(x)$ for all $x \in R$. Then D is easily shown to be a CE-derivation, which for appropriate choices of f is not a derivation.

Example 2.3 Let R_1 be a commutative domain, R_2 a noncommutative prime ring with derivation d and $R = R_1 \oplus R_2$. Define $D\colon R \to R$ by $D((x, y)) = (g(x), d(y))$, where $g\colon R_1 \to R_1$ is any map which is not a derivation. Then R is a semiprime ring and D is a CE-derivation which is not a derivation. Moreover, $R_1 \oplus \{0\}$ is a central ideal of R.

It is no accident that nonzero central ideals play a prominent role in these examples, as the following two theorems show.

Theorem 2.4 *Let R be any ring with no nonzero central ideals. Then every CE-derivation D on R is additive.*

Proof Let x and y be fixed elements of R, and let

$$D(x+y) = D(x) + D(y) + a, \quad a \in Z. \tag{2.1}$$

For arbitrary $t \in R$, we have $b \in Z$ such that

$$\begin{aligned} D(t(x+y)) &= tD(x+y) + D(t)(x+y) + b \\ &= t(D(x) + D(y) + a) + D(t)(x+y) + b \\ &= tD(x) + tD(y) + D(t)x + D(t)y + ta + b. \end{aligned} \tag{2.2}$$

Calculating in a different way, we have

$$\begin{aligned} D(t(x+y)) &= D(tx+ty) \\ &= D(tx) + D(ty) + c \\ &= tD(x) + D(t)x + b_1 + tD(y) + D(t)y + c_1 + c, \end{aligned} \tag{2.3}$$

where $b_1, c_1, c \in Z$.

Comparing (2.2) and (2.3) gives $ta + b = b_1 + c_1 + c$, hence Ra is a central ideal and therefore $Ra = \{0\}$. Thus, letting $A(R)$ be the two-sided annihilator of R, we have $a \in A(R)$. But $A(R)$ is a central ideal, so $a = 0$ and by (2.1) $D(x+y) = D(x)+D(y)$. □

Theorem 2.5 *If R is a semiprime ring with no nonzero central ideals, then every CE-derivation D is a derivation.*

Proof Let $x, y, t \in R$ be arbitrary elements. Then $D((xy)t) - xyD(t) - D(xy)t \in Z$ and $D(x(yt)) - xD(yt) - D(x)yt \in Z$. Subtracting, we get

$$-xyD(t) - D(xy)t + xD(yt) + D(x)yt \in Z. \tag{2.4}$$

Let

$$\begin{aligned} D(xy) &= xD(y) + D(x)y + z_1, \text{ and} \\ D(yt) &= yD(t) + D(y)t + z_2, \quad \text{where } z_1, z_2 \in Z. \end{aligned} \tag{2.5}$$

Then from (2.4),

$$-xyD(t) - xD(y)t - D(x)yt - z_1t + xyD(t) + xD(y)t + xz_2 + D(x)yt \in Z,$$

which reduces to

$$-z_1t + xz_2 \in Z. \tag{2.6}$$

Thus $[xz_2, t] = [x, t]z_2 = 0$. Replacing x by xr, $r \in R$, and recalling (2.5), we have

$$[x, t]R(D(yt) - yD(t) - D(y)t) = 0 \quad \text{for all } x, y, t \in R. \tag{2.7}$$

Let $\{P_\alpha | \alpha \in \Lambda\}$ be a family of prime ideals of R such that $\bigcap P_\alpha = \{0\}$, and let P denote a typical P_α. Let $\bar{R} = R/P$ and $\bar{Z}$ the center of $\bar{R}$, and let $\bar{x} = x + P$ be a typical element of $\bar{R}$.

Fix y and t above, and let x vary. Then z_2 is fixed but z_1 varies with x. Now from (2.7) we have that either $(i)[x, t] \in P$ for all $x \in R$ or $(ii) z_2 = D(yt) - yD(t) - D(y)t \in P$, hence $\bar{t} \in \bar{Z}$ or $\bar{z_2} = \bar{0}$. It follows from (2.6) that for each $x \in R$, $-\bar{z_1}\bar{t} + \bar{x}\bar{z_2} \in \bar{Z}$, so that if $\bar{t} \in \bar{Z}$, $\bar{R}\bar{z_2} \subseteq \bar{Z}$. On the other hand, if $\bar{z_2} = \bar{0}$, it is certainly true that $\bar{R}\bar{z_2} \subseteq \bar{Z}$. Thus, $[rz_2, u] \in P$ for all $r, u \in R$; and since $\bigcap P_\alpha = \{0\}$, this gives the result that Rz_2 is a central ideal of R. As in the proof of Theorem 2.4, we conclude that $z_2 = 0$, i.e., $D(yt) = yD(t) + D(y)t$. Since D is additive by Theorem 2.4, our proof is complete. □

Combining Theorem 2.5 and Example 2.2(b) gives

Theorem 2.6 *A semiprime ring R admits a CE-derivation which is not a derivation if and only if R has a nonzero central ideal.*

CE-epimorphisms are easily treated by the same methods, so we present our results without proof.

Theorem 2.7 *If R is a ring with no nonzero central ideals, every CE-epimorphism on R is additive.*

Theorem 2.8 *If R is a semiprime ring with no nonzero central ideals, then every CE-epimorphism is an epimorphism.*

3 On the invariance problem for Z

We say that a map $f : R \to R$ preserves the subset $S \subseteq R$ if $f(S) \subseteq S$. It is well known that derivations and epimorphisms preserve Z, and the purpose of this section is to study preservation of Z by CE-derivations and CE-epimorphisms.

The CE-derivations of Example 2.2(b) all preserve Z, and so do the CE-epimorphisms of Example 2.2(a) for which t is an epimorphism. However, there do exist CE-derivations and CE-epimorphisms which do not preserve Z, as the following examples show.

Example 3.1 We give an example of a CE-derivation D with $D(Z) \nsubseteq Z$. Let R_2 be a noncommutative ring with $R_2^2 \subseteq Z(R_2)$, for example a noncommutative ring with $R_2^3 = \{0\}$. Let R_1 be a zero ring with $(R_1, +) \cong (R_2, +)$ and let $f : (R_1, +) \to (R_2, +)$ be an isomorphism. Define R to be $R_1 \oplus R_2$, and let $D : R \to R$ be given by $D((x, y)) = (0, f(x))$. It is easily verified that D is a CE-derivation on R. Moreover $R_1 \oplus \{0\} \subseteq Z(R)$ and $D(R_1 \oplus \{0\}) \nsubseteq Z(R)$.

Example 3.2 Let R_1, R_2, f and R be as in Example 3.1. Define $T((x, y)) = (f^{-1}(y), f(x))$ for all $(x, y) \in R$. It is easy to show that T is a CE-endomorphism; moreover, T is surjective, since for any $(u, v) \in R$, $T(f^{-1}(v), f(u)) = (u, v)$. Thus, T is a CE-epimorphism. Again, $R_1 \oplus \{0\} \subseteq Z(R)$ and $T(R_1 \oplus \{0\}) \nsubseteq Z(R)$.

Theorem 3.3 *Let R be a ring with $Z \cap N = \{0\}$. Then every CE-derivation D on R preserves Z.*

Proof Let $z \in Z$ and $x \in R$. Then $D(zx) - D(z)x - zD(x) \in Z$ and $D(xz) - D(x)z - xD(z) \in Z$, and by subtracting we obtain

$$[x, D(z)] \in Z \quad \text{for all } x \in R. \tag{3.1}$$

Replacing x by $xD(z)$ in (3.1) gives $[x, D(z)]D(z) \in Z$, so

$$[[x, D(z)]D(z), x] = 0 = [x, D(z)]^2 \quad \text{for all } x \in R. \tag{3.2}$$

Since $Z \cap N = \{0\}$, (3.1) and (3.2) give $[x, D(z)] = 0$ for all $x \in R$, i.e., $D(z) \in Z$. □

Theorem 3.4 *If R is a ring with $Z \cap N = \{0\}$, then every CE-epimorphism on R preserves Z.*

Proof If $z \in Z$, $T(zx) - T(z)T(x) \in Z$ and $T(xz) - T(x)T(z) \in Z$ for all $x \in R$, and by subtraction we get $[T(x), T(z)] \in Z$ for all $x \in R$. Since T is surjective, this yields $[x, T(z)] \in Z$ for all $x \in R$. Proceeding as in the previous proof, we get $T(z) \in Z$. □

Corollary 3.5 *If R is a semiprime ring, Z is preserved by every CE-derivation and by every CE-epimorphism.*

CE-derivations and CE-epimorphisms which preserve Z may also preserve subsets of Z, in particular the set $K(R)$, defined as $\{x \in Z | xR \subseteq Z\}$. It is easily shown that $K(R)$ is a central ideal containing all central ideals, i.e., the maximal central ideal.

Theorem 3.6 *If D is a CE-derivation on a ring R which preserves $Z(R)$, then D preserves $K(R)$.*

Proof Let $x \in K(R)$. Since $K(R) \subseteq Z$, $D(x) \in Z$. For arbitrary $r \in R$, $D(xr) - xD(r) - D(x)r \in Z$; and since $D(xr) \in Z$ and $xD(r) \in Z$, $D(x)r \in Z$. Therefore $D(x) \in K(R)$. □

A similar argument establishes the following theorem.

Theorem 3.7 *If T is a CE-epimorphism on a ring R which preserves $Z(R)$, then T preserves $K(R)$.*

4 Commutativity results

We begin this section with a very easy result.

Theorem 4.1 *Let R be a prime ring and D (resp. T) be a CE-derivation (resp. a CE-epimorphism). If $D(0) \neq 0$ (resp. $T(0) \neq 0$), then R is commutative.*

Proof We give the proof for CE-derivations; the proof for CE-epimorphisms is similar. Let D be a CE-derivation with $D(0) \neq 0$. Since $D(0+0) - D(0) - D(0) \in Z$, we have $D(0) \in Z$. Since $D(0x) - D(0)x - 0D(x) \in Z$, we now get $D(0)x \in Z$ for all $x \in R$. But Z contains no nonzero divisors of zero, hence $x \in Z$ for all $x \in R$, i.e., R is commutative. □

Theorems 2.6 and 2.8 enable us to replace derivations and epimorphisms by CE-derivations and CE-epimorphisms in certain commutativity theorems. We give two examples.

Theorem 4.2 *Let R be a prime ring and U a nonzero ideal of R. If R admits a non-identity CE-epimorphism T which is strong-commutativity preserving on U, then R is commutative.*

Proof If T is an epimorphism, R is commutative by Bell and Daif (1994), Corollary 2. If T is not an epimorphism, by Theorem 2.8 R contains a nonzero central ideal—a condition well known to imply commutativity in a prime ring. □

Theorem 4.3 *Let R be a semiprime ring and U a nonzero left ideal of R. If R admits a CE-derivation which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.*

Proof By Theorem 2.6, R has a nonzero central ideal or D is a derivation; and if D is a derivation, our theorem reduces to Bell and Martindale (1987), Theorem 3. □

In general, commutativity theorems with hypotheses involving CE-derivations or CE-epimorphisms seem harder to prove than those with hypotheses involving derivations or epimorphisms. However, there are some possibilities. We conclude with an example, which is a partial generalization of the result that a semiprime ring R must be commutative if it admits a derivation d such that $[x, y] = [d(y), d(x)]$ for all $x, y \in R$. (See Ali and Huang 2012, Theorem 3.3; Liu 2013, Corollary 1.3.)

Theorem 4.4 *Let R be a semiprime ring and D a CE-derivation on R such that $[x, y] = [D(y), D(x)]$ for all $x, y \in R$. If R is reduced or D is centralizing on R, then R is commutative.*

Proof We are assuming

$$[x, y] = [D(y), D(x)] \quad \text{for all } x, y \in R. \tag{4.1}$$

Replacing x by xy in (4.1) and using (4.1), we obtain

$$D(x)[y, D(y)] + [x, D(y)]D(y) = 0 \quad \text{for all } x, y \in R; \tag{4.2}$$

and replacing x by yx in (4.1), we get

$$D(y)[D(y), x] + [D(y), y]D(x) = 0 \quad \text{for all } x, y \in R. \tag{4.3}$$

Taking $x = D(y)$ in (4.2) and (4.3), we have for all $y \in R$

$$D^2(y)[y, D(y)] = [y, D(y)]D^2(y) = 0 = [D^2(y), [y, D(y)]]. \tag{4.4}$$

We now replace x by xw in (4.2), thereby obtaining $z_1 \in Z$ such that $(D(x)w + xD(w) + z_1)[y, D(y)] + [xw, D(y)]D(y) = 0$, i.e., $D(x)w[y, D(y)] + xD(w)$

$[y, D(y)] + z_1[y, D(y)] + x[w, D(y)]D(y) + [x, D(y)]wD(y) = 0$; and applying (4.2), we get

$$D(x)w[y, D(y)] + z_1[y, D(y)] + [x, D(y)]wD(y) = 0. \tag{4.5}$$

Taking $x = D(y)$, we get $z_2 \in Z$ such that

$$D^2(y)w[y, D(y)] + z_2[y, D(y)] = 0. \tag{4.6}$$

It follows that

$$[D^2(y)w[y, D(y)], [y, D(y)]] = 0 \quad \text{for all } y, w \in R,$$

which reduces for all $y, w \in R$ to

$$[D^2(y)w, [y, D(y)]][y, D(y)] = 0, \text{ or}$$
$$D^2(y)[w, [y, D(y)]][y, D(y)] + [D^2(y), [y, D(y)]]w[y, D(y)] = 0.$$

Using (4.4), we now get

$$D^2(y)w[y, D(y)]^2 = 0 \quad \text{for all } y, w \in R. \tag{4.7}$$

From this equation we obtain

$$[D^2(y), D(y)]w[y, D(y)]^2 = 0 \quad \text{for all } y, w \in R,$$

which by (4.1) is

$$[y, D(y)]w[y, D(y)]^2 = 0 \quad \text{for all } y, w \in R;$$

and invoking semiprimeness of R, we conclude that

$$[y, D(y)]^2 = 0 \quad \text{for all } y \in R. \tag{4.8}$$

If R is reduced, it is obvious that $[y, D(y)] = 0$; and if D is centralizing on R, $[y, D(y)] = 0$ because $[y, D(y)] \in Z \cap N$. Thus,

$$[y, D(y)] = 0 \quad \text{for all } y \in R. \tag{4.9}$$

It follows from (4.5) and (4.9) that $[x, D(y)]w[x, D(y)] = 0$ for all $x, y, w \in R$, hence $D(R) \subseteq Z$ and therefore R is commutative by (4.1). □

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